Nearly Quaternionic Manifold(s)

Óscar Maciá

University of Valencia & Polytechnic University of Turin
oscarmacia@calvino.polito.it

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References

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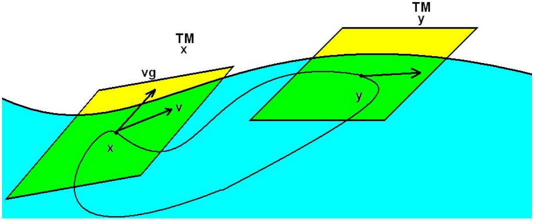
S.Chiossi, O.M., $\mathrm{SO}(3)$-structures on 8-manifolds,
to appear.

1. INTRODUCTION: QK GEOMETRY AQH GEOMETRY
2. INTRINSIC TORSION AND IDEAL GEOMETRY
3. NEARLY QUATERNIONIC STRUCTURE

Riemannian Holonomy

Let $\{M, g\}$ be a Riemannian manifold and let $c : [0, 1] \to M$ a smooth curve on M from x to y. The Levi-Civita connection determines horizontal transport of vectors on TM along the curve c. This defines a linear isometry $(T_x M, g_x) \to (T_y M, g_y)$.
For $x = y$ these transformations determine a group that is independent of x for M connected.



Definition

Holonomy Group (Φ): Group of transformations of the fibres of a bundle induced by parallel translation over closed loops in the base manifold.

Berger's List

Theorem

Let M^n be Riemannian n-manifold non locally symmetric, non locally reducible. Then, its holonomy group Φ is contained in the following list $(n = 2m = 4k)$:

$$\mathrm{SO}(n),\ \mathrm{U}(m),\ \mathrm{SU}(m),\ \mathrm{Sp}(k),\ \mathrm{G}_2,\ \mathrm{Spin}(7),\ \mathrm{Sp}(k)\mathrm{Sp}(1)$$

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- G_2, $\mathrm{Spin}(7)$: Exceptional holonomy. Exist only in dimension 7 and 8.
- $\mathrm{Sp}(k)\mathrm{Sp}(1) := \mathrm{Sp}(k) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$ Quaternionic-Kähler geometry.

QK Geometry

- $\Phi \subseteq \mathrm{Sp}(k)\mathrm{Sp}(1) \rightarrow Ric = \lambda g$. **EINSTEIN** ($\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, G_2 *and* $\mathrm{Spin}(7)$ cases are all Ricci-flat).

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    Geneeral QK manifolds are not Kähler $\mathrm{Sp}(k)\mathrm{Sp}(1) \not\subset \mathrm{U}(m)$.

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- In the following, when referring to QK manifolds we mean $\lambda \neq 0$.

Operational definitions for QK

Definition

A QK manifold is a **Riemannian 4k-manifold** $\{M^{4k}, g\}$ equipped with a family of **three compatible almost complex structures** $\mathcal{J} = \{J_i\}_1^3$

$$g(J_i\cdot, J_i\cdot) = g(\cdot,\cdot), \ i = 1,2,3$$

satisfying the algebra of imaginary quaternions

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -\mathbf{1},$$

such that $\mathcal{J}$ is preserved by the Levi–Civita connection

$$\nabla_X^{LC} J_i = \alpha_k(X) J_j - \alpha_j(X) J_k \qquad (i, j, k \ cyclic)$$

for certain 1-forms $\alpha_i, \ \alpha_j, \alpha_k$.

Non-integrable geometries

Definition

A G-structure is a reduction of the bundle of linear frames $L(M)$ to a subbundle with (prescribed) structure group G.

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- Each G-irreducible component of the tensor $\nabla^{LC}\eta$ characterises a family of non-integrable geometries which bare some particular resemblance with the **integrable** case $\Phi \subseteq G$.

Example: Almost Hermitian (AH) Manifolds

- Let $\{M^{2m}, g, J\}$ be an AH manifold, i.e. a Riemannian $2m$-manifold $\{M^{2m}, g\}$ together with a compatible almost complex structure

$$g(J\cdot, J\cdot) = g(\cdot, \cdot).$$

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- Non-integrable Case: $\boxed{\nabla^{LC}\omega \neq 0}$. The tensor $\nabla^{LC}\omega$ decomposes with respect to the action of $U(m)$ in 4 components usually denoted by
$$\nabla^{LC}\omega = [\![T]\!] \oplus [\![\Lambda^{3,0}]\!] \oplus [\![\Lambda_0^{2,1}]\!] \oplus [\![\Lambda^{1,0}]\!]$$

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- $2^4 = 16$ posiblities (Kähler, nearly Kähler, almost Kähler, locally conformal to Kähler, quasi Kähler, semi-Kähler, etc...)

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A Riemannian $4k$-manifold with $\mathrm{Sp}(k)\mathrm{Sp}(1)$-structure is called Almost Quaternionic Hermitian (AQH).

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- Ω can be written in terms of the (local) Kähler 2-forms ω_i associated to the J_i

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Theorem

An AQH manifold is QK if and only if $\nabla^{LC}\Omega = 0$.

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Locally

$$\mathbb{C} \otimes TM = E \otimes H.$$

Intrinsic Torsion of AQH manifolds

Theorem

The intrinsic torsion of an $4k$-manifold, $k \geq 2$ can be identified with an element $\nabla^{LC}\Omega$ in the space

$$(\Lambda^3_0 E \oplus K \oplus E) \otimes (H \oplus S^3 H)$$

ES^3H	$\Lambda^3_0 ES^3H$	KS^3H
EH	$\Lambda^3_0 EH$	KH

For $k = 2$, the intrinsic torsion belongs to

$$E\,S^3H \oplus K\,S^3H \oplus KH \oplus EH$$

ES^3H	KS^3H
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A. Swann, (1989).

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Theorem

An AQH 4k-manifold, $4k \geq 12$ is QK if and only if $d\Omega = 0$

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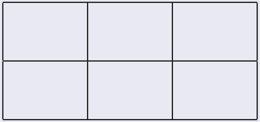

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The Kähler 2-forms $\{\omega_i\}$ *of an AQH 8-manifold generate a differential ideal if and only if* $\nabla^{LC}\Omega \in ES^3H \oplus EH$,

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An AQH 8-manifold is QK iff

1. $d\Omega = 0$
2. $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$

	KS^3H

$\bigcap$

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Existence question

Do actually exist non-QK AQH 8-manifolds satisfying (1) or (2) only?

Theorem

There exists a closed 4-form Ω with stabilizer $\mathrm{Sp}(2)\mathrm{Sp}(1)$ on a compact nilmanifold of the form $M^6 \times T^2$. The associated Riemannian metric g is reducible and is not therefore quaternionic Kähler.

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$\Longrightarrow$ Relation between AQH & QK geometry in 8 dimensions is special.

- NON-QK AQH8 : $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$ (Satisfies condition 2, not 1) ?

Ideal condition

$$\boxed{\mathrm{d}\omega_i = \sum_j \beta_i^j \wedge \omega_j \ : \ \beta_i^j \in \Lambda^1 M}$$

CHANGE OF BASE: $\{\omega_i\} \mapsto \{\widetilde{\omega}_i\}$

$$\widetilde{\omega}_i = \sum_{j=1}^{3} A_i^j \omega_j, \qquad A = (A_i^j) \in \mathrm{SO}(3)$$

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- However, this connection does not reduce to $\mathrm{SO}(3)$ unless β is anti-symmetric.
- Consider the decomposition

$$\beta = \alpha + \sigma, \qquad \alpha_i^j = \frac{1}{2}(\beta_i^j - \beta_j^i) \qquad \sigma_i^j = \frac{1}{2}(\beta_i^j + \beta_j^i)$$

- The symmetric part σ transforms as a tensor:

$$\widetilde{\sigma} = \mathrm{Ad}(A^{-1})\sigma = A^{-1}\sigma A.$$

- The tensor σ can be identified with the remaining non-zero components of intrinsic torsion

$$ES^3H \oplus EH$$

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If an $\mathrm{Sp}(2)\mathrm{Sp}(1)$*-structure satisfies the ideal condition then its intrinsic torsion belongs to* ES^3H *if and only if* $\mathrm{tr}(\beta) = \beta_1^1 + \beta_2^2 + \beta_3^3 = 0.$

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Corollary

Let $\{M, g, \mathcal{J}\}$ *be an AQH 8-manifold. It is QK if and only if generates a differential ideal with* $\sigma = 0$*, so that the ideal condition applies with* $\beta_i^j = -\beta_j^i$.

Geometry of the ideal condition

Consider the matrix $B = (B_i^j)$ of curvature 2-forms associated to the connection defined through β .

$$0 = d^2\omega_i = \sum_j (d\beta_i^j -_k \beta_i^k \wedge \beta_k^j) \wedge \omega_k = \sum_j B_i^j \wedge \omega_j$$

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- In contrast to the QK case, there will in general be a component of B_i^j in $\Lambda_0^2 E\, S^2 H$.

Nearly Quaternionic Manifold(s)

1 INTRODUCTION: QK GEOMETRY AQH GEOMETRY

2 INTRINSIC TORSION AND IDEAL GEOMETRY

3 NEARLY QUATERNIONIC STRUCTURE

Factorisation of $SO(3) \subset SO(8)$

$SO(3) \subset SO(8)$ factors through $Sp(2)Sp(1) \equiv Sp(2) \times_{\mathbb{Z}_2} Sp(1)$ in a unique way.

$$\begin{array}{ccc} SO(3) & \longrightarrow & SO(8) \\ {\scriptstyle [\rho,\mathbf{1}]}\downarrow & & \Big\| \\ Sp(2)Sp(1) & \longrightarrow & SO(8) \end{array}$$

$$\mathbf{1} : SO(3) \simeq SU(2) \simeq Sp(1)$$

$$\rho : SO(3) \simeq Sp(1) \overset{\textit{irreducible}}{\longrightarrow} Sp(2)$$

$$X \in SO(3) \longmapsto (\rho(X), \mathbf{1}(X)) \in Sp(2) \times Sp(1)$$

$$[\rho(X), \mathbf{1}(x)] \in Sp(2)Sp(1)$$

SO(3) Intrinsic torsion from Sp(2)Sp(1)

- Let H denote the basic representation of SO(3) identified with Sp(1), the irreducible action ρ of Sp(1) embedded on Sp(2) gives the identification

$$E = S^3H$$

- The Sp(2)-modules are reducible with respect to the action of SO(3).

$$EH \oplus KH \oplus ES^3H \oplus KS^3H$$

$$W_1 := ES^3H \longrightarrow S^6H \oplus S^4H \oplus S^2H \oplus \mathbb{R}$$

$$W_2 := KS^3H \longrightarrow S^{10}H \oplus 2S^8H \oplus 2S^6H \oplus 3S^4H \oplus 2S^2H$$

$$W_3 := KH \longrightarrow S^8H \oplus 2S^6H \oplus S^4H \oplus S^2H \oplus \mathbb{R}$$

$$W_4 := EH \longrightarrow S^4H \oplus S^2H$$

- The SO(3)-structure described has intrinsic torsion obstructions on

$$KH \oplus ES^3H.$$

Action of SO(3) on SU(3)

- If $\mathrm{SO}(3) \to \mathrm{Sp}(2)\mathrm{Sp}(1) \to \mathrm{SO}(8)$ then,

$$\mathbb{C} \otimes TM = E \otimes H = S^3 H \otimes H = S^4 H \oplus S^2 H$$

- The $\mathrm{SO}(3)$ action leads to a $\mathfrak{so}(3)$ family of endomorphisms

$$\mathfrak{so}(3) \simeq S^2 H \subset End(T)$$

- Take the manifold

$$M = SU(3) \to T_x M \simeq \mathfrak{su}(3)$$

$$\mathfrak{su}(3) = \mathfrak{b} \oplus \mathfrak{p} : \begin{cases} \mathfrak{b} \simeq \mathfrak{so}(3) \subset \mathfrak{su}(3), \quad \mathfrak{b} \simeq S^2 H \\ \mathfrak{p} \simeq Span\{iS : \ S = S^t, \ Tr(S) = 0\} \simeq S^4 H. \end{cases}$$

- Then the action of $\mathrm{SO}(3)$ on $\mathrm{SU}(3)$ is given on tangent space as the action of $S^2 H \simeq \mathfrak{so}(3) \subset End(T)$ on $\mathfrak{su}(3) = S^4 H \oplus S^2 H$

$$\phi : S^2 H \otimes (\mathfrak{b} \oplus \mathfrak{p}) \to \mathfrak{b} \oplus \mathfrak{p}$$

The mapping ϕ

$$\boxed{\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4}$$

$$\begin{aligned}
\phi_1 : (S^2H \otimes \mathfrak{b}) = S^4H \otimes S^2H \otimes S^0H &\longrightarrow S^2H = \mathfrak{b} \\
(A, B) &\longmapsto [A, B]
\end{aligned}$$

$$\begin{aligned}
\phi_2 : (S^2H \otimes \mathfrak{b}) = S^4H \otimes S^2H \otimes S^0H &\longrightarrow S^4H = \mathfrak{p} \\
(A, B) &\longmapsto i\left(\{A, B\} - \frac{2}{3}Tr(AB)\mathbf{1}\right)
\end{aligned}$$

$$\begin{aligned}
\phi_3 : (S^2H \otimes \mathfrak{p}) = S^6H \otimes S^4H \otimes S^2H &\longrightarrow S^2H = \mathfrak{b} \\
(A, C) &\longmapsto i\{A, C\}
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\phi_4 : (S^2H \otimes \mathfrak{p}) = S^6H \otimes S^4H \otimes S^2H &\longrightarrow S^4H = \mathfrak{p} \\
(A, C) &\longmapsto [A, C]
\end{aligned}$$

AQ Action of SO(3) on SU(3)

- Denote the action defined by ϕ with the dot-product

$$A \cdot X = \lambda_1 [A, X^a] + i\lambda_2 \left(\{A, X^a\} - \frac{2}{3} Tr(AX^a) \right) + i\lambda_3 \{A, X^s\} + \lambda_4 [A, X^s].$$

for $A \in \mathfrak{so}(3),\ X \in \mathfrak{su}(3)$.

- Taking $\mathcal{J} = \{J_1, J_2, J_3\} \in S^2 H$ and asking the previous equation to satisfy

$$J_i \cdot (J_i \cdot X) = -X \qquad J_1 \cdot (J_2 \cdot X) = J_3 \cdot X$$

one obtains

$$\boxed{\lambda_1 = \tfrac{1}{2}, \qquad \lambda_3 = -\tfrac{3}{4}\lambda^{-1}, \qquad \lambda_4 = -\tfrac{1}{2}}$$

where $\lambda = \lambda_2$ is a real parameter. This is a 1-parameter family of almost quaternionic actions of SO(3) on SU(3).

AQH structure on $\mathrm{SU}(3)$

- Let $\{e_i\}_1^8$ be a base for $\mathfrak{su}(3)$, orthonormal for a multiple of the Killing metric.
- $\mathfrak{b} = \mathfrak{so}(3) = \mathrm{Span}\{e_6, e_7, e_8\}$.
- Identify $\mathcal{J}_\lambda = \{e_6, e_7, e_8\}$ acting through the 1-parameter family of AQ $\mathrm{SO}(3)$ actions defined by ϕ.
- Define a new metric by rescaling the $\mathfrak{b}$ subspace

$$g_\lambda = \sum_{i=1}^{i=5} e^i \otimes e^i + \frac{4\lambda^2}{3} \sum_{i=6}^{i=8} e^i \otimes e^i .$$

Theorem

$\mathcal{J}_\lambda$ is compatible with g_λ

- $\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$ is a 1-parameter family of AQH 8-manifolds

Ideal AQH structure on SU(3)

Theorem

A set of λ-dependent Kähler 2-forms $\{\omega_i\}_\lambda$ associated to the AQH 8-manifold $\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$ is given by

$$\omega_1 = \frac{1}{2}\left(e^{15} + \sqrt{3}e^{25} + e^{34}\right) + \lambda\left(\frac{1}{\sqrt{3}}e^{28} - e^{46} + e^{37} - e^{18}\right) - \frac{2}{3}\lambda^2 e^{67},$$

$$\omega_2 = -e^{14} - \frac{1}{2}e^{35} + \lambda\left(\frac{2}{\sqrt{3}}e^{27} - e^{38} - e^{56}\right) - \frac{2}{3}\lambda^2 e^{68},$$

$$\omega_3 = \frac{1}{2}\left(e^{13} - \sqrt{3}e^{23} + e^{45}\right) + \lambda\left(\frac{1}{\sqrt{3}}e^{26} - e^{48} + e^{57} + e^{16}\right) - \frac{2}{3}\lambda^2 e^{78}$$

Theorem

AQH $\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$ satisfies the ideal condition $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$ if and only if

$$\lambda^2 = \frac{3}{20}.$$

Nearly quaternionic structure on $\mathrm{SU}(3)$

Corollary

$\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$ *is not QK for any choice of* λ*.*

Due to the topology of $\mathrm{SU}(3)$,

$$\boxed{b_4(\mathrm{SU}(3)) = 0}.$$

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• We call this case **Nearly Quaternionic** (NQ) (by the analogy with the Nearly Kähler case).

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> Is there any other case apart from $\{\mathrm{SU}(3), \mathcal{J}_{\sqrt{3/20}}, g_{\sqrt{3/20}}\}$?

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- In our case, SU(3) cannot admit an SO(3)-invariant quaternionic structure.

SO(3)-structures with invariant torsion

- The SO(3)-intrinsic torsion is a 200-dimensional space with 3-invariants

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S.Chiossi & O.M. (in progress)

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- The example $\{\mathrm{SU}(3), \mathcal{J}_{\sqrt{3/20}}, g_{\sqrt{3/20}}\}$ has one of these invariant SO(3)-structures.
- The SO(3)-structure is determined by six forms of different degrees $\{\alpha^3, \beta^3, \gamma^4, \delta^4, *\alpha^5, *\beta^5\}$ (instead of the only 4-form Ω) and the curvature 2-form B of the connection β can be written in terms of these invariant forms.

S.Chiossi & O.M. (in progress)

Thank You