# Creatable universes

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(Received 29 January 2007; published 5 June 2007)

We consider the question of properly defining energy and momenta for nonasymptotic Minkowskian spaces in general relativity. Only spaces of this type, whose energy, linear 3-momentum, and intrinsic angular momentum vanish, would be candidates for creatable universes, that is, for universes which could have arisen from a vacuum quantum fluctuation. Given a universe, we completely characterize the family of coordinate systems for which one could sensibly say that this universe is a creatable universe.

## I. INTRODUCTION: GENERAL CONSIDERATIONS

Which is the most general universe with null energy, null linear 3-momentum, and null intrinsic 3-angular momentum, and why could such a question be of interest?

From the early seventies, people have speculated about a Universe which could have arisen from a quantum vacuum fluctuation [1,2]. If this were the case, one could expect this Universe to have zero energy.

But, then, why should we consider only the energy? Why not expect that the linear 3-momentum and angular intrinsic 3-momentum, of a Universe arising from a vacuum fluctuation, to be zero too? And finally: why not to expect both, linear 4-momentum and angular intrinsic 4-momentum, to be zero?

So, in the present paper, we will consider both: linear 4-momentum, $P^\alpha = (P^0, P^i)$, and angular 4-momentum, $J^{\alpha\beta} = (J^{0i}, J^{ij})$. In all: it could be expected that only those universes with $P^\alpha = 0$, and $J^{\alpha\beta} = 0$, could have arisen from a quantum vacuum fluctuation. Then, we could say that only these ones would be ''creatable universes''.

Now, as it is well known (see, for example, [3] or [4]), when dealing with an asymptotically flat space-time, one can define in a unique way its linear 4-momentum, provided that one uses any coordinate system which goes fast enough to a Minkowskian coordinate system in the 3-space infinity.

Nevertheless, if, to deal with the Universe as such, we consider nonasymptotically flat space-times, in such space-times these Minkowskian coordinate systems do not exist. Then, we will not know in advance which coordinate systems, if any, should be used, in order to properly define the linear and angular 4-momentum of the Universe. This is, of course a major problem, since, as we will see, and it is well known, $P^\alpha$ and $J^{\alpha\beta}$ are strongly coordinate dependent, and it is so whatever it be the energy-momentum complex we use (the one of Weinberg [3], or Landau [5], or any other one).

As we have just said, this strong coordinate dependence of $P^\alpha$ and $J^{\alpha\beta}$ is very well known, but, in spite of this, in practice, it is not always properly commented or even taken properly into account. This can be seen by having a look at the different calculations of the energy of some universes, which have appeared in the literature (see, for example, among other references [6,7]) since the pioneering papers by Rosen [8] and Cooperstock [9].

Even Minkowski space can have non-null energy if we take non-Minkowskian coordinate systems. This non-null energy would reflect the energy of the *fictitious gravitational field* induced by such non-Minkowskian coordinates, or in other words the energy tied to the family of the corresponding accelerated observers. So, in particular, to define the proper energy and momentum of a universe, we would have to use coordinate systems adapted, in some sense, to the symmetries of this universe, in order to get rid of this spurious energy supply. We will address this question in some detail in the present paper, the summary of which follows.

First, in Secs. II and III, we look for the family of good coordinate systems in order to properly define the energy and momenta of the considered universe. Then, given an arbitrary spacelike 3-surface, we uniquely determine the family of coordinate systems, which are, in principle, good coordinate systems corresponding to this spacelike 3-surface. In Sec. IV, under reasonable assumptions, we show that if a given universe has zero energy and momenta for one coordinate system of the family, then, it has zero energy and momenta for all coordinate systems of the family. Furthermore, in Sec. V, under reasonable assumptions, we show that this ''creatable'' character of a given universe is independent of the above chosen spacelike 3-surface. In Secs. VI and VII we consider some simple examples in which we calculate the universe energy and momenta: the Friedmann-Robertson-Walker (FRW) universes, on one hand, and a nontilted Bianchi V universe, on the other hand. Finally, in Sec. VIII, we summarize the main results and conclude with some comments on open perspectives.

Some, but not all, of these results have been presented with hardly any calculation in the meeting ERE-2006 [10].

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## II. WHICH COORDINATE SYSTEMS?

We expect any well behaved universe to have well defined energy and momenta, i. e., $P^\alpha$ and $J^{\alpha\beta}$ would be finite and conserved in time. So, in order for this conservation to make physical sense, we need to use a *physical* and *universal* time. Then, as we have done in [10], we will use Gauss coordinates:

$$ds^2 = -dt^2 + dl^2, \qquad dl^2 = g_{ij}dx^i dx^j,$$
$$i, j = 1, 2, 3. \qquad (1)$$

In this way, the time coordinate is the proper time and so a *physical* time. Moreover, it is an everywhere synchronized time (see for example [5]) and so a *universal* time.

Obviously, we have as many Gauss coordinate systems in the considered universe (or in part of it) as we have spacelike 3-surfaces, $\Sigma_3$. Then, $P^\alpha$ and $J^{\alpha\beta}$ will depend on $\Sigma_3$ (as the energy of a physical system in the Minkowski space-time does, which depends on the chosen $\Sigma_3$, i.e., on the chosen Minkowskian coordinates).

Now, in order to continue our preliminary inquiry, we must choose one energy-momentum complex. Since besides linear momentum we will also consider angular momentum, we will need a symmetric energy-momentum complex. Then, we will take the Weinberg one [3]. This complex has the property that it allows us to write energy and momenta as some integrals over the boundary 2-surface, $\Sigma_2$, of $\Sigma_3$. Then, any other symmetric complex with this property, like for example the one from Landau [5], will enable us to obtain essentially the same results as the ones we will obtain in the present paper.

Then, taking the above Weinberg complex, one obtains, in Gauss coordinates, for the linear 4-momentum, $P^\alpha = (P^0, P^i)$, and the angular one, $J^{\alpha\beta} = (J^{0i}, J^{ij})$, the following expressions [3]:

$$P^0 = \frac{1}{16\pi G} \int (\partial_j g_{ij} - \partial_i g) d\Sigma_{2i}, \qquad (2)$$

$$P^i = \frac{1}{16\pi G} \int (\dot{g}\delta_{ij} - \dot{g}_{ij}) d\Sigma_{2j}, \qquad (3)$$

$$J^{jk} = \frac{1}{16\pi G} \int (x_k \dot{g}_{ij} - x_j \dot{g}_{ki}) d\Sigma_{2i}, \qquad (4)$$

$$J^{0i} = P^i t - \frac{1}{16\pi G} \int [(\partial_k g_{kj} - \partial_j g)x_i + g\delta_{ij} - g_{ij}] d\Sigma_{2j}, \qquad (5)$$

where we have used the following notation, $g \equiv \delta^{ij}g_{ij}$, $\dot{g}_{ij} \equiv \partial_t g_{ij}$, and where $d\Sigma_{2i}$ is the surface element of $\Sigma_2$. Further, notice, that without losing generality, the angular momentum has been taken with respect to the origin of coordinates.

There is an apparent inconsistency in Eqs. (2)–(5), since we have upper indices in the left hand and lower ones in the right side. This comes from the fact that, when deducing these equations (see Ref. [3]), starting with the Einstein equations in its covariant form, $G_{\alpha\beta} = \chi T_{\alpha\beta}$, indices are raised with the contravariant Minkowski tensor, $\eta^{\alpha\beta}$. Then, in the right side, one can use indistinctly upper or lower space indices.

The area of $\Sigma_2$ could be zero, finite or infinite. In the examples considered next, in Secs. VI and VII, we will deal with the last two possibilities. In the first case, when the area is zero, the energy and momenta would be trivially zero (provided that the metric remains conveniently bounded when we approach $\Sigma_2$).

## III. MORE ABOUT THE GOOD COORDINATE SYSTEMS

From what has been said in the above section, one could erroneously conclude that, in order to calculate the energy and momenta of a universe, one needs to write the metric in all $\Sigma_3$, in Gauss coordinates. Nevertheless, since, according to Eqs. (2)–(5), $P^\alpha$ and $J^{\alpha\beta}$ can be written as surface integrals on $\Sigma_2$, all we need is this metric, in Gauss coordinates, on $\Sigma_2$ and its immediate neighborhood (in this neighborhood too, since the space derivatives on $\Sigma_2$ of the metric appear in some of these integrals).

Furthermore, since $P^\alpha$ and $J^{\alpha\beta}$ are supposed to be conserved, we would only need this metric for a given time, say $t = t_0$. Nevertheless, since in (3)–(5) the time derivatives of the metric appear, we actually need this metric in the elementary vicinity of $\Sigma_3$, whose equation, in the Gaussian coordinates we are using, is $t = t_0$. Thus, we do not need our Gauss coordinate system to cover the whole life of the universe. Nevertheless, in order to be consistent, we will need to check that the conditions for this conservation are actually fulfilled (see next the end of Sec. IV in relation to this question).

Now, the surface element $d\Sigma_{2i}$, which appears in the above expressions of $P^\alpha$ and $J^{\alpha\beta}$, is defined as if our space Gauss coordinates, $(x^i)$, were Cartesian coordinates. Thus, it has not any intrinsic meaning in the event of a change of coordinates in the neighborhood of $\Sigma_2$. So, what is the correct family of coordinate systems we must use in this neighborhood to properly define the energy and momentum of the universe? In order to answer this question, we will first prove the following result:

On $\Sigma_2$, in any given time instant $t_0$ there is a coordinate system such that

$$dl_0^2|_{\Sigma_2} = f\delta_{ij}dx^i dx^j, \qquad i, j = 1, 2, 3, \qquad (6)$$

where $f$ is a function defined on $\Sigma_2$. That is, the restriction to $\Sigma_2$ of the 3-metric $dl_0^2 \equiv dl^2(t = t_0)$ may be expressed in conformally flat form.

The different coordinate systems, in which $dl_0^2|_{\Sigma_2}$ exhibits explicitly its conformal form, are connected to each other by the conformal group in three dimensions. Then, one or some of these different conformal coordinate sys-

tems are to be taken as the good coordinate systems to properly define the energy and momenta of the considered universe. This is a natural assumption since the conformal coordinate systems allow us to write explicitly the space metric on $\Sigma_2$ in the most similar form to the explicit Euclidean space metric. But, which of all the conformal coordinates should be used? We will not try to answer this question here in all its generality, since our final goal in the present paper is to consider universes with zero energy and momenta. Instead of this, we will give some natural conditions to make sure that, when the energy and momenta of the universe are zero in one of the above conformal coordinate systems, these energy and momenta are zero in any other conformal coordinate system.

So, according to what we have just stated, we must prove that $dl_0^2|_{\Sigma_2}$ has a conformally flat form. In order to do this, let us use Gaussian coordinates, $(y^i)$ in $\Sigma_3$, based on $\Sigma_2$. Then, we will have

$$dl_0^2 = (dy^3)^2 + g_{ab}(y^3, y^c)dy^a dy^b, \qquad a, b, c = 1, 2. \tag{7}$$

In the new $(y^i)$ coordinates the equation of $\Sigma_2$ is then $y^3 = L$, where $L$ is a constant.

Then, taking into account that every 2-dimensional metric is conformally flat, we can always find a new coordinate system $(x^a)$ on $\Sigma_2$, such that we can write $dl_0^2$ on $\Sigma_2$, that is to say, $dl_0^2|_{\Sigma_2}$, as:

$$dl_0^2|_{\Sigma_2} = (dy^3)^2|_{\Sigma_2} + f(L, x^a)\delta_{ab}dx^a dx^b. \tag{8}$$

Finally, we introduce the new coordinate

$$x^3 = \frac{y^3 - L}{f^{1/2}(L, x^a)} + C, \tag{9}$$

with $C$ an arbitrary constant, which can be seen to allow us to write $dl_0^2|_{\Sigma_2}$ in the form (6), as we wanted to prove. [Notice that even though, in the general case, $f$ depends on $x^a$, by differentiating Eq. (9), one obtains on $\Sigma_2$, that is, for $y^3 = L$, $dy^3|_{\Sigma_2} = f^{1/2}(L, x^a)dx^3$].

Furthermore, if $r^2 \equiv \delta_{ij}x^i x^j$ in the coordinate system of Eq. (6), and we assume that the equation of $\Sigma_2$ in spherical coordinates is $r = R(\theta, \phi)$, we can expect to have in the elementary vicinity of $\Sigma_2$:

$$dl^2 = [^0g_{ij}(r - R)^n + \cdots]dx^i dx^j, \tag{10}$$

where $n$ is an integer greater than or equal to zero and where $^0g_{ij}$ are functions which do not depend on $r$. Furthermore, according to Eq. (6), on $\Sigma_3$, that is, for $t = t_0$, it must be

$$^0g_{ij}(r - R)^n|_{t=t_0} = f\delta_{ij}. \tag{11}$$

If, leaving aside a boundary at $r = 0$, the equation of the boundary, $\Sigma_2$, is $r = \infty$, we must put $1/r$ where we have written $r - R$ in the above equation, that is, we will have instead of (10) and (11):

$$dl^2 = [^0g_{ij}r^{-n} + \cdots]dx^i dx^j, \qquad ^0g_{ij}r^{-n}|_{t=t_0} = f\delta_{ij}, \tag{12}$$

for $r \to \infty$.

The $^0g_{ij}$ functions will change when we do a conformal change of coordinates. But, this is the only change these functions can undergo. To show this, let us first check which coordinate transformation, if any, could be allowed, besides the conformal transformations, if the explicit conformal form of $dl_0^2|_{\Sigma_2}$ is to be preserved. In an evident notation, these transformations would have the form

$$x^i = x^{i\prime} + y^i(x^j)(t - t_0), \tag{13}$$

in the vicinity of $\Sigma_3$. But it is easy to see that here the three functions $y^i(x^j)$ must all be zero, if the Gaussian character of the coordinates has to be preserved. That is, the only coordinate transformations that can be done on the vicinity of $\Sigma_2$, preserving on it the metric conformal form (6) and the universal character of the Gaussian coordinate time, are the coordinate transformations of the conformal group in the three space dimensions. Thus, we can state the following result.

Given $\Sigma_3$, that is, given the 3-surface which enables us to build our Gauss coordinates, we have defined uniquely $P^\alpha$ and $J^{\alpha\beta}$, according to Eqs. (2)–(5), modulus a conformal transformation in the vicinity of $\Sigma_2$.

So, the question is now: how do $P^\alpha$ and $J^{\alpha\beta}$ change under such a conformal transformation? As we have said above, we are not going to try to answer this general question here. Instead of this, since we are mainly concerned with "creatable universes," we will explore under what reasonable assumptions the energy and momenta of a universe are zero for all the above class of conformal coordinate systems.

## IV. ZERO ENERGY AND MOMENTA IRRESPECTIVE OF THE CONFORMAL COORDINATES

The first thing that can easily be noticed concerning the question is that the global vanishing of $P^\alpha$ and $J^{\alpha\beta}$ is invariant under the action of the groups of dilatations and rotations on $\Sigma_3$.

It is also easy to see that the global vanishing of $P^\alpha$ and $J^{\alpha\beta}$ will be invariant under the translation group on $\Sigma_3$, provided that one assumes the supplementary condition $\int \dot{g}_{ij}d\Sigma_{2j} = 0$, which is slightly more restrictive than $P^i = 0$. Actually, this supplementary condition will be fulfilled in our case, as a consequence of the assumptions we will make below, in the present section, in order to have $P^\alpha = 0$, as we will point out at the end of the section.

In all, we can say that, in the case we are interested here, of vanishing energy and momenta, $P^\alpha$ and $J^{\alpha\beta}$ are invariant under the groups of dilatations, rotations and translations on $\Sigma_3$. But all these three groups are subgroups of the conformal group of coordinate transformations in three

dimensions. Then, we are left with the subgroup of the group elements that have sometimes been called the *essential* conformal transformations. But it is known [11] that these transformations are equivalent to applying an inversion first, that is, $r$ going to $1/r$, then a translation, and finally another inversion. So, in order to see how $P^\alpha$ and $J^{\alpha\beta}$ change when we do a conformal transformation, one only has to see how they change when we apply an inversion, that is, $r$ going to $r'$, such that

$$r' = \frac{1}{r}, \qquad r^2 \equiv \delta_{ij}x^ix^j. \tag{14}$$

Assume as a first case that the equation of the boundary $\Sigma_2$ is $r = \infty$ plus $r = 0$. In this case, the 2-surface element, $d\Sigma_{2i}$, which appears in the Eqs. (2)–(5), can be written as $d\Sigma_{2i} = r^2n_id\Omega$, where $n_i \equiv x^i/r$, and $d\Omega$ is the elementary solid angle.

Now, let us consider the energy first, $P^0$. How does it change when we apply an inversion? This leads us to see how its integrand,

$$I \equiv r^2(\partial_jg_{ij} - \partial_ig)n_id\Omega = r^2(n_i\partial_jg_{ij} - \partial_rg)d\Omega, \tag{15}$$

changes. After some calculation, one sees that the new value, $I'$, for $I$ is

$$I' = r^3(r\partial_rg - rn_i\partial_jg_{ij} + 2n_in_jg_{ij} + 2g)d\Omega. \tag{16}$$

But, the integrands $I$ or $I'$ are both calculated on $\Sigma_2$. Then, according to Eq. (12), $I'$ on $\Sigma_2$ can still be written for $t = t_0$ as

$$I'|_{\Sigma_2} = r^3(r\partial_rg - rn_i\partial_jg_{ij} + 8f)d\Omega. \tag{17}$$

In this expression of $I'$ there is a $r^3$ common factor. Thus, if we want $P^{0'}$ to be zero, it suffices that $r^3f$ goes to zero when $r$ goes to $\infty$ and when $r$ goes to zero. In particular, this means that $f$ must go to zero at least like $r^{-4}$ when $r$ goes to $\infty$. Then, according to Eq. (12), the functions $g_{ij} - f\delta_{ij}$, which must go to zero faster than $f$, will go at least as $r^{-5}$. In a similar way, in order that $r^3f$ goes to zero for $r$ going to zero, $f$ must decrease, or at most cannot grow faster that $r^{-2}$. In a similar way, $g_{ij} - f\delta_{ij}$ must decrease for $r$ going to zero, or at most cannot grow faster than $r^{-1}$. Of course, this asymptotic behavior of $g_{ij}$ makes the original $P^0$ equal zero too. Thus, on the assumption that the equation of $\Sigma_2$ is $r = \infty$ plus $r = 0$, we have proved that this behavior is a sufficient condition in order that $P^0 = 0$ be independent of the conformal coordinate system used.

This natural sufficient condition is not a necessary one, since it is possible that $P^0$ could vanish because of the angular dependence of $I$. An angular dependence which would make zero the integral of $I$ on the boundary 2-surface, $\Sigma_2$, independently of $I$ going to zero or not when $r$ goes to $\infty$. But, in this case, from (15) and (17) one sees that the sufficient and necessary condition to have $P^{0'}$ equal

zero is that the integral of $f$ on $\Sigma_2$ be zero because of the special angular dependence of the function $f$.

Also, one can easily see that, under the above sufficient conditions, that is, $g_{ij}$ goes to zero at least like $r^{-4}$ for $r \rightarrow \infty$, and does not grow faster than $r^{-2}$ for $r \rightarrow 0$, we will have $P^i = 0$ and $J^{\alpha\beta} = 0$, independently of the conformal coordinate system used. This is so, because, according to (12), this asymptotic behavior for $g_{ij}$ entails the same asymptotic behavior for $\dot{g}_{ij}$.

All in all:

Under the assumption that the equation of $\Sigma_2$ *is* $r = \infty$ plus $r = 0$, the linear and angular momenta given by expressions (2)–(5) vanish, irrespective of the conformal coordinates used, if the following sufficient conditions are fulfilled: the metric $g_{ij}$ of Eq. (12) goes to zero at least like $r^{-4}$ *for* $r \rightarrow \infty$ and, on the other hand, the metric does not grow faster than $r^{-2}$ *for* $r \rightarrow 0$.

In Sec. VI, we will see that all this can be applied to the closed and flat Friedmann-Robertson-Walker (FRW) universes, whose energy and momenta then become zero.

Let us continue with the question of the nullity of energy and momenta, leaving now the special case where the equation of $\Sigma_2$ is $r = \infty$ plus $r = 0$ and considering the complementary case where this equation is $r = R(\theta, \phi)$. Then, a natural sufficient condition to have energy zero, irrespective of the conformal system used, is that the exponent $n$ in Eq. (10) be greater or equal to $n = 2$. This is a sufficient condition similar to the one which was present, in a natural way, in the above case, i.e., when the equation of $\Sigma_2$ was $r = \infty$ plus $r = 0$.

But, according to Eq. (10), the above asymptotic behavior, $n \geq 2$, extends to $\dot{g}_{ij}$. Then, it can easily be seen that this entails not only the vanishing of the energy of the considered universe, but also the vanishing of its linear 3-momentum and angular 4-momentum irrespective of the conformal coordinate system used.

All in all, we have established the following result:

Under the assumption that the equation of $\Sigma_2$ is $r = R(\theta, \phi)$, the linear and angular momenta given by expressions (2)–(5) vanish, irrespective of the conformal coordinates used, if the following sufficient condition is fulfilled: the metric $g_{ij}$ of Eqs. (10) and (11) vanishes fast enough in the vicinity of $\Sigma_2$. More precisely, the exponent $n$ in Eq. (10) is greater than or equal to $n = 2$.

In some particular cases, a more detailed analysis, than the one we have just displayed, enables not only sufficient conditions to be given, but also necessary and sufficient ones, to have zero energy and momenta irrespective of the conformal coordinate system used. But we are not going to give these details here since, in any case, the point will always be to write the space metric, $g_{ij}$, in the elementary vicinity of $\Sigma_2$ and $\Sigma_3$, in the form of Eqs. (10) and (11) or, alternatively, in the form of Eq. (12). Once one has reached this point, one could readily say if, irrespective of the conformal coordinate system used, the energy and momenta of the universe vanish or not.

Finally, we must realize that, from the beginning of Sec. III, all what we have said about the proper definition of energy and momenta of a given universe lies on the basic assumption that these are conserved quantities. Then, it can easily be seen that a sufficient condition for this conservation is that the second time-time and time-space derivatives of the space metric $g_{ij}$ vanish on $\Sigma_2$ for the generic constant value, $t_0$ of $t$. But this is entailed by the asymptotic behavior of $g_{ij}$ assumed in Eq. (10) or Eq. (12). This is the answer to the consistency question raised at the end of the second paragraph, at the beginning of Sec. III.

To end the section, notice that the above assumed behavior of $\dot{g}_{ij}(t = t_0)$ near $\Sigma_2$ (going like $r^{-4}$, or like $(r - R)^2$, or even at most like $r^{-2}$ for $r \to 0$, according to the different cases we have considered) makes not only $P^i = 0$, but also $\int \dot{g}_{ij} d\Sigma_{ij} = 0$, as we have announced at the beginning of the section.

## V. THE NULLITY OF ENERGY AND LINEAR MOMENTUM AGAINST A CHANGE OF $\Sigma_3$

Let us look back at Sec. II, where we have selected a spacelike 3-surface, $\Sigma_3$, from which to build a coordinate Gauss system. The energy and momenta of the considered universe are then in relation to the selected 3-surface, that is, depend on this selected 3-surface. This is not a drawback in itself, since, as we put forward in that section, the energy of a given physical system in the Minkowski space also depends on the Minkowskian observer, and so it depends on the spacelike 3-surface associated to the coordinate system used through the equation $t = t_0$. Nevertheless, when this energy and the corresponding linear 3-momentum are both zero for a Minkowskian system, then they are obviously zero for any other Minkowskian system.

Thus, if the definition of null energy-momentum for a given universe that we have given in the last section is correct, one could expect that $P^\alpha = 0$ should remain valid irrespective of the 3-surface $\Sigma_3$ used.

We will prove this, first in the case where the equation of $\Sigma_2$ is $r = \infty$ plus $r = 0$, and then in the complementary case where the equation of $\Sigma_2$ is $r = R(\theta, \phi)$.

In the first case, we will assume that the space metric $g_{ij}$ goes to zero at least like $r^{-3}$ when $r \to \infty$ and that it also behaves *conveniently* for $r = 0$. Here, "conveniently" means that the metric decreases, or at most grows no faster than $r^{-1}$, when $r$ goes to zero. We can take these assumptions for granted since in Sec. IV, in order to have $P^0 = 0$ irrespective of the conformal coordinate system used, we had to assume, as a sufficient condition, the behavior $r^{-4}$ for $r \to \infty$, besides the above convenient behavior for $r = 0$. Notice that the above $r^{-3}$ asymptotic behavior, as any other faster decaying, when completed with that convenient behavior for $r = 0$, allows us to have $P^\alpha = 0$. Indeed, with these assumptions, in Eq. (2), the integrand of $P^0$, for $r$ going to $\infty$, will go like $r^{-4}$, and the one of $P^i$ like $r^{-3}$.

This sort of decaying, plus the above convenient behavior for $r = 0$, will make $P^0$ and $P^i$ vanish.

Now, imagine that we slightly change $\Sigma_3$, from the original $\Sigma_3$ to a new $\tilde{\Sigma}_3 = \Sigma_3 + \delta\Sigma_3$. Then, we will have the corresponding elementary coordinate change between any two Gauss systems associated to $\Sigma_3$ and to $\tilde{\Sigma}_3$, respectively:

$$x^\alpha = x^{\alpha'} + \epsilon^\alpha(x^\beta), \qquad (18)$$

where $|\epsilon^\alpha| \ll |x^\alpha|$, and where the absolute values of all partial derivatives of $\epsilon^\alpha$ are order $|\epsilon| \ll 1$.

Taking into account that $g_{00} = -1$ and $g_{0i} = 0$, we will find for the transformed 3-space metric, to first order in $\epsilon$:

$$g'_{ij} = g_{ij} + g_{ik}\partial_j\epsilon^k + g_{jk}\partial_i\epsilon^k. \qquad (19)$$

Now, to calculate the new energy, $\tilde{P}^0$, corresponding to this transformed metric, we will need $g'_{ij}(t' = t_0)$ in the vicinity of $\tilde{\Sigma}_2$ (the boundary of $\tilde{\Sigma}_3$). According to Eq. (19), we will have to first order

$$g'_{ij}(t' = t_0) = (g_{ij} + \epsilon^0\dot{g}_{ij} + g_{ik}\partial_j\epsilon^k + g_{jk}\partial_i\epsilon^k)(t = t_0), \qquad (20)$$

for any value of $t_0$ and everywhere on $\Sigma_3$.

From this equation we see that $g'_{ij}(t' = t_0)$ goes to zero as least like $r^{-3}$, when we approach $\Sigma_2$ through $r$ going to $\infty$, provided that, as we have assumed, $g_{ij}(t = t_0)$ goes this way to zero. Similarly, for $r \to 0$, $g'_{ij}(t' = t_0)$ will decrease, or at most will grow no faster than $r^{-1}$, provided we have assumed that decreasing or this growing, respectively, for $g_{ij}(t = t_0)$.

Furthermore, one can be easily convinced that $g'_{ij}(t' = t_0)$ will keep the same asymptotic behavior when we approach $\tilde{\Sigma}_2$ instead of $\Sigma_2$. Indeed, in the ancient space coordinates, $x^i$, previous to the infinitesimal coordinate change (18), the equation of $\tilde{\Sigma}_2$ is still $r = \infty$, or more precisely $r = \infty$ plus $t' = t_0$, whereas the equation of $\Sigma_2$ was $r = \infty$ plus $t = t_0$. [The same can be established for the other boundary sheet, $r = 0$. See, next, the case where the equation of $\Sigma_2$ is $r = R(\theta, \phi)$].

Then, as we have said, $g'_{ij}(t' = t_0)$ goes to zero as least like $r^{-3}$, when we approach $\tilde{\Sigma}_2$ through $r$ going to $\infty$. This means that the new energy, $\tilde{P}^0$, corresponding to the new Gauss 3-surface, $\tilde{\Sigma}_3$, is zero, as the original energy was.

On the other hand, because of (12), $\dot{g}_{ij}(t = t_0)$, as $g_{ij}(t = t_0)$, will go to zero like $r^{-3}$ when $r \to \infty$, and will decrease, or at most will grow no faster than $r^{-1}$, when $r \to 0$. Then, also $\tilde{P}^i$, and so the entire 4-momentum, $\tilde{P}^\alpha$, corresponding to the new 3-surface, $\tilde{\Sigma}_3$, is zero, as the original 4-momentum was.

But we can iterate this result along an indefinite succession of similar infinitesimal shifts of $\Sigma_3$. That is, as we wanted to prove, $P^\alpha$ will be also zero for the final 3-surface

$\Sigma_3$, which differs now in a finite amount from the original 3-surface. In this way, we could reach any final $\Sigma_3$, provided that the original and the final metric, in the corresponding Gauss systems, were regular enough (otherwise we could not make sure that in all intermediate infinitesimal steps the above conditions $|\partial_\alpha \epsilon^\beta| \ll 1$ could be satisfied). Here "regular enough" means that the contribution of the neighborhood of any metric singularity, which can appear in the final $\tilde{\Sigma}_3$, to the calculation of $\tilde{P}^\alpha$ goes to zero. In this way, we always could get rid of the difficulty by excluding this neighborhood in the calculation.

Now, we will prove once more that $P^\alpha = 0$ is independent of the chosen 3-surface $\Sigma_3$, this time in the case where the equation of $\Sigma_2$, the boundary of $\Sigma_3$, is $r = R(\theta, \phi)$, plus $t = t_0$, instead of $r = \infty$ plus $t = t_0$. We will prove this under the assumption that the space metric, $g_{ij}$, goes to zero at least like as $(r - R)^2$ as we approach $\Sigma_2$. This assumption plays now the role of the above assumption $g_{ij}$ going like $r^{-3}$ for $r$ going to $\infty$. Again, in Sec. IV, the behavior of $g_{ij}$ and $\dot{g}_{ij}$, going like $(r - R)^2$ in the vicinity of $r = R(\theta, \phi)$, insures that $P^\alpha = 0$ irrespective of the conformal coordinate system used. Notice that this assumption makes zero the original energy-momentum.

Then, as we have done above in the present section, we slightly change $\Sigma_3$, from this original $\Sigma_3$ to a new spacelike 3-surface $\tilde{\Sigma}_3 = \Sigma_3 + \delta\Sigma_3$. Therefore, we will have Eq. (20). But, this equation shows that the domain of variation of the space coordinates for the functions $g'_{ij}$ for $t' = t_0$ is the same that the corresponding domain for the functions $g_{ij}$ at $t = t_0$. That is, the boundary of $\tilde{\Sigma}_3$ is again $r = R(\theta, \phi)$, now for $t' = t_0$, or, in the ancient coordinate time, for $t = t_0 + \epsilon^0$. Of course, to conclude this, we need that the time derivative of the ancient space metric, $\dot{g}_{ij}$, be defined everywhere, that is, be defined all where $g_{ij}$ is defined. But this must be taken for granted if we assume that the metric components are functions of class $C^1$ (i.e., its first derivatives exist and are continuous). This condition holds independently of the coordinate system used if, as usual, the space-time is considered as a differentiable manifold of class $C^2$ (see, for example, Ref. [12]).

The next step in our proof is to show that $g'_{ij}$ goes also like $(r - R)^2$, in the vicinity of $\tilde{\Sigma}_2$. But, this becomes obvious from Eq. (20), once one has proved, as we have just done, that the equation of $\tilde{\Sigma}_2$ is $r = R(\theta, \phi)$ plus $t' = t_0$. Thus, the new energy momentum $\tilde{P}^\alpha$, corresponding to the new 3-surface, $\tilde{\Sigma}_3$, is also zero.

Finally, to end the proof, we need to check that, for any chain of consecutive elementary shifts of the original $\Sigma_3$ spacelike surface, leading to a final new $\tilde{\Sigma}_3$ spacelike surface, we can iterate indefinitely the above procedure of obtaining, each time, a new energy-momentum which vanishes. But, this is again obvious from Eq. (20), since, as we have assumed, our space-time is a differentiable mani-

fold of class $C^2$, which entails that for every shift the time derivative of the space metric, in any admissible coordinate system, is defined wherever the space metric is defined. Thus, iterating indefinitely the above procedure, we find that the final energy-momentum, corresponding to the new spacelike 3-surface, $\tilde{\Sigma}_3$, is also zero, as we wanted to prove.

Let us specify, all the same, that to reach this conclusion we need to assume that the metric is "regular enough." According to what has been explained above, in the present section, a "regular enough" metric is one such that the same metric and its first derivatives have no singularities, or one such that, in the case where some of these singularities are present, the contribution of its neighborhoods to the integrals which define $P^\alpha$ and $J^{\alpha\beta}$ in (2)–(5) goes to zero when the areas of these neighborhoods go to zero.

All in all, under this regularity assumption, we have proved the following proposition:

Let it be any two different spacelike 3-surfaces, $\Sigma_3$ and $\tilde{\Sigma}_3$. Assume that the Gauss metric $g_{ij}$ built from the original 3-surface, $\Sigma_3$, is "regular enough", and that as we approach its boundary $\Sigma_2$ this metric satisfies:

(i) If the equation of $\Sigma_2$ is $r = \infty$ plus $r = 0$, $g_{ij} \to 0$ at least like $r^{-3}$ when $r \to \infty$ and $g_{ij}$ decreases, or at most grows no faster than $r^{-1}$, when $r \to 0$.

(ii) If the equation of $\Sigma_2$ is $r = R(\theta, \phi)$, $g_{ij} \to 0$ at least like $(r - R)^2$.

Then, the original linear 4-momentum corresponding to the 3-surface $\Sigma_3$ vanishes, and the linear 4-momentum corresponding to the other surface, $\tilde{\Sigma}_3$ vanishes too.

By nearly making the same assumptions and by reproducing the same reasoning, we have applied in the case of $P^\alpha$, in the new case of $J^{\alpha\beta}$, one can easily be convinced that, if $J^{\alpha\beta}$ vanishes for a given 3-surface, $\Sigma_3$, it will vanish too for any other spacelike 3-surface $\tilde{\Sigma}_3$. The only change we have to introduce in the above assumptions, to reach this conclusion, is the following one. When the equation of $\Sigma_2$ is $r = \infty$, one has to assume that $g_{ij}(t = t_0)$ goes to zero like $r^{-4}$ instead of $r^{-3}$. Remember, nevertheless, that this $r^{-4}$ behavior for $g_{ij}(t = t_0)$ is already what we had assumed in Sec. IV, in order to have $P^0 = 0$ irrespective of the conformal coordinates used in $\Sigma_3$.

## VI. THE EXAMPLE OF FRW UNIVERSES

As it is well known, in these universes one can use Gauss coordinates such that the 3-space exhibits explicitly its everywhere conformal flat character:

$$dl^2 = \frac{a^2(t)}{[1 + \frac{k}{4}r^2]^2} \delta_{ij} dx^i dx^j, \qquad r^2 \equiv \delta_{ij} x^i x^j, \quad (21)$$

where $a(t)$ is the expansion factor and $k = 0, \pm 1$ is the index of the 3-space curvature.

Then, this conformally flat character will be valid, *a fortiori*, on any vicinity of $\Sigma_3$ and $\Sigma_2$. Therefore, according to Sec. III, we can apply our definitions to the metric (21). Taking into account Eqs. (2)–(5), we will have then:

$$P^0 = -\frac{1}{8\pi G}\int r^2 \partial_r f d\Omega, \qquad (22)$$

$$P^i = \frac{1}{8\pi G}\int r^2 \dot{f} n_i d\Omega, \qquad (23)$$

$$J^{jk} = \frac{1}{16\pi G}\int r^2 \dot{f}(x_k n_j - x_j n_k)d\Omega, \qquad (24)$$

$$J^{0i} = P^i t - \frac{1}{8\pi G}\int r^2 (fn_i - x_i\partial_r f)d\Omega \qquad (25)$$

with $d\Omega = \sin\theta d\theta d\phi$, $n_i \equiv x^i/r$, and where we have put

$$f \equiv \frac{a^2(t)}{[1 + \frac{k}{4}r^2]^2} \qquad (26)$$

which, excluding the limiting case $k = 0$, goes as $1/r^4$ for $r \to \infty$. This is just the kind of behavior that we have assumed in Sec. IV in order to reach the conclusion that $P^\alpha = 0$, $J^{\alpha\beta} = 0$, are conformally invariant. It is also a behavior which allows to make this vanishing of $P^\alpha$ and $J^{\alpha\beta}$ independent of the 3-surface, $\Sigma_3$, chosen.

Then, one can easily obtain the following result, in accord with most literature on the subject (see the pioneering Ref. [8], and also Ref. [13] for a concise account),

$$k = 0, +1: P^\alpha = 0, \qquad J^{\alpha\beta} = 0 \qquad (27)$$

that is, the flat and closed FRW universes have vanishing linear and angular momenta.

Contrary to this, in the case where $k = -1$, one finds for the energy, $P^0 = -\infty$. This is because now the metric is singular for $r = 2$. Thus, in order to calculate its energy, we must consider the auxiliary universe which results from excluding the elementary vicinity $r = 2 \pm \epsilon$. Therefore, we will calculate the energy of this auxiliary universe and then we will take the limit for $\epsilon \to 0$. But now, the boundary of the 3-space universe described by this auxiliary metric is double. On the one hand, we will have, as in the case of $k = 0, +1$, the boundary $r = \infty$, and on the other hand the new boundary $r = 2$ that we can approach from both sides $r = 2 \pm \epsilon$. Both boundaries must be taken into account when doing the calculation of $P^0$ according to the Eq. (22). Then, it can easily be seen that the contribution to the energy calculation from the first boundary, $r = \infty$, vanishes, but further elementary calculation shows that the contribution from the other boundary is $-\infty$. Thus, as we have said, the FRW universes with $k = -1$, have $P^0 = -\infty$.

All in all, the flat and closed FLRW universes are "creatable universes," but the open one is not.

## VII. THE EXAMPLE OF SOME BIANCHI UNIVERSES

Let us consider the case of the family of nontilted perfect fluid Bianchi V universes [14], whose metric can be written as

$$ds^2 = -dt^2 + A^2 dx^2 + e^{2x}(B^2 dy^2 + C^2 dz^2), \qquad (28)$$

where $A$, $B$ and $C$ are functions of $t$.

The first thing one must notice about this universe metric is that, as in the above case of the FRW universes, it is written in Gauss coordinates, which according to Sec. II is the coordinate system family with which to define the proper energy and momenta of a given universe.

Then, for $t = t_0$, we will have

$$dl_0^2 \equiv dl^2(t = t_0) = dx^2 + e^{2\alpha x}(dy^2 + dz^2), \qquad (29)$$

where we have rescaled the original notation $(x, y, z)$ according to $A_0 x \to x$, $B_0 y \to y$, $C_0 z \to z$, and where $\alpha = 1/A_0$, with $A_0 \equiv A(t_0)$, and so on. Now, let us move from the variable $x$ to new variable $x'$: $x' = e^{-\alpha x}/\alpha$. Then, we will have for the instantaneous space metric, $dl_0^2$,

$$dl_0^2 = \frac{1}{\alpha^2 x'^2}(dx'^2 + dy^2 + dz^2), \qquad (30)$$

or changing the above notation such that $x' \to x$:

$$dl_0^2 = \frac{1}{\alpha^2 x^2}\delta_{ij}dx^i dx^j. \qquad (31)$$

This is a conformal flat metric not only in the vicinity of $\Sigma_2$ but everywhere on $\Sigma_3$ (except for $x = 0$). Then, according to Sec. III, we can use this particular expression of $dl_0^2$ to calculate the energy of our family of Bianchi universes, since, in fact, to calculate this energy we only need the instantaneous space metric in the vicinity of $\Sigma_2$.

Now, this metric has a singularity for $x = 0$. Thus, in order to calculate its energy, we must proceed as in the above case of an open FRW universe. So, we consider the auxiliary universe which results from excluding the elementary vicinity of $x = 0$, $x \in (0, +\epsilon)$, where we have taken $\alpha > 0$. Therefore, we will calculate the energy of this auxiliary universe and then we will take the limit for $\epsilon \to 0$. The boundary of the 3-space universe described by this auxiliary metric is double. On the one hand, we will have the boundary $x = +\infty$, and on the other hand the boundary $x = +\epsilon$. Both must be taken into account when doing the calculation of $P^0$ according to the Eq. (22).

Then, it is easy to see that the contribution to $P^0$ of the second boundary, $x = \epsilon$, gives $+\infty$, and that the contribution of the first boundary, $x = +\infty$, gives $+\infty$ too. Therefore, we can conclude that the energy of our Bianchi V family of universes is $P^0 = +\infty$. Then, this family of universes, next to the open FRW universe we have just seen, are examples of non-"creatable universes."

## VIII. DISCUSSION AND PROSPECTS

We have analyzed which family of coordinate systems could be suitable to enable the linear and angular 4-momenta of a nonasymptotically flat universe to be considered as the energy and momenta of the universe itself, without the spurious energy and momenta of the fictitious gravitational fields introduced by accelerated (noninertial) observers. Though we have not been able to uniquely determine this family in the general case, we have been able to do so in a particular but interesting case, where the energy and momenta of the universe vanish. As a consequence, the notion of a universe having zero energy and momenta is unique and so makes sense. This result is in contrast with the exhaustive studies on the energy and momentum of a 3-surface $\Sigma_3$, in general relativity, mainly focussed on the asymptotically flat behavior of $\Sigma_3$ (see [4] and references therein).

Universes whose energy and momenta vanish are the natural candidates for universes that could have risen from a vacuum quantum fluctuation. Here we have called these universes "creatable universes".

Any given universe could be rejected from the very beginning, as a good candidate for representing our real Universe, in the event that it were a noncreatable one. We could reject it either before the inflationary epoch, or after this epoch, or just right now. This could be the main interest of the characterization of the "creatable universes" that we have reached in the present paper. Thus, for example, people have considered the possibility that our present Universe could be represented by Stephani universes [15–18], that is, by a universe which at different times admits homogenous and isotropic spacelike 3-surfaces whose curvature index can change. Such a possibility is a generalization of the FRW universes and could not be easily discarded on the grounds of present cosmic observations. Nevertheless, if all, or some, of these Stephani universes were non-"creatable universes," we could reject them on the grounds of the assumption that all candidate universes able to represent our real Universe should be "creatable universes." This is why it could be interesting to see which Stephani universes have zero energy and momenta. For similar reasons, it could be interesting to make the same analysis in the case of Lemaître-Tolman universes [19–21], and in the case of a particular Bianchi type VII universe [22]. We expect to consider these questions in detail elsewhere shortly.

## ACKNOWLEDGMENTS

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