$SO(3)$, 8-MANIFOLDS AND QUATERNIONIC GEOMETRY

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New Trends in Differential Geometry,
L'Aquila, 2011 September, 8^{th}.

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2. Chiossi, S.G., –, $SO(3)$*-structures on 8-manifolds*, math.DG/1105.1746 (2011).

CONTENTS

1. $SO(3)$-STRUCTURES

• A (linear) **G-structure** is a subbundle of the (linear) frame bundle $L(M)$ with structure group G.

• A **Riemannian metric** g on an n-manifold M^n determines a **$SO(n)$-structure**, where the tangent space T_pM^n behaves as a **representation** for $SO(n)$.
Subgroups $G \subset SO(n)$ determine more restricted Riemannian G-structures; T_pM^n must behave as a representation for G.

• Usually, T_pM^n is regarded as an **irreducible** representation of G. For $G = SO(3)$ this has been the case in 5-dimensions by (Bobienski & Nurowski, 2007), (Chiossi & Fino, 2007) and (Agricola, Becker-Bender & Friedrich, 2011).

2. $\mathscr{G}$-STRUCTURES IN 8-DIMENSIONS

• We will consider **$SO(3)$-structures in Riemannian 8-manifolds** $\{M^8, g\}$ for which T_pM^8 behaves as a REDUCIBLE $SO(3)$-module:

Fix a homomorphism $\rho : SO(3) \longrightarrow SO(8)$ whose image will be called $\mathscr{G} \equiv SO(3)_\rho \equiv \{SO(3), \rho\}$ such that

$$T_pM = V \oplus S_0^2 V = V \oplus W.$$

$$V \cong \mathbf{R}^3, \qquad W = S_0^2 V \cong \mathbf{R}^5$$

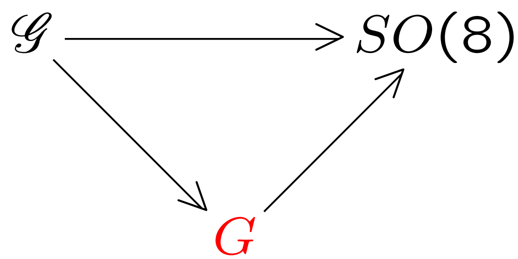
• By notational convenience we use notation of representations of $Sp(1)$, thus

$$V = r(S^2\mathbf{H}), \qquad W = r(S^4\mathbf{H})$$

In what follows $S^k := r(S^k\mathbf{H})$.

3. SUBORDINATE G-STRUCTURES

This particular embedding of $SO(3)$ (or $\mathscr{G}$-structure) factors through other Lie groups G,

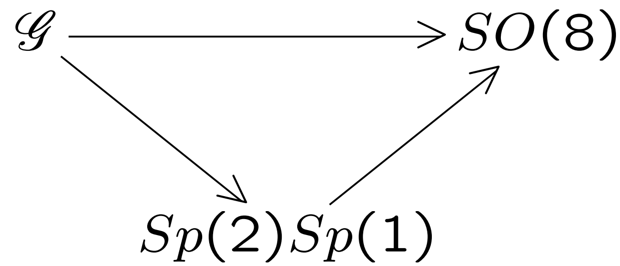


1. $G = Sp(2)Sp(1)$ defines an almost quaternion-Hermitian structure on M^8. In the **integrable** case leads to **quaternion-Kähler geometry** (Salamon 1982,1986, Swann 1989).

2. $G = SO(3) \times SO(5)$ defines an **almost product structure** (Naveira, 1983).

3. $G = PSU(3)$-structure (Hitchin 2001, Witt 2008).

Example. $\mathscr{G} \subset Sp(2)Sp(1) \subset SO(8)$



- Consider the homomorphism

$$\phi : Sp(1) \to Sp(2) \times Sp(1) : g \mapsto (i(g), g)$$

where

$$i : Sp(1) \hookrightarrow Sp(2)$$

is the inclusion whereby $Sp(1)$ acts irreducibly on $\mathbf{E} = \mathbf{C}^4_{(1,0)}$, the fundamental representation of $Sp(2)$.

- By definition

$$Sp(2)Sp(1) := Sp(2) \times_{\mathbf{Z}_2} Sp(1).$$

Therefore ϕ induces an inclusion

$$SO(3) = Sp(1)/\mathbf{Z}_2 \longrightarrow Sp(2)Sp(1) \subset SO(8)$$

• The representation space for $Sp(2)Sp(1)$ is $\mathbf{E} \otimes \mathbf{H} = \mathbf{C}^4_{(1,0)} \otimes \mathbf{C}^2_{(1)}$. From the point of view of $Sp(1)$-representations

$$\mathbf{E} \cong S^3\mathbf{H}, \qquad \mathbf{H} \cong S^1\mathbf{H}.$$

Using Clebsch–Gordan

$$S^3\mathbf{H} \otimes S^1\mathbf{H} \cong S^2 H \oplus S^4\mathbf{H}.$$

Then, passing to real representations

$$r(S^2\mathbf{H}) \oplus r(S^4\mathbf{H}) = S^2 \oplus S^4 \equiv V \oplus W.$$

■

Proposition 1. *A $\mathscr{G}$-structure on a Riemannian 8-manifold $\{M^8, g\}$ induces altogether an almost-product structure, a $PSU(3)$-structure and an almost quaternion-Hermitian structure.*

4. SOME TOPOLOGY

• An oriented $\mathscr{G}$-manifold M^8 is spin. The 2^{nd} Stiefel-Whitney class

$$w_2(M^8) \in H^2(M^8, \mathbf{Z}_2)$$

of a $Sp(n)Sp(1)$-structures satisfies

$$w_2(M^{4n}) = n\epsilon(n),$$

where ϵ represents the Marchiafava-Romani class(1975-76). For $n = 2$

$$w_2(M^8) = 2\epsilon(2) = 0 \; mod 2$$

• From $\mathscr{G} \subset PSU(3)$ and the work by Witt, 2008

$$w_1(M) = w_2(M) = w_3(M) = w_4(M)^2 = 0.$$

• The integral Pontrjagin classes $p_i \in H^{4i}(M, \mathbf{Z}),\ i = 1, 2$ of a $\mathscr{G}$-manifold $\{M^8, g\}$ are related by

$$4p_2(M) = p_1(M) \smile p_1(M)$$

$$p_1^2 \in 8640\mathbf{Z}$$

• A compact $\mathscr{G}$-manifold $\{M^8, g\}$ has vanishing Euler class

$$e(TM^8) = 0.$$

Example: This rules out:

$$Gr_3(\mathbf{R}^8), \qquad G_2/SO(4);$$

But is not enough to rule out the product of odd-dimensional spheres

$$\mathbf{S^3} \times \mathbf{S^5}$$

which does not admit $SO(3)$-structures (Friedrich, 2003) ■

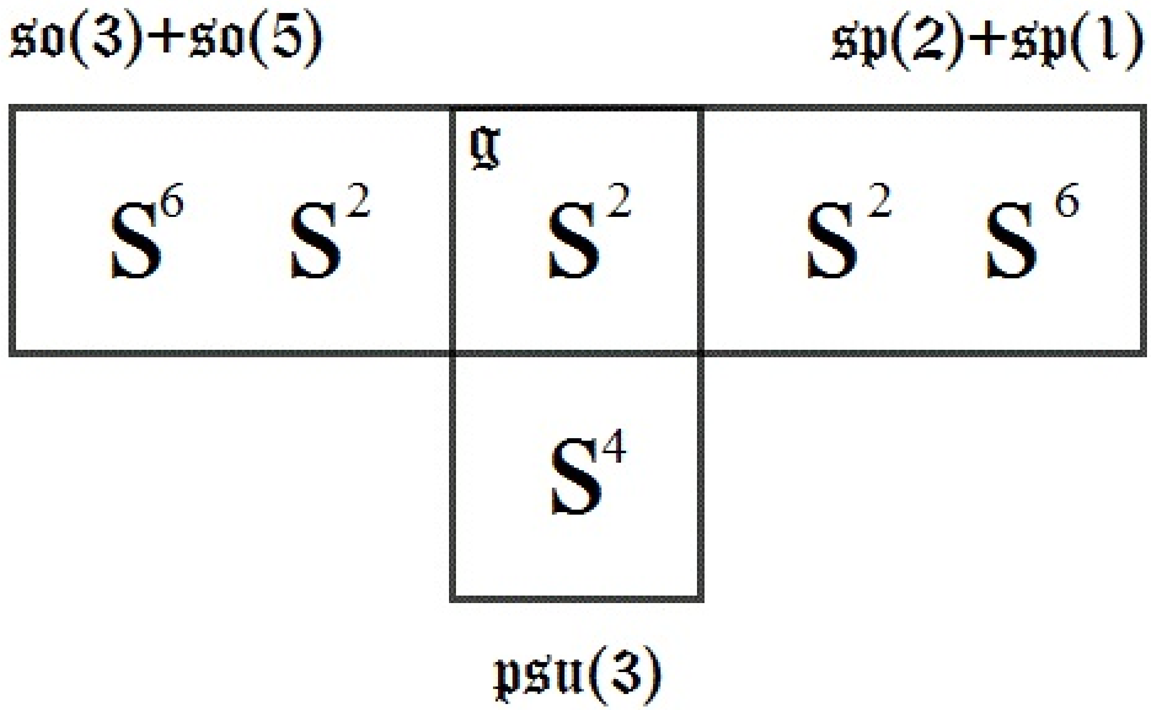
5. TRIPLE TWOFOLD INTERSECTION

Proposition 2. *Let $\mathscr{G} = SO(3)$ be the subgroup of $SO(8)$ acting infinitesimally on a Riemannian 8-manifold $\{M^8, g\}$ by decomposing the tangent spaces as in 1., where $V \cong \mathbf{R}^3$ is the fundamental representation. Then,*

1. $\mathscr{G} = (SO(3) \times SO(5)) \cap PSU(3)$;

2. $\mathscr{G} = PSU(3) \cap Sp(2)Sp(1)$;

3. $\mathscr{G} = Sp(2)Sp(1) \cap (SO(3) \times SO(5))$.


so(3)+so(5)
sp(2)+sp(1)
g
S6 S2
S2
S2 S6
S4
psu(3)

Proposition 3. *Let* $\mathfrak{g}_i$, $i = 1, 2, 3$, *denote the Lie algebras of the groups* $SO(3) \times SO(5)$, $PSU(3)$, $Sp(2)Sp(1)$, $\mathfrak{g}_i^\perp$ *the complements in* $\mathfrak{so}(8)$ *and* $\mathfrak{g}$ *the Lie algebra of* $\mathscr{G} = SO(3)$. *Then*

$$\begin{aligned} \mathfrak{g}_i^\perp &= (\mathfrak{g}_j/\mathfrak{g}) \oplus (\mathfrak{g}_k/\mathfrak{g}), \qquad i \neq j \neq k = 1, 2, 3 \\ \mathfrak{g}^\perp &= \bigoplus_{i=1}^{3} (\mathfrak{g}_i/\mathfrak{g}). \end{aligned}$$

Example

$$(\mathfrak{sp}(2) + \mathfrak{sp}(1))^\perp = \left(\frac{\mathfrak{so}(3) + \mathfrak{so}(5)}{\mathfrak{g}} \right) \oplus \left(\frac{\mathfrak{psu}(3)}{\mathfrak{g}} \right)$$

$$\mathfrak{g}^\perp = \left(\frac{\mathfrak{so}(3) + \mathfrak{so}(5)}{\mathfrak{g}} \right) \oplus \left(\frac{\mathfrak{psu}(3)}{\mathfrak{g}} \right) \oplus \left(\frac{\mathfrak{sp}(2) + \mathfrak{sp}(1)}{\mathfrak{g}} \right)$$

■

6. INTRINSIC TORSION

• Let $\{M^n, g\}$ be a Riemannian n-manifold (thus a $SO(n)$-structure). For some $G \subset SO(n)$ consider the associated G-structure. Then, the **intrinsic torsion** of the G-structure is a tensor τ belonging to

$$T_pM^n \otimes \mathfrak{g}^{\perp}.$$

• The **intrinsic torsion** is the obstruction for a G-structure to reduce the holonomy group of the Levi-Civita connection form $SO(n)$ to G.

Example. $Sp(2)Sp(1)$**-structure in 8-dimensions**, (Salamon 1982,1986; Swann, 1989)

$$T_pM^8 \cong \mathbf{E} \otimes \mathbf{H} \cong \mathbf{C}^4_{(1,0)} \otimes \mathbf{C}^2_{(1)}$$

$$(\mathfrak{sp}(2) + \mathfrak{sp}(1))^\perp \cong \Lambda^2_0\mathbf{E} \otimes S^2\mathbf{H} \cong \mathbf{C}^5_{(1,1)} \otimes \mathbf{C}^3_{(2)}$$

$$\tau \in \left(\mathbf{C}^4_{(1,0)} \otimes \mathbf{C}^5_{(1,1)}\right) \otimes \left(\mathbf{C}^2_{(1)} \otimes \mathbf{C}^3_{(2)}\right)$$

Simplifying, using Clebsch–Gordan,

$$\tau \in \left(\underbrace{\mathbf{C}^4_{(1,0)}}_{\mathbf{E}} \oplus \underbrace{\mathbf{C}^{16}_{(2,1)}}_{\mathbf{K}}\right) \otimes \left(\underbrace{\mathbf{C}^2_{(1)}}_{\mathbf{H}} \oplus \underbrace{\mathbf{C}^4_{(3)}}_{S^3\mathbf{H}}\right)$$

Hence, the space of intrinsic torsion tensors of the $Sp(2)Sp(1)$-structure has 4 $Sp(2)Sp(1)$-irreducible components:

$$\tau_{Sp(2)Sp(1)} \in \underbrace{\mathbf{E}S^3\mathbf{H}}_{(1)} \oplus \underbrace{\mathbf{K}S^3\mathbf{H}}_{(2)} \oplus \underbrace{\mathbf{KH}}_{(3)} \oplus \underbrace{\mathbf{EH}}_{(4)}$$

• This method (Gray & Hervella, 1980) gives rise to the 16 classes of almost quaternion-Hermitian manifolds (Swann, 1991; Martín-Cabrera & Swann, 2007).

• For the intrinsic torsion of a $\mathscr{G}$-structure

$$T_pM^8 \cong S^2 \oplus S^4$$

$$\mathfrak{g}^\perp \cong 2S^6 \oplus S^4 \oplus 2S^2.$$

Proposition 4. *The intrinsic torsion of the $\mathscr{G}$-structure is a tensor belonging to*

$$2S^{10} \oplus 5S^8 \oplus 8S^6 \oplus 10S^4 \oplus 8S^2 \oplus 3\mathbf{R}.$$

This space is 200-dimensional and contains a 3-dimensional subspace of $\mathscr{G}$-invariant tensors.

7. RELATIVE INTRINSIC TORSION

Definition 1. *For any given Lie group G containing $\mathscr{G}$ we denote $\tau^G_{\mathscr{G}}$ or $\tau(G,\mathscr{G})$ the intrinsic torsion of a G-structure decomposed under the action of $\mathscr{G}$, and call it the G-torsion relative to $\mathscr{G}$ or just the relative G-torsion, $\mathscr{G}$ being implicit.*

Example $Sp(2)Sp(1)$**-torsion relative to** $\mathscr{G}$.

$$T_pM^8 \cong \underbrace{\mathbf{EH}}_{Sp(2)Sp(1)} \cong \underbrace{S^2 \oplus S^4}_{\mathscr{G}}$$

$$(\mathfrak{sp}(2)+\mathfrak{sp}(1))^\perp \cong \underbrace{\Lambda^2_0\mathbf{E}S^2\mathbf{H}}_{Sp(2)Sp(1)} \cong \underbrace{S^6 \oplus S^4 \oplus S^2}_{\mathscr{G}}$$

$$\tau_{\mathscr{G}}^{Sp(2)Sp(1)} \in (S^2 \oplus S^4) \otimes (S^6 \oplus S^4 \oplus S^2)$$
$$= S^{10} \oplus 3S^8 \oplus 5S^6 \oplus 6S^4 \oplus 5S^2 \oplus 2\mathbf{R}.$$

■

Proposition 5. *Let* $G = SO(3)\times SO(5), PSU(3), Sp(2)Sp(1)$. *The relative* G*-torsion* $\tau_{\mathscr{G}}^{G}$ *of* $\{M, g\}$ *lives in the direct sum of the following modules:*

	S^{10}	S^8	S^6	S^4	S^2	$\mathbf{R}$	$\dim_{\mathbf{R}}$
$\tau_{\mathscr{G}}^{SO(3)\times SO(5)}$	1	3	5	6	5	2	120
$\tau_{\mathscr{G}}^{PSU(3)}$	2	4	6	8	6	2	158
$\tau_{\mathscr{G}}^{Sp(2)Sp(1)}$	1	3	5	6	5	2	120

8. RELATIVE INTRINSIC TORSION II

• Let P, Q, R denote any of the groups $SO(3)\times SO(5)$, $PSU(3)$, $Sp(2)Sp(1)$, and denote by

> $\tau^P_{\mathscr{G}}(Q)$ the co1mponent of $\tau^P_{\mathscr{G}}$ appearing also in $\tau^Q_{\mathscr{G}}$ but not in $\tau^R_{\mathscr{G}}$.

Algebraically,

$$\tau^P_{\mathscr{G}} \in T_pM^8 \otimes (\mathfrak{p}^\perp) = T_pM^8 \otimes \left(\frac{\mathfrak{q}}{\mathfrak{g}} \oplus \frac{\mathfrak{r}}{\mathfrak{g}}\right)$$

$$\tau^Q_{\mathscr{G}} \in T_pM^8 \otimes (\mathfrak{q}^\perp) = T_pM^8 \otimes \left(\frac{\mathfrak{p}}{\mathfrak{g}} \oplus \frac{\mathfrak{r}}{\mathfrak{g}}\right)$$

$$\tau^R_{\mathscr{G}} \in T_pM^8 \otimes (\mathfrak{r}^\perp) = T_pM^8 \otimes \left(\frac{\mathfrak{p}}{\mathfrak{g}} \oplus \frac{\mathfrak{q}}{\mathfrak{g}}\right)$$

Hence

$$\tau^P_{\mathscr{G}}(Q) \in T_pM^8 \otimes \frac{\mathfrak{r}}{\mathfrak{g}} \qquad \tau^Q_{\mathscr{G}}(R) \in T_pM^8 \otimes \frac{\mathfrak{p}}{\mathfrak{g}}$$

$$\tau^R_{\mathscr{G}}(P) \in T_pM^8 \otimes \frac{\mathfrak{q}}{\mathfrak{g}}$$

Then,

Proposition 6. *The tensor $\tau_{\mathscr{G}}$ of $\{M, g\}$ determines P-, Q-, R-structures whose relative torsion tensors $\tau^P_{\mathscr{G}}, \tau^Q_{\mathscr{G}}, \tau^R_{\mathscr{G}}$ satisfy the cyclic conditions*

$$\tau^P_{\mathscr{G}}(R) = \tau^R_{\mathscr{G}}(P),$$
$$\tau^P_{\mathscr{G}} = \tau^P_{\mathscr{G}}(R) \oplus \tau^P_{\mathscr{G}}(Q),$$
$$\tau_{\mathscr{G}} = \tau^P_{\mathscr{G}}(R) \oplus \tau^R_{\mathscr{G}}(Q) \oplus \tau^Q_{\mathscr{G}}(P).$$

In particular, any two yield the third.

5. $\mathscr{G}$-INVARIANT TORSION

• From now on, let us consider the case $\tau_{\mathscr{G}} \in 3\mathbf{R}$, ie., **$\mathscr{G}$-invariant intrinsic torsion**.

• $\mathscr{G}$ stabilises certain differential forms

$$\{\alpha, \beta\} \in \Lambda^3 \cong S^8 \oplus 3S^6 \oplus 3S^4 \oplus 3S^2 \oplus 2\mathbf{R} \cong \Lambda^5 \ni \{*\alpha, *\beta\},$$

$$\{\gamma, *\gamma\} \in \Lambda^4 \cong 2S^8 \oplus 2S^6 \oplus 6S^4 \oplus 2S^2 \oplus 2\mathbf{R}.$$

The $\mathscr{G}$-invariant forms are two 3-forms, one 4-form and their duals in 8-dimensions, satisfying

$$\begin{array}{ccccc}
\Lambda^3(T^*M)^{\mathscr{G}} & \xrightarrow{d} & \Lambda^4(T^*M)^{\mathscr{G}} & \xrightarrow{d} & \Lambda^5(T^*M)^{\mathscr{G}} \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Span}_{\mathbf{R}}\{\alpha, \beta\} & \xrightarrow{A} & \mathrm{Span}_{\mathbf{R}}\{\gamma, *\gamma\} & \xrightarrow{B} & \mathrm{Span}_{\mathbf{R}}\{*\alpha, *\beta\}
\end{array}$$

• A, B are 2×2 matrices encoding the $\mathscr{G}$-invariant intrinsic torsion, such that

$$BA = 0 \qquad \leftrightarrow \qquad d^2\Phi = 0,\ \forall \Phi \in \Lambda^k(T^*M)$$

Proposition 7. *Let $\{M, g\}$ be a $\mathscr{G}$-manifold equipped with the six $\mathscr{G}$-invariant forms. If the intrinsic torsion is $\mathscr{G}$-invariant, the differential forms satisfy one of the following sets of differential equations*

	$d\alpha$	$d\beta$	$d\gamma$	$d(*\gamma)$
I	$a_1^1\gamma$	$a_2^1\gamma$	0	$ma_1^1(*\alpha) + b_2^2(*\beta)$
II	0	$a_2^1\gamma + a_2^2(*\gamma)$	$b_1^2(*\beta)$	$-((a_2^1 b_1^2)/a_2^2)(*\beta)$
III	0	$a_2^1\gamma$	0	$b_2^2(*\beta)$
IV	0	0	$b_1^2(*\beta)$	$b_2^2(*\beta)$

with the remaining two 5-forms always closed.

10. TYPE-I: NQ EXAMPLE

• $SU(3)$, where $T_p(SU(3)) \cong \mathfrak{su}(3) = \mathfrak{so}(3) + \mathfrak{b}^5$, together with a 1-parameter infinitesimal action of $SO(3)$ in which the basis of $\mathfrak{so}(3)$, $\{e_6, e_7, e_8\}$ behaves as the imaginary quaternions induces a 1-parameter family of almost quaternionic structures on $SU(3)$.

$$A\cdot X = \lambda_1[A, X^a] + i\lambda_2(\{A, X^a\} - \tfrac{2}{3}(AX^a)\mathbb{1}) + i\lambda_3\{A, X^s\} + \lambda_4[A, X^s]$$

$$\lambda_1 = \tfrac{1}{2}, \qquad \lambda_3 = -\tfrac{3}{4}(\lambda_2)^{-1}, \qquad \lambda_4 = -\tfrac{1}{2}, \qquad \lambda := \lambda_2.$$

• The metric

$$g_\lambda = \sum_{i=1}^{5} e^i \otimes e^i + \frac{4\lambda^2}{3} \sum_{i=6}^{8} e^i \otimes e^i$$

is compatible with the almost quaternionic structure. Thus, together with the associated Kähler 2-forms,

$$
\begin{aligned}
\omega_1 &= \frac{1}{2}\left(15+\sqrt{3}\cdot 25+34\right)+\lambda\left(\frac{1}{\sqrt{3}}28-46+37-18\right)-\frac{2\lambda^2}{3}67,\\
\omega_2 &= -14-\frac{1}{2}35+\lambda\left(\frac{2}{\sqrt{3}}27-38-56\right)-\frac{2\lambda^2}{3}68,\\
\omega_3 &= \frac{1}{2}\left(13-\sqrt{3}\cdot 23+45\right)+\lambda\left(\frac{1}{\sqrt{3}}26-48+57+16\right)-\frac{2\lambda^2}{3}78.
\end{aligned}
$$

induces an almost quaternion-Hermitian structure.

- With the parameter

$$\lambda^2 = \frac{3}{20}$$

is *nearly-quaternionic*: the Kähler 2-forms expand a differential ideal but the fundamental 4-form is not closed ($b_4(SU(3)) = 0$)

$$d\omega_i = \sum_{j=1}^{3} \beta_i^j \wedge \omega_j, \qquad \beta_i^j \in \Lambda^1(T^*M)$$

$$\beta = (\beta_i^j) = \begin{pmatrix} s(1 - \frac{1}{\sqrt{3}}2) & s3 + a6 & s4 + a7 \\ s3 - a6 & \frac{2}{\sqrt{3}}s2 & s5 + a8 \\ s4 - a7 & s5 - a8 & -s(1 + \frac{1}{\sqrt{3}}2) \end{pmatrix}$$

β being not antisymmetric, $SU(3)$ with the given almost quaternion-Hermitian structure is not quaternion-Kähler.

• By Swann's theorem (1989): An almost quaternion-Hermitian $4n$-manifold, $n \geq 3$ is quaternion-Kähler if and only if $d\Omega = 0$. For $n = 2$ the following two conditions are required:
1. $d\Omega = 0$;
2. $d\omega_j = \sum_i \beta_j^i \wedge \omega_i$.

• This is the only example (known to the author) of a complete*, almost quaternion-Hermitian 8-manifold of type $W_1 \subset W_{1+4}$ (ie., satisfying condition 2., not 1.) This implies

$$\tau_{Sp(2)Sp(1)} \in \mathbf{E}S^3\mathbf{H} \subset \mathbf{E}S^3\mathbf{H} \oplus \mathbf{EH}.$$

• Examples of manifolds satisfying condition 1., not 2., where found by Salamon 2001, Giovannini 2006.

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