Flat synchronizations in spherically symmetric space-times

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Schwarzschild solution in Painlevé-Gullstrand (PG) coordinates

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + 2\varepsilon\sqrt{\frac{2m}{r}}\, dt\, dr + dr^2 + r^2 d\Omega^2$$

$\swarrow$ $\downarrow$ $\searrow$

non diagonal | asymptotically flat | regular and stationary everywhere

$\Downarrow$

space-time appears foliated by a synchronization of flat instants

$\varepsilon = \pm 1$ and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

In general, for a spherically symmetric space-time (SSST), existence and uniqueness (up to a timelike isometry) of PG synchronizations is usually taken for granted but,

> does every SSST admit a region of physical interest where a synchronization by flat instants exists?

Here we consider the existence of flat synchronizations in SSST

Let us consider a SSST

$$ds^2 = A(T,R)dT^2 + B(T,R)dR^2 + 2C(T,R)dT\,dR + D(T,R)d\Omega^2$$

being $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, $D(T,R) \neq 0$, and $\delta \equiv AB - C^2 < 0$.

In general, these coordinates are not PG coordinates

- The induced metric γ on the 3-surfaces $T = constant$ is

$$\gamma = B(T,R)dR^2 + D(T,R)d\Omega^2$$

so the Ricci tensor and the scalar curvature of γ are

$$\mathcal{Ric}(\gamma) = \left(\frac{B}{2}\mathcal{R} - \frac{B}{D}F\right) dR \otimes dR + \left(\frac{D}{4}\mathcal{R} + \frac{F}{2}\right) h$$

$$\mathcal{R} \equiv \mathcal{R}(\gamma) = \begin{cases} \dfrac{2F}{D} + \dfrac{4\partial_R F}{\partial_R D} & \partial_R D \neq 0 \\ \dfrac{2}{D} & \partial_R D = 0 \end{cases}$$

where

$$F = 1 - \frac{(\partial_R D)^2}{4BD}$$

Then we have that

γ is a flat metric $\Leftrightarrow$ $F = 0$

$(\mathcal{R}ic(\gamma) = 0)$ $\qquad ((\partial_R D)^2 = 4BD)$

- The coordinate transformation

$$T(t,r) \qquad R(t,r)$$

leads to

$$ds^2 = \xi^2\, dt^2 + \chi^2 dr^2 + 2\xi \cdot \chi\, dt\, dr + \mathcal{D}(t,r) d\Omega^2$$

with $\mathcal{D}(t,r) \equiv D(T(t,r), R(t,r))$, and the vector fields

$$\xi \equiv \dot{T}\frac{\partial}{\partial T} + \dot{R}\frac{\partial}{\partial R}, \qquad \chi \equiv T'\frac{\partial}{\partial T} + R'\frac{\partial}{\partial R}$$

where $\quad J \equiv \dot{T}R' - T'\dot{R} \neq 0 \quad$ (to assure coordinate regularity)

(over-dot and prime stand for partial derivative with respect t and r, respectively)

- The induced metric on the 3-surfaces $t = constant$ is flat iff

$$F = 1 - \frac{\mathcal{D}'^2}{4\xi^2\mathcal{D}} = 0 \Leftrightarrow 4\mathcal{D}\xi^2 = \mathcal{D}'^2 \Leftrightarrow (d\sqrt{\mathcal{D}})^2 \leq 1$$

- r is a coordinate of curvature if $\mathcal{D}(t,r) = r^2$ then

$$\chi^2 = 1 \qquad \xi^2 = J^2\delta(dr)^2 \qquad \xi \cdot \chi = \varepsilon J\sqrt{\delta[(dr)^2 - 1]}$$

So, the real function $\mathcal{B}(t,r) \equiv \xi \cdot \chi$ exists if $(dr)^2 \leq 1$ and the metric results

$$ds^2 = \mathcal{A}(t,r)dt^2 + 2\mathcal{B}(t,r)dt\ dr + dr^2 + r^2 d\Omega^2$$

Theorem

Let r be the radius of curvature of the orbits (2-spheres) of the isometry group of a spherically symmetric space-time with metric g. In the region defined by the condition

$$(dr)^2 \equiv g^{\mu\nu}\partial_\mu r \partial_\nu r \leq 1$$

a curvature coordinate system $\{t, r, \theta, \varphi\}$ exists in which the metric line element may be written as

$$ds^2 = \mathcal{A}(t,r)dt^2 + 2\mathcal{B}(t,r)dt\ dr + dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

Given a metric in the form,

$$g = \eta + h$$

with η the Minkowski metric, the Weinberg pseudotensor is defined by

$$2Q^{i0\lambda} = \frac{\partial h^{\mu}_{\mu}}{\partial x_0}\eta^{i\lambda} - \frac{\partial h^{\mu}_{\mu}}{\partial x_i}\eta^{0\lambda} - \frac{\partial h^{\mu 0}}{\partial x^{\mu}}\eta^{i\lambda} + \frac{\partial h^{\mu i}}{\partial x^{\mu}}\eta^{0\lambda} + \frac{\partial h^{0\lambda}}{\partial x_i} - \frac{\partial h^{i\lambda}}{\partial x_0}$$

Let us consider a SSST in PG coordinates

$$ds^2 = \mathcal{A}(t,r)dt^2 + 2\mathcal{B}(t,r)dt\ dr + dr^2 + r^2 d\Omega^2$$

written in the form $g = \eta + h$, then

$$h_{00} = 1 + \mathcal{A} \qquad h_{0i} = \mathcal{B}\frac{x_i}{r} \qquad h_{ij} = 0$$

and the Weinberg pseudotensor in this case is

$$Q^{i00} = 0, \qquad 2Q^{i0j} = \left(\frac{\mathcal{B}}{r} + \mathcal{B}'\right)\delta_{ij} + \left(\frac{\mathcal{B}}{r} - \mathcal{B}'\right)\frac{x_i}{r}\frac{x_j}{r}$$

The derivative of this expression and its contractions of indexes directly lead to

$$\tau^{0\lambda} \equiv -\frac{1}{8\pi G}\frac{\partial Q^{i0\lambda}}{\partial x^i} = 0$$

and then, the angular momentum densities, $j^{i\lambda} = x^i \tau^{0\lambda} - x^\lambda \tau^{0i}$ also vanish.

Theorem

In any spherically symmetric space-time, the energy and momenta densities vanish for every Painlevé-Gullstrand synchronization.

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For a SSST in PG coordinates

$$ds^2 = \mathcal{A}(t,r)dt^2 + 2\mathcal{B}(t,r)dt\ dr + dr^2 + r^2 d\Omega^2$$

the vacuum Einstein equations are

constraint equations

$$\Phi(\Phi + 2\Psi) = 0$$

$$r\Phi' + \Phi - \Psi = 0$$

evolution equations

$$\dot{\Psi} = \Psi(2\Phi - \Psi) + \frac{1}{\mathcal{B}}(\mathcal{B}^2\Psi)'$$

$$\dot{\Phi} = \Psi\Phi + \frac{\mathcal{B}}{r^2}(r^2\Phi)'$$

with

$$\Phi = \frac{1}{\alpha}\frac{\mathcal{B}}{r}, \qquad \Psi = \frac{\mathcal{B}'}{\alpha}, \qquad \alpha^2 = \mathcal{B}^2 - \mathcal{A}$$

Solving these equations we have that

The PG extension of the Schwarzschild solution is obtained from

1. the existence of a flat synchronization in a SSST
2. solving the vacuum Einstein equations

Summary

In this work we have

- studied the existence of a PG synchronization in SSST.
- proved that the energy and momenta densities of a PG synchronization vanish in a SSST.
- obtained the PG extension of the Schwarzschild solution by solving the vacuum Einstein equations for a SSST in PG coordinates.

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- If $\Phi \neq 0 \quad \Rightarrow \quad \Phi = -2\Psi \quad$ and $\quad \mathcal{B} = f(t) r^{-1/2}$ (with $f(t)$ an arbitrary function)

  Then $\alpha' = 0$, we can take $\alpha = 1$ by re-scaling the t coordinate parameter, and

  $$\Phi = f(t) r^{-3/2} = -2\Psi$$

  with $f(t) = constant = \sqrt{2m}$ by the evolution equations. So, we have the Schwarzschild space–time in the PG extension.

- If $\Phi = 0 \quad \Rightarrow \quad$ Minkowski space–time.