# Painlevé–Gullstrand synchronizations in spherical symmetry

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## Abstract

A Painlevé–Gullstrand synchronization is a slicing of the spacetime by a family
of flat space-like 3-surfaces. For spherically symmetric spacetimes, we show
that a Painlevé–Gullstrand synchronization only exists in the region where
$(\mathrm{d}r)^2 \leqslant 1$, $r$ being the curvature radius of the isometry group orbits (2-
spheres). This condition states that the Misner–Sharp gravitational energy
of these 2-spheres is not negative and has an intrinsic meaning in terms of
the norm of the mean extrinsic curvature vector. It also provides an algebraic
inequality involving the Weyl curvature scalar and the Ricci eigenvalues. We
prove that the energy and momentum densities associated with the Weinberg
complex of a Painlevé–Gullstrand slice vanish in these curvature coordinates,
and we give a new interpretation of these slices by using semi-metric Newtonian
connections. It is also outlined that, by solving the vacuum Einstein's equations
in a coordinate system adapted to a Painlevé–Gullstrand synchronization, the
Schwarzschild solution is directly obtained in a whole coordinate domain that
includes the horizon and both its interior and exterior regions.

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## 1. Introduction

It is known that Painlevé [1], Gullstrand [2] and (some years later) Lemaître [3] used a non-
orthogonal curvature coordinate system which allows us to extend the Schwarzschild solution
inside its horizon, see equation (58) below. In this coordinate system, from now on called a
Painlevé–Gullstrand (PG) coordinate system, the metric is not diagonal, but asymptotically flat
and regular across the horizon, and then, everywhere non-singular up to $r = 0$. Furthermore,

one has a very simple spatial 3-geometry: the spacetime appears foliated by a synchronization of flat instants (hereafter called PG synchronizations)[3].

Nowadays, there is an increasing interest in the study of this type of synchronizations, for instance, (i) in connection with astrophysical applications, by taking into account that the dynamics of the gravitational collapse should be pursued beyond the Schwarzschild radius in a PG coordinate system [6–9], (ii) in spherically symmetric spacetimes (SSSTs), as a convenient initial condition preserved under time evolution [10, 11], (iii) in relativistic hydrodynamics, when an effective Lorentzian metric is introduced [12], (iv) in non-relativistic situations admitting a Lorentzian description, namely in 'analog gravity models' (see, for example, [13–18]), and also (v) in modeling the black hole geometry and its associated physics, or to describe some quantum effects by starting from a Hamiltonian formulation [19–22]. More physical issues about the use of PG coordinates and their interpretation can be found in [4, 5, 23, 24]. For a description of the causal character of the geometric elements (coordinate lines, coordinate 2-surfaces and coordinate 3-surfaces) associated with PG coordinates, see [25, 26].

The existence of PG synchronizations in SSSTs has been studied in the static case considering that the induced metric has a vanishing Ricci tensor [27], and some specific constructions are presented in [28]. In more general cases, this existence is usually taken for granted but, recently, a limitation to this ansatz has been pointed out [29]. However, as far as the authors are aware, a definitive interpretation of this limitation as well as the analysis of the domains where a PG synchronization exists has not been performed up to now. Then, some related questions arise: Does every SSST admits a region of physical interest where a synchronization by flat instants exists? and, What are the advantages of adapting coordinates to a flat spatial 3-geometry? The main contribution of this paper is to state the above limitation clearly, in a form that it is coordinate independent, and to provide its physical interpretation.

Generalized (but, in general, non-flat) PG synchronizations have been constructed in Schwarzschild [30], Reissner–Nordström [29] and Kerr [31, 32] geometries, and also in non-vacuum SSSTs, where new insights in the study of gravitational collapse scenarios are achieved (by evolving an initial 3-geometry [33, 34]). However, here we restrict ourselves to flat synchronizations in order to discuss their existence in SSSTs.

The paper is structured as follows. Section 2 is devoted to introduce some general formulae for the induced geometry on space-like hypersurfaces and surfaces in SSSTs. In section 3 the condition for the existence of a PG synchronization in SSSTs is analyzed and physically interpreted. In section 4 we write the components of the Weinberg pseudotensor [35] with respect to a PG synchronization, and we prove that the energy and momentum densities of each PG slice vanish. In section 5 we consider the semi-metric connection (see [36]) associated with a spherically symmetric metric expressed in PG coordinates and provide new insights on the Newtonian interpretation of the properties exhibited by the Schwarzschild field in these coordinates. Section 6 deals with the (3 + 1) decomposition of the Einstein equations with respect to a PG synchronization. By integration of the vacuum equations, one recovers the extended Schwarzschild metric in PG coordinates, including the region inside the horizon. Finally, in section 7 we discuss the role that our results can play for a better understanding of the geometry and physics in SSSTs. Some preliminary results of this work were presented at the Spanish Relativity meeting ERE-2009 [37].

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[3] We use here the abbreviations 'PG coordinates' and 'PG synchronization' for the sake of simplicity, without any intention of misplacing the relevant contribution by Lemaître [3], who clarified the coordinate character of the 'Schwarzschild singularity' obtaining an extended metric form for the Schwarzschild solution. In fact, the main motivation in [1–3] is rather different. For historical remarks about this subject and for some physical interpretations, see [4, 5].

Let us precis the used notation. The curvature tensor $R^k_{lij}$ of a symmetric connection $\nabla$ is defined according to the identity $\nabla_i\nabla_j\xi^k - \nabla_j\nabla_i\xi^k = R^k_{lij}\xi^l$ for the vector field $\xi$ and $R_{ij} \equiv R^k_{ikj}$ is the Ricci tensor. We take natural units in which $c = G = 1$ and the Einstein constant is $\kappa = 8\pi$. We say that $\{t, r, \theta, \varphi\}$ is a curvature coordinate system if for constant $t$ and $r$ the line element is $dl^2 = r^2\,d\Omega^2$ (with $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\,d\varphi^2$ being the metric of the unit sphere). In these coordinates, the metric form is, in general, non-diagonal. When, in addition, the 3-surfaces defined by $t = $ constant are flat, the curvature coordinate system is called a PG coordinate system. From now on, it will be understood that the title of the sections and the results of this work always concern spacetimes with spherical symmetry. The agreement for spacetime signature is $(- + ++)$.

## 2. Some geometrical relations

In this section we present the geometric background needed in the following sections: expressions for the Ricci tensor and the extrinsic curvature of a spherically symmetric synchronization, as well as, the mean curvature vector and the Gauss identity for the 2-spheres of a SSST. Of course, this material is not new and it may be bypassed or used as a glossary of formulae, which are conveniently referred throughout the text of the remaining sections. For an account on 2+2 warped spacetimes properties allowing to intrinsically characterize SSST see [38, 39].

Let $(V_4, g)$ be a SSST, and let us consider a canonical coordinate system $\{T, R, \theta, \varphi\}$ of $V_4$, which is adapted to the symmetries of the metric $g$. Then we may express the metric in the following general form [40–42]:

$$g = A\,dT \otimes dT + B\,dR \otimes dR + C(dT \otimes dR + dR \otimes dT) + D\,\sigma, \tag{1}$$

with $A(T, R), B(T, R), C(T, R)$ verifying the condition $\delta \equiv AB - C^2 < 0, D(T, R) \neq 0$ and $\sigma \equiv d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi$ being the metric on the unit 2-sphere.

### 2.1. Ricci tensor

The induced metric $\gamma$ on the 3-surfaces $T = $ constant is written as

$$\gamma = B\,dR \otimes dR + D\sigma, \tag{2}$$

with $B \neq 0$. The Ricci tensor of $\gamma$, $\mathcal{R}ic(\gamma)$, is given by

$$\mathcal{R}ic(\gamma) = \left(\frac{B}{2}\mathcal{R} - \frac{B}{D}F\right)dR \otimes dR + \left(\frac{D}{4}\mathcal{R} + \frac{F}{2}\right)\sigma, \tag{3}$$

where

$$F = 1 - \frac{(\partial_R D)^2}{4BD} \tag{4}$$

and $\mathcal{R} \equiv \mathcal{R}(\gamma)$, the scalar curvature of $\gamma$, results

$$
\begin{aligned}
\mathcal{R} &= \frac{2}{D}\left(1 + \frac{\partial_R B\,\partial_R D}{2B^2} + \frac{(\partial_R D)^2}{4BD} - \frac{\partial_R^2 D}{B}\right) \\
&= \begin{cases} \dfrac{2F}{D} + \dfrac{4\partial_R F}{\partial_R D} & \text{if } \partial_R D \neq 0, \\[2mm] \dfrac{2}{D} & \text{if } \partial_R D = 0. \end{cases}
\end{aligned}
\tag{5}
$$

Note that the 3-surfaces $T = $ constant are conformally flat[4] but, in general, they are not flat. From equations (3)–(5), we see that $\gamma$ is a flat metric if, and only if, $F = 0$, that is

$$\mathcal{R}ic(\gamma) = 0 \quad \Longleftrightarrow \quad 4BD = (\partial_R D)^2. \tag{6}$$

### 2.2. Extrinsic curvature

The extrinsic curvature of the slicing $T = $ constant is defined as $K = -\frac{1}{2}\mathcal{L}_n \gamma$, where $\mathcal{L}_n$ is the Lie derivative along the unit normal vector $n$,

$$n = \frac{1}{\alpha}\left(\frac{\partial}{\partial T} - \frac{C}{B}\frac{\partial}{\partial R}\right) \tag{7}$$

with $\alpha^2 = -\dfrac{\delta}{B}$. From (2), we obtain

$$K = \Psi B\, \mathrm{d}R \otimes \mathrm{d}R + \Phi D\, \sigma, \tag{8}$$

where

$$\Psi = K_R^R = \frac{1}{2B\alpha}\left(2\partial_R C - \frac{C}{B}\partial_R B - \partial_T B\right) \tag{9}$$

$$\Phi = K_\theta^\theta = K_\varphi^\varphi = \frac{1}{2D\alpha}\left(\frac{C}{B}\partial_R D - \partial_T D\right) \tag{10}$$

are the eigenvalues of $K$. Developing the Lie derivative of $K$ with respect to the shift vector $\beta = \frac{C}{B}\frac{\partial}{\partial R}$, we arrive to the expression

$$\mathcal{L}_\beta K = \frac{B}{C}\,\partial_R\left[\left(\frac{C}{B}\right)^2 \Psi B\right]\mathrm{d}R \otimes \mathrm{d}R + \frac{C}{B}\,\partial_\theta(\Phi D)\,\sigma, \tag{11}$$

which will be needed in section 6 to split the Einstein evolution equations with respect to a PG synchronization.

### 2.3. Mean curvature vector

The mean curvature vector $H$ of a 2-sphere $S$ defined by constant $T$ and $R$ is given by

$$H = -\frac{1}{D\delta}\left[(B\partial_T D - C\partial_R D)\frac{\partial}{\partial T} + (A\partial_R D - C\partial_T D)\frac{\partial}{\partial R}\right]. \tag{12}$$

This expression directly follows by taking the trace (with respect to the induced metric $D\sigma$) of the extrinsic curvature tensor $\mathcal{K}$ of each $S$, which is defined by

$$\mathcal{K}(e_a, e_b) \equiv -\left(\nabla_{e_a} e_b\right)^\perp = -\left(\Gamma_{ab}^T \frac{\partial}{\partial T} + \Gamma_{ab}^R \frac{\partial}{\partial R}\right), \tag{13}$$

where the minus sign is taken as a matter of convention. $\nabla$ is the Levi-Civita connection of $g$, $\{e_i \equiv \frac{\partial}{\partial x^i}\}$ is a coordinate basis of $S$, $i = a, b = \theta, \varphi$, and the symbol $\perp$ stands for the projection on $S^\perp$: the time-like 2-surface orthogonal to $S$. For a detailed study of $\mathcal{K}$ with applications in relativity see, for example, [44–46]. Then, it results

$$\mathcal{K} = \frac{1}{2}D\sigma \otimes H. \tag{14}$$

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[4] Note that the Cotton tensor of $\gamma$, $\mathcal{C}_{ijk}(\gamma) \equiv D_i Q_{jk} - D_j Q_{ik}$ ($D_i$ is the covariant derivative with the Levi-Civita connection of $\gamma$, and $Q_{ij} \equiv \mathcal{R}_{ij} - \frac{\mathcal{R}}{4}\gamma_{ij}$) identically vanishes, $\mathcal{C}_{ijk}(\gamma) = 0$. This is the result that one could expect ought to the algebraic properties of the Cotton tensor and the assumed spherical symmetry. This means that $\gamma$ is a conformally flat metric. Then, $\gamma$ may always be written in isotropic form by making a coordinate change on each 3-surface $T = $ constant. An interesting summary on the Cotton tensor properties is given in [43].

The one-form $\Gamma$ metrically equivalent to $H$, $\Gamma_\alpha = g_{\alpha\beta}H^\beta$, is written as

$$\Gamma = -\frac{1}{D}(\partial_T D \, \mathrm{d}T + \partial_R D \, \mathrm{d}R) = -d \ln D, \tag{15}$$

which can also be obtained from the general expressions presented in [47].

### 2.4. Gauss identity

Given a space-like 2-surface $\Sigma$ of a spacetime $(V_4, g)$, the Gauss identity provides a scalar relation involving the background geometry and the intrinsic and extrinsic properties of $\Sigma$, and it may be expressed as[5]

$$R(h) = \tfrac{2}{3}R(g) + g(H, H) + 2\operatorname{tr}(K_l \times K_k) + 2\operatorname{Ric}(l, k) - 2W(l, k, l, k), \tag{16}$$

where $R(h)$ is the scalar curvature of the induced metric $h$ on $\Sigma$, $R(g)$, Ric and $W$ are, respectively, the scalar curvature, the Ricci and the Weyl tensor of $g$, and $l$ and $k$ are two independent future-pointing null vectors normal to $\Sigma$ satisfying $g(l, k) = -1$. Now, $H$ is the mean curvature of $\Sigma$, that is $H = \operatorname{tr}\mathcal{K}$, with $\mathcal{K}$ being the extrinsic curvature tensor of $\Sigma$ (as defined by the first equality of (13), taking $e_a$ and $e_b$ tangent to $\Sigma$); $K_l$ and $K_k$ are, respectively, the second fundamental forms of $\Sigma$ relative to $l$ and $k$, that is, $K_l(e_a, e_b) = g(l, \mathcal{K}(e_a, e_b))$, and similarly for $K_k$. The trace is taken with respect to $h$, that is, $\operatorname{tr}(K_l \times K_k) = h^{ad}h^{bc}(K_l)_{ab}(K_k)_{cd}$.

For $\Sigma = S$, a 2-sphere (orbit) in a SSST, the Gauss identity may be written as

$$\rho = 2\lambda + \tfrac{1}{3}(4\mu - \mu_1 - \mu_2) + \tfrac{1}{2}H^2, \tag{17}$$

with $\rho = 2/D$ being the scalar curvature of $D\sigma$, $\lambda$ the simple eigenvalue[6] of $W$ (provided that $W \neq 0$) and $\{\mu, \mu_1, \mu_2\}$ stand for the Ricci eigenvalues. More precisely, the tangent space to $S$ is the eigenplane associated with the eigenvalue $\mu$. The eigenvalues $\mu_1$ and $\mu_2$ may be real (simple or multiple) or complex, and have associated the invariant 2-plane tangent to $S^\perp$. Here, we have considered the algebraic classification of Churchill–Plebański for a symmetric 2-tensor in Lorentzian geometry, and its peculiarities in spherical symmetry (see [48], and also [49] for an intrinsic approach). We have taken into account that $R(g) = 2\mu + \mu_1 + \mu_2$ and $2\operatorname{Ric}(l, k) = -(\mu_1 + \mu_2)$. In addition $H^2 \equiv g(H, H) = -4\operatorname{tr}(K_l \times K_k)$ because, according to (14), the second fundamental forms $K_l$ and $K_k$ are both proportional to $h = D\sigma$.

## 3. Painlevé–Gullstrand slicings

In this section we find the condition ensuring the existence of a PG synchronization in spherical symmetry, and we discuss its invariant meaning in terms of the eigenvalues of the Weyl and Ricci tensors. Also, using a radial curvature coordinate $r$, we provide a physical interpretation of this condition in terms of the Misner–Sharp gravitational energy of a 2-sphere of radius $r$. As commented below, this result shows an interesting interconnection between the study and classification of trapped surfaces (see [50]) and the existence of flat slicings.

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[5] The Gauss identity is usually given in terms of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ (see, for instance [45, 46]) from which expression (16) follows by taking into account the algebraic decomposition:

$$R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha} - \frac{R}{3}g_{\alpha[\gamma}g_{\delta]\beta},$$

where the bracket stands for antisymmetrization of the index pairs, $T_{[\alpha\beta]} \equiv \tfrac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})$.

[6] One has

$$\frac{1}{2}W_{\mu\nu\alpha\beta}(l^\alpha k^\beta - l^\beta k^\alpha) = \lambda(l_\mu k_\nu - l_\nu k_\mu),$$

and then $W(l, k, l, k) \equiv W_{\mu\nu\alpha\beta}l^\mu k^\nu l^\alpha k^\beta = -\lambda$.

### 3.1. Existence condition

Let us start from the general metric form (1). Exploring the gauge freedom to make coordinate transformations of the form $T = T(t, r)$, $R = R(t, r)$, we look for a function $t(T, R)$ whose level hypersurfaces $t = $ constant are Euclidean, i.e. the induced metric is positive and flat. Under such a transformation, metric (1) is expressed as

$$\mathrm{d}s^2 = \xi^2 \, \mathrm{d}t^2 + \chi^2 \mathrm{d}r^2 + 2\xi \cdot \chi \, \mathrm{d}t \, \mathrm{d}r + \mathcal{D}(t, r) \, \mathrm{d}\Omega^2, \tag{18}$$

with $\mathcal{D}(t, r) \equiv D(T(t, r), R(t, r))$, and the vector fields $\xi$ and $\chi$ are defined by

$$\xi \equiv \dot{T}\frac{\partial}{\partial T} + \dot{R}\frac{\partial}{\partial R}, \qquad \chi \equiv T'\frac{\partial}{\partial T} + R'\frac{\partial}{\partial R}. \tag{19}$$

Over-dot and prime stand for partial derivative with respect $t$ and $r$, respectively, and $J \equiv \dot{T}R' - T'\dot{R} \neq 0$ assures coordinate regularity.

The scalar products $\xi^2 \equiv g(\xi, \xi)$, $\chi^2 \equiv g(\chi, \chi)$ and $\xi \cdot \chi \equiv g(\xi, \chi)$ can be written as

$$\delta (dD)^2 \, \dot{T}^2 + 2\dot{\mathcal{D}} \, Z \, \dot{T} + B\dot{\mathcal{D}}^2 = (\partial_R D)^2 \, \xi^2, \tag{20}$$

$$\delta (dD)^2 \, T'^2 + 2\mathcal{D}' \, Z \, T' + B\mathcal{D}'^2 = (\partial_R D)^2 \, \chi^2, \tag{21}$$

$$(\partial_R D)^2 \, \xi \cdot \chi = \delta (dD)^2 \, \dot{T} \, T' + Z \, (\mathcal{D}' \dot{T} + \dot{\mathcal{D}} T') + B\dot{\mathcal{D}} \mathcal{D}', \tag{22}$$

where we have used the relations $\dot{\mathcal{D}} = \dot{T}\partial_T D + \dot{R}\partial_R D$ and $\mathcal{D}' = T'\partial_T D + R'\partial_R D$ to substitute $\dot{R}$ and $R'$ in terms of $\dot{T}$ and $T'$. We have denoted $Z \equiv C\partial_R D - B\partial_T D$. Then, due to the Lorentzian character of the metric ($\delta < 0$), equations (20) and (21) lead to the real values for $\dot{T}$ and $T'$ if, and only if, the inequalities

$$\xi^2 (d\mathcal{D})^2 \leqslant \dot{\mathcal{D}}^2, \qquad \chi^2 (d\mathcal{D})^2 \leqslant \mathcal{D}'^2 \tag{23}$$

are satisfied. Now, looking for a flat synchronization, we have that the induced metric on the 3-surfaces $t = $ constant is flat if, and only if,

$$4\mathcal{D} \, \chi^2 = \mathcal{D}'^2 \tag{24}$$

according to equation (6). Consequently, in the case of a flat synchronization, the second inequality in (23) is equivalent to

$$(d\sqrt{\mathcal{D}})^2 \leqslant 1. \tag{25}$$

So, under the assumed spherical symmetry, equation (25) provides the necessary and sufficient condition to be fulfilled for the existence of a flat slicing. The first inequality in (23) guarantees that the slices are space-like, that is the slicing is a PG synchronization.

### 3.2. Geometric interpretation

In terms of the scalar curvature $\rho = 2/\mathcal{D}$ of the metric $\mathcal{D} \, \mathrm{d}\Omega^2$, the above condition (25) may be expressed as follows:

$$(\mathrm{d}\rho)^2 \leqslant 2\rho^3, \tag{26}$$

which involves the sole invariant $\rho$. On the other hand, according to equation (15), $H^2 = \Gamma^2 = (d \ln \mathcal{D})^2$, and then equation (25) gives an upper bound for the norm of the mean extrinsic curvature $H$ of the group orbits:

$$H^2 \leqslant \frac{4}{\mathcal{D}} = 2\rho. \tag{27}$$

Moreover, from the Gauss relation (17), we arrive to the following result.

**Proposition 1.** *In a SSST the following conditions are equivalent.*

*(i) There exists a PG synchronization.*

*(ii)* $(\mathrm{d}\rho)^2 \leqslant 2\rho^3$, *where $\rho$ is the scalar curvature of the 2-spheres.*

*(iii)* $H^2 \leqslant 2\rho$, *where $H$ is the mean curvature vector of the 2-spheres.*

*(iv)* $\mu_1 + \mu_2 - 4\mu \leqslant 6\lambda$, *where $\mu_1$, $\mu_2$ and $\mu$ (double) are the Ricci eigenvalues, and $\lambda$ is the simple eigenvalue of the Weyl tensor, or $\lambda = 0$ when the spacetime is conformally flat.*

Note that this is a geometric result, which will be physically interpreted in the next subsection. Taking into account the Einstein equations, in the above item (iv), the Ricci eigenvalues, $\{\mu_1, \mu_2, \mu\}$ may by substituted by the corresponding energy tensor eigenvalues, $\{e_1, e_2, e\}$, giving

$$e_1 + e_2 - e \leqslant 3\lambda. \tag{28}$$

### 3.3. Physical interpretation

By definition, see [40, 41], $r$ is a coordinate of curvature for the spherically symmetric metric form (18) if $\mathcal{D}(t, r) = r^2$, so that $\mathcal{D}' = 2r$ and $\dot{\mathcal{D}} = 0$. Then, (25) states that $(\mathrm{d}r)^2 \leqslant 1$, and taking into account that the Misner–Sharp gravitational energy $E$ of a 2-sphere of radius $r$ is expressed as (see [50, 51])

$$E = \frac{r}{2}(1 - (\mathrm{d}r)^2), \tag{29}$$

we arrive to the following result.

**Proposition 2.** *Any SSST admits a PG synchronization in the region where the Misner–Sharp gravitational energy is non-negative, $E \geqslant 0$.*

The Misner–Sharp energy has been painstakingly analyzed in [50], providing useful criteria to study trapped surfaces. The main novelty here has been to relate this concept and the existence of PG synchronizations[7].

Moreover, the flatness condition (24) implies that $\chi^2 = 1$, and metric (18) is written as

$$\mathrm{d}s^2 = \xi^2 \, \mathrm{d}t^2 + 2\xi \cdot \chi \, \mathrm{d}t \, \mathrm{d}r + \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2. \tag{30}$$

Then, accordingly to (20) and (22), the following relations must occur:

$$\begin{aligned}
\mathcal{A}(t, r) &\equiv \xi^2 = J^2 \, \delta(\mathrm{d}r)^2 \\
\mathcal{B}(t, r) &\equiv \xi \cdot \chi = \varepsilon J \sqrt{\delta[(\mathrm{d}r)^2 - 1]},
\end{aligned} \tag{31}$$

where $\varepsilon = \pm 1$. So, the real function $\mathcal{B}$ exists in the region where $(\mathrm{d}r)^2 \leqslant 1$, and we have the following result.

**Proposition 3.** *Let $r$ be the radius of curvature of the orbits (2-spheres) of the isometry group of a SSST with metric g. In the region defined by the condition*

$$(\mathrm{d}r)^2 \equiv g^{\mu\nu}\partial_\mu r \partial_\nu r \leqslant 1, \tag{32}$$

*the Misner–Sharp energy is not negative and a curvature coordinate system $\{t, r, \theta, \varphi\}$ exists in which the metric line element may be written as*

$$\mathrm{d}s^2 = \mathcal{A}(t, r) \, \mathrm{d}t^2 + 2\mathcal{B}(t, r) \, \mathrm{d}t \, \mathrm{d}r + \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2, \tag{33}$$

*where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2$.*

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[7] Marc Mars inspired us in obtaining this relation.

　　　The Lorentzian character of the metric imposes that the functions $\mathcal{A}$ and $\mathcal{B}$ must satisfy the sole restriction $\mathcal{A} < \mathcal{B}^2$, which is implied by (31).

　　　The Misner–Sharp energy is a geometric invariant that may be physically interpreted as an effective gravitational energy whose origin is the interaction between the energetic content and its associated field (see [50]). Given that the intrinsic and extrinsic scalar curvatures of the 2-spheres are $\rho = 2/r^2$ and $H^2 = 4(\mathrm{d}\ln r)^2$, and according to (29) one has the invariant expression

$$E = \frac{1}{\sqrt{2\rho}}\left(1 - \frac{1}{2\rho}H^2\right).\tag{34}$$

Finally, note that equation (27) does not constraint the causal character of the mean curvature vector $H$, which might be time-like, light-like or space-like. This is a remarkable property because a 2-sphere is said to be trapped, marginal or untrapped if $H$ is, respectively, time-like, light-like or space-like (see e.g. [45, 46, 50, 52]).

## 4. Energy and momentum densities of a Painlevé–Gullstrand slice

In this section we establish the following result.

**Proposition 4.** *In any SSST, the Weinberg energy and momentum densities vanish for every PG synchronization.*

　　　Of course, to find a coordinate system in which the Weinberg densities vanish is not a surprising property, due to the non-tensorial character of them. However, the novelty here is to show that, for every SSST, such a vanishing property occurs in PG coordinates.

　　　In order to proof the above result, let us consider metric (33) written in a quasi-Minkowskian form, that is $g = \eta + h$ with $\eta$ being the Minkowski metric, $h_{00} = 1 + \mathcal{A}$, $h_{0i} = \mathcal{B}x_i/r$ and $h_{ij} = 0$.

　　　We start from the expression of the Weinberg pseudo-tensor [35]

$$2Q^{i0\lambda} = \frac{\partial h_\mu^\mu}{\partial x_0}\eta^{i\lambda} - \frac{\partial h_\mu^\mu}{\partial x_i}\eta^{0\lambda} - \frac{\partial h^{\mu 0}}{\partial x^\mu}\eta^{i\lambda} + \frac{\partial h^{\mu i}}{\partial x^\mu}\eta^{0\lambda} + \frac{\partial h^{0\lambda}}{\partial x_i} - \frac{\partial h^{i\lambda}}{\partial x_0},$$

where Latin and Greek indices vary from 1 to 3 and from 0 to 3, respectively, and all indices are raised and lowered with the flat metric $\eta$. In this case, it results $Q^{i00} = 0$ (according to [53]) and

$$2Q^{i0j} = \left(\frac{\mathcal{B}}{r} + \mathcal{B}'\right)\delta_{ij} + \left(\frac{\mathcal{B}}{r} - \mathcal{B}'\right)\frac{x_i}{r}\frac{x_j}{r}.\tag{35}$$

The derivative of this expression leads to

$$2\frac{\partial Q^{i0j}}{\partial x^k} = \left(\frac{\mathcal{B}'}{r} - \frac{\mathcal{B}}{r^2} + \mathcal{B}''\right)\delta_{ij}\frac{x_k}{r} + \left(\frac{\mathcal{B}}{r^2} - \frac{\mathcal{B}'}{r}\right)\left(\delta_{ik}\frac{x_j}{r} + \delta_{jk}\frac{x_i}{r}\right)$$
$$+ \left(3\frac{\mathcal{B}'}{r} - 3\frac{\mathcal{B}}{r^2} - \mathcal{B}''\right)\frac{x_i}{r}\frac{x_j}{r}\frac{x_k}{r},$$

and by contraction of the indices, it directly follows that $\frac{\partial Q^{i0j}}{\partial x^i} = 0$. Then, the four-momentum density vanishes

$$\tau^{0\lambda} \equiv -\frac{1}{8\pi G}\frac{\partial Q^{i0\lambda}}{\partial x^i} = 0\tag{36}$$

and hence, the angular momentum densities $j^{i\lambda} = x^i\tau^{0\lambda} - x^\lambda\tau^{0i}$ also vanish, according to the announced conclusion.

For the special case of the Schwarzschild geometry, the vanishing of the energy density may intuitively be understood invoking the Einstein equivalence principle. Taking $\epsilon = 1$ in the extended form (58) of the Schwarzschild metric, $t$ represents the proper time of a radial geodesic observer which initially stays, in $r = \infty$, at rest with respect to a static observer. Locally, such an observer does not feel any gravitational effect.

## 5. Painlevé–Gullstrand slicings and semi-metric connections

In the 1980s, Bel proposed an extended Newtonian theory of gravitation based on a semi-metric connection associated with an observer congruence and a flat spatial 3-metric [36]. In a spacetime, with the metric $g_{\mu\nu}$, which admits a spatially flat slicing given by the coordinate hypersurfaces $x^0 = $ constant, the connection coefficients of the aforementioned semi-metric connection are written as [36]

$$\Lambda^k = -\Gamma^k_{00} = \tfrac{1}{2}\delta^{ki}(\partial_i g_{00} - 2\partial_0 g_{0i}) \tag{37}$$

$$\Omega^k_j = -2\Gamma^k_{0j} = \delta^{ki}(\partial_i g_{0j} - 2\partial_j g_{0i}). \tag{38}$$

Consequently, a SSST metric admits a Newtonian interpretation when it is written in PG coordinates and it is considered in the above context. In fact, taking into account expression (33) of the metric, we have $g_{00} = \mathcal{A}$, $g_{0i} = \mathcal{B}x_i/r$ and then

$$\partial_i g_{0j} = \frac{\mathcal{B}}{r}\delta_{ij} + \left(\mathcal{B}' - \frac{\mathcal{B}}{r}\right)\frac{x_i}{r}\frac{x_j}{r} = \partial_j g_{0i}.$$

Then, the connection coefficients result

$$\Lambda^k = \frac{1}{2}(\mathcal{A}' - 2\dot{\mathcal{B}})\frac{x^k}{r} \tag{39}$$

$$\Omega^k_j = 0, \tag{40}$$

which means that, in the region of a SSST where a PG synchronization exists, the gravitational field may be interpreted as an inertial field of radial accelerations and vanishing rotation.

In particular, for the case of the Schwarzschild metric, we have $\mathcal{A} = -\left(1 - \frac{2m}{r}\right)$, $\mathcal{B} = \epsilon\sqrt{\frac{2m}{r}}$, and then the vector component of the connection reduces to

$$\vec{\Lambda} = -\frac{m}{r^2}\vec{e}_r, \tag{41}$$

where $\vec{e}_r$ is the unit vector in the radial direction. The above expression (41) gives the acceleration of a unit mass particle radially falling in the Newtonian field of a mass $m$. Similar Newtonian interpretations have been considered from a different point of view (see, for example, [4, 5, 23]).

## 6. Painlevé–Gullstrand slicings and Einstein equations

In general relativity, when dealing with the evolution (or 3 + 1) formalism (see [54, 55], and [56] for a recent review), one introduces a vorticity-free observer $n$, $n^2 = -1$, and Einstein equations are decomposed into the following set of constraint equations ($\kappa$ is the Einstein constant):

$$\mathcal{R}(\gamma) + (\operatorname{tr} K)^2 - \operatorname{tr} K^2 = 2\kappa\,\tau \tag{42}$$

$$\nabla \cdot (K - \operatorname{tr} K \gamma) = \kappa q \tag{43}$$

and this set of evolution equations

$$\partial_t \gamma = -2\alpha K + \mathcal{L}_\beta \gamma \tag{44}$$

$$\partial_t K = -\nabla\nabla\alpha - \kappa\alpha\left[\Pi + \tfrac{1}{2}(\tau - p)\gamma\right] + \alpha[\mathcal{R}ic(\gamma) + \operatorname{tr} KK - 2K^2] + \mathcal{L}_\beta K. \tag{45}$$

Here, $\gamma$ and $K$ are, respectively, the metric and the extrinsic curvature of the space-like slices whose normal vector is $n$; $\nabla$ is the Levi-Civita connection of $\gamma$, and the Ricci tensor and scalar curvature of $\gamma$ are denoted by $\mathcal{R}ic(\gamma)$ and $\mathcal{R}(\gamma)$, respectively; the trace operator associated with $\gamma$ is denoted by tr, so that, $(\nabla \cdot K)_a \equiv (\operatorname{tr}\nabla K)_a \equiv \gamma^{ij}\nabla_i K_{ja}$ is the divergence of $K$ with respect to $\gamma$. In the usual evolution formalism notation, $n$ is written as $n = \alpha^{-1}\left(\frac{\partial}{\partial t} - \beta\right)$, where $\alpha$ is the lapse function and $\beta$ is the shift vector.

The energy content $\mathcal{T} \equiv \{\tau, q, p, \Pi\}$ has been decomposed relatively to $n$, that is

$$\mathcal{T} = \tau n \otimes n + n \otimes q + q \otimes n + \Pi + p\gamma, \tag{46}$$

with $\tau \equiv \mathcal{T}(n, n)$, $q \equiv -\perp\mathcal{T}(n, \cdot)$, $p$ and $\Pi$ being the energy density, the energy flux, the mean pressure and the traceless anisotropic pressure as measured by $n$, respectively; $\perp$ is the projector on the 3-space orthogonal to $n$ associated with the 3-metric $\gamma \equiv g + n \otimes n$.

### 6.1. Spherical symmetry

In the case of a SSST, using expression (8) of the extrinsic curvature, the constraint equations (42) and (43) are equivalent to

$$\Phi(\Phi + 2\Psi) = \kappa\tau - \frac{\mathcal{R}}{2} \tag{47}$$

$$2\partial_R\Phi + \frac{\partial_R D}{D}(\Phi - \Psi) = -\kappa q_R, \tag{48}$$

where $q_R$ is now the radial component of the energy flux. For the evolution equation (45), taking into account expression (11), we have $\Pi_{\varphi\varphi} = \Pi_{\theta\theta}\sin^2\theta$ and

$$\partial_T(\Psi B) = -\sqrt{B}\partial_R\left(\frac{\partial_R\alpha}{\sqrt{B}}\right) - \kappa\alpha\left(\Pi_{RR} + \frac{1}{2}(\tau - p)B\right)$$
$$+ \alpha\left(\frac{B}{2}\mathcal{R} - \frac{B}{D}F + B\Psi(2\Phi - \Psi)\right) + \frac{B}{C}\partial_R\left[\left(\frac{C}{B}\right)^2 B\Psi\right] \tag{49}$$

$$\partial_T(\Phi D) = -\frac{\partial_R D}{2B}\partial_R\alpha - \kappa\alpha\left(\Pi_{\theta\theta} + \frac{1}{2}(\tau - p)D\right)$$
$$+ \alpha\left(\frac{D}{4}\mathcal{R} + \frac{F}{2} + \Phi\Psi D\right) + \frac{C}{B}\partial_R(D\Phi). \tag{50}$$

For metric (1), equations (47)–(50) are the 3 + 1 splitting of the Einstein equations with respect to a vorticity-free observer. The proper space of such an observer is Euclidean if, and only if, $F = 0$, and then $\mathcal{R} = 0$. The integration of these equations for simple energetic contents (for instance, a dust model) should provide the corresponding metric form in PG coordinates. In the next section, the vacuum case is considered: the extended form of Schwarzschild solution in PG coordinates is obtained from the sole consideration of the field equations.

### 6.2. Schwarzschild vacuum solution

The extended Painlevé–Gullstrand–Lemaître metric form of the Schwarzschild solution may be obtained assuming spherical symmetry and the existence of a flat synchronization, $\mathcal{R}ic(\gamma) = 0$, and then, solving the vacuum Einstein equations in a coordinate system adapted to such a synchronization. So, let us take $\tau = p = \Pi_{RR} = \Pi_{\theta\theta} = F = \mathcal{R} = 0$. Then, for the metric expression (33), the lapse function is given by $\alpha^2 = \mathcal{B}^2 - \mathcal{A}$ and the constraint equations (47) and (48) result

$$\Phi(\Phi + 2\Psi) = 0 \tag{51}$$

$$r\Phi' + \Phi - \Psi = 0 \tag{52}$$

with

$$\Phi = \frac{1}{\alpha}\frac{\mathcal{B}}{r}, \qquad \Psi = \frac{\mathcal{B}'}{\alpha}. \tag{53}$$

When $\Phi = 0$, taking into account also the evolution equations, we recover the Minkowski spacetime. In the generic case, $\Phi = -2\Psi \neq 0$, equation (53) leads to

$$\mathcal{B} = f(t)r^{-1/2} \tag{54}$$

with $f(t)$ being an arbitrary function. Substituting equation (54) into the momentum constraint (52), it reduces to $\alpha' = 0$. Consequently, the lapse is a function of the sole variable $t$, $\alpha(t)$, and we can take $\alpha = 1$ by re-scaling the coordinate $t$. Then, we have

$$\Phi = f(t)r^{-3/2} = -2\Psi. \tag{55}$$

Next, the evolution equations (49) and (50) are written as

$$\dot{\Psi} = \Psi(2\Phi - \Psi) + \frac{1}{\mathcal{B}}(\mathcal{B}^2\Psi)'$$

$$\dot{\Phi} = \Psi\Phi + \frac{\mathcal{B}}{r^2}(r^2\Phi)'.$$

Given that $\Phi = -2\Psi$, these last equations are equivalent to

$$\dot{\Psi} = \Psi^2 + \frac{\mathcal{B}}{r^2}(r^2\Psi)' \tag{56}$$

$$3\Psi + \frac{\mathcal{B}}{r} - \mathcal{B}' = 0. \tag{57}$$

By using expressions (54) and (55), equation (56) leads to $f(t) = $ constant and equation (57) is identically satisfied. Finally, by taking $f = \epsilon\sqrt{2m}$, we obtain

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + 2\epsilon\sqrt{\frac{2m}{r}}\,dt\,dr + dr^2 + r^2\,d\Omega^2, \tag{58}$$

which is the extended form of the Schwarzschild solution obtained by Painlevé, Gullstrand and Lemaître [1–3]. The positive parameter $m$ is the Schwarzschild energy. The sign $\epsilon$ provides two coordinate branches for the solution; the Kruskal–Szekeres black and white hole regions being described by the above metric with $\epsilon = 1$ and $\epsilon = -1$, respectively, see [21]. Note that the $r$ coordinate can take any positive value, $0 < r < +\infty$. In fact, from (58), we have $(dr)^2 = g^{rr} = 1 - \frac{2m}{r} < 1$, and the domain of a PG chart extends for every value of $r \neq 0$. Note that (29) implies that $E = m$, which provides the physical interpretation of the parameter

$m$ as an effective energy [50]. Moreover, writing $dt = dt_S + \epsilon\sqrt{\frac{2m}{r}}\left(1 - \frac{2m}{r}\right)^{-1} dr$, one recovers the usual Schwarzschild metric form

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt_S^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{59}$$

where $t_S$ is the coordinate time of the static observer ($-\infty < t_S < \infty$) and the rank of the $r$ coordinate is restricted to be $r > 2m$. According to the Jebsen–Birkhoff theorem (see [57] and references therein), we recover the Schwarzschild metric as the sole spherically symmetric solution of the vacuum Einstein equations.

Other derivations of the Schwarzschild solution providing improvements of the original proof of the Jebsen–Birkhoff theorem have been achieved by solving the field equations in null coordinates (see [50, 58] and reference therein). From the conceptual point of view, any of these derivations makes it unnecessary to get a coordinate transformation allowing us to extend the domain of Schwarzschild chart from the outer to the inner horizon regions.

## 7. Discussion

In this work we have analyzed the existence of flat synchronizations in SSSTs. Condition (27) provides an upper bound for the norm of the mean extrinsic curvature vector of the isometry group orbits which, using the Gauss identity, may be expressed in terms of curvature invariants. Moreover, the associated flat slices have vanishing Weinberg energy and momentum densities. We have seen that any spherically symmetric metric admits a Newtonian interpretation in the context of the Bel-extended Newtonian theory of gravitation. Our study offers a new perspective about the meaning of the PG coordinates. This study applies for any SSST in the region where these coordinates exist. In this region, the gradient of the radial PG coordinate $r$ may be space-like, light-like, or time-like, according to the condition $(dr)^2 \leqslant 1$, which means that the Misner–Sharp gravitational energy of a sphere of radius $r$ is non-negative. This condition may be tested for any SSST, starting from the general metric form (1). For instance, it occurs elsewhere in the Schwarzschild geometry, as it has been pointed out at the end of section 6. Moreover, one has that $(dr)^2 \leqslant 1$ everywhere for any Robertson–Walker metric with an energetic content which satisfies the usual energy conditions. In fact, if we put in (1), $A = -1$, $B = a^2(t)\big/\left(1 + \frac{k}{4}r^2\right)^2$ (with $k = 1, 0, -1$ being the universe curvature index), $C = 0$ and $D = r^2 B$, we obtain that, in this case, (27) is equivalent to $k + \dot{a}^2 \geqslant 0$, which means that the proper energy density of the cosmological fluid is non-negative. Consequently, any Robertson–Walker spacetime that satisfies this energy condition admits a PG synchronization. This property is also obtained directly from inequality (28). In this case, $\tau = -e_1$ and $p = e_2 = e$ are, respectively, the energy density and the pressure of the cosmological fluid, and $\lambda = 0$, because the Robertson–Walker metric is conformally flat. We leave for a future work the results of the PG form of these Robertson–Walker cosmological models.

Finally, we have presented an improved proof of the Jebsen–Birkhoff theorem by expressing and solving the vacuum Einstein equations in PG coordinates. So, the extended Painlevé–Gullstrand–Lemaître metric form of the Schwarzschild solution is directly obtained.

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