ELEMENTARY DEFORMATIONS AND THE HYPERKAEHLER / QUATERNIONIC KAEHLER CORRESPONDENCE

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Aarhus, DK

March 2, 2017

1 Planning

Introduction and motivation. The c-map

Interlude: Differential-geometric construction of the c-map
(pre-twist version)

Twist and hK/qK correspondence

Planning 1/3

Introduction and motivation. The c-map

3 Berger's List

Theorem

Let M be a Riemannian, oriented, simply-connected n-dimensional manifold, which is not locally a product, nor symmetric. Then its holonomy group belongs to the following list:

M. Berger (1955)

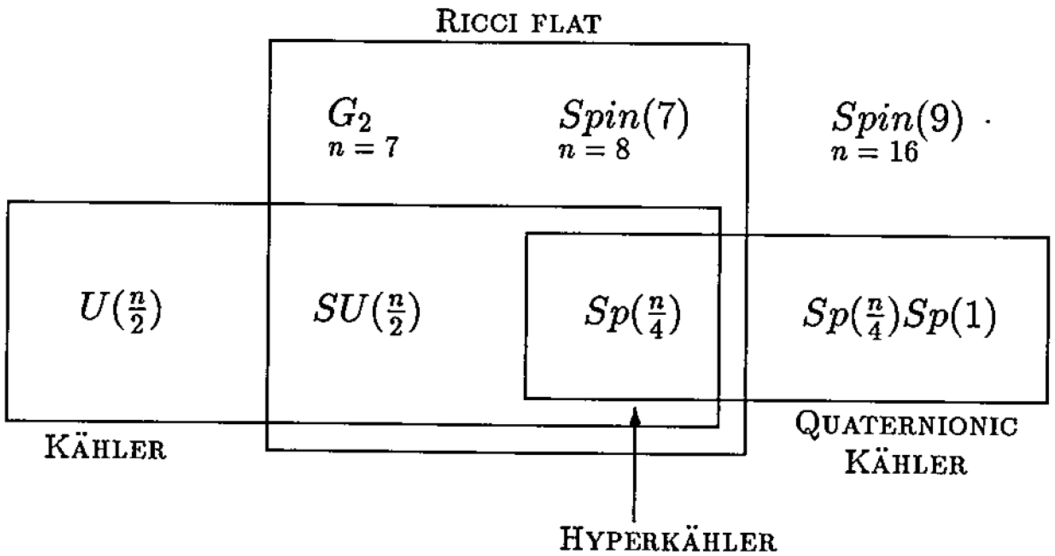

image credit: S.M. Salamon (1989)

4 Kähler manifolds (K)

$$(M^{2m}, g) + \{\mathrm{Hol} \subseteq \mathrm{U}(m)\} \Rightarrow \mathrm{K}$$

$$\mathrm{U}(m) = \mathrm{SO}(2m) \cap \mathrm{Sp}(m, \mathbb{R}) \subset \mathrm{GL}(m, \mathbb{C})$$

$$\mathrm{U}(m) \subset \mathrm{GL}(m, \mathbb{C}) \Rightarrow \exists I \in End(TM) : I^2 = -Id$$

$$\mathrm{d}\Lambda^{1,0} \subset \Lambda^{2,0} \oplus \Lambda^{1,1}$$

$$\mathrm{U}(m) \subset \mathrm{SO}(2m) \Rightarrow g(IX, IY) = g(X, Y)$$

$$\mathrm{U}(m) \subset \mathrm{Sp}(m, \mathbb{R}) \Rightarrow \omega(X, Y) = g(IX, Y) \in \Lambda^2 T^* M$$

$$\mathrm{d}\omega = 0$$

5 Quaternion-Kähler manifolds (QK)

$$(M^{4k}, g) + \{\mathrm{Hol} \subseteq \mathrm{Sp}(k)\mathrm{Sp}(1)\} \Rightarrow \mathrm{QK}$$

$$\exists I, J, K \in \Gamma \mathrm{End} TM \ : \ I^2 = J^2 = K^2 = IJK = -Id$$

$$g(AX, AY) = g(X, Y) \qquad A = I, J, K$$

$$\omega_A(X, Y) = g(AX, Y) \in \Lambda^2 T^* M$$

$$\Omega = \sum_A \omega_A^2 \in \Lambda^4 T^* M \qquad \nabla \Omega = 0.$$

$$\mathrm{Sp}(k)\mathrm{Sp}(1) \not\subset \mathrm{U}(m) \Rightarrow \mathrm{QK} \not\subset \mathrm{K}$$

$$\mathrm{Hol} \subsetneq \mathrm{Sp}(k)\mathrm{Sp}(1) \Rightarrow \begin{cases} *\mathrm{Hol} \subseteq \mathrm{Sp}(k) \subset \mathrm{Sp}(k)\mathrm{Sp}(1) \Rightarrow \mathrm{HK} \\ *M \text{ symmetric space} \end{cases}$$

6 Hyper-Kähler manifolds (HK)

$$(M^{4k}, g) + \{\mathrm{Hol} \subseteq \mathrm{Sp}(k)\} \Rightarrow \mathrm{HK}$$

$$\mathrm{Sp}(k) \subset \mathrm{U}(m) \Rightarrow \mathrm{HK} \subset \mathrm{K}$$

$$\exists I, J, K \in \Gamma \mathrm{End} TM : I^2 = J^2 = K^2 = IJK = -Id$$

$$\omega_A \in \Lambda^2 T^* M \qquad \mathrm{d}\omega_I = \mathrm{d}\omega_J = \mathrm{d}\omega_K = 0.$$

Curvature

$$\mathrm{QK} \Rightarrow \mathrm{Einstein}$$

$$\mathrm{QK} + \{s = 0\} \Rightarrow \mathrm{HK}$$

7 (First) Motivation

Wolf spaces

$$\mathbb{HP}^n, \quad \mathrm{Gr}_2(\mathbb{C}^{n+2}), \quad \mathrm{Gr}_4(\mathbb{R}^{n+4})$$

$$\frac{\mathrm{G}_2}{\mathrm{SO}(4)}, \ \frac{\mathrm{F}_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}, \ \frac{\mathrm{E}_6}{\mathrm{SU}(6)\mathrm{Sp}(1)}, \ \frac{\mathrm{E}_7}{\mathrm{Spin}(12)\mathrm{Sp}(1)}, \ \frac{\mathrm{E}_8}{\mathrm{E}_7\mathrm{Sp}(1)}$$

Alekseevsky spaces

$\exists$ Homogeneous, non-symmetric QK, with $s < 0$.

(First) Motivation

Wolf spaces

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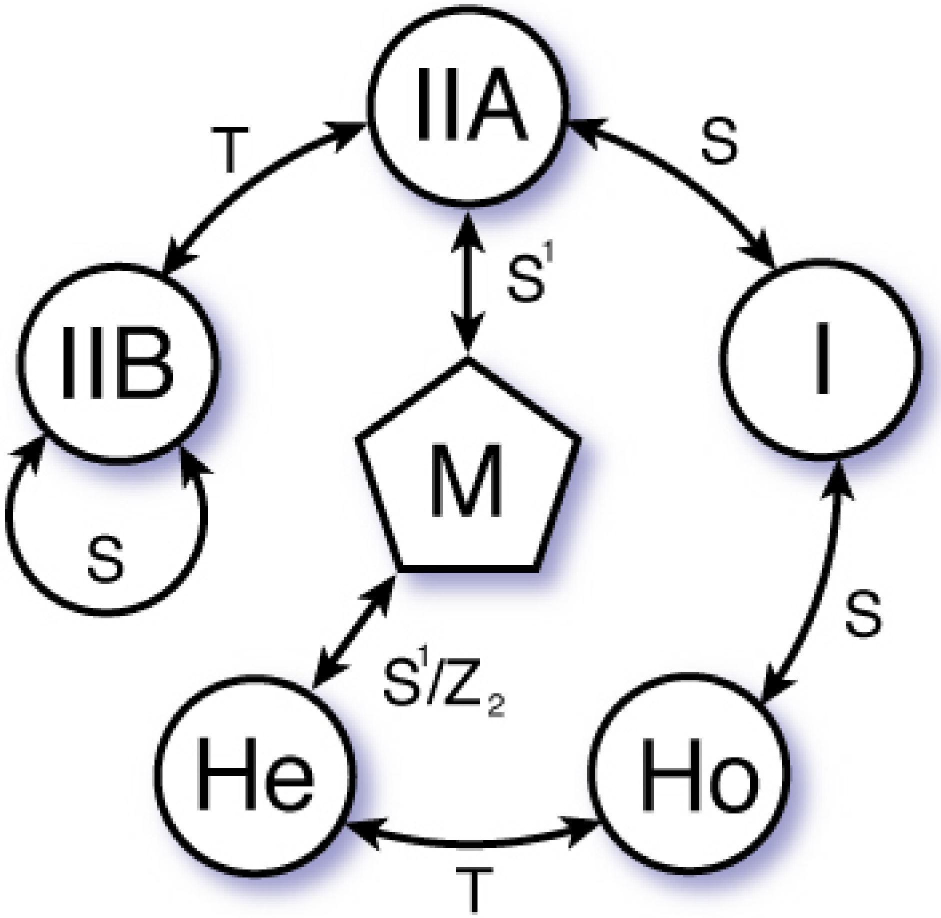
Alekseevsky spaces

$\exists$ Homogeneous, non-symmetric QK, with $s < 0$.

1. Find explicit examples of QK manifolds

> "*The mathematical problems that have been solved or techniques that have arisen out of physics in the past have been the lifeblood of mathematics.*"
>
> Sir Michael F. Atiyah *Collected Works Vol. 1 (1988), 19, p.13*

9 M-Theory



10 T-Duality & c-map

T-Duality

$$\mathrm{IIA} \longleftrightarrow \mathrm{IIB}$$

c-map

$$\mathrm{K}_1 \times \mathrm{QK}_2 \times \mathbb{C}\mathrm{H}(1) \longrightarrow \mathrm{K}_2 \times \mathrm{QK}_1 \times \mathbb{C}\mathrm{H}(1)$$

$$\mathrm{K}^{2m} \overset{\text{c-map}}{\dashrightarrow} \mathrm{QK}^{4(m+1)}$$

Cecotti, Ferrara & Girardello (1989), Ferrara & Sabharwal (1990)

Rigid c-map

$$\mathrm{K}^{2m} \overset{\text{rigid c-map}}{\dashrightarrow} \mathrm{HK}^{4m}$$

c-map & homogeneous QK classification

1975

1. Completely solvable Lie groups admitting QK metrics.
2. All known homogeneous non-symmetric QK manifolds.

Alekseevsky (1975)

1989, 1990

$$\mathrm{K}^{2m} \xrightarrow{c} \mathrm{QK}^{4(m+1)}$$

Cecotti, Ferrara & Girardello (1989), Ferrara & Sabharwal (1990)

1992

Use c-map to complete Alekseevsky's classification.

de Witt & Van Proeyen (1992)

1996

Complete classification (without c-map).

Cortes (1996)

12 Field content of $N = 2,\ D = 4$ SIMPLE SUGRA

$$a = 0, \ldots, 3, \qquad \hat{\mu} = 0, \ldots 3, \qquad \Lambda = 1, 2$$

Gravity supermultiplet

$$(V^a_{\hat{\mu}},\ \psi^\Lambda,\ A^0_{\hat{\mu}})$$

m Vector supermultiplets

$$(A^i_{\hat{\mu}},\ \lambda^{i\,\Lambda},\ \phi^i), \qquad i = 1, \ldots, m$$

ϕ^i

m $\mathbb{C}$-scalars $\cong$ **2m** $\mathbb{R}$-scalars

Coordinates on a $2m$-dimensional, PSK Σ-model

13 $4D \to 3D$ Kaluza–Klein compactification, I

$$\hat\mu = 0, \dots 3 \longmapsto (0, \mu),\ \ \mu = 1, 2, 3$$

Metric

$$\eta_{ab} V^a_{\hat\mu} V^b_{\hat\nu} = g_{\hat\mu\hat\nu} \overset{\text{KK}}{=} \left(\begin{array}{c|c} e^{2\sigma} & e^{2\sigma} A_\nu \\ \hline e^{2\sigma} A_\mu & e^{i\sigma} g_{\mu\nu} + e^{2\sigma} A_\mu A_\nu \end{array} \right)$$

4-vectors

$$A^0_{\hat\mu} \overset{\text{KK}}{=} (A^0_0, A^0_\mu) \equiv (\zeta_0, A^0_\mu)$$

$$A^i_{\hat\mu} \overset{\text{KK}}{=} (A^i_0, A^i_\mu) \equiv (\zeta_i, A^i_\mu)$$

$\sigma,\ \zeta_0,\ \zeta_i$ — $\mathbf{m} + \mathbf{2}$ extra scalar fields

$A_\mu,\ A^0_\mu,\ A^i_\mu$ — $\mathbf{m} + \mathbf{2}$ extra $3D$-vector vields.

$4D \to 3D$ Kaluza–Klein compactification, II

Dualization of $3D$-vector fields

$$(A_\mu,\ A^0_\mu,\ A^i_\mu) \xrightarrow{Duality} (a, \widetilde{\zeta}_0,\ \widetilde{\zeta}_i)$$

$a, \widetilde{\zeta}_0,\ \widetilde{\zeta}_i$ $\quad$ $\mathbf{m}+\mathbf{2}$ extra scalar fields.

15 **SUGRA** $N = 2, D = 4$

Kaluza-Klein 4D→3D

Gravity supermultiplet

$4D$	#s	$3D$	#s	*	#s
$V^i_{\hat{\mu}}$		$V^i_\mu,\ A_\mu,\ e^{2\sigma}$	**1**	$V^i_\mu,\ a,\ \phi$	**2**
ψ^Λ		ψ^Λ		ψ^Λ	
$A^0_{\hat{\mu}}$		$A^0_0,\ A^0_\mu$	**1**	$\zeta^0,\ \widetilde{\zeta}^0$	**2**

m Vector supermultiplet

$4D$	#s	$3D$	#s	*	#s
$A^i_{\hat{\mu}}$		$A^i_0,\ A^i_\mu$	**m**	$\zeta^i,\ \widetilde{\zeta}^i$	**2m**
λ^i_Λ		λ^i_Λ		λ^i_Λ	
ϕ^i	$2\mathbf{m}, \mathbb{R}$	ϕ^i	**2m**	ϕ^i	**2m**

Planning 2/3

Introduction and motivation: The c-map.

Interlude: Differential-Geometric construction of the c-map (pre-twist version)

(Second) Motivation

2. To provide a geometric construction to the c-map.

$$\mathrm{K}^{2m} \overset{}{\underset{\text{c-map}}{\dashrightarrow}} \mathrm{QK}^{4(m+1)}$$

Actually ...

$$\mathrm{PSK}^{2m} \overset{}{\underset{\text{c-map}}{\dashrightarrow}} \mathrm{QK}^{4(m+1)}$$

Special Kähler manifolds, I

Special Kähler manifolds, SK

$$\mathrm{SK} = (\mathrm{K},\ \nabla^s)$$

1. $\nabla^s \omega = 0$,
2. $\mathrm{R}(\nabla^s) = \mathrm{T}(\nabla^s) = 0$,
3. $\nabla^s_X IY = -\nabla^s_Y IX$ ("special condition")

Conic special Kähler manifolds, CSK

$$\mathrm{CSK} = (\mathrm{SK}, X) \qquad X \in \mathfrak{X}M$$

1. $g(X, X) \neq 0$;
2. $\nabla^\omega X = -I = \nabla^g X$.

Special Kähler manifolds, II

* Moment mapping

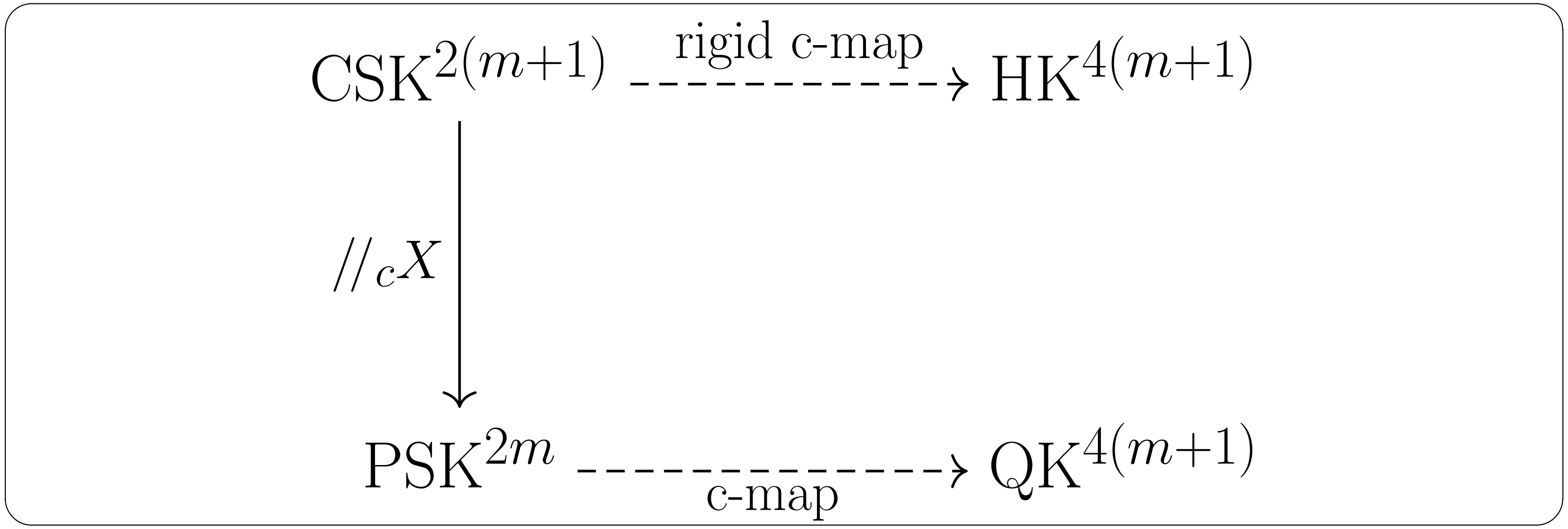
$$\mu : M \to \mathbb{R} : p \mapsto ||X_p||^2$$

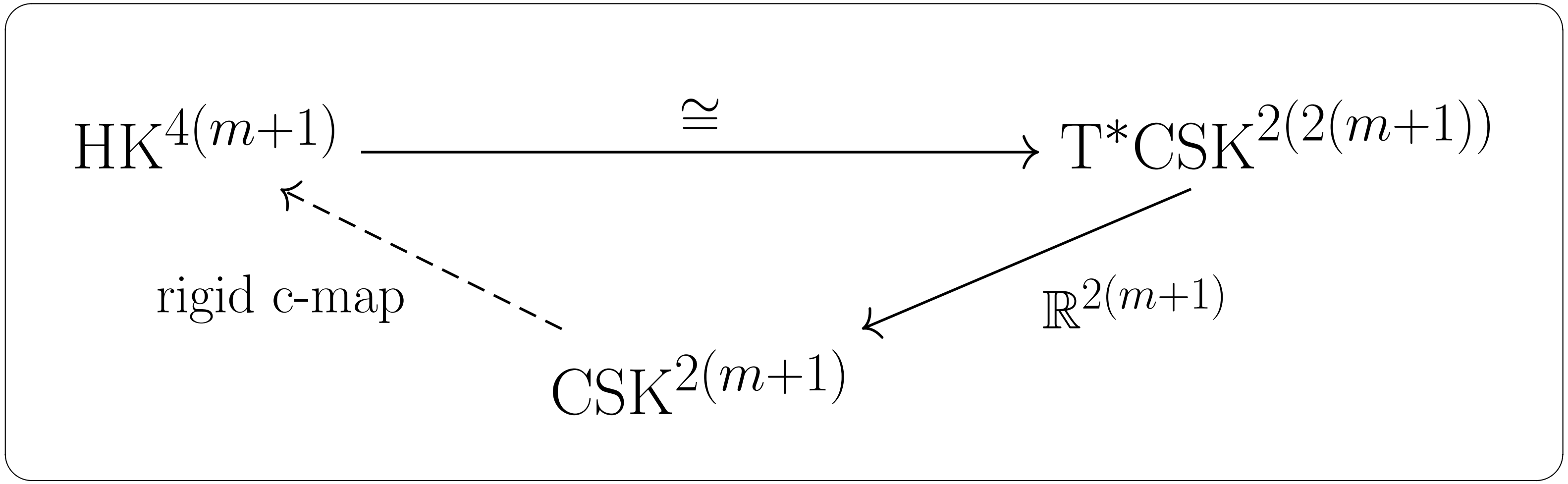
Projective special Kähler manifolds, PSK

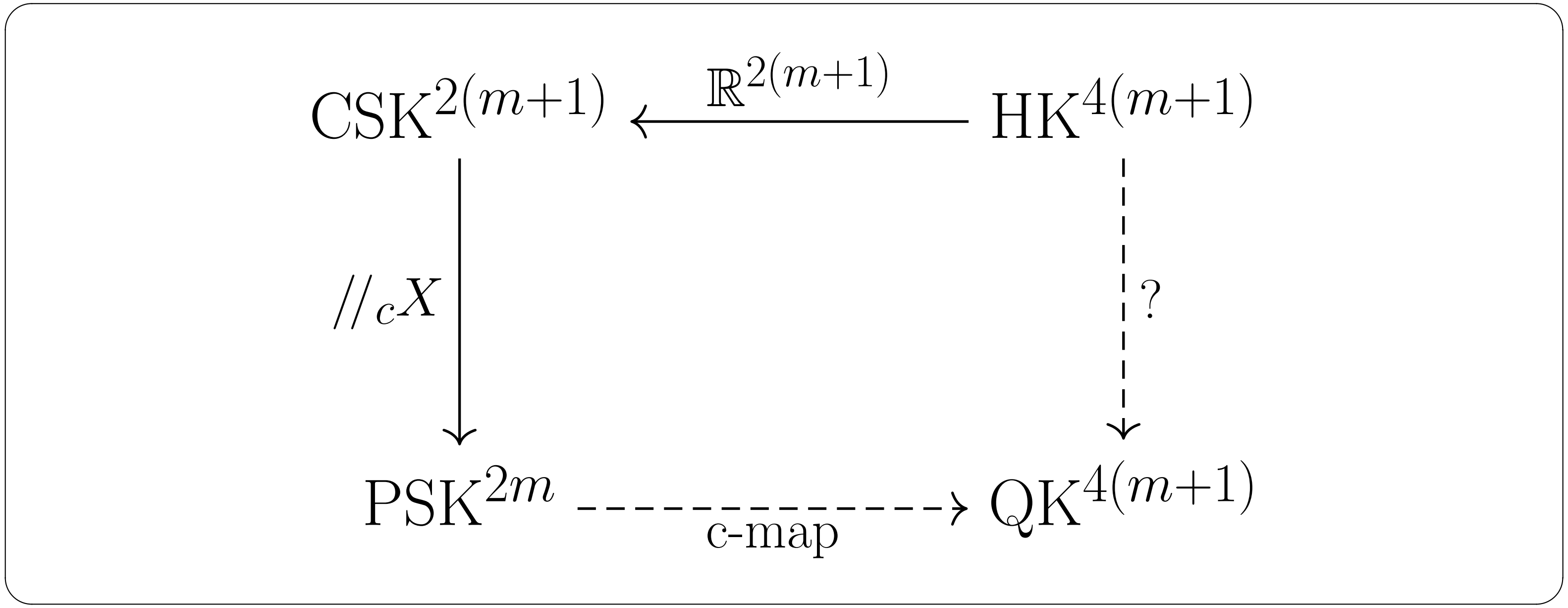
$$\mathrm{PSK} = \mathrm{CSK}//_c X = \mu^{-1}(c)/X \qquad c \in \mathbf{R}$$

$$\begin{array}{ccc} \mathrm{CSK}_0 = \mu^{-1}(c) & \longrightarrow & \mathrm{CSK} \\ \downarrow & & \\ \mathrm{CSK}//_c X = \mathrm{PSK} & & \end{array}$$

20 General picture arising from physics







$\mathrm{CSK}^{2(m+1)}$
$\mathbb{R}^{2(m+1)}$
$\mathrm{HK}^{4(m+1)}$
$//_c X$
?
PSK^{2m}
c-map
$\mathrm{QK}^{4(m+1)}$

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Introduction and Motivation. The c-map.

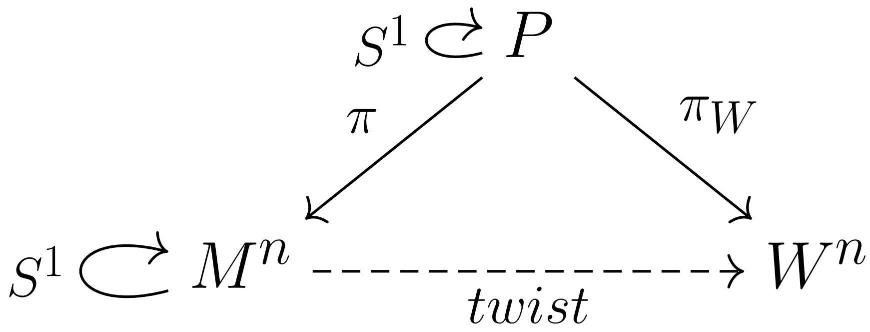
Interlude: Differential-Geometric construction of the c-map.
(pre-twist version)

Twist and HK/QK correspondence

The Twist construction (sketch)

The twsit construction associates to a manifold M with a S^1-action, a new space W of the same dimension, with a distinguished vector field.

This construction fits into a double fibration



so W is M twisted by the S^1-bundle P.

A. Swann, (2007, 2010)

23 Twists & HK/QK correspondence

1992, 2001

Instanton twists (Hypercomplex, Quaternionic, HKT)

D.Joyce (1992), G.Grantcharov & Y.S. Poon (2001)

2007, 2010

General twists (T-duality, HKT, KT, SKT, ...)

A.F. Swann (2007, 2010)

2008

HK/QK correspondence

$$\{\text{HK} + \text{symmetry fixing one } \omega\} \Leftrightarrow \{\text{QK} + \text{circle action}\}$$

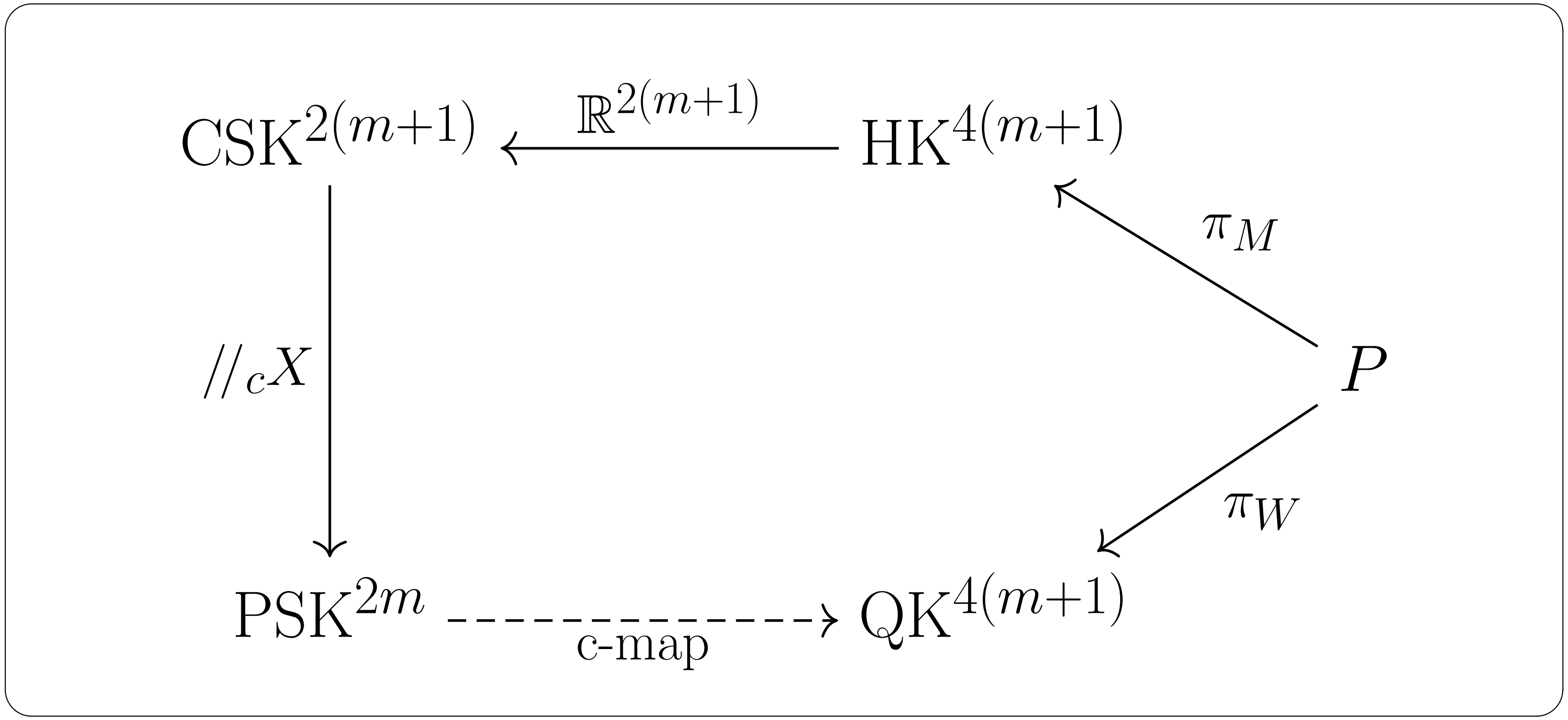
A. Haydys (2008)

2013

Twistor interpretation

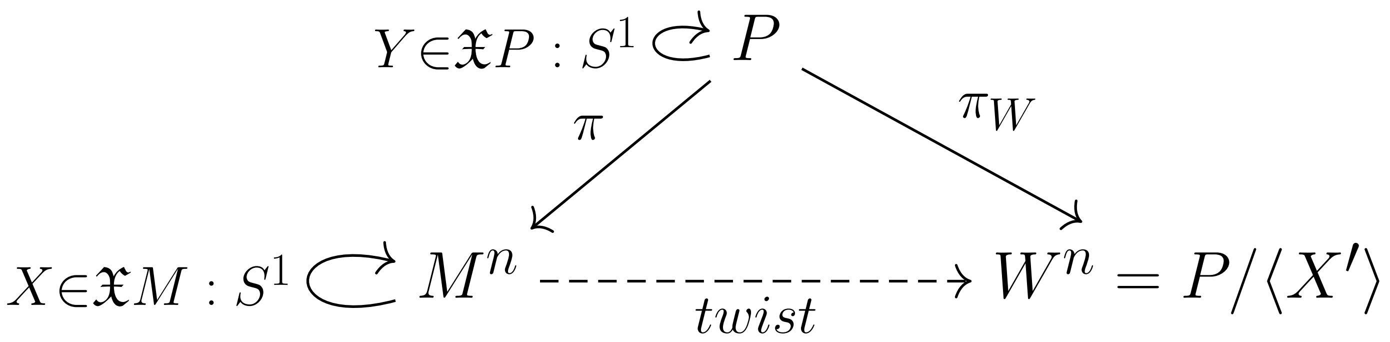
N.J. Hitchin (2013)

24 Idea



$\mathrm{CSK}^{2(m+1)}$
$\mathbb{R}^{2(m+1)}$
$\mathrm{HK}^{4(m+1)}$
π_M
$//_c X$
P
π_W
PSK^{2m}
$\mathrm{QK}^{4(m+1)}$
c-map

The Twist construction (in detail)



A. Swann, (2007, 2010)

1. $P(M, S^1)$: connection θ, curvature $\pi_M^* F = d\theta$.
2. $L_X F = 0$
3. $X' = X^\theta + aY \in \mathfrak{X}P$: such that $L_{X'}\theta = L_{X'}Y = 0$.
4. $W = P/\langle X' \rangle$ with induced action by Y.

26 Twist data

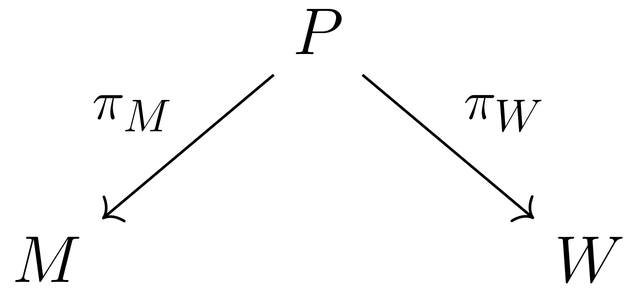
$$(M, X, F, a) \Longrightarrow \quad \begin{array}{ccc} & P & \\ {\scriptstyle \pi_M} \swarrow & & \searrow {\scriptstyle \pi_W} \\ M & & W \end{array}$$

Twist data

1) M, a C^∞ manifold.
2) $X \in \mathfrak{X}M$, generating the S^1-action.
3) $F \in \Omega^2 M$, X-invariant, with integral periods.
4) $a \in C^\infty M$ such that

$$\mathrm{d}a = -X \lrcorner F$$

 $\mathcal{H}$-related tensors



$$T_pP = \mathcal{H}_p + \mathcal{V}_p$$
$$\mathcal{H}_p \cong T_{\pi(p)}M \cong T_{\pi_W(p)}W$$

$\sim_{\mathcal{H}}$

$$\alpha \in \mathbf{T}M, \qquad \alpha_W \in \mathbf{T}W$$
$$\alpha \sim_{\mathcal{H}} \alpha_W \Longleftrightarrow (\pi_M^*\alpha)|_{\mathcal{H}} = (\pi_W^*\alpha_W)|_{\mathcal{H}}$$

Lemma

$$\alpha \in \Omega^p M^X \Rightarrow \exists! \alpha_W \in \Omega^p W \;:\; \alpha_W \sim_{\mathcal{H}} \alpha$$
$$\pi_W^*\alpha_W = \pi^*\alpha - \theta \wedge \pi^*(a^{-1} X \lrcorner \alpha)$$

Computing the Twist

$d\alpha_W$

$\alpha \in \Omega^p M^X$,

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - \frac{1}{a} F \wedge (X \lrcorner\, \alpha) \,.$$

***Twisted exterior differential,* d_W**

$$\alpha_W \sim_{\mathcal{H}} \alpha, \qquad d\alpha_W \sim_{\mathcal{H}} d_W \alpha$$

$$d_W := d - \frac{1}{a} F \wedge X \lrcorner$$

Twist vs Integrability

Complex case

Let I an invariant complex structure on M, $\mathcal{H}$-related to an almost-complex structure I_W on W, I_W is integrable iff $F \in \Omega_I^{1,1} M$.

Kähler case K $\to \mathbb{C}$

$$\begin{array}{ccc} & S^{2n+1} \times T^2 & \\ \swarrow^{\pi_M} & & \searrow^{\pi_W} \\ \mathbb{C}\mathrm{P}^n \times T^2 & & S^{2n+1} \times S^1 \end{array}$$

30 Need for a deformation

Problem

$$d\alpha = 0 \not\to d\alpha_W = 0$$

$$d\alpha_W \sim d_W\alpha = d\alpha - \frac{1}{a}F\wedge(X\lrcorner\,\alpha) = -\frac{1}{a}F\wedge(X\lrcorner\,\alpha) \neq 0$$

31 Simetríes of the structure $\mathrm{HK} = (M, g, I, J, K)$

Rotating symmetry

$X \in \mathfrak{X}M$

1. $L_X(g) = 0.$

2. $L_X(I) = \langle I, J, K\rangle$

3. (a) $L_X I = 0$, (b) $L_X J = K$, (c) $L_X K = -J$

Elementary deformations of the HK metric

g_α

$$\mathbb{H}X = \langle X, IX, JX, KX \rangle$$

$$\alpha_0 = g(X, \cdot), \qquad \alpha_A = -g(AX, \cdot) \ \ (A = \mathrm{I}, \mathrm{J}, \mathrm{K})$$

$$g_\alpha := \alpha_0^2 + \sum_A \alpha_A^2 \equiv g|_{\mathbb{H}X}$$

g^N, Elementary deformation of the metric

$$g^N = fg + hg_\alpha, \qquad f, g \in \mathcal{C}^\infty M$$

33 Uniqueness

Theorem

(M^{4k}, g, I, J, K) HK, $k \geq 2$

$X \in \mathfrak{X}M$ Rotating symmetry

μ Moment mapping

∃! Elementary deformation

$$g^N = -\frac{1}{\mu - c} g + \frac{1}{(\mu - c)^2} g_\alpha$$

∃! Twist data

$$F = kG = k(d\alpha_0 + \omega_I), \qquad a = k(g(X,X) - \mu + c).$$

W QK

O.Macia, A.F.Swann (2015)

Idea of proof

1. From g^N and (I, J, K), construct ω_A^N y Ω^N.
2. Impose arbitrary twist of Ω^N to be QK.
3. Decompose equation with respect to the splitting $TM = \langle \mathbb{H}X \rangle \oplus \langle \mathbb{H}X \rangle^{\perp}$.
4. All this leads to $f = f(\mu),\ h = h(\mu)$ y $h = f'$.
5. Impose $da = -X \lrcorner F$ to determine a.
6. Imposing $dF = 0$ leads to ODE's which determine f.

MANGE TAK

Planning 4/3 (!)

Introduction and Motivation. The c-map.

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(pre-twist version)

Twist and HK/QK correspondence

Bonus Track: A worked-out example

$\mathbb{R}$H(2)

- $\mathbb{C}$H(1) : Real, 2-dimensional solvable Lie group with Kähler metric of constant curvature.

- Local basis of one-forms on $S \subset \mathbb{C}\mathrm{H}(1)$: $\{a, b\} \in \Omega^1_S$:

$$da = 0, \quad db = -\lambda a \wedge b$$

- Metric:

$$g_S = a^2 + b^2$$

- Almost complex structure:

$$Ia = b$$

- Kähler two-form:

$$\omega_S = a \wedge b.$$

37 Local cone structure

- The PSK S is the Kaehler quotient of a CSK manifold $C \equiv (C, g, \omega, \nabla, X)$

$$\begin{array}{ccc} C_0 & \xrightarrow{\ i\ } & C \\ \Big\downarrow \pi & & \\ S & & \end{array} \qquad\qquad S = C//_c X$$

- Locally $C = \mathbf{R}_{>0} \times C_0$.
- C_0 is the level set $\mu^{-1}(c)$ for the moment map of X.
- $C_0 \to S$ is a bundle with connection 1-form φ :

$$d\varphi = 2\pi^* \omega_S$$

38 Metric and Kaehler form on C

- Write t for the standard coordinate on $\mathbf{R}_{>0}$ and $\hat{\psi} = dt$.
- Write $\hat{a} = t\pi^* a,\ \hat{b} = t\pi^* b,\ \hat{\varphi} = t\varphi$

Lemma

The $(2,2)$ pseudo–Riemannian metric and the Kaehler form of $C = \mathbf{R}_{>0} \times C_0$ are

$$g_C = \hat{a}^2 + \hat{b}^2 - \hat{\varphi}^2 - \hat{\psi}^2 = -dt^2 + t^2 g_{C_0}$$

$$\omega_C = \hat{a} \wedge \hat{b} - \hat{\varphi} \wedge \hat{\psi}.$$

The conic isometry satisfies

$$IX = t\frac{\partial}{\partial t}$$

39 LC connection

- Coframe $s^*\theta = (\hat{a}, \hat{b}, \hat{\varphi}, \hat{\psi})$

Exterior differential $d(s^*\theta)$

$$d\hat{a} = \frac{1}{t} dt \wedge \hat{a} \qquad d\hat{b} = \frac{1}{t}(dt \wedge \hat{b} - \lambda \hat{a} \wedge \hat{b})$$

$$d\hat{\varphi} = \frac{1}{t}(dt \wedge \hat{\varphi} + 2\hat{a} \wedge \hat{b}) \quad d\hat{\psi} = 0$$

LC connection $s^*\omega_{LC}$

$$s^*\omega_{LC} = \frac{1}{t} \begin{pmatrix} 0 & \hat{\varphi} + \lambda \hat{b} & \hat{b} & \hat{a} \\ -\hat{\varphi} - \lambda \hat{b} & 0 & -\hat{a} & \hat{b} \\ \hat{b} & -\hat{a} & 0 & \hat{\varphi} \\ \hat{a} & \hat{b} & -\hat{\varphi} & 0 \end{pmatrix}$$

Curvature 2-form

Curvature 2-form $s^*\Omega_{LC}$

$$s^*\Omega_{LC} = \frac{4-\lambda^2}{t^2}\begin{pmatrix} 0 & \hat{a}\wedge\hat{b} & 0 & 0 \\ -\hat{a}\wedge\hat{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lemma

The pseudo–Riemannian metric g_C is flat iff $\lambda^2 = 4$.

Conic special Kaehler condition

Symplectic connection in terms of LC connection

$$s^*\omega_\nabla = s^*\omega_{LC} + \eta.$$

Conditions on η.

1. $\eta \wedge s^*\theta$.
2. $\mathbf{i}\,\eta = -\eta\,\mathbf{i}$.
3. ${}^t\eta\,\mathbf{s} = -\mathbf{s}\,\eta$.
4. $X \lrcorner\, \eta = IX \lrcorner\, \eta = 0$.

Proposition

The cone (C, g_C, ω_C) over $S \subset \mathbf{R}\mathrm{H}(2)$ is CSK iff $\lambda^2 = \frac{4}{3}$ or $\lambda^2 = 4$.

HK structure on T*CSK

HK metric

$$g_{HK} = \left(\hat{a}^2 + \hat{b}^2 - \hat{\varphi}^2 - \hat{\psi}^2\right) + \left(\hat{A}^2 + \hat{B}^2 - \hat{\Phi}^2 - \hat{\Psi}^2\right)$$

HK structure

$$\begin{aligned}
\omega_I &= \hat{a} \wedge \hat{b} - \hat{\varphi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \Phi \wedge \hat{\Psi} \\
\omega_J &= \hat{A} \wedge \hat{a} + \hat{B} \wedge \hat{b} + \hat{\Phi} \wedge \hat{\varphi} + \hat{\Psi} \wedge \hat{\psi} \\
\omega_K &= \hat{A} \wedge \hat{b} - \hat{B} \wedge \hat{a} + \hat{\Phi} \wedge \hat{\psi} - \hat{\Psi} \wedge \hat{\varphi}.
\end{aligned}$$

Elementary deformation of g_{HK}

$\alpha_0, \ldots \alpha_K$

$$\alpha_I = I\widetilde{X} \lrcorner g_H = -t\hat{\psi}, \qquad \alpha_0 = -I\alpha_I = -t\hat{\varphi}$$

$$\alpha_J = I\widetilde{X} \lrcorner g_H = -t\hat{\Phi}, \qquad \alpha_K = I\alpha_J = -t\hat{\Psi}$$

μ

$$\mu = \frac{1}{2}||\widetilde{X}||^2 = -\frac{t^2}{2}$$

Elementary deformation

$$\begin{aligned} g_N &= -\frac{1}{\mu} g_H + \frac{1}{\mu^2}(\alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2) \\ &= \frac{2}{t^2}(\hat{a}^2 + \hat{b}^2 + \hat{\varphi}^2 + \hat{\psi}^2 + \hat{A}^2 + \hat{B}^2 + \hat{\Phi}^2 + \hat{\Psi}^2) \end{aligned}$$

Twist data

Twisting two-form

$$F = -\hat{a} \wedge \hat{b} + \hat{\varphi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \hat{\Phi} \wedge \hat{\Psi}$$

Twisting function

$$a = \mu = -\frac{t^2}{2}$$

45 Flat case $\lambda^2 = 4$.

- The LC and the special connection coincide.
- X-invariant coframe on CSK

$$\gamma = s^*\theta/t = (\tilde{a}, \tilde{b}, \varphi, \tilde{\psi})$$

(we can compute the twisted differentials immediatelly)

$$d_W \tilde{a} = 0 \qquad d_W \tilde{b} = 2\tilde{b} \wedge \tilde{a}$$
$$d_W \varphi = 2\tilde{a} \wedge \tilde{b} + \frac{2}{t^2} F \quad d_W \tilde{\psi} = 0$$

- The (vertical) coframe

$$\tilde{\delta} = (s^*\alpha)/t = (\tilde{A}, \tilde{B}, \tilde{\Phi}, \tilde{\Psi})$$

is NOT $\tilde{X}$-invariant.

$$\delta = \tilde{\delta} e^{\mathbf{i}\tau}$$

$$\epsilon = \frac{1}{2}(\delta_1 + \delta_4, \delta_2 - \delta_3, -\delta_2 - \delta_3, -\delta_1 + \delta_4)$$

$$d_W \epsilon = \epsilon \wedge \begin{pmatrix} \psi - \tilde{a} & 0 & 2\tilde{b} & 0 \\ 0 & \tilde{\psi} - \tilde{a} & 0 & -2\tilde{b} \\ 0 & 0 & \tilde{\psi} - \tilde{a} & 0 \\ 0 & 0 & 0 & \tilde{\psi} + \tilde{a} \end{pmatrix}$$

$$d_W \varphi = 2(\varphi \wedge \tilde{\psi} + \epsilon_{13} + \epsilon_4 \wedge \epsilon_2)$$

The resulting Lie algebra is isomorphic to the non-compact symmetric space

$$Gr_2^+(\mathbf{C}^{2,2}) = \frac{U(2,2)}{U(2) \times U(2)}$$

46 Non-flat case $\lambda^2 = \frac{4}{3}$

- Same g_{HK}, g_N and same twist data F, a.

- Twisted differentials for $\gamma = (\tilde{a}, \tilde{b}, \varphi, \tilde{\psi})$:

$$d_W \tilde{a} = 0 \qquad d_W \tilde{b} = -\frac{2}{3}\tilde{a} \wedge \tilde{b} \qquad d_W \tilde{\psi} = 0$$

$$d_W \varphi = 2(\varphi \wedge \tilde{\psi} - \tilde{A} \wedge \tilde{B} + \tilde{\Phi} \wedge \tilde{\Psi})$$

- Adjusting the vertical coframe we arrive to

$$d_W\epsilon = \epsilon \wedge \tilde{\psi}\mathbf{Id_4} + \frac{1}{\sqrt{3}}\epsilon \wedge \tilde{a} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$+\frac{2}{\sqrt{3}}\epsilon \wedge \tilde{b} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We see the structure of solvable algebra associated to

$$\frac{G_2^*}{SO(4)}$$

MANGE TAK