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Covariant determination of the Weyl tensor geometry

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Abstract

We give a covariant and deductive algorithm to determine, for every Petrov type, the geometric elements associated with the Weyl tensor: principal and other characteristic 2-forms, Debever null directions and canonical frames. We show the usefulness of these results by applying them in giving the explicit characterization of two families of metrics: static type I spacetimes and type III metrics with a hypersurface-orthogonal Killing vector.

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1. Introduction

The Weyl tensor plays an essential role in gravitational physics because it contains the main conceptual differences between the Newton and Einstein theories of gravitation [1]. Its basic constituents (namely the canonical structure of eigenbivectors, Debever 2-forms and their related principal directions and frames) provide information with which to travel over vacuum spacetime geometries and to carry them out. It is likely that they are pieces of a geometrical version of the Szekeres gravitational compass, which is a simple idealized mechanical device for visualizing spacetime curvature effects and sensitive to the different types of gravitational fields [1, 2]. The wealth of the geometric structure associated with the Weyl tensor has been extensively analysed for years using different approaches. However, obtaining the above-mentioned constituents and their explicit expression in a coordinate-free way has been little investigated so far. In this paper, we deal with the question of obtaining and expressing the referred geometric objects. To understand better what it means, how it appears and why it claims our attention, a more detailed explanation will be helpful.

The first significant aspect is that the Weyl and Ricci tensors are, algebraically, complementary parts of the Riemann curvature tensor. Einstein field equations place Ricci and energy tensors together, connecting them at the same point of the spacetime, and then the Weyl tensor is usually identified as the free gravitational field, that is, the remainder part of the curvature that is not determined by the energy content at this point. Nevertheless, as a consequence of the differential Bianchi identities, spacetimes rates of the Weyl and Ricci tensors are not independent of each other [1]. In fact, for a four-dimensional geometry these identities state that the divergence of the Weyl tensor equals the Cotton tensor which physically seems like a tensorial gravitational current. For three-dimensional geometries, the vanishing of the Cotton tensor characterizes (locally) conformal flatness [3]. In contrast, for dimension $n > 3$, the nullity of the Weyl tensor gives the necessary and sufficient condition for a geometry to be conformally flat. Consequently, a non-vanishing Weyl tensor on a four–dimensional spacetime expresses its conformal flatlessness.

A second thing to bear in mind concerns the physical meaning usually given to the curvature tensor: it governs the equation of geodesic deviation producing relative acceleration between near-test particles [4] or optical effects on light test congruences [5]. More specifically, in the analysis of the transport equations of optical scalars, the Weyl and Ricci tensors act, respectively, as a direct source of shear and expansion of light beams [6]. In vacuum, the peculiarities of these effects depend exclusively on the different algebraic types associated with the invariant classification of the Weyl tensor.

This algebraic classification (the details of which will be commented on later) was developed independently by several authors (Bel [7], Debever [8], Géhéniau [9], Penrose [10], Petrov [11], Pirani [12], Sachs [5]) in the mid-1950s and early 1960s, and has been frequently designed as the Petrov classification of the gravitational fields. At that time, these studies were mainly aimed at obtaining invariant criteria for the existence of gravitational radiation [7, 12, 13], and also at analysing the asymptotic expansion of bounded source gravitational fields [5, 14]. The principal null directions of the Weyl tensor provide an elegant approach to these questions. For these and related issues we refer to a monograph by Zakharov [15] where a wide bibliography has been registered.

It is worth mentioning here that the Petrov classification is a useful tool in searching for families of exact solutions of the Einstein equations, giving separate geometric criteria to classify and characterize them [16]. In particular, for vacuum or Ricci structureless metrics, the different algebraic types of the Petrov classification provide the first basic ingredients needed to distinguish non-equivalent solutions. Hence obtaining Weyl constituents explicitly is of interest in dealing with the equivalence problem of metrics and in analysing other issues that require the determination of a distinctive reference frame. For instance, in studying the properties of the infinitesimal holonomy group at a spacetime domain, which are closely related to the Petrov type at this domain [17].

In the algebraic study of the Ricci and Weyl tensors three complementary issues should be considered: the first consists in classifying the different types taking into account the eigenvalue multiplicity and the degree of the minimal polynomial of an associated linear map; secondly, we can give algorithms that can be used to distinguish the algebraic type of a given tensor; finally, it is also necessary to obtain covariant and explicit expressions for the eigenspaces or for other associated characteristic directions, as well as to give conditions that determine its causal character. For the Ricci tensor case this last aspect has been studied by Bona *et al* [18], and their results are a necessary tool in dealing with the characterization of spacetimes obeying their energetic content. In particular, its utility has been shown in building a Rainich theory for the thermodynamic perfect fluid [19]. In a similar way, in addition to knowing its classification and giving algorithms to distinguish every case, a complete algebraic study of the Weyl tensor implies knowing the covariant determination of the Weyl eigen 2-forms. Here we present a full analysis of this subject, considering all the Petrov types, thus completing previous results for the type I case [20].

In order to obtain the algebraic classification of the Weyl tensor, Petrov [11] studied the eigenvalue and eigenvector problem for the Weyl tensor regarded as an endomorphism on the 2-forms space. This approach was completed by Géhéniau [9] and Bel [21], considering not only the number of independent invariant subspaces but also the eigenvalue multiplicity. In this framework, in a natural way, we find the notion of principal 2-form which was analysed widely by Bel [7] for the different algebraic types. An alternative viewpoint [8, 22] is the study of the relative positions between the null cones determined by the canonical metric and the Weyl tensor as quadratic forms on the 2-form space. From this angle, which is equivalent to the spinorial approach [10, 23], the Weyl tensor classification implies studying the roots of a fourth-degree algebraic equation with complex coefficients. A Debever null direction of the Weyl tensor corresponds to every root of this equation [10, 24, 25].

After the study by Sachs [5], where he gave the hierarchy of equations that characterizes the multiplicity of the Debever directions, and the publication of the d'Inverno and Russell–Clark algorithm to obtain the Petrov type [26], the null direction approach has been the most widely considered in literature, and since then, the principal 2-forms unrelated to Debever directions have seldom been taken into account. Therefore, although the geometric richness of both points of view was underlined and widely studied in pioneering papers by Bel [7], Debever [8] and Penrose [10] some years ago, until now the relation between them has not been sufficiently analysed. However, concerning this subject we must quote the results by Penrose and Rindler [23] analysing the Weyl geometry in spinorial formalism, and the papers by Trümper [37] and Narain [28], or the more recent ones by McIntosh *et al* [29] and Bonanos [30], where we can find expressions that relate Debever directions and the orthonormal canonical frame in particular type I cases: purely electric, purely magnetic and when the four Debever directions span a 3-plane. Nevertheless, it is necessary to look for covariant and explicit relations between both geometries and, in particular, for the unknown expressions giving simple Debever directions in terms of principal 2-forms. In our above-mentioned study we analyse this subject for a generic type I metric [20]. Now, in this paper, we consider the algebraically special types, and for them we study the relationship between Debever directions and principal 2-forms accurately. In this analysis the concept of the *unitary Debever bivector* that we introduce here plays a fundamental role.

The Cartan formalism [31] is the most suitable tool to attempt the metric equivalence problem and, after Karlhede's study [32], this subject became more popular within the general relativity framework. The Cartan–Karlhede method is based on working in an orthonormal (or a null) frame, fixed by the underlying geometry of the Riemann tensor. Dealing with vacuum solutions, or with metrics in which the Ricci tensor has a high algebraic degeneration, it may be convenient, or even necessary, to choose a Weyl canonical frame. A covariant determination of the Weyl canonical frames in Petrov type I spacetimes has been presented [20], and in the present study, we obtain the Weyl canonical frames with a covariant, explicit and deductive algorithm for all the Petrov types.

The paper is organized as follows. In section 2 we summarize the notation and formalism used which are basically those that are applied in [20]. We introduce basic concepts on 2-forms and on the self-dual 2-form space, and give covariant expressions for the one-to-one map between orthonormal (or null) frames on the tangent space and orthonormal (or null) bases on the self-dual 2-form space. These preliminary results are necessary in order to obtain the Weyl canonical frames later.

In section 3 we give a short survey about Petrov classification. We briefly present the different algebraic types that appear when classifying the linear map defined by the self-dual Weyl tensor, and offer an alternative algorithm to determine the Weyl algebraic type that is arranged starting from the minimal polynomial and exclusively uses tensorial expressions on the Weyl tensor. Here we also summarize the basic elements of the Debever approach and the concept of the unitary Debever bivector is introduced.

Section 4 is devoted to the main goal of this paper: the covariant and deductive determination of the principal bivectors, the Debever directions and the canonical frames of an arbitrary Weyl tensor. Every Petrov type is studied in a subsection with four points. Firstly, we start from the canonical form of the self-dual Weyl tensor and point out the eigenbivectors and other characteristic bivectors that appear in a natural way, as well as their associated spacetime directions. In the second step we analyse the underlying Debever geometry and offer, when this geometry differs from that presented in the first point, an alternative expression for the Weyl tensor that is adapted to the Debever bivectors; we show how both canonical forms allow us to understand the degeneration of a Petrov type along the arrows of a Penrose diagram. At the third level we give deductive algorithms that determine the geometric elements presented in the two previous steps, namely eigenbivectors and Debever directions. It is important to note that the term deductive means that these geometric elements are obtained through coordinate-free expressions and without solving any algebraic equation. Finally, we consider the Weyl canonical frames by studying their relation with the previously presented eigenbivectors and Debever directions. An algorithm to determine these frames is also offered.

Finally, section 5 is devoted to presenting two examples where we apply our results: the type I static spacetimes and the type III metrics with a hypersurface-orthogonal Killing vector.

2. Notes on the 2-forms space

Let (V_4, g) be an oriented and time-oriented spacetime of signature $\{-,+,+,+\}$. On the six-dimensional vectorial space of the 2-forms, two different metrics can be considered: the usual one, $G = \frac{1}{2} g \wedge g$, $(g \wedge g)_{\alpha\beta\mu\nu} = 2(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$, induced by the spacetime metric, and that defined by the metric volume element η, i.e.

$$G(F, H) \equiv (F, H) = \tfrac{1}{4} G_{\alpha\beta\mu\nu} F^{\alpha\beta} H^{\mu\nu} \qquad \eta(F, H) \equiv (*F, H) = \tfrac{1}{4} \eta_{\alpha\beta\mu\nu} F^{\alpha\beta} H^{\mu\nu}.$$

The scalar invariants of a 2-form F are defined by its squares calculated with these two metrics: (F, F) and $(*F, F)$. A 2-form is usually named null or singular when $(F, F) = (*F, F) = 0$ and regular in other cases. Simple 2-forms are those that satisfy $(*F, F) = 0$.

The *principal directions* $l_\pm$ of a regular 2-form F are the common (null) eigenvectors to F and $*F$:

$$l_\pm \wedge F(l_\pm) = 0 \qquad l_\pm \wedge *F(l_\pm) = 0. \tag{1}$$

In the following we note U as a timelike unitary simple 2-form: $(U, *U) = 0$, $(U, U) = -1$. Moreover, we use a deductive and covariant method to obtain the principal directions of a 2-form without solving any equations [33]; on this count, the principal directions of U are obtained as

$$l_\pm \propto [U^2 \pm U](x) \tag{2}$$

where x is an arbitrary timelike direction and $U^2 = U \times U$, the symbol $\times$ being the cross product, contraction of the adjacent spaces of the tensor product.

We choose and parametrize the principal directions in such a way that $U = l_- \wedge l_+$. Moreover, if l_- and l_+ are future-pointing, we refer to them as the first and the second principal vectors of U. The principal vectors satisfy $(l_-, l_+) = -1$, and they are fixed up to change $l_\pm \hookrightarrow e^{\pm\lambda} l_\pm$. But if the first principal vector l_- is given, we get the (unique) second principal vector l_+ as

$$l_+ = -\frac{[U^2 + U](x)}{2(x, l_-)} \tag{3}$$

x being an arbitrary timelike future-pointing vector.

A singular 2-form H and its dual $*H$ admit a unique common (null) eigendirection l with zero associated eigenvalue: $H(l) = 0$, $*H(l) = 0$. It is named the *fundamental direction* of H. This fundamental direction is obtained as [33]

$$l \propto H^2(x) \tag{4}$$

x being an arbitrary timelike direction. A parametrization of l exists such that it is future-pointing and $H = l \wedge e_2$, where e_2 is a spacelike unitary vector orthogonal to l, and fixed up to change $e_2 \hookrightarrow e_2 + \mu l$. With this *canonical parametrization* we name l as the *fundamental vector* of H. It is obtained as

$$l = \frac{H^2(x)}{\sqrt{-H^2(x,x)}} \tag{5}$$

x being an arbitrary timelike future-pointing vector.

A self-dual 2-form is a complex 2-form $\mathcal{F}$ such that $*\mathcal{F} = \mathrm{i}\mathcal{F}$. We can associate biunivocally with every real 2-form F the self-dual 2-form $\mathcal{F} = \frac{1}{\sqrt{2}}(F - \mathrm{i} * F)$. Here we refer to a self-dual 2-form as a *bivector*. The endowed metric on the three-dimensional complex space of the bivectors is $\mathcal{G} = \frac{1}{2}(G - \mathrm{i}\,\eta)$. Then, for every bivector $\mathcal{F}$, the complex scalar invariant $\mathcal{G}(\mathcal{F}, \mathcal{F})$ gives the scalar invariants of the associated real 2-form F: $\mathcal{G}(\mathcal{F}, \mathcal{F}) \equiv (\mathcal{F}, \mathcal{F}) = (F, F) - \mathrm{i}(*F, F)$. A self-dual 2-form $\mathcal{H} = \frac{1}{\sqrt{2}}(H - \mathrm{i} * H)$ is a null vector for $\mathcal{G}$ when H is singular, and a $\mathcal{G}$-unitary self-dual 2-form $\mathcal{U} = \frac{1}{\sqrt{2}}(U - \mathrm{i} * U)$ corresponds to every timelike unitary simple 2-form U. Thus, we call principal those directions (or vectors) of $\mathcal{U}$ and fundamental direction (or fundamental vector) of $\mathcal{H}$ associated with the corresponding real 2-forms U and H.

Let $\{e_\alpha\}_{\alpha=0}^3$ be an oriented and orthochronous orthonormal frame. Then, if we define $U_i = e_0 \wedge e_i$, we obtain a self-dual orthonormal frame $\{\mathcal{U}_i\}_{i=1}^3$ which has an induced orientation given by

$$\mathcal{U}_k = \mathrm{i}\,\sqrt{2}\,\epsilon_{ijk}\,\mathcal{U}_i \times \mathcal{U}_j. \tag{6}$$

Conversely, every oriented (by (6)) orthonormal frame $\{\mathcal{U}_i\}$ on the bivectors space determines a unique oriented and orthochronous orthonormal frame $\{e_\alpha\}$ such that $U_i = e_0 \wedge e_i$. This one-to-one map follows from the isomorphism between the proper orthochronous Lorentz group, $\mathcal{L}_+^{\nearrow}$, and the group $SO(3, \mathbb{C})$ of the proper orthogonal transformations on the bivector space. In order to obtain the Weyl canonical frames in the next section, we need the explicit and covariant expressions which provide $\{e_\alpha\}$ in terms of $\{\mathcal{U}_i\}$. They are given in the following [20]:

Lemma 1. *If $\{e_\alpha\}$ is an oriented and orthochronous orthonormal frame and $U_i = e_0 \wedge e_i$, then the bivectors $\mathcal{U}_i = \frac{1}{\sqrt{2}}(U_i - \mathrm{i} * U_i)$ define an oriented orthonormal frame $\{\mathcal{U}_i\}$ on the bivector space. This is a one-to-one map and its inverse is given by*

$$e_0 = \frac{-P_0(x)}{\sqrt{P_0(x,x)}} \qquad e_i = U_i(e_0)$$

with $P_0 \equiv \frac{1}{2}\left(g - \sum_{i=1}^3 U_i^2\right)$, *where x is an arbitrary future-pointing vector.*

An oriented and orthochronous real null frame $\{l_-, l_+, e_2, e_3\}$ can biunivocally be associated with every oriented and orthochronous orthonormal frame $\{e_\alpha\}_{\alpha=0}^3$

$$l_\epsilon = \frac{1}{\sqrt{2}}(e_0 + \epsilon e_1) \qquad e_0 = \frac{1}{\sqrt{2}}(l_+ + l_-) \qquad e_1 = \frac{1}{\sqrt{2}}(l_+ - l_-) \tag{7}$$

In a similar way, a self-dual oriented null frame $\{\mathcal{U}, \mathcal{H}_-, \mathcal{H}_+\}$ biunivocally corresponds to every self-dual oriented orthonormal frame $\{\mathcal{U}_i\}_{i=1}^3 : \mathcal{U} = \mathcal{U}_1, \mathcal{H}_\epsilon = \mathcal{U}_2 - \epsilon\, \mathrm{i}\, \mathcal{U}_3$. This self-dual frame is characterized by $\mathcal{U}$ to be unitary, $\mathcal{H}_\mp$ to be null, and the relations

$$(\mathcal{H}_+, \mathcal{H}_-) = -2 \qquad \mathcal{U} \times \mathcal{H}_\epsilon = \frac{\epsilon}{\sqrt{2}}\, \mathcal{H}_\epsilon \tag{8}$$

The fundamental vectors $l_\mp$ of $\mathcal{H}_\mp$ are the principal vectors of $\mathcal{U}$ and this fact is a necessary and sufficient condition for a null bivector $\mathcal{H}_\mp$ to be orthogonal to a unitary bivector $\mathcal{U}$.

At this point it is worth pointing out a difference between the oriented orthonormal frames and the null ones. In the first case we have a simple way of obtaining one element of the frame in terms of the other elements. For example, in an orthonormal frame $\{e_\alpha\}$, we have $e_0 = *(e_1 \wedge e_2 \wedge e_3)$. Expression (6) gives a similar property for a self-dual orthonormal frame $\{\mathcal{U}_i\}_{i=1}^3$. Thus, in both circumstances we are considering a *deductive* algorithm to complete the frame. Obviously this procedure does not work for the null frames. Then, we could in this case opt for solving the equations that restrict the unknown element. But if we also look for a *deductive* algorithm for this case, we can make use of expression (3) that determines a principal vector of a timelike unitary simple 2-form U. More precisely, for an oriented and orthochronous real null frame $\{l_-, l_+, e_2, e_3\}$, l_+ is given, in terms of the other three elements, by (3), with $U = *(e_2 \wedge e_3)$.

But, in order to determine the Weyl canonical frames in the next section, it will be more useful to obtain for the self-dual null frames a result similar to that given in lemma 1 for the orthogonal ones. In fact, making use of the above results for 2-forms it is easy to prove the following:

Lemma 2. *Let $\mathcal{U}$ and $\mathcal{H}$ be bivectors, unitary and null, respectively, verifying $\mathcal{U} \times \mathcal{H} = \frac{-1}{\sqrt{2}}\mathcal{H}$. Then, there exists a unique oriented and orthochronous real null frame $\{l_-, l_+, e_2, e_3\}$ such that the associated real 2-forms are $U = l_- \wedge l_+$ and $H = l_- \wedge e_2$. This frame is given by*

$$l_- = \frac{H^2(x)}{\sqrt{-H^2(x,x)}} \qquad l_+ = -\frac{[U^2 + U](x)}{2(x, l_-)} \qquad e_2 = H(l_+) \qquad e_3 = -*H(l_+) \tag{9}$$

x being an arbitrary timelike future-pointing vector.

3. A brief summary about Petrov classification

The algebraic classification of the Weyl tensor W can be obtained [7, 9, 11] by studying the traceless linear map defined by the self-dual Weyl tensor $\mathcal{W} = \frac{1}{2}(W - \mathrm{i}{*}W)$ on the bivectors space. In terms of its complex scalar invariants, $a = \mathrm{tr}\,\mathcal{W}^2$, $b = \mathrm{tr}\,\mathcal{W}^3$, the characteristic equation reads

$$x^3 - \tfrac{1}{2}ax - \tfrac{1}{3}b = 0. \tag{10}$$

The Petrov classification follows taking into account both the eigenvalue multiplicity and the degree of the minimal polynomial. The algebraically regular case (type I) occurs when the characteristic equation (10) admits three different roots ρ_k

$$\rho_k = \beta \mathrm{e}^{\frac{2\pi k}{3}\mathrm{i}} + \frac{a}{6\beta}\mathrm{e}^{-\frac{2\pi k}{3}\mathrm{i}} \qquad \beta = \sqrt[3]{\frac{1}{6}\left(b + \sqrt{b^2 - \frac{a^3}{6}}\right)}. \tag{11}$$

If there is a double root $\rho = -\frac{b}{a}$ and a simple one -2ρ, the minimal polynomial distinguishes between types D and II. Finally, if all the roots are equal, and therefore zero, the Weyl tensor is of type O, N or III, depending on the minimal polynomial. We summarize Petrov

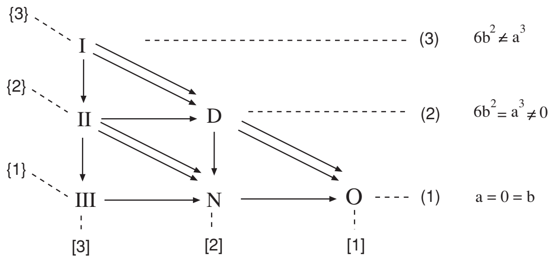

Figure 1. Penrose diagram.

classification in an extended-type Penrose diagram (see figure 1): the horizontal arrows mean a simple degeneration in the algebraic type without changing the number (n) of different eigenvalues; the vertical arrows also have associated simple degenerations, but now the degree of the minimal polynomial [r] is fixed. Finally, the double oblique arrows indicate that both degenerations are present but the number of independent invariant subspaces $\{s\}$ remains.

From all these considerations about the Petrov classification we can obtain algorithms, that use exclusively covariant conditions on the self-dual Weyl tensor $\mathcal{W}$, to determine the Weyl algebraic type. When the multiplicity of the eigenvalues is considered before the degree of the minimal polynomial, we obtain an algorithm equivalent to the d'Inverno and Russell-Clark algorithm [26] as for the number of steps and the decision scheme (see [34] and the references therein for the improvements of this algorithm). But if the algorithm is arranged starting from the minimal polynomial we obtain an alternative scheme that we present later (see figure 2). We put $\mathcal{P}^T$ to indicate the traceless part of a tensor. In particular, if $\mathcal{P}$ is a self-dual symmetric double 2-form,

$$[\mathcal{P}]^T{}_{\alpha\beta\epsilon\delta} = \mathcal{P}_{\alpha\beta\epsilon\delta} - \tfrac{1}{3}\left(\mathcal{G}^{\mu\nu\lambda\delta}\mathcal{P}_{\mu\nu\lambda\delta}\right)\mathcal{G}_{\alpha\beta\epsilon\delta}.$$

An alternative viewpoint to classify the Weyl tensor consists of studying the relative positions between the 'null cones' determined on the 2-form space by the canonical metric and the Weyl tensor as a quadratic form [24]. These 'null cones' defined by the metric $\mathcal{G}$ and the self-dual Weyl tensor $\mathcal{W}$ cut, generically, on four null bivectors $\mathcal{H}$ that are the solution of the equations

$$\mathcal{G}(\mathcal{H},\mathcal{H}) = 0 \qquad \mathcal{W}(\mathcal{H},\mathcal{H}) = 0. \tag{12}$$

We name them *null Debever bivectors*. The fundamental direction k of each one defines a null direction on the spacetime, usually called the *Debever direction* [8]. If we denote $W(k;k)_{\alpha\delta} = k^\beta k^\epsilon\ W_{\beta\alpha\epsilon\delta}$, the second equation in (12) for the Debever direction k is written as [5]

$$k \wedge W(k;k) \wedge k = 0 \tag{13}$$

Given a null frame $\{\mathcal{U}, \mathcal{H}_-, \mathcal{H}_+\}$ we can always write a null bivector $\mathcal{H}$ (with a suitable parametrization) in the form $\mathcal{H} = 2\nu\mathcal{U} - \nu^2\mathcal{H}_+ + \mathcal{H}_-$. Then, the second condition in (12) leads to a complex fourth-degree algebraic equation on ν. The alternative approach to the Weyl algebraic classification follows by analysing the multiplicity of the roots of this equation [8]. Type I appears as a case where four simple Debever directions exist, a double direction and two simple directions exist in type II, type D is a case with two different double Debever direction, a Weyl tensor of type III has a simple direction and a triple one and finally, a quadruple Debever direction exists in type N. The Penrose diagram shown in figure 1 suitably summarizes this approach, where the arrows indicate a degeneration in the multiplicity.

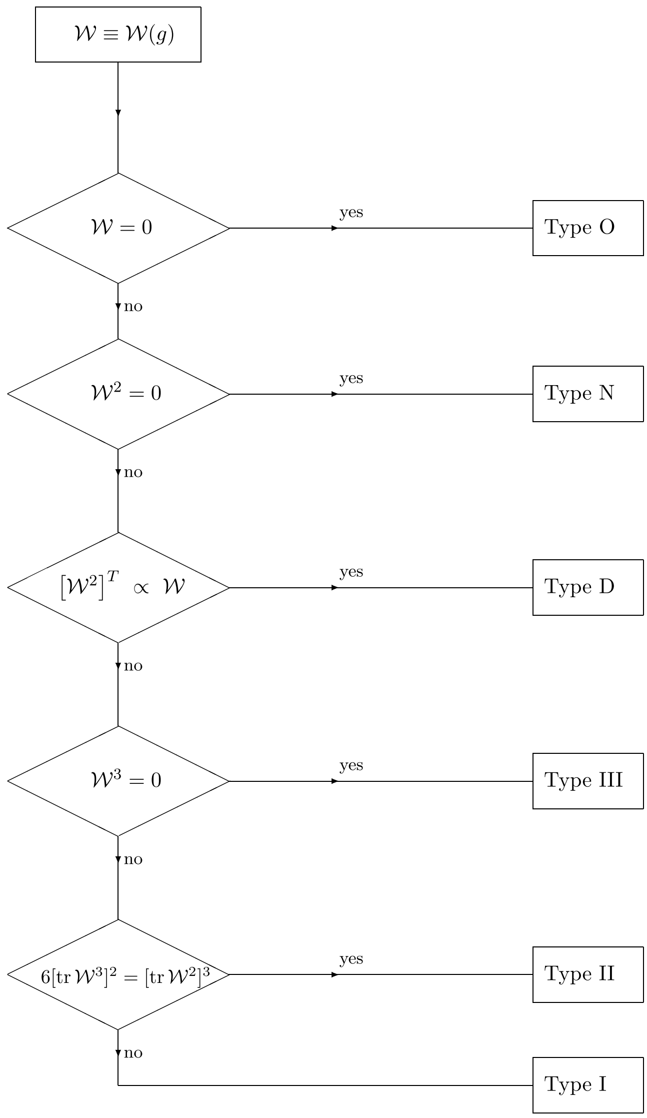

Figure 2. Algorithm to determine the Petrov type.

Depending on its multiplicity, a Debever direction k satisfies an equation of the Sachs [5] hierarchy, an equation that has an equivalent condition on the corresponding null Debever bivector $\mathcal{H}$ [35]. In the case of a simple direction we have equation (13) for k and the second in (12) for $\mathcal{H}$. The double, triple or quadruple character is characterized, respectively, by

$$k \wedge W(k; k) = 0 \qquad \text{or} \qquad \mathcal{H} \times \mathcal{W}(\mathcal{H}) = 0 \tag{14}$$

$$k \wedge i(k)W = 0 \qquad \text{or} \qquad \mathcal{W}(\mathcal{H}) = 0 \tag{15}$$

$$i(k)W = 0 \qquad \text{or} \qquad \mathcal{H} \times \mathcal{W} = 0 \tag{16}$$

Taking into account these conditions that characterize the multiple null Debever 2-forms we arrive easily at the following known result: the multiple Debever directions are the fundamental directions of null eigenbivectors of $\mathcal{W}$.

For a given Weyl tensor, every pair $\mathcal{H}_{\mp}$ of null Debever bivectors defines a 2-plane in the bivector space with a non-null orthogonal direction. The unitary bivector $\mathcal{V}$ in this direction has the fundamental directions of $\mathcal{H}_{\mp}$ as the principal directions, and so $\mathcal{V}$ has two Debever directions as principal directions. When this occurs we say that $\mathcal{V}$ is a *unitary Debever bivector*. In the type N case no unitary Debever bivector exists. Only one unitary Debever bivector can be associated with a Petrov type III Weyl tensor, and the same occurs in the case of type D. When the Weyl tensor is type II we have three unitary Debever bivectors, one of them with the simple Debever directions as principal directions and the other with the double Debever direction and each of the simple ones. Finally, when the Weyl tensor is Petrov type I, we can form six pairs with the four simple Debever directions that this case admits.

4. Weyl tensor geometry

Once the basic notation and concepts concerning to the Weyl tensor and its algebraic classification have been introduced, we come to the main goal of this work: a covariant and deductive determination of all the geometric objects that a given Weyl tensor defines. The richness of these objects depends strongly on the Petrov type. Thus, we separately study the different algebraic classes by considering four steps for every one. We begin with the canonical form of the self-dual Weyl tensor and examine the geometric elements induced by it, like the eigenbivectors and other characteristic bivectors, and their associated spacetime directions. Secondly, we analyse the different Petrov types taking into account Debever directions, present an alternative canonical form which is better adapted to the Debever bivectors and analyse its relation with those in the first step. At the third level we give the deductive algorithms that determine the geometric objects presented in the two previous steps. Finally, we accurately define the canonical frames for every Petrov type and offer an algorithm to determine them.

4.1. Type N

(a) We start with the N Petrov type, the more algebraically degenerated case for a non-identically zero Weyl tensor. From its minimal equation, $\mathcal{W}^2 = 0$, it follows that the image of the self-dual Weyl tensor $\mathcal{W}$ is a bivector null direction. Thus, there exists a null bivector $\mathcal{H}$ such that

$$\mathcal{W} = \mathcal{H} \otimes \mathcal{H}. \tag{17}$$

We name $\mathcal{H}$ as the *canonical bivector* and it is determined up to the sign. We denote l as the fundamental vector of $\mathcal{H}$ that is named the *fundamental vector* of a type N Weyl tensor. The real associated 2-forms H and $*H$ define two null 2-planes that we name *canonical 2-planes*.

From (17) one can see that the Weyl eigenbivectors constitute the 2-plane orthogonal to $\mathcal{H}$. The unique null eigendirection is that defined by $\mathcal{H}$, and the other (unitary) eigenbivectors are

$$\mathcal{U} = \tilde{\mathcal{U}} + \nu\mathcal{H} \tag{18}$$

where $\nu \in \mathbb{C}$ and $\tilde{\mathcal{U}}$ is a unitary bivector orthogonal to $\mathcal{H}$. Thus, the null eigenbivectors have l as the fundamental direction and the unitary ones have l as a principal direction.

(b) A type N Weyl tensor may also be characterized by admitting a quadruple Debever direction, or equivalently, a null Debever bivector exists verifying equation (16). From (17) the canonical bivector $\mathcal{H}$ is a solution to this equation. Thus:

Proposition 1. *The fundamental vector l of a type N Weyl tensor determines the quadruple Debever direction.*

We observe that, in this more degenerate case, the Debever viewpoint does not give other geometric elements than those that may be found in the Jordan canonical form (17). However, the more regular the Weyl tensor is, the more the geometries of both approaches differ. In order to clarify the relation between them, when we next consider less degenerate cases, we eventually offer alternative canonical forms adapted to the Debever bivectors. Moreover, this helps us understand the way every Petrov type degenerates along the arrows of a Penrose diagram.

(c) Now we carry out the covariant and deductive determination of the main geometric elements associated with a type N Weyl tensor that we have presented above: the canonical bivector $\mathcal{H}$ and the fundamental vector l. Taking into account (17) and expression (5) that gives the fundamental vector of a singular 2-form, we have the following:

Proposition 2. *Let $\mathcal{W}$ be a type N self-dual Weyl tensor. Then, its canonical bivector $\mathcal{H}$ may be obtained as*

$$\mathcal{H} = \frac{\mathcal{W}(\mathcal{X})}{\sqrt{\mathcal{W}(\mathcal{X}, \mathcal{X})}} \tag{19}$$

where $\mathcal{X}$ is an arbitrary bivector such that $\mathcal{W}(\mathcal{X}) \neq 0$. The fundamental vector l is given by

$$l = \frac{H^2(x)}{\sqrt{-H^2(x, x)}} \tag{20}$$

where H is the real 2-form associated with $\mathcal{H}$, and x is an arbitrary timelike future-pointing vector.

(d) We now look for the two-parameter family of canonical frames that in the type N case exists. We can obtain them from lemma 2 taking $\mathcal{H}$ as the canonical bivector and $\mathcal{U}$ as every unitary eigenbivector. From the expression (18), these last have associated real 2-forms given by $U = \tilde{U} + \lambda H + \mu * H$, with $\lambda, \mu \in \mathbb{R}$, and $\tilde{U}$ is a unitary 2-form admitting l as a principal direction. We build $\tilde{U}$ using a timelike future-pointing vector x, $\tilde{U} = -\frac{1}{(l,x)} l \wedge x$. Then, applying lemma 2 and taking into account that the canonical bivector is fixed up to the sign, we finally get to

Proposition 3. *If the Weyl tensor is of Petrov type N, let $\mathcal{H}$ and l be the canonical bivector and the fundamental vector given in (19) and (20), respectively. Then, a two-parameter family of oriented and orthochronous canonical null frames $\{l, l', e_2, e_3\}$ exist such that $H = \epsilon\, l \wedge e_2$, $*H = -\epsilon\, l \wedge e_3$. These frames are given by*

$$l' = -\frac{[U^2 + U](x)}{2(x, l)} \qquad e_2 = \epsilon H(l') \qquad e_3 = -\epsilon * H(l') \tag{21}$$

where ϵ takes the values ± 1, and where

$$U = \tilde{U} + \lambda H + \mu * H \qquad \tilde{U} = -\frac{l \wedge x}{(l, x)} \qquad \lambda, \mu \in \mathbb{R} \tag{22}$$

x being an arbitrary timelike future-pointing vector.

We will have a better understanding of the two-parameter freedom in choosing a canonical frame if we observe that only the fundamental vector l and the 2-planes H and $*H$ are outlined in this case. Then, we can take two spacelike unitary vectors e_2 and e_3, in H and $*H$, respectively, and complete a real null frame $\{l, l', e_2, e_3\}$. This frame is not unique because e_2 and e_3 are fixed up to change $e_2 = \epsilon \tilde{e}_2 + \lambda l$, $e_3 = \epsilon \tilde{e}_3 + \mu l$, with $\lambda, \mu \in \mathbb{R}$. An alternative and equivalent algorithm to obtain the frames follows from this view because a sample of unitary vectors in the planes H and $*H$ may be obtained, respectively, as

$$\tilde{e}_2 = \frac{H(x)}{\sqrt{-H^2(x, x)}} \qquad \tilde{e}_3 = \frac{*H(x)}{\sqrt{-H^2(x, x)}}.$$

Also, for every pair e_2, e_3 generated by two real numbers λ, μ, the null vector l' that completes the frame is given by the first expression in (21) with $U = *(e_2 \wedge e_3)$.

4.2. Type III

(a) In type III, the minimal equation, $\mathcal{W}^3 = 0$, implies that there exists a null bivector $\mathcal{H}$ such that $\mathcal{W}^2 = -\mathcal{H} \otimes \mathcal{H}$. But $\mathcal{W}^2 \neq 0$ and, consequently, the image of $\mathcal{W}$ is the 2-plane orthogonal to $\mathcal{H}$ that is also an invariant subspace. Thus, we can take a unitary bivector $\mathcal{U}$ (orthogonal to $\mathcal{H}$) such that

$$\mathcal{W} = \mathcal{U} \tilde{\otimes} \mathcal{H}. \tag{23}$$

We name $\mathcal{H}$ as the *canonical null bivector* and its fundamental vector l is named the *fundamental vector* of a type III Weyl tensor. The real associated 2-forms H and $*H$ define the *null canonical 2-planes*. The canonical null bivector $\mathcal{H}$ determines the unique eigendirection that the Weyl tensor admits in this case, but in the invariant subspace the outlined bivector $\mathcal{U}$ exists. We name it the *canonical unitary bivector* and it admits l as a principal direction. The pair of canonical bivectors $\{\mathcal{H}, \mathcal{U}\}$ is fixed up to the change of sign in both elements. However, the spacetime orientation allows us to discriminate between the two possibilities, and we can always choose that verifying $\mathcal{U} \times \mathcal{H} = -\frac{1}{\sqrt{2}} \mathcal{H}$.
(b) When the Weyl tensor is Petrov type III, there exists a triple Debever direction and a simple one. It is evident that the null canonical bivector $\mathcal{H}$ verifies equation (15) and, consequently, the fundamental vector defines the triple Debever direction. On the other hand, the other null bivector $\mathcal{H}'$, which is orthogonal to the unitary canonical bivector $\mathcal{U}$, is a solution to equation (13) that characterizes a simple Debever direction. Therefore, the two Debever directions are just the principal directions of $\mathcal{U}$ and so the unitary canonical bivector $\mathcal{U}$ is the *unitary Debever bivector* that exists in this case. Thus, we have the following result:

Proposition 4. *The fundamental vector l of a type III Weyl tensor determines the triple Debever direction. This, and the simple one, are the principal directions of the unitary canonical bivector.*

We have seen that, like the type N case, when the Weyl tensor is of Petrov type III, the associated Debever geometric elements follow from the Jordan canonical form. Thus, if we compare the expression (23) with the corresponding one for type N (17), we can understand the arrow between these two types in a Penrose diagram. A null bivector $\mathcal{H}$ in the canonical form indicates a double multiplicity as the Debever vector for its fundamental direction. In type N, its canonical bivector $\mathcal{H}$ appears twice, and so we have a quadruple Debever direction. Nevertheless, in the canonical form of a type III Weyl tensor we have a null bivector $\mathcal{H}$ and the unitary Debever bivector $\mathcal{U}$, a principal direction of $\mathcal{U}$ being the fundamental direction of $\mathcal{H}$. This indicates a triple Debever direction and a simple one. The degeneration from a type III to a type N Weyl tensor happens, therefore, when the regular Debever bivector $\mathcal{U}$ changes to become the null Debever bivector $\mathcal{H}$.
(c) We have outlined above the geometric objects defined by a type III Weyl tensor: the canonical bivectors $\mathcal{H}$ and $\mathcal{U}$, the fundamental vector l and the simple Debever direction, which we denote l' when we consider the canonical parametrization $(l, l') = -1$. Starting from the canonical form (23) we give covariant expressions for $\mathcal{H}$ and $\mathcal{U}$ in terms of the self-dual Weyl tensor and arbitrary bivectors. On the other hand, from these expressions and using (5) and (3), the null vectors l and l' are obtained. Thus, we state

Proposition 5. *Let $\mathcal{W}$ be a type III Weyl tensor. Then, the canonical null bivector $\mathcal{H}$ and the canonical unitary bivector $\mathcal{U}$ may be obtained as*

$$\mathcal{H} = -\frac{\mathcal{W}^2(\mathcal{X})}{\sqrt{-\mathcal{W}^2(\mathcal{X},\mathcal{X})}} \qquad \mathcal{U} = \frac{1}{2(\mathcal{H},\mathcal{X})^2}[2(\mathcal{H},\mathcal{X})\mathcal{W}(\mathcal{X}) - \mathcal{W}(\mathcal{X},\mathcal{X})\,\mathcal{H}] \tag{24}$$

where $\mathcal{X}$ is an arbitrary bivector such that $\mathcal{W}^2(\mathcal{X}) \neq 0$.

The fundamental vector l and the simple Debever vector l' are given by

$$l = \frac{H^2(x)}{\sqrt{-H^2(x,x)}} \qquad l' = -\frac{[U^2 + \epsilon U](x)}{2(x,l)} \tag{25}$$

where H and U are the real 2-forms associated with $\mathcal{H}$ and $\mathcal{U}$, ϵ is such that $\mathcal{U} \times \mathcal{H} = -\frac{\epsilon}{\sqrt{2}}\,\mathcal{H}$, and x is an arbitrary timelike future-pointing vector.

(d) In the Petrov type III case a unique canonical frame can be selected. Indeed, we can apply lemma 2 taking $\mathcal{U}$ and $\mathcal{H}$ as the canonical bivectors. Then, taking into account its sign in order to satisfy the hypothesis of this lemma, we arrive at the following:

Proposition 6. *If the Weyl tensor is of Petrov type III, let $\mathcal{H}$ and $\mathcal{U}$ be the canonical bivectors given in (24), and let us consider ϵ such that $\mathcal{U} \times \mathcal{H} = -\frac{\epsilon}{\sqrt{2}}\,\mathcal{H}$. Then, we have the oriented and orthochronous canonical frame $\{l, l', e_2, e_3\}$ such that $U = \epsilon l \wedge l'$, $H = l \wedge e_2$, $*H = -l \wedge e_3$, where l and l' are given in (25), and*

$$e_2 = \epsilon\; H(l') \qquad e_3 = -\epsilon * H(l'). \tag{26}$$

4.3. Type D

(a) The minimal equation in this case, $(\mathcal{W} + 2\rho\mathcal{G})(\mathcal{W} - \rho\mathcal{G}) = 0$, implies that a non-null eigendirection is associated to a simple eigenvalue -2ρ, and the 2-plane orthogonal to this direction is the eigenspace corresponding to a double eigenvalue $\rho = -\frac{\operatorname{tr}\mathcal{W}^3}{\operatorname{tr}\mathcal{W}^2}$. If we denote $\mathcal{U}$ as the unitary eigenbivector corresponding to the simple eigenvalue, and because the induced metric on the orthogonal space is $\mathcal{G} + \mathcal{U} \otimes \mathcal{U}$, we have the following canonical form for the self-dual Weyl tensor:

$$\mathcal{W} = 3\rho\mathcal{U} \otimes \mathcal{U} + \rho\mathcal{G}. \tag{27}$$

We name $\mathcal{U}$ as the *canonical bivector* of a type D Weyl tensor, and the two 2-planes defined by the associated real 2-forms U and $*U$ are its *principal 2-planes*. The bivector $\mathcal{U}$ is fixed up to the sign and its principal directions $l_\pm$ have not outlined any canonical parametrization.
(b) In the eigen 2-plane orthogonal to the canonical bivector $\mathcal{U}$ there exist two null eigendirections that we denote $\mathcal{H}_\pm$. Their fundamental directions are the principal directions of $\mathcal{U}$. It is easy to show that the null bivectors $\mathcal{H}_\pm$ are the solution to the equations (14) and, consequently, they are the two null Debever bivectors that a type D Weyl tensor admits. Thus $\mathcal{U}$ is the unique unitary Debever bivector that exists in this case. In conclusion we have

Proposition 7. *The double Debever directions of a type D Weyl tensor are the principal directions of the canonical bivector $\mathcal{U}$.*

In a Penrose diagram the Petrov type D can degenerate to type N through a simple arrow, or to type O with a double degeneration. This evidently follows taking $\rho = 0$ in the Jordan canonical form (27) which implies that $W = 0$. But type N has zero scalar invariants too, and its canonical form (17) cannot be achieved from (27). However, we can overcome this shortcoming by considering an alternative canonical expression for a type D Weyl tensor. Indeed, if we parametrize the null Debever bivectors in the form $(\mathcal{H}_+, \mathcal{H}_-) = 6\rho$, the self-dual metric becomes $\mathcal{G} = -\mathcal{U} \otimes \mathcal{U} - \frac{1}{6\rho}\mathcal{H}_+ \tilde{\otimes} \mathcal{H}_-$. Then we can eliminate the canonical bivector $\mathcal{U}$ of (27) and we arrive at

$$\mathcal{W} = -2\rho\,\mathcal{G} + \tfrac{1}{2}\mathcal{H}_+ \tilde{\otimes} \mathcal{H}_-. \tag{28}$$

From this expression for a type D Weyl tensor we arrive at the type N canonical form making $\rho = 0$ and $\mathcal{H}_+ = \mathcal{H}_-$.

(c) The characteristic geometric elements in the Petrov type D case are the canonical form $\mathcal{U}$ and its principal directions, the double Debever directions $l_\pm$. From (27) it results that $\mathcal{W} - \rho\mathcal{G}$ is the projector on the simple eigendirection $\mathcal{U}$. Furthermore, if we consider expression (2), which determines the principal directions of a unitary 2-form, we have shown:

Proposition 8. *Let $\mathcal{W}$ be a type D Weyl tensor. Then, the canonical bivector $\mathcal{U}$ is obtained as*

$$\mathcal{U} = \frac{\mathcal{P}(\mathcal{X})}{\sqrt{-\mathcal{P}^2(\mathcal{X}, \mathcal{X})}} \qquad \mathcal{P} \equiv \mathcal{W} - \rho\,\mathcal{G} \tag{29}$$

where $\mathcal{X}$ is an arbitrary bivector such that $\mathcal{P}(\mathcal{X}) \neq 0$.

The double Debever directions (or principal null directions) $l_\pm$ are given by

$$l_\pm \propto [U^2 \pm U](x) \tag{30}$$

where U is the real 2-form associated with $\mathcal{U}$, and x is an arbitrary timelike future-pointing vector.

(d) In the type D case orthonormal frames formed with eigenbivectors exist: we can consider the canonical bivector $\mathcal{U}$ and choose a unitary bivector $\mathcal{U}_2$ in the orthogonal eigen 2-plane, the third element of the oriented frame then being given by (6). But from lemma 1 we know that every oriented frame on the bivector space determines an oriented and orthochronous orthonormal frame $\{e_\alpha\}$ on the spacetime. For this reason, in this case we have a two-parameter family of canonical frames because the choice of $\mathcal{U}_2$ depends on a complex rotation. All these frames satisfy that $U = \pm e_0 \wedge e_1$ and $*U = \mp e_2 \wedge e_3$ and, consequently, the two parameter freedom permits a boost on the 2-plane U and a rotation on the 2-plane $*U$. Furthermore, spacelike directions in these two 2-planes can be determined as $U(x)$ and $*U(y)$, with x and y arbitrary vectors, the first being timelike. Then, if we also consider that the canonical bivector $\mathcal{U}$ is fixed up to the sign, we arrive at the following.

Proposition 9. *If the Weyl tensor is of Petrov type D, let $\mathcal{U}$ be the canonical bivector given in (29). Then, a two-parameter family of oriented and orthochronous canonical frames $\{e_\alpha\}$ exists such that $U = \epsilon e_0 \wedge e_1$ and $*U = -\epsilon e_2 \wedge e_3$. These frames are given by*

$$\begin{aligned} e_0 &= \cosh\psi\,\tilde{e}_0 + \epsilon \sinh\psi\,\tilde{e}_1 \qquad & e_1 &= \sinh\psi\,\tilde{e}_0 + \epsilon\cosh\psi\,\tilde{e}_1 \\ e_2 &= \cos\phi\,\tilde{e}_2 - \epsilon\sin\phi\,\tilde{e}_3 \qquad & e_3 &= \sin\phi\,\tilde{e}_2 + \epsilon\cos\phi\,\tilde{e}_3 \end{aligned} \tag{31}$$

with $\psi \in \mathbb{R}$ and $\phi \in [0, 2\pi)$, where ϵ takes the values ± 1, and where

$$\tilde{e}_1 = \frac{U(x)}{\sqrt{-U^2(x,x)}} \qquad \tilde{e}_0 = U(\tilde{e}_1)$$
$$\tilde{e}_2 = \frac{*U(y)}{\sqrt{-*U^2(y,y)}} \qquad \tilde{e}_3 = *U(\tilde{e}_2) \tag{32}$$

x being an arbitrary timelike vector, and y an arbitrary vector verifying $*U(y) \neq 0$.

4.4. Type II

(a) A Petrov type II has, like the type D, a simple eigenvalue -2ρ and a double one ρ. However, now $(\mathcal{W} - \rho\,\mathcal{G})(\mathcal{W} + 2\rho\,\mathcal{G}) \neq 0$ and the minimal polynomial coincides with the characteristic one. This implies that the restriction of the self-dual Weyl tensor on the invariant 2-plane orthogonal to the simple eigendirection has non-zero traceless part, that is to say, $\mathcal{W} - 3\rho\,\mathcal{U} \otimes \mathcal{U} - \rho\mathcal{G} \neq 0, \mathcal{U}$ being the simple unitary eigenbivector. But the square of this tensor is zero and, consequently, there exists a null bivector $\mathcal{H}$ such that the self-dual Weyl tensor takes the canonical form

$$\mathcal{W} = 3\rho\mathcal{U} \otimes \mathcal{U} + \rho\mathcal{G} + \mathcal{H} \otimes \mathcal{H}. \tag{33}$$

We will name $\mathcal{H}$ as the *canonical null bivector* and its fundamental vector l is named the *fundamental vector* of a type II Weyl tensor. The canonical null bivector $\mathcal{H}$ determines the unique eigendirection associated with the double eigenvalue ρ. In this case we have also outlined the *canonical unitary bivector* $\mathcal{U}$, an eigenbivector that also admits l as the principal direction. Both canonical bivectors, $\mathcal{H}$ and $\mathcal{U}$, are fixed up to the change of sign. However, the spacetime orientation permits us to discriminate two cases among the four possible. Indeed, given $\mathcal{H}$, we can always choose $\mathcal{U}$ verifying $\mathcal{U} \times \mathcal{H} = -\frac{1}{\sqrt{2}}\,\mathcal{H}$.

(b) When the Weyl tensor is Petrov type II there exist two simple Debever directions and a double one. Thus, we can consider three unitary Debever bivectors. One of them has the simple Debever directions as the principal directions, and we name it the *(1, 1)-Debever bivector*. The other two are *(2, 1)-Debever bivectors* with principal directions the double Debever direction and a simple one. From the Jordan canonical expression (33) it is easy to show that the canonical null bivector $\mathcal{H}$ verifies equation (14) and so it is a null Debever bivector. Consequently, the fundamental vector l is the double Debever direction that a type II Weyl tensor admits. We also know that l is a principal direction of the canonical unitary bivector $\mathcal{U}$, but the other null bivector $\mathcal{H}'$ on the invariant 2-plane is not a solution of (12) and so $\mathcal{U}$ is not a unitary Debever bivector. Hence, in this case the canonical form (33) only renders the Debever geometry partially explicit. But, what about the two simple Debever directions? In order to detect them we look for the (2,1)-Debever bivectors $\mathcal{V}_\pm$. They have l as a principal direction and so belong to the 2-plane orthogonal to $\mathcal{H}$. Therefore, they have the form $\mathcal{V}_\pm = \mathcal{U} + \lambda_\pm\mathcal{H}$. Then, taking $\lambda_\pm = \pm\frac{\mathrm{i}}{\sqrt{3\rho}}$, the expression (33) of the Weyl tensor turns out to be

$$\mathcal{W} = \rho\mathcal{G} + \tfrac{3}{2}\rho\mathcal{V}_+ \tilde{\otimes} \mathcal{V}_-. \tag{34}$$

This Debever canonical form for the Weyl tensor shows that the null bivectors orthogonal to $\mathcal{V}_\pm$ are Debever null bivectors which are the solutions of (12) and so $\mathcal{V}_\pm$ are the (2,1)-Debever bivectors. We can summarize these results as follows:

Proposition 10. *The fundamental vector l of a type II Weyl tensor determines the double Debever direction. This is also a principal direction of the Debever bivectors $\mathcal{V}_\pm$ which are written in terms of canonical bivectors:*

$$\mathcal{V}_\pm = \mathcal{U} \pm \frac{\mathrm{i}}{\sqrt{3\rho}}\mathcal{H}. \tag{35}$$

The simple Debever directions are the other principal directions of $\mathcal{V}_\pm$.

The Jordan canonical form (33) of a Petrov type II collapses to the canonical form of the type N case if we make $\rho = 0$. Thus, we appreciate the double degeneration that a type II Weyl tensor can suffer in a Penrose diagram. To understand the simple decays to types D or III we must again make use of the Debever canonical expressions. In fact we recover the expression (27) of type D taking $\mathcal{V}_+ = \mathcal{V}_- = \mathcal{U}$ in (34). The simple arrow towards type III can be comprehended considering other canonical expressions adapted to the (1,1)-Debever bivector $\mathcal{V}$. Indeed, in the 2-plane orthogonal to $\mathcal{U}$ we can find the unitary bivector $\mathcal{V}$ such that a type II Weyl tensor takes the form

$$\mathcal{W} = -2\rho\,\mathcal{G} + \hat{\mathcal{H}} \tilde{\otimes} \mathcal{V} \tag{36}$$

where $\hat{\mathcal{H}}$ is an adequate parametrization of the null canonical bivector $\mathcal{H}$. So, making $\rho = 0$, we recover the canonical expression (23) of type III.

(c) Now we will go to the algorithms to obtain the canonical bivectors, fundamental direction and other Debever directions of a type II Weyl tensor. From the characteristic equation, $(\mathcal{W} - \rho\mathcal{G})^2(\mathcal{W} + 2\rho\mathcal{G}) = 0$, it follows that $(\mathcal{W} - \rho\,\mathcal{G})^2(\mathcal{X})$ has the direction of the unitary canonical bivector $\mathcal{U}$. Likewise, the null canonical direction can be explicitly obtained as $(\mathcal{W} - \rho\mathcal{G})(\mathcal{W} + 2\rho\mathcal{G})(\mathcal{X})$. Then, taking into account the expression (35) of the Debever bivectors and the results in section 2 about 2-forms, we can enunciate

Proposition 11. *Let $\mathcal{W}$ be a type II self-dual Weyl tensor. Then, its canonical bivectors $\mathcal{U}$ and $\mathcal{H}$ may be obtained as*

$$\mathcal{U} = \frac{\mathcal{P}(\mathcal{X})}{\sqrt{-\mathcal{P}^2(\mathcal{X},\mathcal{X})}} \qquad \mathcal{H} = \frac{\mathcal{Q}(\mathcal{Y})}{\sqrt{\mathcal{Q}(\mathcal{Y},\mathcal{Y})}} \tag{37}$$

with $\mathcal{P} = (\mathcal{W} - \rho\mathcal{G})^2$ and $\mathcal{Q} = \frac{1}{3\rho}[(\mathcal{W} - \rho\mathcal{G})(\mathcal{W} + 2\rho\mathcal{G})]$, and where $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary bivectors such that $\mathcal{P}(\mathcal{X}) \neq 0$, $\mathcal{Q}(\mathcal{Y}) \neq 0$, respectively.

The fundamental vector l and the simple Debever directions $l_\pm$ are given by

$$l = \frac{H^2(x)}{\sqrt{-H^2(x,x)}} \qquad l_\pm = -\frac{\left[V_\pm^2 + \epsilon V_\pm\right](x)}{2(x,l)} \tag{38}$$

where H and $V_\pm$ are, respectively, the real 2-forms associated with the canonical bivector $\mathcal{H}$ and the Debever bivector $\mathcal{V}_\pm$ given in (35), ϵ is such that $\mathcal{V}_\pm \times \mathcal{H} = -\frac{\epsilon}{\sqrt{2}}\,\mathcal{H}$ and x is an arbitrary timelike future-pointing vector.

(d) In the Petrov type II we have the orthogonal canonical bivectors $\mathcal{U}$ and $\mathcal{H}$ that are fixed up to the sign. Then, if we consider the two pairs $\{\mathcal{U}, \mathcal{H}\}$ that verify the hypothesis of lemma 2, we obtain the two canonical frames for type II

Proposition 12. *If the Weyl tensor is of Petrov type II, let $\mathcal{H}$ and $\mathcal{U}$ be the canonical bivectors given in (37), and let us consider ϵ such that $\mathcal{U} \times \mathcal{H} = -\frac{\epsilon}{\sqrt{2}}\,\mathcal{H}$. Then, we have the oriented and orthochronous canonical frames $\{l, l', e_2, e_3\}$ and $\{l, l', -e_2, -e_3\}$ such that $U = \epsilon l \wedge l'$, $H = l \wedge e_2$, $*H = l \wedge e_3$, where l is given in (38), and*

$$l' = -\frac{[U^2 + \epsilon\, U](x)}{2(x,l)} \qquad e_2 = H(l') \qquad e_3 = -{*H}(l') \tag{39}$$

where x is an arbitrary timelike future-pointing vector.

4.5. Type I

(a) Finally, we study the algebraically general case. The Weyl tensor is of Petrov type I when $6b^2 \neq a^3$ or, equivalently, when the characteristic equation admits three different roots $\{\rho_i\}$.

Then there exists an orthonormal frame of eigenbivectors $\{\mathcal{U}_i\}$, and so the Weyl tensor is written as

$$\mathcal{W} = -\sum_{i=1}^{3} \rho_i \mathcal{U}_i \otimes \mathcal{U}_i. \tag{40}$$

Thus, in this case we have three *canonical bivectors* $\mathcal{U}_i$, fixed up to the sign, and six *canonical 2-planes* defined by the associated 2-forms U_i and $*U_i$.

(b) Type I can be also characterized as admitting four simple Debever directions. Let $\mathcal{H}_{i\pm}$ be the null bivectors that are orthogonal to the canonical bivector $\mathcal{U}_i$. It is evident that no $\mathcal{H}_{i\pm}$ verify Debever equation (12), and so the principal directions of the canonical bivectors $\mathcal{U}_i$ do not coincide with any Debever direction. Thus, the canonical form (40) does not give, at first view, information about the Debever geometry of a type I Weyl tensor. In consequence, as in the case of type II, we can only achieve the Debever directions using the unitary Debever bivectors.

In [20] we have shown that for every canonical bivector $\mathcal{U}_i$, choosing for example $i = 3$, we can find in its orthogonal 2-plane two unitary bivectors without common principal directions $\mathcal{V}_\pm$ such that the Weyl tensor is written as

$$\mathcal{W} = \rho_3\, \mathcal{G} + \frac{\rho_2 - \rho_1}{2}\, \mathcal{V}_+ \tilde{\otimes} \mathcal{V}_-. \tag{41}$$

This expression shows that the null bivectors orthogonal to $\mathcal{V}_\pm$ are Debever null bivectors which are solutions of (12) and so $\mathcal{V}_\pm$ are unitary Debever bivectors. They are defined as the unique directions such that their bisectors are the canonical bivectors $\mathcal{U}_1$ and $\mathcal{U}_2$ and, moreover, they form an angle depending on scalar invariants. The results can be summarized in the following [20]:

Proposition 13. *The four Debever directions of a type I Weyl tensor are the principal directions of the bivectors*

$$\mathcal{V}_\epsilon = \cos\Omega\, \mathcal{U}_1 + \epsilon\, \sin\Omega\, \mathcal{U}_2 \qquad \epsilon = \pm 1 \tag{42}$$

where Ω *is the invariant scalar such that* $\cos^2\Omega = \frac{\rho_3 - \rho_1}{\rho_2 - \rho_1}$.

It is inportant to mention that, if we consider $i = 2, 3$, we can obtain similar expressions for the other four unitary Debever bivectors that the type I admits.

Petrov type I turns to type D through a double arrow in a Penrose diagram. This degeneration can be appreciated making $\rho_2 = \rho_3 = \rho$ and $\rho_1 = -2\rho$ in type I canonical expression (40), and thus obtaining the form (27) of type D. Imposing identical degeneration on the eigenvalues in the Debever canonical form (41) we arrive at (34) and so master the simple arrow between type I and type II.

(c) Now we bring up the determination of the geometric elements outlined above for a type I Weyl tensor: canonical bivectors $\{\mathcal{U}_i\}$ and Debever directions. From the characteristic equation, $\prod_{i=1}^{3}(\mathcal{W} - \rho_i\mathcal{G}) = 0$, it follows that $(\mathcal{W} - \rho_i\mathcal{G})(\mathcal{P}_i(\mathcal{X})) = 0$ for every bivector $\mathcal{X}$, where $\mathcal{P}_i = \prod_{j\neq i}(\mathcal{W} - \rho_j\mathcal{G})$. Thus $\mathcal{P}_i(\mathcal{X})$ belongs to the eigenspace corresponding to the eigenvalue ρ_i, that is to say, $\mathcal{P}_i$ is the projection map on this eigenspace, and it permits the canonical bivectors to be obtained. Unitary Debever bivectors can then be determined using (42). Finally, putting in use expression (2) which gives the principal directions of a unitary bivector, we are led to the following:

Proposition 14. *Let* $\mathcal{W}$ *be a type I self-dual Weyl tensor. Then, its canonical bivector* $\mathcal{U}_i$ *corresponding to the eigenvalue* ρ_i *may be obtained as*

$$\mathcal{U}_i = \frac{\mathcal{P}_i(\mathcal{X})}{\sqrt{-\mathcal{P}_i^2(\mathcal{X}, \mathcal{X})}} \tag{43}$$

with $\mathcal{P}_i = \mathcal{W}^2 + \rho_i \mathcal{W} + \left(\rho_i^2 - \frac{1}{2}a\right)\mathcal{G}$*, and where* $\mathcal{X}$ *is an arbitrary bivector such that* $\mathcal{P}_i(\mathcal{X}) \neq 0$*. The four Debever directions are given by*

$$l_{\epsilon\,\pm} \propto \left[V_\epsilon^2 \pm V_\epsilon\right](x) \tag{44}$$

where V_ϵ *is the real 2-form associated with the Debever bivector* $\mathcal{V}_\epsilon$ *given in (42), and x is an arbitrary timelike future-pointing vector.*

(d) We finish with the canonical frames of a Petrov type I Weyl tensor. The last proposition gives its canonical bivectors $\mathcal{U}_i$ that define an orthonormal frame on the bivector space. They are determined up to the sign and permutation. Thus, we can consider 24 oriented eigenframes $\{\mathcal{U}_i\}$: for every permutation, the sign of two of them gives us four possibilities, the third being given by (6). Thus, as a consequence of lemma 1, we have 24 *canonical frames* of the Weyl tensor, all of them formed with the four *principal directions* of a type I Weyl tensor. We have specifically [20]

Proposition 15. *If the Weyl tensor is Petrov type I, let* $\mathcal{U}_i$ *be the canonical bivectors given in (43), chosen in such a way that (6) is satisfied. The Weyl canonical frames* $\{e_\alpha\}$ *of a Petrov type I spacetime may be determined as*

$$e_0 = \frac{-P_0(x)}{\sqrt{P_0(x,x)}} \qquad e_i = U_i(e_0) \tag{45}$$

with $P_0 \equiv \frac{1}{2}\left(g - \sum_{i=1}^{3} U_i^2\right)$*, and* U_i *the 2-forms associated with the canonical bivector* $\mathcal{U}_i$ *and where x is an arbitrary future-pointing vector.*

From the results presented in the three last propositions concerning the geometric elements associated with a type I Weyl tensor we can obtain the Debever directions $l_{\epsilon\,\pm}$ in terms of the principal directions $\{e_\alpha\}$ and *vice versa*. In [20] we have used these explicit expressions to analyse when the Debever directions expand a 3-plane, and the consequences of this property on the scalar invariants.

5. Some suggested applications

In the introduction of this paper we have pointed out several situations where our results can be applied. In this section we show their usefulness in looking for explicit characterization of metrics, and more specifically, of spacetimes admitting isometries.

It is known that the metrics admitting a maximal group of isometries may be identified as having constant curvature. Thus this family of spacetimes, defined by a condition that involves, in principle, other elements than the metric tensor (Killing vectors of the Poincaré, de Sitter or anti-de Sitter algebras), also admits an intrinsic local description involving the metric tensor exclusively, namely $\mathrm{Riem}(g) \propto g \wedge g$. The interest of getting this *explicit* characterization of a given family of spacetimes is evident because they may be verified by substituting the metric tensor directly. Elsewhere, an intrinsic and explicit description of the Schwarszchild spacetime, as well as of every type D static vacuum solution, has been given [36], and in this issue the principal structure defined by a type D Weyl tensor plays a fundamental role. Likewise, in looking for an intrinsic and explicit characterization for the spacetimes admitting non-maximal isometry groups may be necessary to make use of invariant geometric elements associated with the curvature tensor and, in particular, it may be useful to know the covariant expressions for the Weyl geometry presented in this paper. As a sample of this fact, we present here an intrinsic and explicit characterization of the type I static spacetimes as well as of the type III spacetimes admitting a hypersurface-orthogonal Killing vector.

5.1. Type I static spacetimes: an explicit characterization

Static spacetimes are defined by the existence of a hypersurface-orthogonal timelike Killing vector ξ. Then, the timelike unitary vector u collinear with ξ satisfies, and only satisfies, the equations

$$\nabla u = -u \otimes \dot{u} \qquad \mathrm{d}\dot{u} = 0 \tag{46}$$

where $\dot{u} = i(u)\nabla u$. These conditions imply that u defines a timelike shear-free and vorticity-free congruence and, according to a result by Trümper [37], a static spacetime is Petrov type I, D or O with real eigenvalues, and u is a timelike principal direction of the Weyl tensor. In type D metrics, in order to detect the Killing direction it is necessary to analyse in detail the relative position between it and the second fundamental forms of the Weyl principal 2-planes. Dealing with conformally flat metrics we must take into account the algebraic properties of the Ricci tensor. Both cases require the study of several possibilities that would make it too long to cover in this study and will be considered elsewhere [38]. However, a type I metric has a unique timelike principal direction that determines its *Weyl principal observer* . We have already employed this fact to give an intrinsic description of type I spacetimes admitting isotropic radiation for a vorticity-free observer [20]. A similar attempt can be made here for the type I static spacetimes. Firstly, we can state the following.

Lemma 3. *A Petrov type I spacetime is static if, and only if, its Weyl principal observer satisfies conditions (46).*

This statement provides an *invariant characterization* of static type I spacetimes because it asserts conditions on an invariant direction of the Weyl tensor. Moreover, we can obtain an *intrinsic and explicit characterization* of these spacetimes because the first expression in (45) allows us to calculate the timelike principal direction in terms of the Weyl tensor.

Indeed, we can denote the projector tensor in the u direction, $v = -u \otimes u$, in terms of the Weyl eigenbivectors by using (45) and, after proposition 14, we obtain an expression of v solely in terms of the metric tensor, Weyl concomitants and arbitrary directions: $v = v(W)$. On the other hand, conditions (46) may be easily written as equations on v. Then, $\nabla \cdot v$ being $(\nabla \cdot v)_\alpha = \nabla_\lambda v^\lambda_\alpha$, we can conclude.

Proposition 16. *A Petrov type I spacetime is static if, and only if, the Weyl tensor satisfies*

$$(g - v) \times \nabla v = 0 \qquad \mathrm{d}\,[\nabla \cdot v] = 0 \tag{47}$$

with $v = v(W) \equiv \frac{1}{2}\left[\sum_{i=1}^{3} U_i^2 - g\right]$, *and* $U_i = U_i(W)$ *the real 2-form associated with the eigenbivectors* $\mathcal{U}_i$ *determined in proposition 14. Then, the hypersurface-orthogonal timelike Killing direction is given by*

$$\xi \propto v(x) \tag{48}$$

where x *is an arbitrary future-pointing vector.*

5.2. Type III spacetimes with a hypersurface-orthogonal Killing vector

If a spacetime admits a timelike shear-free and vorticity-free congruence u, the metric is Petrov type I, D or O with real eigenvalues, and u is a timelike principal direction of the Weyl tensor. This result by Trümper [37] has been widely considered in literature and, in particular, we have used it in the previous section to give an intrinsic description of type I static spacetimes.

We can distinguish two levels in the Trümper proof: at first, Ricci identities are used to prove that the magnetic part $*W(u, u)$ of the Weyl tensor vanishes. From here, the Petrov matrix for u is a real symmetric matrix in a three-dimensional Euclidean space and it always diagonalizes.

If we have a spacelike congruence s such that it is shear-free and vorticity-free the first step in the proof by Trümper still holds true, that is, $*W(s; s) = 0$. But as the space orthogonal to s is not a Euclidean space the Petrov matrix could be non-diagonal, and so we cannot conclude anything about the Petrov type.

Elsewhere [39], all the congruences satisfying the equation $*W(q; q) = 0$ have been found for every Petrov type, and the causal character of the solutions has also been studied. Let us consider the Petrov III case now. We know from the Trümper result that no timelike solutions exist. Nevertheless, if we do not restrict their causal character, the congruences verifying $*W(q; q) = 0$ are those defined by the fundamental direction l and the intersection e_3 of the spacelike 2-plane $*U$ with the null 2-plane $*H$ [39]. Moreover, the results in the last section about type III metrics allow us to give these solutions in an explicit and covariant manner. More precisely, we have the following:

Proposition 17. *In a Petrov type III spacetime the equation $*W(q; q) = 0$ always admits solutions. The explicit expressions of these solutions are the (null) fundamental direction l given in proposition 5 and the (spacelike) direction e_3 given in proposition 6.*

This result can be used to obtain an intrinsic characterization of type III spacetimes that admit a hypersurface-orthogonal Killing vector field. As happens with timelike ones, conditions for a spacelike direction to be a hypersurface-orthogonal Killing field can be stated in terms of the unitary field s collinear to it. We have in this case the same expressions (46) up to the change of sign in the first one, that is

$$\nabla s = s \otimes \dot{s} \qquad \mathrm{d}\dot{s} = 0. \tag{49}$$

The unitary congruence s is shear-free and vorticity-free, so it is a solution of the equation $*W(s; s) = 0$. From proposition 17 it must be e_3.

Let us consider now a null and hypersurface-orthogonal Killing vector field ζ:

$$\mathcal{L}_\zeta g \equiv \nabla\zeta + {}^t\nabla\zeta = 0 \qquad \mathrm{d}\zeta \wedge \zeta = 0 \tag{50}$$

These two conditions can be collected as $\nabla\zeta = p \wedge \zeta$. Then, for an arbitrary function λ, the null field $l = \lambda\zeta$ satisfies $\nabla l = (p + \mathrm{d}\ln\lambda) \otimes l - l \otimes p$. This expression shows that the necessary and sufficient conditions for a null field l to be collinear with a hypersurface-orthogonal Killing vector are

$$\nabla l = a \otimes l + l \otimes b \qquad \mathrm{d}(a + b) = 0. \tag{51}$$

The first equation in (51) is equivalent to $l \wedge \nabla l \wedge l = 0$. Moreover, if this condition holds, we have $L \equiv \nabla l + {}^t\nabla l = (a + b)\tilde{\otimes} l$ and so, for an arbitrary timelike direction x, we obtain

$$a + b = \frac{1}{(l, x)}\left[L(x) - \frac{1}{2}\frac{L(x, x)}{(l, x)} l\right].$$

These results provide a parameter non-dependent characterization for a null direction to be defined by a hypersurface-orthogonal Killing vector field. We can summarize it as

Lemma 4. *A null vector l is collinear with a hypersurface-orthogonal Killing field if, and only if,*

$$l \wedge \nabla l \wedge l = 0 \qquad \mathrm{d}\left(\frac{1}{(l, x)}\left[L(x) - \frac{1}{2}\frac{L(x, x)}{(l, x)} l\right]\right) = 0$$

where $L = \nabla l + {}^t\nabla l$, and x is an arbitrary timelike vector.

On the other hand, the Ricci identities allow us to prove that every null and hypersurface-orthogonal Killing vector field is an eigendirection of the Ricci tensor and satisfies $*W(l;l) = 0$, so that, in a type III spacetime it must be the multiple Debever direction that these spaces admit. From these considerations, we get an intrinsic characterization of type III spacetimes that admit a hypersurface-orthogonal Killing field that we can state as

Proposition 18. *For a type III spacetime, the following statements hold:*

1. *There exists a null and hypersurface-orthogonal Killing vector field if, and only if, the (null) vector $H^2(x)$ satisfies the conditions of lemma 4 and H is the real 2-form associated with the canonical null bivector given in proposition 5 and x is an arbitrary timelike vector.*
2. *There exists a spacelike and hypersurface-orthogonal Killing vector field if, and only if, the (spacelike) vector e_3 given in proposition 6 satisfies conditions (49).*

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