HOLOMORPHIC ISOMETRIC EMBEDDINGS FROM $\mathbb{C}\mathrm{P}^1$ TO COMPLEX QUADRICS

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Based on:

[MNT] Macia, Nagatomo, Takahashi
" *Holomorphic isometric embeddings from the projective line into quadrics*"

[N] Nagatomo
"*Harmonic maps into Grassmannians*"

[MN] Macia, Nagatomo,
"*Einstein–Hermitian harmonic maps from the projective line into quadrics*"

1 Planning: 3 things

Characterisation of harmonic maps to Grassmannians in terms of vector bundles

Characterisation of the moduli space (holomorphic case)

Description of the moduli space of holomorphic isometric embeddings $\mathbb{C}\mathrm{P}^1 \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$

2 Planning: 1/3

Characterisation of harmonic maps to Grassmannians in terms of vector bundles

3 Minimal immersions of Riemannian manifolds

Theorem (Takahashi, J. Math. Soc. Japan, 1966)

$f : M \to S^{N-1} \subset \mathbb{R}^N$ Mapping

$x^k \qquad (k = 1, \ldots, N)$ Coordinates

The following conditions are equivalent:

1.

f is a harmonic map.

2.

$$\exists h \in C^\infty(M) \ : \ \Delta x^k = h \cdot x^k$$

Then,

$$|df|^2 = \sum_{i=1}^{m} |df(e_i)|^2 = h.$$

4 Geometry of Grassmannians

W : $\mathbb{R}$ or $\mathbb{C}$ vector space of dimension N

$\mathrm{Gr}_p(W)$: Grassmannian of p-planes in W

Exact sequence of bundles

$$0 \longrightarrow S \xrightarrow{i_S} \underline{W} \xrightarrow{\pi_Q} Q \longrightarrow 0$$

$\underline{W} \to \mathrm{Gr}_p(W)$: $\mathrm{Gr}_p(W) \times W \to \mathrm{Gr}_p(W)$

$S \to \mathrm{Gr}_p(W)$: Tautological bundle over $\mathrm{Gr}_p(W)$.

$Q \to \mathrm{Gr}_p(W)$: Universal quotient bundle.

5 Induced fibre metrics

Fix an inner product ($\mathbb{R}$) or a Hermitian product ($\mathbb{C}$) on W.

$$0 \longrightarrow S \underset{\pi_S}{\overset{i_S}{\rightleftarrows}} \underline{W} \underset{i_Q}{\overset{\pi_Q}{\rightleftarrows}} Q \longrightarrow 0$$

$S \to \mathrm{Gr}_p(W)$: fibre metric g_S.

$Q \to \mathrm{Gr}_p(W)$: fibre metric g_Q.

6 Connections and Second Fundamental Forms

$$s \in \Gamma(S) \Rightarrow i_S(s) \in \Gamma(\underline{W}) \Rightarrow di_S(s) \in \Omega^1(\underline{W})$$
$$di_S(s) = \pi_S di_S(s) + \pi_Q di_S(s) = \nabla^S s + Hs$$

Connection on $S \to \mathrm{Gr}_p(W)$

$$\nabla^S = \pi_S di_S \in \Omega^1(\mathrm{Hom}(S,S))$$

2nd fundamental form of $S \to \mathrm{Gr}_p(W)$

$$H = \pi_Q di_S \in \Omega^1(\mathrm{Hom}(S,Q))$$

$$t \in \Gamma(Q) \Rightarrow i_Q(t) \in \Gamma(\underline{W}) \Rightarrow di_Q(t) \in \Omega^1(\underline{W})$$

$$di_Q(t) = \pi_S di_Q(t) + \pi_Q di_Q(t) = Kt + \nabla^Q t$$

Connection on $Q \to \mathrm{Gr}_p(W)$

$$\nabla^Q = \pi_Q di_Q \in \Omega^1(\mathrm{Hom}(Q, Q))$$

2nd fundamental form of $Q \to \mathrm{Gr}_p(W)$

$$K := \pi_S di_Q \in \Omega^1(\mathrm{Hom}(Q, S))$$

7 Pull-backs

(M, g) Riemannian manifold (RM)

$(\mathrm{Gr}_p(W), g_{Gr})$ Grassmannian as RM.

$$f : M \longrightarrow \mathrm{Gr}_p(W)$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{i_U} & \underline{W}' & \xrightarrow{\pi_V} & V & \longrightarrow & 0 \\
 & & \big\uparrow f^* & & \big\downarrow f & & \big\uparrow f^* & & \\
0 & \longrightarrow & S & \xrightarrow[i_S]{} & \underline{W}'' & \xrightarrow[\pi_Q]{} & Q & \longrightarrow & 0
\end{array}$$

$\underline{W}' \to M$ $\quad M \times W \to M$

$\underline{W}'' \to \mathrm{Gr}_p(W)$ $\quad \mathrm{Gr}_p(W) \times W \to \mathrm{Gr}_p(W)$

8 Fullness

a. Grassmannian case

$$W \hookrightarrow \Gamma(Q) \ : \ w \mapsto \pi_Q(w)$$

$$W \subset \Gamma(Q)$$

b. Riemannian manifold case

$$W \to \Gamma(V) \ : \ w \mapsto \pi_V(w)$$

Definition

A map $f : M \to \mathrm{Gr}_p(W)$ is called *full* if $W \subset \Gamma(V)$.

Mean curvature operator

Definition

The bundle homomorphism $A \in \Gamma(\mathrm{Hom}\, V)$ defined as

$$A := \sum_{i=1}^{n} H^U_{e_i} \circ K^V_{e_i},$$

where $e_1, \cdots, e_n$ is an orthonormal basis of T_xM is called the *mean curvature operator of* f.

Lemma

The mean curvature operator A is a non-positive Hermitian operator and we have

$$|df|^2 = -\mathrm{trace} A.$$

10 Laplacian acting on sections

Laplace operator acting on $\Gamma(V)$

$$\Delta t = \nabla^{V^*} \nabla^V t = -\sum_{i=1}^{n} \nabla^V_{e_i} \left(\nabla^V t \right) (e_i), \; t \in \Gamma(V).$$

11 Generalisations of Takahashi's Theorem

Theorem

(M, g) RM

$f : M \to \mathrm{Gr}_p(W)$ Smooth map between RM's

The following conditions are equivalent:

1. f is a harmonic.

2. $$\Delta t = -At, \qquad \forall t \in W \subset \Gamma(V)$$

Theorem

$(M,g):$ RM

$f: M \to \mathrm{Gr}_p(W)$ Smooth map between RM's

The following conditions are equivalent:

1.

f is harmonic and $\exists h \in C^\infty(M):$

$$A_x = -h(x) Id_V \qquad \forall x \in M$$

2.

$\exists h \in C^\infty(M):$

$$\Delta t = ht \quad \forall t \in W \subset \Gamma(V)$$

Moreover,

$$|df|^2 = qh, \qquad q := \mathrm{rnk}\, V$$

12 Recovering the original Takahashi's theorem

$$
\begin{array}{ccc}
V & \xleftarrow{f^{-1}} & Q \\
\downarrow & & \downarrow \\
M & \xrightarrow[f]{} & \mathrm{Gr}_{N-1}(\mathbb{R}^N)
\end{array}
$$

$$\mathrm{Gr}_{N-1}(\mathbb{R}^N) = \frac{SO(N)}{SO(N-1)} = S^{N-1}$$

$$A \in \Gamma(\mathrm{Hom}(V)) \quad + \quad \{\mathrm{rnk}V = \mathrm{rnk}Q = 1\} \quad \Rightarrow \quad A = -hId_V$$

Planning: 2/3

Characterisation of harmonic maps to Grassmannians in terms of vector bundles

Characterisation of the moduli space (holomorphic case)

14 Evaluation & globally generated vector bundles

$V \to M$ VB

$W \subset \Gamma(V)$ finite-dimensional vector space

$\underline{W}$ $M \times W \to M$

Evaluation homomorphism

$$\mathrm{ev} : \underline{W} \longrightarrow V$$

$$\mathrm{ev}_x(t) = t(x) \in V_x,\ \ t \in W,\, x \in M$$

Definition

The vector bundle $V \to M$ is said to be *globally generated by* W if $\mathrm{ev} : \underline{W} \to V$ is surjective.

15 Map to a Grassmannian induced by a VB

$V \to M$ VB globally generated by W

$\dim W = N$

Induced map by $(V \to M, W)$

$$f : M \longrightarrow \mathrm{Gr}_p(W)$$

$$f(x) := \ker \mathrm{ev}_x$$

where

$$p = N - \mathrm{rnk}\, V$$

Lemma

$V \to M$ can be naturally identified with $f^*Q \to M$

$\underline{W}' \to M$ $\qquad$ $M \times W \to M$

$\underline{W}'' \to \mathrm{Gr}_p(W)$ $\qquad$ $\mathrm{Gr}_p(W) \times W \to \mathrm{Gr}_p(W)$

Natural identification ϕ

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\ \mathrm{ev} & \longrightarrow & \underline{W}' & \longrightarrow & V & \longrightarrow & 0 \\
 & & {\scriptstyle i}\big\downarrow & & \big\| & & \big\downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & f^*S & \longrightarrow & \underline{W}' & \longrightarrow & f^*Q & \longrightarrow & 0 \\
 & & {\scriptstyle f^*}\big\uparrow & & \big\downarrow{\scriptstyle f} & & \big\uparrow{\scriptstyle f^*} & & \\
0 & \longrightarrow & S & \xrightarrow[i_S]{} & \underline{W}'' & \xrightarrow[\pi_Q]{} & Q & \longrightarrow & 0
\end{array}
$$

16 Standard maps

(M, g) compact RM

$(V \to M,\ h_V,\ \nabla)$ VB + fibre metric + connection

Eigenspaces of the Laplacian (acting on sections)

$$\Gamma(V) = \bigoplus_{\mu} W_{\mu}, \quad W_{\mu} := \{t \in \Gamma(V) \,|\, \Delta t = \mu t\}\,.$$

Standard induced map by W_{μ}

Suppose $V \to M$ globally generated by W_{μ}

$$f_0 : M \longrightarrow \mathrm{Gr}_p(W_{\mu})$$

$$f_0(x) = \ker\ \mathrm{ev}_x | W_{\mu}$$

17 Homogeneous VBs

$M = G/K$ Compact reductive homogeneous space.

V_0 q-dimensional K-module

$V \to M$ homogeneous VB, standard fibre V_0

$$G \times_K V_0 \longrightarrow G/K$$

$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ Lie algebra decomposition

∇ Canonical connection

$\mathcal{H}_g = \{(L_g)_* \mathfrak{m} | g \in G\} \subset T_g G$ Horizontal distribution

Consquences of homogeneity

$V \to M$ Homogeneous VB.

Lemma

$W \subset W_\mu$ globally generates $V \to M$.
Then

$$V_0 \subset W$$

$U_0 := V_0^\perp \subset W$ Orthogonal complement to V_0

Lemma

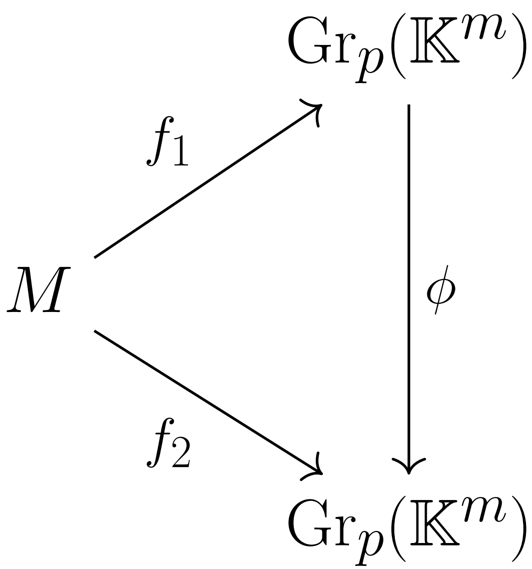
$$f^*\nabla^Q \overset{\mathrm{G}}{\equiv} \nabla \longleftrightarrow \mathfrak{m}V_0 \subset U_0$$

19 Image equivalence of maps

$f_1, f_2 : M \to \mathrm{Gr}_p(\mathbb{K}^m)$ mappings

Image equivalence of maps

f_1 is *image equivalent* to f_2, if $\exists$ isometry ϕ of $\mathrm{Gr}_p(\mathbb{K}^m)$ such that the diagram commutes



$$f_2 = \phi \circ f_1$$

20 Gauge equivalence of maps

$V \to M$ VB

$f_i : M \to \mathrm{Gr}_p(\mathbb{K}^m)$ mappings

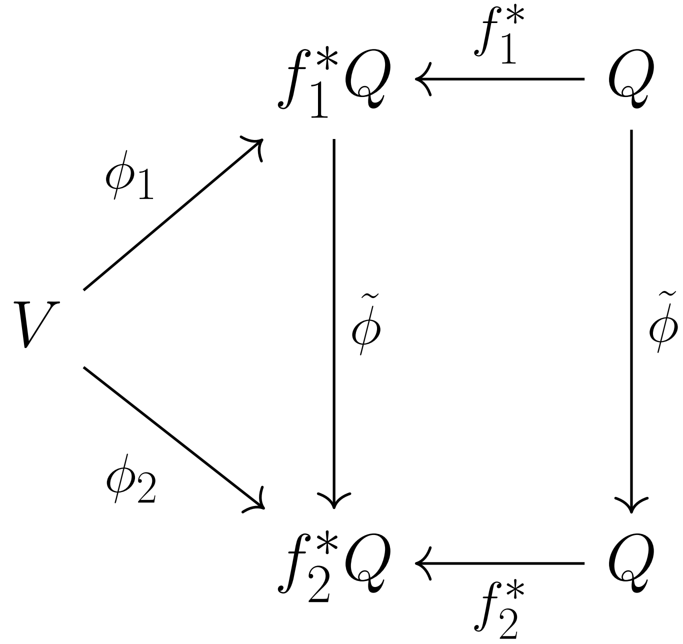
$\phi_i : V \to f_i^* Q$ VB isomorphisms

Gauge equivalence of maps

Two couples (f_i, ϕ_i),$(i = 1, 2)$ are called *gauge equivalent*, if $\exists$ isometry ϕ of $\mathrm{Gr}_p(\mathbb{K}^m)$ such that

$$f_2 = \phi \circ f_1, \qquad \phi_2 = \tilde{\phi} \circ \phi_1$$

where $\tilde{\phi}$ is the bundle automorphism of $Q \to \mathrm{Gr}_p(\mathbb{K}^m)$ covering ϕ.



$$\phi_2 = \tilde{\phi} \circ \phi_1$$

Generalisation of the do Carmo – Wallach theory (Holomorphic case)

Theorem Hypothesis

$M = G/K$	Cpct. irr. Hermitian symm. space
$V \to M$	Complex(*) homogeneous line bundle
$\{h, \nabla, J\}$	metric, can. connection, CS
$f : M \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$	Full holomorphic map satisfying

Gauge condition

$$\left(f^*Q \to M, f^*g_Q, f^*\nabla^Q, J^Q\right) \overset{\mathrm{G}}{\equiv} (V \to M, h, \nabla, J)$$

Einstein–Hermitian condition

$$A = -\mu Id_V, \qquad \mu \in \mathbb{R}_+$$

$$e(f) = 2\mu$$

Theorem Thesis (I)

0.

$W := H^0(V) \subset \Gamma(V)$ space of holomorphic sections is eigenspace of Laplacian with eigenvalue μ.

Regard W as real $W_{\mathbb{R}}$ + L_2-inner product.

1.

$$\iota : \mathbb{R}^{n+2} \longrightarrow W_{\mathbb{R}}$$

$\mathbb{R}^{n+2} \subset W_{\mathbb{R}}$, and $V \to M$ is globally generated by $\mathbb{R}^{n+2}$.

Theorem Thesis (II)

$\exists T \in S(W_{\mathbb{R}}) \in \mathrm{End}(W_{\mathbb{R}})$ positive semi-definite

2.

$\mathbb{R}^{n+2} = (\ker\ T)^{\perp}$, and $T|_{\mathbb{R}^{n+2}}$ is positive definite.

3. (Orthogonality conditions)

$$(T^2 - Id_W, \mathrm{GS}(V_0, V_0))_S = (T^2, \mathrm{GS}(\mathfrak{m}V_0, V_0))_S = 0$$

4.

T provides holomorphic embedding

$$\mathrm{Gr}_n(\mathbb{R}^{n+2}) \longrightarrow \mathrm{Gr}_{n'}(W)$$

$$n' = n + \dim\ker T$$

and a bundle isomorphism

$$\phi : V \to f^*Q$$

Theorem Thesis (III)

a.

$f : M \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$ can be expressed as

$$f([g]) = (\iota^* \mathrm{T} \iota)^{-1} \left(f_0([g]) \cap (\ker\ T)^\perp \right)$$

where

$$f_0([g]) = gU_0 \subset W_{\mathbb{R}}$$

is the standard map.

b.

The correspondence

$$[f] \longleftrightarrow T$$

is one-to-one, where $[f]$ is the gauge equivalence class of maps represented by $\iota^* \mathrm{T} \iota$ and ϕ.

Planning: 3/3

Characterisation of harmonic maps in terms of vector bundles

Characterisation of the moduli space (holomorphic case)

Description of the moduli space of holomorphic isometric embeddings $\mathbb{C}\mathrm{P}^1 \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$

Holomorphic isometric embeddings of degree k

$\mathbb{C}\mathrm{P}^1$	Complex projective line $+ g + J \sim \omega_0$
$\mathrm{Gr}_n(\mathbb{R}^{n+2})$	Grassmannian $+ g + J \sim \omega_Q$

Holo.emb.

$$f : \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{n+2}) \subset \mathbb{C}\mathrm{P}^{n+1}$$

Iso. of $\deg\ k$

$$f^*\omega_Q = k\omega_0, \qquad k \in \mathbb{N}$$

24 Holo.iso.emb. of $\deg\ k$ & Gauge condition

$\mathcal{O}(1) \to \mathbb{C}\mathrm{P}^1$ Hyperplane section bundle

$$\mathcal{O}(k) = \mathcal{O}(1) \otimes \overset{k}{\cdots} \otimes \mathcal{O}(1)$$

Lemma

$f : \mathbb{C}\mathrm{P}^1 \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$ holo.emb. is k-holo.iso.emb. iff

$$\begin{array}{ccccc}
\mathcal{O}(k) & \xleftrightarrow{\ \ (\mathrm{G})\ \ } & f^*Q & \xleftarrow{\ f^*\ } & Q \\
 & \searrow & \swarrow & & \downarrow \\
 & & \mathbb{C}\mathrm{P}^1 & \xrightarrow[f]{\qquad} & \mathrm{Gr}_n(\mathbb{R}^{n+2})
\end{array}$$

Holo.iso.emb. of $\deg\ k$ & EH condition

Lemma

$f : \mathbb{C}\mathrm{P}^1 \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$ k-holo.iso.emb., then

$$\text{(EH)} \quad A = -\mu Id, \qquad \mu \in \mathbb{R}_{>0}$$

Remark

a. Holo.emb.

$$\text{(G)} \to \text{(EH)}$$

b. Harmonic maps, minimal immersions

$$\text{(G)} \not\to \text{(EH)}$$

Complex representations

$\mathcal{O}(k) \to \mathbb{C}\mathrm{P}^1$ $\qquad$ $\mathrm{SU}(2) \times_{\mathrm{U}(1)} V_0 \to \mathrm{SU}(2)/\mathrm{U}(1)$

V_0 $\qquad$ $\mathrm{U}(1)$-module

Holomorphic sections + Borel–Weil Thm.

$$W := H^0(\mathcal{O}(k)) = S^k\mathbb{C}^2 \qquad \mathrm{SU}(2) - \mathrm{IRREP}$$

$$\dim_{\mathbb{C}} W = k + 1$$

$\mathrm{SU}(2)|\mathrm{U}(1)$

$$S^k\mathbb{C}^2 = \mathbb{C}_{-k} \oplus \mathbb{C}_{-k+2} \oplus \cdots \oplus \mathbb{C}_k \qquad \mathrm{U}(1) - \mathrm{IRREPs}$$

$$V_0 = \mathbb{C}_{-k} \qquad \mathfrak{m}V_0 = \mathbb{C}_{-k+2} \subset U_0$$

27 Real representations

$$\mathrm{SU}(2) \xrightarrow{2:1} \mathrm{SO}(3)$$

$$S_0^l\mathbb{R}^3(\dim = 2l+1) \qquad \mathrm{SO}(3) - \mathrm{IRREP}$$

$k = 2l$

$$W_{\mathbb{R}} = (S^{2l}\mathbb{C}^2)_{\mathbb{R}} = \mathbb{R}^{4l+2} = 2S_0^{2l}\mathbb{R}^3$$

$k = 2l+1$

$$W_{\mathbb{R}} = (S^{2l+1}\mathbb{C}^2)_{\mathbb{R}} = \mathbb{R}^{4l+4}$$

$$\mathrm{H}(W) \subset \mathrm{S}(W_{\mathbb{R}}) \subset \mathrm{End}(W_{\mathbb{R}})$$

Lemma

$$\mathrm{S}(W_{\mathbb{R}}) = \mathrm{H}_+(W) \oplus \mathrm{H}_-(W) \oplus \sigma\mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)$$

Spectral formulae

Lemma

$W_{\mathbb{R}} = \left(S^{2l}\mathbb{C}^2\right)_{\mathbb{R}} = \mathbb{R}^{4l+2}$

$$S^2\mathbb{R}^{4l+2} = 3\left(\bigoplus_{r=0}^{l} S_0^{2l-2r}\mathbb{R}^3\right) \oplus \left(\bigoplus_{r=0}^{l-1} S_0^{(2l-1)-2r}\mathbb{R}^3\right)$$

$W_{\mathbb{R}} = \left(S^{2l+1}\mathbb{C}^2\right)_{\mathbb{R}} = \mathbb{R}^{4l+4}$

$$S^2\mathbb{R}^{4l+4} = 3\left(\bigoplus_{r=0}^{l} S_0^{(2l+1)-2r}\mathbb{R}^3\right) \oplus \left(\bigoplus_{r=0}^{l-1} S_0^{2l-2r}\mathbb{R}^3\right)$$

Spaces of Hermitian operators

Proposition

$S^{2l}\mathbb{C}^2$

$$\mathrm{H}_+(S^{2l}\mathbb{C}^2) = \bigoplus_{r=0}^{l} S_0^{2l-2r}\mathbb{R}^3$$

$$\mathrm{H}_-(S^{2l}\mathbb{C}^2) = \bigoplus_{r=0}^{l-1} S_0^{(2l-1)-2r}\mathbb{R}^3$$

$S^{2l+1}\mathbb{C}^2$

$$\mathrm{H}_+(S^{2l+1}\mathbb{C}^2) = \bigoplus_{r=0}^{l} S_0^{(2l+1)-2r}\mathbb{R}^3$$

$$\mathrm{H}_-(S^{2l+1}\mathbb{C}^2) = \bigoplus_{r=0}^{l} S_0^{2l-2r}\mathbb{R}^3$$

Finding T via orthogonality relations

Lemma

(a) $\mathrm{H}(W) \subset \mathrm{GS}(\mathfrak{m}V_0, V_0)$

(b) $\mathrm{GS}(\mathfrak{m}V_0, V_0) \cap (\sigma H_+(W) \oplus J\sigma H_+(W))$ is the highest-weight representation.

$$\mathrm{GS}(\mathfrak{m}V_0, V_0) = \mathrm{H}(W) \oplus S_0^k\mathbb{R}^3 \oplus S_0^k\mathbb{R}^3$$

Corollary

$$\mathcal{M}_k \cong (\mathrm{GS}(\mathfrak{m}V_0, V_0) \oplus \mathbb{R}Id)^\perp \in \mathrm{S}(W_\mathbb{R})$$

$$\mathcal{M}_k \cong 2\bigoplus_{r=1}^{k\geq 2r} S_0^{k-2r}\mathbb{R}^3, \qquad \dim_\mathbb{R} \mathcal{M}_k = k(k-1)$$

Main Results about Moduli Spaces

$f : \mathbf{C}P^1 \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$ Full Holo.Iso.Emb of $\deg\ k$

$\mathcal{M}_k$ Moduli by gauge equivalence.

Theorem 1

$$n \leqq 2k.$$

Theorem 2

If $n = 2k$ (target $\mathrm{Gr}_{2k}(\mathbb{R}^{2k+2})$) then $\mathcal{M}_k$ can be regarded as an open convex body in

$$\bigoplus_{l=1}^{k \geqq 2l} S^{2k-4l}\mathbb{C}^2.$$

$\overline{\mathcal{M}_k}$ Compactification of $\mathcal{M}_k$

Theorem 3

Boundary points correspond to maps whose images are included in totally geodesic submanifolds

$$\mathrm{Gr}_p(\mathbb{R}^{p+2}) \subset \mathrm{Gr}_{2k}(\mathbb{R}^{2k+2}), \qquad p < 2k.$$

$\mathbf{M}_k$ Moduli space by image equivalence

Theorem 4

If $n = 2k$ (target $\mathrm{Gr}_{2k}(\mathbb{R}^{2k+2})$)

$$\mathbf{M}_k = \mathcal{M}_k / S^1.$$

MUCHAS GRACIAS