$SO(3)$-structures on AQH 8-manifolds

Óscar Maciá

Polytechnic University of Turin
oscarmacia@calvino.polito.it

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References

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1 INTRODUCTION: QK GEOMETRY AQH GEOMETRY

2 INTRINSIC TORSION AND IDEAL GEOMETRY

3 NEARLY QUATERNIONIC STRUCTURE

RIEMANNIAN HOLONOMY

Let $\{M, g\}$ be a Riemannian manifold.
Let $c : [0, 1] \to M$ a smooth curve on M from x to y. The Levi-Civita connection determines horizontal transport of vectors on TM along the curve c

Transport.png

Berger's List

Theorem

(Berger, 1955) Let M^n be Riemannian n-manifold non locally symmetric, non locally reducible. Then, its holonomy group Φ is contained in the following list list $(n = 2m = 4k)$:

$$\mathrm{SO}(n),\ \mathrm{U}(m),\ \mathrm{SU}(m),\ \mathrm{Sp}(k),\ \mathrm{G}_2,\ \mathrm{Spin}(7),\ \mathrm{Spin}(9)^{\dagger},\ \mathrm{Sp}(k)\mathrm{Sp}(1)$$

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- $\mathrm{SO}(n)$: Generic Riemannian geometry
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- $\mathrm{Spin}(9)^{\dagger}$: D=16. ALWAYS SYMMETRIC (ruled out from the list).

QK manifolds and geometry in Berger's list

BERGERLIST.png

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- **NICE NAME...**

groucho.png

Operational definitions for QK

In the following, when referring to QK manifolds we mean $\lambda \neq 0$.

Definition

A QK manifold is a **Riemannian 4n-manifold** $\{M^{4n}, g\}$ equipped with a family of **three compatible almost complex structures** $\mathcal{J} = \{J_i\}_1^3$

$$g(J_i\cdot, J_i\cdot) = g(\cdot,\cdot),\ i = 1, 2, 3$$

satisfying the algebra of imaginary quaternions

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -\mathbf{1},$$

such that it is preserved by the Levi–Civita connection

$$\nabla_X^{LC} J_i = \alpha_k(X) J_j - \alpha_j(X) J_k$$

for certain 1-forms $\alpha_i,\ \alpha_j, \alpha_k$.

G-Structures

The theory of G-structures allows to work with general (non necessarily torsion-free) connections on Riemannian manifolds.

Definition

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- A G-structure is defined through a distinguished G-invariant tensor η.

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  - Example 1: A n-manifold M with $\mathrm{SO}(n)$-structure is equivalent to define a metric on M.

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  - Example 2: A Riemannian $2n$-manifold $\{2n\}$ with $\mathrm{U}(n)$-structure is equivalent to define a compatible almost complex structure J

$$\boxed{\{\mathrm{U}(n) - \textit{structure on } M\} \longleftrightarrow \textit{Almost Hermitian manifold } \{M, g, J\}}$$

Connections on Riemannian G-structures

Theorem

$$\nabla^{LC}\eta = 0 \longleftrightarrow \Phi \subseteq \mathrm{G}$$

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$$\xi := \nabla^{LC}\eta \subset T^*M \otimes \mathfrak{g}^{\perp}$$

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Theorem

If G is a closed and connected subgroup of $\mathrm{SO}(n)$ there is a unique (non-torsion free) metric G-connection (minimal G-connection) satisfying

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•The decomposition of ξ in irreducible components w.r.t. the action of G classifies all possible minimal G-connections.

Almost Quaternionic Hermitian (AQH) manifolds

Definition

A Riemannian $4n$-manifold with $\mathrm{Sp}(n)\mathrm{Sp}(1)$-structure is called Almost Quaternionic Hermitian (AQH).

Definition

An AQH manifold is a **Riemannian** $4n$**-manifold** $\{M, g\}$ equipped with a family **three compatible almost complex structures** $\{J_i\}_1^3$

$$g(J_i\cdot, J_i\cdot) = g(\cdot,\cdot), \;\; i = 1, 2, 3$$

satisfying the algebra of imaginary quaternions

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -\mathbf{1}.$$

- The difference between AQH and QK definitions involves the **relations between tensors** $\{J_i\}$ **and the connection**.

The fundamental 4-form Ω

- The AQH structure is defined by the distinguished $\mathrm{Sp}(k)\mathrm{Sp}(1)$ invariant 4-form $\Omega \in \Lambda^4 M$.

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- Ω can be written in terms of the Kähler 2-forms associated to J

$$\Omega = \sum_i \omega_i^2 = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

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Theorem

An AQH manifold is QK if and only if $\nabla^{LC}\Omega = 0$.

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EH-Formalism

Representation theory notation

- $E \simeq \mathbb{C}^4$ irreducible basic complex representation of $\mathrm{Sp}(2)$. (Highest weight $[1, 0]$).
- $H \simeq \mathbb{C}^2 \simeq \mathbb{H}$ irreducible basic complex represtentation of $\mathrm{Sp}(1)$. (Highest weight $[1]$).

Locally

$$\mathbb{C} \otimes TM = E \otimes H.$$

Other important $\mathrm{Sp}(2)$ representations

- $K \simeq \mathbb{C}^{16}$ irreducible complex representation of $\mathrm{Sp}(2)$. (Highest weight $[2, 1]$ in the basis of roots).
- $\Lambda_0^3 E \simeq \mathbb{C}^3$ irreducible complex representation of $\mathrm{Sp}(2)$. (Highest weight $[3, 3]$ in the basis of roots).

$$\Lambda_0^n E = Coker\{L : \Lambda^{n-2} E \to \Lambda^n E : \alpha \mapsto \omega_E \wedge \alpha\}.$$

Intrinsic Torsion of AQH manifolds

Theorem

(Swann, 1989) The intrinsic torsion of an $4n$-manifold, $n \geq 2$ can be identified with an element $\nabla\Omega$ in the space

$$(\Lambda^3_0 E \oplus K \oplus E) \otimes (H \oplus S^3 H)$$

ES^3H	$\Lambda^3_0 ES^3H$	KS^3H
EH	$\Lambda^3_0 EH$	KH

For $n = 2$, the intrinsic torsion belongs to

$$E\,S^3H \oplus K\,S^3H \oplus KH \oplus EH$$

ES^3H	KS^3H
EH	KH

$AQH8 \xrightarrow{?} QK8$

Corollary

The fundamental 4-form Ω *of an 8-manifold is closed* $d\Omega = 0$*, i.e.,* M *is* **almost parallel** *if and only if* $\nabla\Omega \in KS^3H$.

$d\Omega = 0 \longleftrightarrow$

$\bullet$	KS^3H
$\bullet$	$\bullet$

Corollary

The Kähler 2-forms $\{\omega_i\}$ *of an 8-manifold generate a differential ideal if and only if* $\nabla\Omega \in ES^3H \oplus EH$,

$d\omega_i = \sum_j \beta_i^j \wedge \omega_j \ : \ \beta_i^j \in \Lambda^1 M \longleftrightarrow$

ES^3H	$\bullet$
EH	$\bullet$

$AQH(4n) \xrightarrow{?} QK(4n)$

Theorem

(Swann, 1989)

An AQH 4n-manifol, $4n \geq 12$ *is QK if and only if*

1. $d\Omega = 0$

•	•	•
•	•	•

An AQH 8-manifold is QK if and only if

1. $d\Omega = 0$
2. $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$

•	KS^3H
•	•

$\bigcap$

ES^3H	•
EH	•

$=$

•	•
•	•

Existence question

Do actually exist non-QK AQH 8-manifolds satisfying (1) or (2) only?

Theorem

(Salamon, 2001)
There exists a closed 4-form Ω *with stabilizer* $\mathrm{Sp}(2)\mathrm{Sp}(1)$ *on a compact nilmanifold of the form* $M^6 \times T^2$*. The associated Riemannian metric* g *is reducible and is not therefore quaternionic Kähler.*

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$\Longrightarrow$ Relation between AQH & QK geometry in 8 dimensions is special.

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$\Longrightarrow$ Relation between AQH & QK geometry in 8 dimensions is <u>special</u>.

- NON-QK AQH8 : $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$ (Satisfies condition 2, not 1) ?

Ideal condition

$$\boxed{\mathrm{d}\omega_i = \textstyle\sum_j \beta_i^j \wedge \omega_j \ : \ \beta_i^j \in \Lambda^1 M}$$

CHANGE OF BASE: $\{\omega_i\} \mapsto \{\widetilde{\omega}_i\}$

$$\widetilde{\omega}_i = \sum_{j=1}^{3} A_i^j \omega_j, \qquad A = (A_i^j) \in \mathrm{SO}(3)$$

- The matrix β transforms as a connection

$$d\widetilde{\omega}_i = \sum_{j=1}^{3} \widetilde{\beta}_i^j \wedge \widetilde{\omega}_j \qquad : \qquad \widetilde{\beta} = A^{-1} dA + \mathrm{Ad}(A^{-1})\beta.$$

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- However, this connection does not reduce to $\mathrm{SO}(3)$ unless β is anti-symmetric.
- Consider the decomposition

$$\beta = \alpha + \sigma, \qquad \alpha_i^j = \frac{1}{2}(\beta_i^j - \beta_j^i) \qquad \sigma_i^j = \frac{1}{2}(\beta_i^j + \beta_j^i)$$

- The symmetric part σ transforms as a tensor:

$$\widetilde{\sigma} = \mathrm{Ad}(A^{-1})\sigma = A^{-1}\sigma A.$$

- The tensor σ can be identified with the remaining non-zero components of intrinsic torsion

$$ES^3H \oplus EH$$

- $$d\Omega = 2\sum_{i=1}^{3} d\omega_i \wedge \omega_i = 2\sum_{i,j=1}^{3} \sigma_i^j \wedge \omega_i \wedge \omega_j.$$

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Lemma

If an $\mathrm{Sp}(2)\mathrm{Sp}(1)$-structure satisfies the ideal condition then its intrinsic torsion belongs to ES^3H if and only if $\mathrm{tr}(\beta) = \beta_1^1 + \beta_2^2 + \beta_3^3 = 0.$

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Corollary

Let $\{M, g, \mathcal{J}\}$ be an AQH 8-manifold. It is QK if and only if generates a differential ideal with $\sigma = 0$, so that the ideal condition applies with $\beta_i^j = -\beta_j^i$.

Geometry of the ideal condition

Consider the matrix $B = (B_i^j)$ of curvature 2-forms associated to the connection defined through β .

$$0 = d^2\omega_i = \sum_j (d\beta_i^j -_k \beta_i^k \wedge \beta_k^j) \wedge \omega_k = \sum_j B_i^j \wedge \omega_j$$

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- In particular, they have no S^2E component, because

  $$S^2E\, S^2H \subset \Lambda^4 T^* M;$$

  thus

  $$B_i^j \in S^2H \oplus \Lambda_0^2 E\, S^2H \subset \Lambda^2 T^* M.$$

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  $$B_i^j \in S^2H \oplus \Lambda_0^2 E\,S^2H \subset \Lambda^2 T^*M.$$

- In contrast to the QK case, there will in general be a component of B_i^j in $\Lambda_0^2 E\,S^2H$.

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Factorisation of $SO(3) \subset SO(8)$

$SO(3) \subset SO(8)$ factors through $Sp(2)Sp(1) \equiv Sp(2) \times_{\mathbb{Z}_2} Sp(1)$ in a unique way.

$$\begin{array}{ccc} SO(3) & \longrightarrow & SO(8) \\ {\scriptstyle [\rho,\mathbf{1}]}\big\downarrow & & \big\| \\ Sp(2)Sp(1) & \longrightarrow & SO(8) \end{array}$$

$$\mathbf{1} : SO(3) \simeq SU(2) \simeq Sp(1)$$

$$\rho : SO(3) \simeq Sp(1) \overset{\textit{irreducible}}{\longrightarrow} Sp(2)$$

$$X \in SO(3) \longmapsto (\rho(X), \mathbf{1}(X)) \in Sp(2) \times Sp(1)$$

$$[\rho(X), \mathbf{1}(x)] \in Sp(2)Sp(1)$$

SO(3) Intrinsic torsion from Sp(2)Sp(1)

- Let H denote the basic representation of SO(3) identified with Sp(1), the irreducible action ρ of Sp(1) embedded on Sp(2) gives the identification

$$E = S^3 H$$

- The Sp(2)-modules are reducible with respect to the action of SO(3).

$$EH \oplus KH \oplus ES^3H \oplus KS^3H$$

$$\begin{array}{rcl} EH & \longrightarrow & S^4 \oplus S^2 \\ KH & \longrightarrow & S^8 \oplus 2S^6 \oplus S^4 \oplus S^2 \oplus S^0 \\ ES^3H & \longrightarrow & S^6 \oplus S^4 \oplus S^2 \oplus S^0 \\ KS^3H & \longrightarrow & S^{10} \oplus 2S^8 \oplus 2S^6 \oplus 3S^4 \oplus 2S^2 \end{array}$$

- The SO(3)-structure described has intrinsic torsion obstructions on

$$KH \oplus ES^3H.$$

Action of SO(3) on SU(3)

- If $SO(3) \to \mathrm{Sp}(2)\mathrm{Sp}(1) \to \mathrm{SO}(8)$ then,

$$\mathbb{C} \otimes TM = E \otimes H = S^3 H \otimes H = S^4 H \oplus S^2 H$$

- The SO(3) action leads to a $\mathfrak{so}(3)$ family of endomorphisms

$$\mathfrak{so}(3) \simeq S^2 H \subset End(T)$$

- Take the manifold

$$M = SU(3) \to T_x M \simeq \mathfrak{su}(3)$$

$$\mathfrak{su}(3) = \mathfrak{b} \oplus \mathfrak{p} : \begin{cases} \mathfrak{b} \simeq \mathfrak{so}(3) \subset \mathfrak{su}(3), \quad \mathfrak{b} \simeq S^2 H \\ \mathfrak{p} \simeq Span\{iS : \ S = S^t, \ Tr(S) = 0\} \simeq S^4 H. \end{cases}$$

- Then the action of SO(3) on SU(3) is given on tangent space as the action of $S^2 H \simeq \mathfrak{so}(3) \subset End(T)$ on $\mathfrak{su}(3) = S^4 H \oplus S^2 H$

$$\phi : S^2 H \otimes (\mathfrak{b} \oplus \mathfrak{p}) \to \mathfrak{b} \oplus \mathfrak{p}$$

The mapping ϕ

$$\boxed{\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4}$$

$$\begin{aligned}\phi_1 : (S^2H \otimes \mathfrak{b}) = S^4H \otimes S^2H \otimes S^0H &\longrightarrow S^2H = \mathfrak{b} \\ (A, B) &\longmapsto [A, B]\end{aligned}$$

$$\begin{aligned}\phi_2 : (S^2H \otimes \mathfrak{b}) = S^4H \otimes S^2H \otimes S^0H &\longrightarrow S^4H = \mathfrak{p} \\ (A, B) &\longmapsto i\left(\{A, B\} - \frac{2}{3} Tr(AB)\mathbf{1}\right)\end{aligned}$$

$$\begin{aligned}\phi_3 : (S^2H \otimes \mathfrak{p}) = S^6H \otimes S^4H \otimes S^2H &\longrightarrow S^2H = \mathfrak{b} \\ (A, C) &\longmapsto i\{A, C\}\end{aligned}$$

$$\begin{aligned}\phi_4 : (S^2H \otimes \mathfrak{p}) = S^6H \otimes S^4H \otimes S^2H &\longrightarrow S^4H = \mathfrak{p} \\ (A, C) &\longmapsto [A, C]\end{aligned}$$

- Denote the action defined by ϕ with the dot-product

$$A \cdot X = \lambda_1 [A, X^a] + i\lambda_2 \left(\{A, X^a\} - \frac{2}{3} Tr(AX^a) \right) + i\lambda_3 \{A, X^s\} + \lambda_4 [A, X^s].$$

for $A \in \mathfrak{so}(3),\ X \in \mathfrak{su}(3)$.

- Taking $\mathcal{J} = \{J_1, J_2, J_3\} \in S^2 H$ and asking the previous equation to satisfy

$$J_i \cdot (J_i \cdot X) = -X \qquad J_1 \cdot (J_2 \cdot X) = J_3 \cdot X$$

one obtains

$$\boxed{\lambda_1 = \tfrac{1}{2}, \qquad \lambda_3 = -\tfrac{3}{4}\lambda^{-1}, \qquad \lambda_4 = -\tfrac{1}{2}}$$

where $\lambda = \lambda_2$ is a real parameter. This is a 1-parameter family of almost quaternionic actions of SO(3) on SU(3).

AQH structure on SU(3)

- Let $\{e_i\}_1^8$ be a base for $\mathfrak{su}(3)$, orthonormal for a multiple of the Killing metric.
- $\mathfrak{b} = \mathfrak{so}(3) = \mathrm{Span}\{e_6, e_7, e_8\}$.
- Identify $\mathcal{J}_\lambda = \{e_6, e_7, e_8\}$ acting through the 1-parameter family of AQ SO(3) actions defined by ϕ.
- Define a new metric by rescaling the $\mathfrak{b}$ subspace

$$g_\lambda = \sum_{i=1}^{i=5} e^i \otimes e^i + \frac{4\lambda}{3} \sum_{i=6}^{i=8} e^i \otimes e^i .$$

Theorem

$\mathcal{J}_\lambda$ is compatible with g_λ

- $\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$ is a 1-parameter family of AQH 8-manifolds

Ideal AQH structure on SU(3)

Theorem

A set of λ-dependent Kähler 2-forms $\{\omega_i\}_\lambda$ associated to the AQH 8-manifold $\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$ is given by

$$\begin{aligned}
\omega_1 &= \frac{1}{2}(e^{15} + \sqrt{3}e^{25} + e^{34}) + \lambda(\frac{1}{\sqrt{3}}e^{28} - e^{46} + e^{37} - e^{18}) - \frac{2}{3}\lambda^2 e^{67}, \\
\omega_2 &= -e^{14} - \frac{1}{2}e^{35} + \lambda(\frac{2}{\sqrt{3}}e^{27} - e^{38} - e^{56}) - \frac{2}{3}\lambda^2 e^{68}, \\
\omega_3 &= \frac{1}{2}(e^{13} - \sqrt{3}e^{23} + e^{45}) + \lambda(\frac{1}{\sqrt{3}}e^{26} - e^{48} + e^{57} + e^{16}) - \frac{2}{3}\lambda^2 e^{78}
\end{aligned}$$

Theorem

AQH $\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$ satisfies the ideal condition $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$ if and only if

$$\lambda^2 = \frac{3}{20}.$$

Nearly quaternionic structure on $\mathrm{SU}(3)$

Corollary

$\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$ *is not QK for any choice of* λ*.*

Due to the topology of $\mathrm{SU}(3)$,

$$b_4(\mathrm{SU}(3)) = 0.$$

Nearly quaternionic structure on $\mathrm{SU}(3)$

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Hence, for $\lambda^2 = \frac{3}{20}$, $\{\mathrm{SU}(3), \mathcal{J}_\lambda, g_\lambda\}$

- NON-QK AQH8 : $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$ (Satisfies condition 2, not 1) $\boxed{\checkmark}$.

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$$Tr(\beta) = 0 \longrightarrow \nabla\Omega \in ES^3H$$

ES^3H	•
•	•

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•	•

THANK YOU.