Emission coordinates in Minkowski space-time

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Abstract. The theory of relativistic positioning systems and their natural associated emission coordinates are essential ingredients in the analysis of navigation systems and astrometry. Here we study emission coordinates in Minkowski space-time. For any choice of the four emitters (arbitrary space-time trajectories) the relation between the corresponding emission coordinates and the inertial ones are explicitly given.

Keywords: Relativistic positioning systems, emission coordinates.
PACS: 04.20.-q

Navigation systems (GPS, Galileo, ...) use as starting point Newtonian conceptions, the role of the Relativity theory being banished to monitor and correct, by trial and error, the running of the system. In order to make progress in the theoretical understanding of the current navigation systems a fully relativistic approach to positioning systems has been proposed [1]. Here, we present some results to move forward in this proposal.

A relativistic positioning system is basically defined by four emitters broadcasting their own proper time by means of electromagnetic signals. Then, at each reception event, the received four times define the emission coordinates of this event. The associated coordinate hypersurfaces are four foliations of future light cones based on the emitter's trajectories. Alternatively, the past light cone of a given event P cuts the emitter world lines at the points $\gamma^A(\tau^A)$, the proper times (τ^A) being the emission coordinates of P $(A = 1,2,3,4)$. Notice that it is possible to construct emission coordinates in any space-time and that, in order to define them, the emitters don't need to be synchronized. The basic properties of the emission coordinates have been analyzed elsewhere [2, 3]. The coordinate covectors, $d\tau^A$, are null and future-directed, and the coordinate vectors, as well as the 2-planes they expand, are space-like.

Here we study emission coordinates in Minkowski space-time when the world-lines of the emitters $\gamma^A(\tau^A)$ are known for an inertial system. Our goal is to find the inertial coordinates $\{x^\alpha\}$ of a given event in terms of its emission coordinates $\{\tau^A\}$. Applying these expressions, a user of the positioning system (that knows his location in emission coordinates) is able to determine his coordinates with respect to the given inertial system. We write the main relations of a positioning system in flat metric, and then, we present the transformation from emission to inertial coordinates, for any choice of the world-lines of the four emitters. For some particular examples, see the recent paper by Bini et al. [4].

The electromagnetic signal received at the event x of inertial coordinates $\{x^\alpha\}$ from the emitter γ^A follows the future directed null direction l^A, connecting γ^A and x. The system of quadratic equations

$$(x-\gamma^A)\cdot(x-\gamma^A) = 0 \quad (A = 1,2,3,4) \tag{1}$$

describing four families of such null cones are here called the *null propagation equations* of the positioning system. We want to solve this algebraic system for the unknown x when the trajectories of the emitters γ^A are known data, no matter how these trajectories might be. In order to do that, we are going to split the system in a sole quadratic equation and a degenerate linear system for the unknown $y \equiv l^4 = \gamma^4 - x$,

$$(x-\gamma^A)\cdot(x-\gamma^A) = 0 \quad \Longleftrightarrow \quad \begin{cases} y^2 = 0\,, & A = 4 \\ e^a \cdot y = \Omega^a\,, & A = a \end{cases} \tag{2}$$

All the quantities are referred to the emitter γ^4. The vector $e^a \equiv \gamma^a - \gamma^4 = l^4 - l^a$ $(a = 1,2,3)$ gives the relative position of the a-emitter with respect the reference emitter. By construction e^a is space-like, $\Omega^a = (e^a)^2/2$ being the Ruse-Synge world function of the referred and reference emitters [5]. Furthermore, given an arbitrary observer u $(u^2 = 1)$ we have the conditions (emission inequalities)

$$\forall A \qquad (x-\gamma^A)\cdot u > 0 \tag{3}$$

insuring that the four null directions are future directed. On the other hand, the regularity of the coordinate system, $d\tau^1 \wedge d\tau^2 \wedge d\tau^3 \wedge d\tau^4 \neq 0$, implies that $\chi \equiv *(e^1 \wedge e^2 \wedge e^3)$ is not zero ($*$ denotes the Hoodge dual operator). Consequently, the above considered linear system has rank three. The equations (2), jointly with the emission inequalities and the regularity condition, constitute the *main relations* of a positioning system in Minkowski space-time. The solution of the linear system is written as $y = y_* + \lambda\chi$ where y_* is a particular solution and λ a real parameter. The condition $y^2 = 0$ gives

$$\chi^2\lambda^2 + 2(\chi\cdot y_*)\,\lambda + y_*^2 = 0\,, \tag{4}$$

which allows to determine λ.

The four events $\gamma^A(\tau^A)$ define the internal configuration of the emitters for the event whose emission coordinates are $\{\tau^A\}$. The geometric elements associated with an internal configuration are the vectors e^a, the 2-planes and the 3-plane that they expand. The sign of χ^2 provides the causal character (space-like, light-like or time-like) of this 3-plane. In the coordinate domain Υ of the emission coordinates, one can distinguish three disjoint regions according with the causal character of the emitter configuration of their events, $\Upsilon_s \equiv \{(\tau^A)\,|\,\chi^2 > 0\}$, $\Upsilon_l \equiv \{(\tau^A)\,|\,\chi^2 = 0\}$ and $\Upsilon_t \equiv \{(\tau^A)\,|\,\chi^2 < 0\}$, so that $\Upsilon \equiv \Upsilon_s \cup \Upsilon_l \cup \Upsilon_t$. Here, the important fact is that no discontinuity nor lake of differentiability will be reflected in the solution when we go from an event with a given causal configuration to another event with different causal configuration. In order to obtain a *sole* expression relating emission and inertial coordinates in *all the domain* Υ of emission coordinates, an additional element ξ is also required. This element ξ is an arbitrary vector transversal to the configuration, $\xi\cdot\chi \neq 0$. A particular solution of the linear system is then given by

$$y_* \equiv \frac{1}{\xi\cdot\chi}\, i(\xi)H\,, \tag{5}$$

where $H \equiv *(\Omega^1\, e^2 \wedge e^3 + \Omega^2\, e^3 \wedge e^1 + \Omega^3\, e^1 \wedge e^2)$ is a function of the configuration of the emitters. Note that $\xi\cdot y_* = 0$. To reach the solution y, let us consider the straight

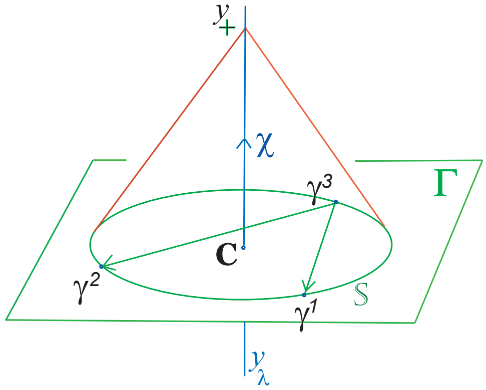

FIGURE 1. A space-like configuration ($\chi^2 > 0$). The event **C** where y_λ and Γ intersect is the center of a space-like 2-sphere $\mathbb{S}$ containing the four configuration events; **C** is the particular solution $y_* = \mathbf{C}$ when $\xi \equiv \chi$. It has been plotted the case when χ is future-directed. Under this condition, the emission solution corresponds to take the root λ_+. Otherwise (when χ is past-directed) we must take the root λ_- in order to obtain the required emission solution.

line y_λ defined by the equidistant points from the four configuration events defining the hyperplane Γ. Any point of the line y_λ is a solution of the linear system. Figures 1, 2 and 3 are $2+1$ diagrams of this geometric construction . Of course, the solution is at zero (Minkowskian) distance from the four configuration events. Then, looking for a sole expression of λ that solves (4) for any value of χ^2, we arrive to the following result.

Proposition In all the domain of emission coordinates, the transformation to inertial coordinates, $y = y(\tau^A)$, is given by

$$y = y_* + \lambda \chi$$

where y_*, χ and λ are given in terms of the emitter configuration as

$$y_* = \frac{1}{\xi \cdot \chi} i(\xi) H, \qquad H \equiv *(\Omega^1 e^2 \wedge e^3 + \Omega^2 e^3 \wedge e^1 + \Omega^3 e^1 \wedge e^2)$$

$$\Omega^a = \frac{1}{2}(e^a)^2, \qquad e^a = \gamma^a - \gamma^4, \qquad \chi \equiv *(e^1 \wedge e^2 \wedge e^3)$$

$$\lambda = -\frac{y_*^2}{y_* \cdot \chi \pm \sqrt{\Delta}}, \qquad \Delta \equiv (y_* \cdot \chi)^2 - \chi^2 y_*^2$$

and where ξ is any vector transversal to χ, $\xi \cdot \chi \neq 0$.

Finally, a last comment is in order. The control segment of the current navigation systems is based on the Earth surface. In contrast, in a fully relativistic scheme, it is a secondary segment which might be referred to the satellite constellation [1]. Our present result opens the door to more investigation in this direction by taking into account the Earth gravitational field.

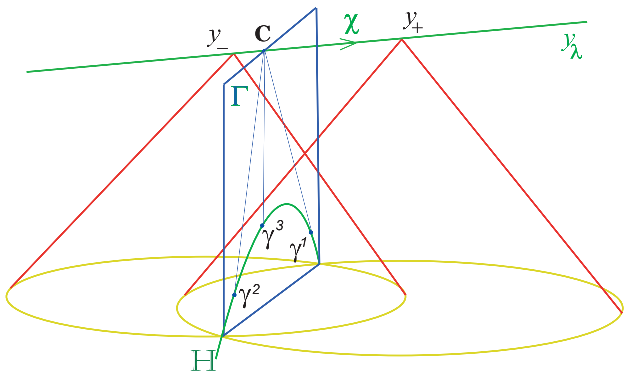

FIGURE 2. A time-like configuration ($\chi^2 < 0$). There are two distinct points having the same emission coordinates but different Cartesian representation, y_+ and y_-, respectively. Both roots of the quadratic equation provide emission solutions. The straight line y_λ is space-like and cut the time-like hyperplane Γ at the event **C**, which is the center of a space-like 2-hyperboloid $\mathbb{H}$ containing these four configuration events.

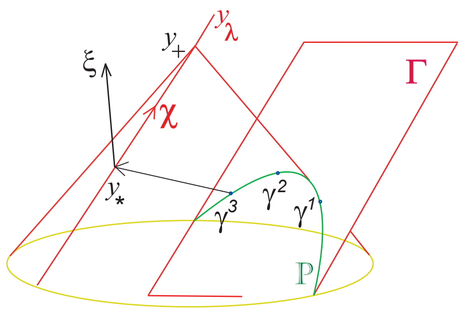

FIGURE 3. A light-like configuration ($\chi^2 = 0$). There is no center. We choice an arbitrary vector ξ, external to the configuration, and select the particular solution y_* that is orthogonal to the chosen vector. The straight line y_λ is now null as well as the hyperplane Γ, and they don't cut. The four configuration events stay on the space-like paraboloid $\mathbb{P}$.

ACKNOWLEDGMENTS

This work has been supported by the Spanish Ministerio de Educación y Ciencia, MEC-FEDER project FIS 2006-06062.

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