Triangular Bézier Surfaces of Minimal Area*

A. Arnal[1], A. Lluch[1], and J. Monterde[2]

[1] Dep. de Matemàtiques,Universitat Jaume I
Castelló, Spain
parnal@mat.uji.es,lluch@mat.uji.es
[2] Dep. de Geometria i Topologia, Universitat de València,
Burjassot (València), Spain
monterde@uv.es

Abstract. We study some methods of obtaining approximations to surfaces of minimal area with prescribed border using triangular Bézier patches. Some methods deduced from a variational principle are proposed and compared with some masks.

1 Introduction

In this note we address the problem of finding the triangular Bézier patch minimizing the area, and saving then material costs, of the corresponding triangular Bézier surface with prescribed border. As it is well known, the border of a triangular Bézier surface is determined by the border control points. So, the problem can be reformulated as follows: Given the exterior points of a triangular control net, find out the inner ones in such a way that the resulting triangular Bézier surface had minimal area among all the triangular Bézier surfaces with the same border. Let us call this problem as the *triangular Bézier-Plateau problem.*

The theory of minimal surfaces shows that in order to prove the existence of minimal surfaces, one can replace the area functional, a highly non linear functional, by another one having the same extremals. The common substitute is the Dirichlet functional, so called in the mathematical literature, also called the stretch functional in the CAGD literature (see [6]).

The advantage of the use of the Dirichlet functional is that the determination of extremals becomes just a linear problem. Moreover, if a triangular Bézier chart is harmonic and isothermal it is extremal both of the area and Dirichlet functionals. At this point, we can compute the extremal for cubical triangular Bézier surfaces and to use the solution as a mask for obtaining approximations in higher degrees, or we can try to obtain the extremals of the Dirichlet functional in higher degrees directly.

The use of masks is due to the fact that the system of linear equations to solve is easier with their use, it is a sparse system, than the system deduced from the Dirichlet equations, which has no null coefficients.

* This work has been partially supported by Spanish Ministry of Sci. and Tech. BFM2002-00770 and Bancaixa-Universitat Jaume I, P1-1A2002-110.

V. Kumar et al. (Eds.): ICCSA 2003, LNCS 2669, pp. 366–375, 2003.

There is a difference with the same problem for rectangular Bézier surfaces (which has been treated in [5]). For usual Bézier surfaces, there is a very clear association between rectangular control nets and charts of the Bézier surface. In the triangular case, things are not so easy. The most direct association between triangular control nets and charts of triangular Bézier surfaces is the one obtained after the substitution of one of the barycentric coordinates by an expression depending of the other two having in mind the relation $u+v+w = 1$. For example, if $\vec{y}(u, v, w)$ is the chart in barycentric coordinates, then we shall work with the chart $\vec{x}(u, v) = \vec{y}(u, v, 1 - u - v)$. Moreover, if we suppose now that (u, v) are cartesian coordinates, or equivalently, that the triangle used to define the barycentric coordinates is the non equilateral one whose vertices are $(0, 0), (1, 0)$ and $(0, 1)$, then this association has a serious drawback: the breakdown of the symmetry. Even if the control net is symmetric ($P_{\sigma(i)\sigma(j)\sigma(k)} = P_{ijk}$ for any permutation σ), the Bézier surface does not preserve the symmetry.

Nevertheless we have followed this non symmetric approach due to the following facts:

1. Our approximations, although the use of non symmetric methods, seem to be at least as good as than other results deduced from symmetric methods.
2. We have checked that when the control net verifies some condition at the three corner points (isothermality) then our method shows, in general, a significative improvement.
3. For degree 3, there is a well known polynomial minimal surface, the Enneper's surface. It is possible to show that fixing the border control points as the ones of an arbitrary triangular piece of the Enneper surface, our asymmetric mask gives always exactly the inner control point. Moreover this is no longer true for any symmetric mask, and even it can be shown that there are pieces of the Enneper surface for which the inner control point cannot be obtained by applying a symmetric mask to the exterior control points.
4. Symmetric masks are deduced after some arguments on the control net, but not on the Bézier surface. Our methods are based directly on the Bézier surface because what we want is to minimize some functional directly related with the surface.

When isothermality at the three corner points is not satisfied, then the Dirichlet extremal considered as an approximation to the area extremal, presents an intrinsic error due to the method. In the last section we propose an improvement of the approximation based on geometric principles which maintain the use of linear systems. Now, the surfaces we obtain are of lesser area than the surfaces obtained by the other methods for the same border configurations.

2 Notation, Definitions, and Preliminary Results

2.1 Triangular Bézier Surfaces

Consider a triangle with vertices A, B, C and a fourth point P, all in $\mathbb{R}^3$. Then, it is always possible to write P as a barycentric combination of A, B, C

$$P = uA + vB + wC \quad \text{requiring that} \quad u + v + w = 1.$$

The coefficients $\mathbf{u} = (u, v, w)$ are called *barycentric coordinates* of P respect to A, B, C. To build a triangular Bézier surface of degree n we have to repeat the barycentric interpolation analogously as we repeat the bilinear interpolation in the Casteljau Algorithm for constructing a Bézier surface. The control net for a triangular Bézier surface of degree n consists of $\frac{(n+1)(n+2)}{2}$ points arranged in a triangular grid. If $I = (i, j, k)$ each point of the triangular control net will be denoted by P_I. We will use also the following notation $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ and $|I| = i + j + k$.

We can express a triangular surface in terms of trivariate Bernstein polynomials: if $|I| \neq n$, then $B_I^n(\mathbf{u}) = 0$, else, i.e, if $|I| = n$ then

$$B_I^n(\mathbf{u}) = \binom{n}{I} u^i\, v^j\, w^k = \frac{n!}{i!j!k!}\, u^i\, v^j\, w^k.$$

These Bernstein polynomials, although they look trivariate, they are not, since $u+v+w=1$. We will denote by $\mathcal{R}$ the region $\mathcal{R} = \{\mathbf{u} = (u,v,w)/u+v+w = 1 \text{ and } u,v,w \geq 0\}$ and by $\mathcal{T}$ the region $\mathcal{T} = \{(u,v) \in \mathbb{R}^2 : 0 \leq u, 0 \leq v, u+v \leq 1\}$.

Definition 1. *Given a triangular control net in* $\mathbb{R}^3$, $\mathcal{P} = \{P_I\}_{|I|=n}$, *the triangular Bézier surface of degree n associated to* $\mathcal{P}$, $\overrightarrow{x} : \mathcal{T} \longrightarrow \mathbb{R}^3$ *is given by:*

$$\overrightarrow{x}(\mathbf{u}) = \sum_{|I|=n} P_I B_I^n(\mathbf{u}).$$

A surface S is minimal if its mean curvature vanishes. Equivalently, S is a minimal surface iff for each point $p \in S$ one can chose a neighborhood, U_p, which has minimal area among other patches V having the same boundary as U_p.

A chart $\overrightarrow{x} : U \longrightarrow S$ of a surface is said to be isothermal if $E = G, F = 0$, being E, F, G the coefficients of the first fundamental form associated to the chart $\overrightarrow{x}$.

Proposition 1. *If a chart,* $\overrightarrow{x} : U \to S$, *of a surface,* S, *is isothermal, then* $\overrightarrow{x}(U)$ *is minimal if and only if the chart is harmonic, i.e.,* $\Delta\overrightarrow{x} = 0$.

3 The Dirichlet Functional Results

To solve the triangular Bézier-Plateau problem we have to try to minimize the functional area among all the triangular Bézier Surfaces with prescribed border determined by the exterior control points. Nevertheless, due to its high non linearity, the problem of minimizing the area functional is hard to manage with, so we shall work instead with the Dirichlet functional:

$$D(\mathcal{P}) = \frac{1}{2}\int_{\mathcal{T}} (\|\overrightarrow{x_u}\|^2 + \|\overrightarrow{x_v}\|^2)du\, dv.$$

There are two reasons for doing such a substitution: the first one is given by the following fact relating the area and the Dirichlet functional:

$$(EG - F^2)^{\frac{1}{2}} \leq (EG)^{\frac{1}{2}} \leq (E + G)/2. \qquad (1)$$

Therefore, for any triangular control net $\mathcal{P}$, $A(\mathcal{P}) \leq D(\mathcal{P})$. Moreover, equality can occur only if $E = G$ and $F = 0$, i.e.: for isothermal charts.

The second one is related with the Euler-Lagrange equation associated to the Dirichlet functional defined not on control nets, but on charts

$$\overrightarrow{x} \to \frac{1}{2}\int_{\mathcal{T}} (\|\overrightarrow{x_u}\|^2 + \|\overrightarrow{x_v}\|^2) du\, dv.$$

This equation is just $\Delta\overrightarrow{x} = 0$. Therefore, if the extremal of the Dirichlet functional is an isothermal chart, it is automatically a harmonic chart, and then the surface is minimal. Nevertheless, we are not working with charts. We are working instead with triangular control nets. So, our aim is to find the minimum of the real function $\mathcal{P} \to \mathcal{D}(\overrightarrow{x}_{\mathcal{P}})$, being $\overrightarrow{x}_{\mathcal{P}}$ the triangular Bézier chart associated to the control net $\mathcal{P}$.

Proposition 2. *A triangular control net, $\mathcal{P} = \{P_I\}_{|I|=n}$, is an extremal of the Dirichlet functional with prescribed border if and only if:*

$$0 = \sum_{|I|=n} \frac{\binom{n-1}{I}}{\binom{2n-2}{I+I_0}} (a_1 + a_2 + 2a_3 - b_{13} - b_{23})\ P_I$$

for all $|I_0 = (i_0, j_0, k_0)| = n$ with $i_0, j_0, k_0 > 0$ and where:

$$a_1 = \begin{cases} 0 & i = 0, \\ \frac{i_0 i}{(i+i_0)(i+i_0-1)} & i > 0 \end{cases} \quad a_2 = \begin{cases} 0 & j = 0, \\ \frac{j_0 j}{(j+j_0)(j+j_0-1)} & j > 0 \end{cases} \quad a_3 = \begin{cases} 0 & k = 0, \\ \frac{k_0 k}{(k+k_0)(k+k_0-1)} & k > 0 \end{cases}$$

$$b_{12} = \frac{i_0 j + j_0 i}{(i+i_0)(j+j_0)} \qquad b_{13} = \frac{i_0 k + k_0 i}{(i+i_0)(k+k_0)} \qquad b_{23} = \frac{j_0 k + k_0 j}{(j+j_0)(k+k_0)}$$

In particular we give the general result for the case $n = 3$.

Proposition 3. *A triangular Bézier surface of degree 3 is an extremal of the Dirichlet functional with prescribed border if and only if*

$$P_{111} = \frac{1}{4}(2\,P_{003} - P_{021} + P_{030} + P_{120} - P_{201} + P_{210} + P_{300}).$$

4 Triangular Permanence Patches Related with the Bézier-Plateau Problem

Farin and Hansford define in [2] the triangular permanence patches as those triangular patches that satisfy the *permanence principle*: given a triangle $\mathcal{T}$ in the domain U of a triangular Coons patch, the three boundaries of this subpatch will map to three curves on the Coons patch, then the triangular Coons patch of those three boundaries is the original triangular Coons patch. This permanence principle can be established by using a mask like:

$$\begin{array}{ccccccc} \alpha & & \beta & & \beta & & \alpha \\ & \beta & & \star & & \beta & \\ & & \beta & & \beta & & \\ & & & \alpha & & & \end{array}$$

with $3\alpha + 6\beta = 1$ (i.e.: $\alpha = \frac{1-6\beta}{3}$). Let us denote this mask by M_α.

It can be found in [2] that the mask M_0 is the discrete form of the Laplacian operator when the control net is considered as a discretization of the Bézier surface. Such a mask is used in the cited reference to obtain control nets resembling minimal surfaces that fit between given boundary polygons.

Another main masks are: $M_{\frac{1}{9}}$ that can be deduced by asking the quadrilaterals associated to the interior edges of the triangular patch to be as close as possible to parallelograms and mask $M_{\frac{1}{3}}$ that is the dual of M_0 in the sense that for $\alpha = \frac{1}{3}$ we have $\beta = 0$.

From the condition obtained in Proposition 3 we can try to generate, given the exterior control points the whole triangular net by solving a linear system where the equations are:

$$\begin{aligned} 4P_{i,j,k} = {} & 2\,P_{i+2,j-1,k-1} - P_{i,j-1,k+1} - P_{i,j+1,k-1} + P_{i-1,j+2,k-1} \\ & + P_{i-1,j+1,k} + P_{i-1,j,k+1} + P_{i-1,j-1,k+2}, \end{aligned}$$

being $P_{i,j,k}$ a inner control point. This equation can be expressed as the following mask, which will be called *the Dirichlet mask*:

$$\frac{1}{4} \times \begin{array}{ccccccc} 1 & & 1 & & 1 & & 1 \\ & -1 & & \star & & -1 & \\ & & 0 & & 0 & & \\ & & & 2 & & & \end{array}$$

As we can see, the Dirichlet mask is not a mask like Farin-Hansford's mask because it is not symmetric. The asymmetry of the Dirichlet mask is due to the fact that the triangle on which we define the Bernstein polynomials is not an equilateral triangle, (see [7]). Applying a symmetrization process to the Dirichlet mask we obtain one of the masks worked in [2], that with $\alpha = \frac{1}{3}$.

5 Comparison between Masks

As it is said in [3], the natural question: *there is or not a better mask*, has a negative answer. It depends on the boundary conditions. In this section we will show some examples with simple boundary curves.

Case $n = 3$. Let us start the comparison by studying some examples in the cubic case. We fix the three boundary curves with its control points and we construct the triangular Bézier surface computing the inner control point using the masks M_α and the Dirichlet mask. We have chosen some examples with their border control points: along the border of a piece of the Enneper's surface in the first example; along two straight lines and a circle of radius 1 in NI1; along three circles of radius 1 in NI2, border Is is built in such a way that at the corner points any associated chart would be isothermal and finally border HNI is such that the isothermality conditions at its corners are far of being fulfilled. The following figures show the borders and the triangular Bézier surfaces constructed by means of the Dirichlet mask.

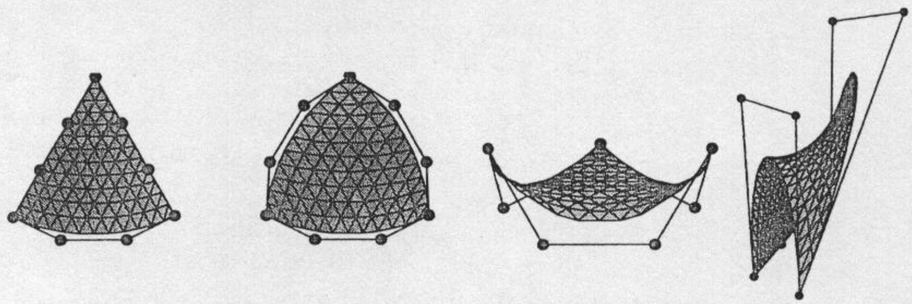

Fig.I: Surfaces NI1, NI2, Is and HNI by its Dirichlet Extremal.

In table I the areas of the corresponding triangular Bézier Surfaces are shown:

Method	Enneper	NI1	NI2	Is	HNI
M_0	4.67858	0.99685	1.21350	2.99046	13.22692
$M_{\frac{1}{9}}$	4.67835	0.99631	1.20844	2.88558	12.66618
$M_{\frac{1}{3}}$	4.67899	0.99563	1.20275	2.76656	11.67948
Dirichlet mask	4.67778	0.99793	1.20277	2.76957	12.22934
Dirichlet Correction	4.67778	0.99546	1.20216	2.75167	11.36520

Table I: Comparison between different masks for degree 3.

As we will introduce later the method that we have called Dirichlet correction, let us first analyze the results for the M_α masks and the Dirichlet mask. We can find that for these cubical examples the lesser areas are obtained by $M_{\frac{1}{3}}$ mask with the exception of the Enneper case that we will study in a later section.

Case $n = 10$. Let us see in this section how things change with more degrees of freedom. The following examples for the case $n = 10$ are similar to the cubical examples NI2, Is and HNI. In NI2 we have chosen equally spaced border control points along the circles described before. At the other cases the choice of the borders has been done with the same configuration than before but with a slight modification in order to assure the isothermality at the corners in Is and the non isothermality in HNI. Table II shows the areas of the corresponding triangular Bézier surfaces by using the masks M_α, the Dirichlet mask and the area of the Dirichlet extremal, that is, the area of the triangular Bézier surface which inner control points are obtained by applying the Dirichlet equations in Prop. 2.

Method	NI2	Is	HNI
M_0 mask	1.34247	3.54592	12.61296
$M_{\frac{1}{9}}$ mask	1.34009	3.49978	12.53044
$M_{\frac{1}{3}}$ mask	1.33864	3.47307	12.47569
Dirichlet mask	1.33961	3.43799	12.68629
Dirichlet extremal	1.33963	3.43659	12.68513
Dirichlet Correction	1.33623	3.41091	12.42494
Second step	1.33625	3.37410	12.25581

Table II: Different masks and the Dirichlet extremal areas for $n = 10$.

In the NI2 and the HNI cases the best area is the one obtained using the $M_{\frac{1}{3}}$ mask, but now when we have the isothermality at the corners, case Is, the Dirichlet extremal is the one that give us the lesser area, even the use of the Dirichlet mask represents a significative improvement. An explanation of why the Dirichlet extremal has less area in Is will be given in section 7.

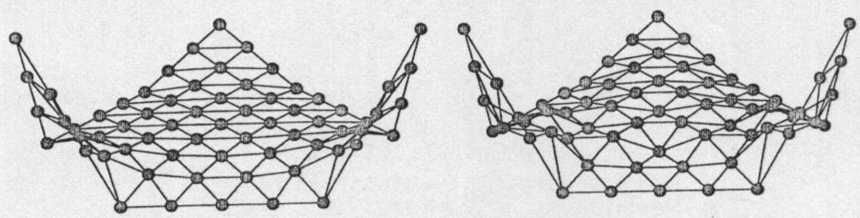

Figure II: Left M_0 and right Dirichlet mask control nets for Is surface.

Note the non regular shape of the control net of the right hand figure, the one obtained using the Dirichlet mask for degree 10, in comparison with the left hand figure, the one obtained by using the mask M_0, that is the mask for the discrete form of the Laplacian operator on the control net. The control net is not regular, but the associated Bézier surface is a better approximation to the minimal surface. Recall that we are looking for triangular Bézier surface minimizing some functional related with the surface, and not for triangular control nets minimizing some functional related with the net. The same fact also happens for rectangular Bézier surfaces.

6 The Enneper's Surface as a Testing Model

The first non trivial example of minimal surface with polynomial coordinate functions is the Enneper's surface (see [3] or [1] for some plots of this surface), $\overrightarrow{x} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by

$$\overrightarrow{x}(u,v) := (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2). \qquad (2)$$

Therefore, it can be used to test the masks we have used. Moreover, as the chart (2) is isothermal, then it is an extremal no just of the area functional, but also of the Dirichlet functional. This means that, if we take a triangular piece of the Enneper's surface, we determine its control net, and we look for the extremal of the Dirichlet functional with that border control points, then the inner control point P_{111} is always given by the formula given in Proposition 3.

Nevertheless, there are cases where all symmetric masks fail to reobtain the inner control points. For example, let us consider the chart

$$\overrightarrow{y}(u,v) = \overrightarrow{x}(u+1,v), \qquad (u,v) \in T.$$

It is again an isothermal and harmonic chart, so it is a chart of a minimal triangular surface and the area of this triangular Bézier surface is 4.67778. The control net is

$$\begin{array}{ccccccc}
(\frac{2}{3},0,1) & & (\frac{2}{3},\frac{2}{3},1) & & (1,\frac{4}{3},\frac{2}{3}) & & (\frac{5}{3},\frac{5}{3},0) \\
& (\frac{2}{3},0,\frac{5}{3}) & & (\frac{2}{3},1,\frac{5}{3}) & & (\frac{4}{3},2,\frac{4}{3}) & \\
& & (\frac{1}{3},0,\frac{8}{3}) & & (\frac{1}{3},\frac{5}{3},\frac{8}{3}) & & \\
& & & (-\frac{2}{3},0,4) & & &
\end{array}$$

It is easy to check that this control net verifies the formula in Proposition 3. So, the inner control point P_{111} can be reobtained using the corresponding asymmetric mask.

Nevertheless, for a symmetric mask, the computation of the inner control points gives

$$P_{111}^{\alpha} = (\frac{13-9\alpha}{18}, \frac{17-21\alpha}{18}, \frac{5}{3}),$$

and there is no value for α such that $P_{111} = (\frac{2}{3}, 1, \frac{5}{3})$.

Moreover, the minimum of the area of the associated triangular Bézier surface with inner control point P_{111}^{α} is attained at $\alpha = 0.12833$ and its value is 4.67834. A possible explanation of the fact that in this case a non symmetric method gives better results that any symmetric method is the following: Let us recall first that the definition of triangular Bézier patches heavily depends on the triangle defining the barycentric coordinates. In this case, the coordinates u, v in the chart of the Enneper's surface are rectangular coordinates. After polarization, coordinates u, v, w are the barycentric coordinates with respect to triangle $\mathcal{T}$ which is not an equilateral triangle, and this fact breaks down the symmetry between the barycentric coordinates.

7 Correction of the Dirichlet Extremal

The obtainment of an approximation of the minimal Bézier surface according to the previous method has a serious drawback: the first fundamental form at the corners of any triangular Bézier surface with prescribed border is determined by the border control points. For example, at the point $\overrightarrow{x}(0,0)$ the three coefficients of the first fundamental form are determined by the control points $P_{0,0,n}, P_{0,1,n-1}$ and $P_{1,0,n-1}$. Therefore, since the three points are border control points, the coefficients E, F and G at $\overrightarrow{x}(0,0)$ of any triangular Bézier chart with the same border will be always the same, even for the Dirichlet extremal, no matter which are the inner control points.

Let us recall that the Dirichlet method is based in the fact that the substitution of the area functional by the Dirichlet one will cause a negligible error. Moreover, both functional agree for isothermal charts. But if the configuration of the border control points is such that at the corner points the chart is always non isothermal, then the inequalities in (1) are strict. The non isothermality at corners points will produce an error when substituting the area functional by the Dirichlet one. At different points from the corner points, the configuration of the Dirichlet extremal tends to the isothermality of the chart. But at the corners points, isothermality or not is fixed from the border control points and it can not be modified. This is why the Dirichlet extremal does not improve the results of the harmonic mask in some cases.

We will propose along this section a method that will obtain, from the Dirichlet extremal as a first approximation to the minimal Bézier surface, a new and better approximation trying to avoid this problem and maintaining the fact that the new approximation is computed thanks to a system of linear equations.

Let us recall the following fact about the Dirichlet functional. As we have mentioned before, the Euler-Lagrange equation of the Dirichlet functional defined on the set of all differentiable charts with prescribed border is

$$\Delta \overrightarrow{x} = 0,$$

where Δ is the usual Laplacian operator. This equation is related to minimal surfaces thanks to Proposition 1. But there is another main result that does not mention the isothermality condition.

Proposition 4. *A chart $\overrightarrow{x}$ is minimal iff $\Delta^g \overrightarrow{x} = 0$ where g represents the first fundamental form of $\overrightarrow{x}$ and Δ^g is the associated Laplacian operator: for a function f:*

$$\Delta^g f = \left(\frac{f_u G - f_v F}{\sqrt{EG-F^2}}\right)_u + \left(\frac{-f_u F + f_v G}{\sqrt{EG-F^2}}\right)_v .$$

It is easy to check that, for a given metric, g, with coefficients E, F and G, the equation $\Delta^g \overrightarrow{x} = 0$ is the Euler-Lagrange equation of the functional

$$\begin{aligned} \mathcal{D}^g(\overrightarrow{x}) &= \int_T \left(\tfrac{\|\overrightarrow{x}_u\|^2 G - 2<\overrightarrow{x}_u, \overrightarrow{x}_v> F + \|\overrightarrow{x}_v\|^2 E}{\sqrt{EG-F^2}}\right) dudv \\ &= \int_T g^{-1}(d\overrightarrow{x}, d\overrightarrow{x}) \mu_g, \end{aligned}$$

where $\mu_g = \sqrt{EG-F^2}dudv$ is the metric volume element.

Note that for a given g, the extremal of the functional $\mathcal{D}^g$ is a chart that can be computed thanks to a linear system. Therefore, the correction of the Dirichlet method is the following: let $\vec{x}_0$ be the Dirichlet extremal and let g_0 be its first fundamental form. The new approximation is the extremal of the functional $\mathcal{D}^{g_0}$, this is, using the Dirichlet extremal as the fixed metric. Note that the functional $\vec{x} \to \mathcal{D}^{g_0}(\vec{x})$ is quadratic in $\vec{x}$. Therefore the extremal equations are linear.

Proposition 5. *A triangular control net, $\mathcal{P} = \{P_I\}_{|I|=n}$, is an extremal of the functional $\mathcal{D}^{g_0}$ with prescribed border if and only if:*

$$0 = \sum_{|I|=n} \frac{\binom{n-1}{I}}{\binom{2n-2}{I+I_0}} \Big(\int_T \frac{G_0}{\mu_{g_0}} a_1\, B^{2n-2}_{I_0+I-2e_1} + \int_T \frac{E_0}{\mu_{g_0}} a_2\, B^{2n-2}_{I_0+I-2e_2}$$
$$+ \int_T \frac{G_0 - 2F_0 + E_0}{\mu_{g_0}} a_3\, B^{2n-2}_{I_0+I-2e_3} - \int_T \frac{G_0 - F_0}{\mu_{g_0}} b_{13}\, B^{2n-2}_{I_0+I-e_1-e_3}$$
$$- \int_T \frac{F_0}{\mu_{g_0}} b_{12}\, B^{2n-2}_{I_0+I-e_1-e_2} - \int_T \frac{E_0 - F_0}{\mu_{g_0}} b_{23}\, B^{2n-2}_{I_0+I-e_2-e_3} \Big)\, P_I$$

for all $|I_0 = (i_0, j_0, k_0)| = n$ with $i_0, j_0, k_0 > 0$ and where a_s and b_{rt} were defined in Proposition 2.

The formulas obtained in Prop. 5 give us a system of linear equations for the interior points of the triangular net given its border. Now if we have a look at the last row in Table I and Table II, we can see that this method improves the results obtained through all the other methods, and moreover we get this improvement for all the examples, even when we deal with non isothermal charts.

Finally we have gone one step forward, if $\vec{x}_1$ is the Dirichlet extremal of the functional $\mathcal{D}^{g_0}$ and g_1 is its first fundamental form the new approximation is the extremal of the functional $\mathcal{D}^{g_1}$. The results obtained from this last method are shown in the last row at Table II, and from them we conjecture that, specially for highly non isothermal charts, the improvement given by the correction method can be even enhanced by repeating the process.

References

[1] C. Cosín, J. Monterde, *Bézier surfaces of minimal area*, Proceedings of the Int. Conf. of Computational Science, ICCS'2002, Amsterdam, eds. Sloot, Kenneth Tan, Dongarra, Hoekstra, LNCS 2330, vol II, pages 72–81, Springer-Verlag, (2002).

[2] G. Farin, D. Hansford, *Discrete Coons patches*, CAGD 16, 691–700 (1999).

[3] J. Gallier, *Curves and surfaces in Geometric Modeling*, Morgan Kaufmann publishers, S.Francisco, California, (2000).

[4] J.Hoscheck and D.Lasser, *Fundamentals of Computer Aided Geometric Design*, A.K.Peters, Wellesley,(1993).

[5] J. Monterde, *Bézier surfaces of minimal area: The Dirichlet approach*, preprint.

[6] R. Veltkamp, W. Wesselink, *Variational Modeling of triangular Bézier Surfaces*, pr.