ARMA manuscript No.
(will be inserted by the editor)

Finite Propagation Speed for Limited Flux Diffusion Equations

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Abstract

We prove that the support of solutions of a limited flux diffusion equation known as a relativistic heat equation evolves at constant speed, identified as light's speed c. For that we construct entropy sub- and super-solutions which are fronts evolving at speed c and prove the corresponding comparison principle between entropy solutions and sub- and super-solutions, respectively. This enables us to prove the existence of discontinuity fronts moving at light's speed.

1. Introduction

To limit the speed of propagation of different types of waves which are solutions of nonlinear degenerate parabolic equations some mechanisms of saturation of the flux as the gradient becomes unbounded have been proposed by different authors [22, 15, 23].

The speed of light c is the highest admissible velocity for transport of radiation in transparent media, and, to ensure it, J.R. Wilson (in an unpublished work, see [22]) proposed to use a flux limiter. The flux limiter merely enforces the physical restriction that the flux cannot exceed energy density times the speed of light, that is, the flux cannot violate causality. The basic idea is to modify the diffusion-theory formula for the flux in a way that gives the standard result in the high opacity limit, while simulating free streaming (at light speed) in transparent regions. As an example, one of the expressions suggested for the flux of the energy density u is

$$F = -\nu u \frac{Du}{u + \nu c^{-1}|Du|} \tag{1}$$

(where ν is a constant representing a kinematic viscosity and c the speed of light) which yields in the limit $\nu \to \infty$ the flux $F = -cu\frac{Du}{|Du|}$. Observe also that when $c \to \infty$, the flux tends to $F = -\nu Du$, and the corresponding diffusion equation becomes the heat equation, which has an infinite speed of propagation.

The diffusion equation corresponding to (1) is

$$u_t = \nu \,\mathrm{div}\left(\frac{uDu}{u + \frac{\nu}{c}|Du|}\right) \tag{2}$$

and is one among the various *flux limited diffusion equations* used in the theory of radiation hydrodynamics [22]. Indeed, the same effects can be guaranteed for a similar equation [13]

$$u_t = \nu \,\mathrm{div}\left(\frac{|u|Du}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}}\right). \tag{3}$$

Y. Brenier ([13]) was able to derive (3) from Monge-Kantorovich's mass transport theory and described it as a *relativistic heat equation*. Both equations, (2) and (3), interpolate ([13]) between the usual heat equation (when $c \to \infty$) and the diffusion equation in transparent media (when $\nu \to \infty$) with constant speed of propagation c

$$u_t = c \,\mathrm{div}\left(u\frac{Du}{|Du|}\right). \tag{4}$$

Many other models of nonlinear degenerate parabolic equations with flux saturation as the gradient becomes unbounded have been proposed by Rosenau and his coworkers [15,23]. In [15] the authors exhibited some models for which initial conditions of compact support, even if smooth, develop a discontinuity in finite time and displayed experiments to show the evolution of the support of its solutions. The same phenomenon of breaking of solutions and apparition of discontinuities was proved in [12,17] for equations of type $u_t = (\varphi(u)\mathbf{b}(u_x))_x$, where $\varphi : I\!R \to I\!R^+$ is smooth and strictly positive, and $\mathbf{b} : I\!R \to I\!R$ is a smooth odd function such that $\mathbf{b}' > 0$ and $\lim_{s\to\infty} \mathbf{b}(s) = \mathbf{b}_\infty$, which model heat and mass transfer in turbulent fluids [12].

Our main purpose in this paper is to study the evolution of the support of solutions of (3) which we take as a model of flux limited diffusion equation. Let us first mention that the well-posedness of (3) for initial conditions in $(L^1(I\!R^N) \cap L^\infty(I\!R^N))^+$ was proved in [7]. Indeed, existence and uniqueness of entropy solutions of the Cauchy problem for a general class of equations including (3) and (2) were proved in [7]. These results will be reviewed in Section 3. We shall prove in this paper that the support of entropy solutions of (3) evolves at finite speed c. For that we construct sub- and super-solutions of (3) which move at speed c and prove comparison principles between solutions and sub- and super-solutions, respectively. The sub-solutions permit also to prove that discontinuity fronts evolve at speed

c and live for ever. This has an interesting consequence since it proves that u is not smooth and we can only expect that u_t is a Radon measure. We do not know if this is true in general, though there are radial solutions for which it is true [8]. This explains in an indirect way the complexity of some of the requirements of the notion of entropy solution. The study of equation (4) and the convergence of solutions of (3) to solutions of (4) will not be considered here and will be the object of a subsequent work.

Let us explain the plan of the paper. In Section 2 we recall some basic facts about functions of bounded variation, denoted by BV, functionals defined on BV, and Green's formula. In Section 3 we review the notion of entropy solution for a class of degenerate parabolic equations including (3) and the existence and uniqueness results proved in [7]. We shall also explain the main ingredients of the notion of entropy solution. In Section 4 we introduce the concepts of entropy sub and super-solutions and we prove comparison principles between solutions and sub- and super-solutions, respectively. Finally, in Section 5 we construct sub- and super-solutions of (3) which permit us to prove that the support of solutions evolves at light's speed c, and the existence of discontinuity fronts moving at speed c.

2. Preliminaries

2.1. Functions of bounded variations and some generalization

Let us start with some notation. We denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $I\!R^N$, respectively. Given an open set Ω in $I\!R^N$ we denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω. The space of continuous functions with compact support in $I\!R^N$ will be denoted by $C_c(I\!R^N)$.

Due to the linear growth condition on the Lagrangian, the natural energy space to study the problems we are interested in is the space of functions of bounded variation. Recall that if Ω is an open subset of $I\!R^N$, a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. For $u \in BV(\Omega)$, the vector measure Du decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \; \mathcal{L}^N$, where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$. We also split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$. It is well known (see for instance [1]) that

$$D^j u = (u_+ - u_-)\nu_u \mathcal{H}^{N-1} \llcorner J_u,$$

where J_u denotes the set of approximate jump points of u, and $\nu_u(x) = \frac{Du}{|Du|}(x)$, $\frac{Du}{|Du|}$ being the Radon-Nikodym derivative of Du with respect to its total variation $|Du|$. For further information concerning functions of bounded variation we refer to [1], [20] or [25].

We need to consider the following truncature functions. For $a < b$, let $T_{a,b}(r) := \max(\min(b, r), a)$. As usual, we denote $T_k = T_{-k,k}$. We also consider truncature functions of the form $T^l_{a,b}(r) := T_{a,b}(r) - l$ ($l \in I\!R$). We denote

$$\mathcal{T}_r := \{T_{a,b} \ : \ 0 < a < b\},$$

$$\mathcal{T}^+ := \{T^l_{a,b} \ : \ 0 < a < b, \ l \in I\!R, \ \ T^l_{a,b} \geq 0\},$$

and

$$\mathcal{T}^- := \{T^l_{a,b} \ : \ 0 < a < b, \ l \in I\!R, \ \ T^l_{a,b} \leq 0\}.$$

Given any function u and $a, b \in I\!R$ we shall use the notation $[u \geq a] = \{x \in I\!R^N : u(x) \geq a\}$, $[a \leq u \leq b] = \{x \in I\!R^N : a \leq u(x) \leq b\}$, and similarly for the sets $[u > a]$, $[u \leq a]$, $[u < a]$, etc. We denote $u^+ := \max\{u, 0\}$, and $u^- := \min\{u, 0\}$.

We need to consider the function space

$$TBV^+(I\!R^N) := \left\{u \in L^1(I\!R^N)^+ \ : \ \ T(u) \in BV(I\!R^N), \ \ \forall\, T \in \mathcal{T}_r\right\},$$

and to give a sense to the Radon-Nikodym derivative (with respect to the Lebesgue measure) ∇u of Du for a function $u \in TBV^+(I\!R^N)$. Using chain's rule for BV-functions (see for instance [1]), with a similar proof to the one given in Lemma 2.1 of [11], we obtain the following result.

Lemma 1. *For every $u \in TBV^+(I\!R^N)$ there exists a unique measurable function $v : I\!R^N \to I\!R^N$ such that*

$$\nabla T_{a,b}(u) = v\chi_{[a<u<b]} \qquad \mathcal{L}^N - \text{a.e.}, \ \ \forall\, T_{a,b} \in \mathcal{T}_r. \tag{5}$$

Thanks to this result we define ∇u for a function $u \in TBV^+(I\!R^N)$ as the unique function v which satisfies (5). This notation will be used throughout in the sequel.

We denote by $\mathcal{P}$ the set of Lipschitz continuous functions $p : [0, +\infty[\to I\!R$ satisfying $p'(s) = 0$ for s large enough. We write $\mathcal{P}^+ := \{p \in \mathcal{P} \ : \ p \geq 0\}$. We have the following result ([7]).

Lemma 2. *If $u \in TBV^+(I\!R^N)$, then $p(u) \in BV(I\!R^N)$ for every $p \in \mathcal{P}$ such that there exists $a > 0$ with $p(r) = 0$ for all $0 \leq r \leq a$. Moreover,* $\nabla p(u) = p'(u)\nabla u \ \ \mathcal{L}^N$-a.e.

2.2. Functionals defined on BV

Let Ω be an open subset of $\mathbb{R}^N$. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$ be a Borel function such that

$$C(x)\|\xi\| - D(x) \le g(x, z, \xi) \le M'(x) + M\|\xi\| \tag{6}$$

for any $(x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, $|z| \le R$, where M is a positive constant and $C, D, M' \ge 0$ are bounded Borel functions which may depend on R. Assume that $C, D, M' \in L^1(\Omega)$.

Following Dal Maso [16] we consider the following functional for $u \in BV(\Omega) \cap L^\infty(\Omega)$:

$$\begin{aligned} \mathcal{R}_g(u) &:= \int_\Omega g(x, u(x), \nabla u(x))\, dx + \int_\Omega g^0\left(x, \tilde{u}(x), \frac{Du}{|Du|}(x)\right) |D^c u| \\ &+ \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} g^0(x, s, \nu_u(x))\, ds \right) d\mathcal{H}^{N-1}(x), \end{aligned} \tag{7}$$

where the recession function g^0 of g is defined by

$$g^0(x, z, \xi) = \lim_{t \to 0^+} t g\left(x, z, \frac{\xi}{t}\right), \tag{8}$$

and is convex and homogeneous of degree 1 in ξ, and $\tilde{u}$ is the approximated limit of u (see [1]).

In case that Ω is a bounded set, and under standard continuity and coercivity assumptions, Dal Maso proved in [16] that $\mathcal{R}_g(u)$ is L^1-lower semi-continuous for $u \in BV(\Omega)$. Recently, De Cicco, Fusco, and Verde [18], have obtained a very general result about the L^1-lower semi-continuity of $\mathcal{R}_g$ in $BV(\mathbb{R}^N)$.

Assume that $g : \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$ is a Borel function such that

$$C\|\xi\| - D \le g(z, \xi) \le M(1 + \|\xi\|) \qquad \forall (z, \xi) \in \mathbb{R}^N,\ |z| \le R, \tag{9}$$

for some constants $C, D, M \ge 0$ which may depend on R. Given a function $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we define the Radon measure $g(u, Du)$ in $\mathbb{R}^N$ by

$$\langle g(u, Du), \phi \rangle := \mathcal{R}_{\phi g}(u) \qquad \phi \in C_c(\mathbb{R}^N)^+. \tag{10}$$

If $\phi \in C_c(\mathbb{R}^N)$, we write $\phi = \phi^+ - \phi^-$ with $\phi^+ = \max(\phi, 0)$, $\phi^- = -\min(\phi, 0)$, and we define $\langle g(u, Du), \phi \rangle := \mathcal{R}_{\phi^+ g}(u) - \mathcal{R}_{\phi^- g}(u)$.

Let us observe that if $g^0(z, \xi) = \varphi(z)\psi^0(\xi)$, where φ is Lipschitz continuous and ψ^0 is an homogeneous function of degree 1, by applying the chain rule for BV-functions (see [1]), we have

$$\mathcal{R}_{\phi g}(u) = \int_{\mathbb{R}^N} \phi(x) g(u, \nabla u) dx + \int_{\mathbb{R}^N} \phi(x) \psi^0\left(\frac{Du}{|Du|}\right) |D^s J_\varphi(u)|, \tag{11}$$

where, for any function q, $J_q(r)$ denotes the primitive of q, i.e., $J_q(r) = \int_0^r q(s)\,ds$. In this case we have

$$g(u, Du)^s = \psi^0 \left(\frac{Du}{|Du|} \right) |D^s J_\varphi(u)|. \tag{12}$$

2.3. A generalized Green's formula

We shall need several results from [9] (see also [21]) in order to give a meaning to integrals of bounded vector fields with divergence in L^1 integrated with respect to the gradient of a BV function. Following [9], we denote

$$X_1(\mathbb{R}^N) = \left\{ \mathbf{z} \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) \ : \ \operatorname{div}(\mathbf{z}) \in L^1(\mathbb{R}^N) \right\}. \tag{13}$$

If $\mathbf{z} \in X_1(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ we define the functional $(\mathbf{z}, Dw) : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Dw), \varphi \rangle := - \int_{\mathbb{R}^N} w\, \varphi \operatorname{div}(\mathbf{z})\, dx - \int_{\mathbb{R}^N} w\, \mathbf{z} \cdot \nabla \varphi\, dx. \tag{14}$$

Then $(\mathbf{z}, Dw)$ is a Radon measure in $\mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} (\mathbf{z}, Dw) = \int_{\mathbb{R}^N} \mathbf{z} \cdot \nabla w\, dx, \qquad \forall\ w \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{15}$$

Moreover, $(\mathbf{z}, Dw)$ is absolutely continuous with respect to $|Dw|$.

We have the following *Green's formula* for $\mathbf{z} \in X_1(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ ([9]):

$$\int_{\mathbb{R}^N} w \ \operatorname{div}(\mathbf{z}) \ dx + \int_{\mathbb{R}^N} (\mathbf{z}, Dw) = 0. \tag{16}$$

3. The notion of solution, existence and uniqueness results: a review

In this section, following [7], we recall the concept of entropy solution and the existence and uniqueness result for the Cauchy problem

$$\begin{cases} u_t = \operatorname{div} \mathbf{a}(u, Du) & \text{in} \quad Q_T = (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in} \quad x \in \mathbb{R}^N. \end{cases} \tag{17}$$

Even if the main purpose of the paper is the study of equation (3), the general notation is convenient when writing the proof of the comparison result proved in Section 4. Moreover, the general statement of it will be useful for later reference.

3.1. Basic assumptions

Assume that the Lagrangian $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ satisfies the following assumptions, which we shall refer collectively as (H):

(H_1) f is continuous on $\mathbb{R} \times \mathbb{R}^N$ and is a convex differentiable function of ξ such that $\nabla_\xi f(z,\xi) \in C(\mathbb{R} \times \mathbb{R}^N)$. Further we require f to satisfy the linear growth condition

$$C_0(z)\|\xi\| - D_0(z) \leq f(z,\xi) \leq M_0(z)(\|\xi\| + 1) \tag{18}$$

for any $(z,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and some positive and continuous functions C_0, D_0, M_0, such that $C_0(z) > 0$ for any $z \neq 0$. Let f^0 denote the recession function of f.

We consider the function $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$ associated to the Lagrangian f. By the convexity of f, we have

$$\mathbf{a}(z,\xi) \cdot (\eta - \xi) \leq f(z,\eta) - f(z,\xi), \tag{19}$$

and the following monotonicity condition is satisfied

$$(\mathbf{a}(z,\eta) - \mathbf{a}(z,\xi)) \cdot (\eta - \xi) \geq 0. \tag{20}$$

Moreover, it is easy to see that for each $R > 0$, there is a constant $M = M(R) > 0$, such that

$$\|\mathbf{a}(z,\xi)\| \leq M \qquad \forall\ (z,\xi) \in \mathbb{R} \times \mathbb{R}^N,\ |z| \leq R. \tag{21}$$

We also assume that $\mathbf{a}(z,0) = 0$ for all $z \in \mathbb{R}$, and $\mathbf{a}(z,\xi) = z\mathbf{b}(z,\xi)$ with

$$\|\mathbf{b}(z,\xi)\| \leq M_0 \qquad \forall\ (z,\xi) \in \mathbb{R} \times \mathbb{R}^N,\ |z| \leq R. \tag{22}$$

We consider the function $h : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$h(z,\xi) := \mathbf{a}(z,\xi) \cdot \xi.$$

By (20), we have

$$h(z,\xi) \geq 0 \qquad \forall \xi \in \mathbb{R}^N,\ z \in \mathbb{R}. \tag{23}$$

Moreover we assume that

$$h(z,\xi) \leq M(z)\|\xi\| \tag{24}$$

for some positive continuous function $M(z)$ and for any $(z,\xi) \in \mathbb{R} \times \mathbb{R}^N$. On the other hand, from (19) and (18), it follows that

$$C_0(z)\|\xi\| - D_1(z) \leq h(z,\xi) \tag{25}$$

for any $(z,\xi) \in \mathbb{R} \times \mathbb{R}^N$ where $D_1(z) = D_0(z) + f(z,0)$. We assume that there exist constants $A, B > 0$ and $\alpha, \beta \geq 1$, such that

$$|D_1(z)| \leq A|z|^\alpha + B|z|^\beta \quad \text{for any } z \in \mathbb{R}^N. \tag{26}$$

As we noticed in [7], this condition was used in only to prove some estimates during the proof of existence, and a more general condition could be used.

(H_2) We assume that $\frac{\partial \mathbf{a}}{\partial \xi_i}(z,\xi) \in C(\mathbb{R} \times \mathbb{R}^N)$ for any $i = 1, \dots, N$.

(H_3) $h(z,\xi) = h(z,-\xi)$, for all $z \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ and h^0 exists.

Observe that we have

$$C_0(z)\|\xi\| \le h^0(z,\xi) \le M\|\xi\| \quad \text{for any } (z,\xi) \in \mathbb{R} \times \mathbb{R}^N,\ |z| \le R.$$

(H_4) $f^0(z,\xi) = h^0(z,\xi)$, for all $\xi \in \mathbb{R}^N$ and all $z \in \mathbb{R}$.

(H_5) $\mathbf{a}(z,\xi) \cdot \eta \le h^0(z,\eta)$ for all $\xi, \eta \in \mathbb{R}^N$, and all $z \in \mathbb{R}$.

(H_6) We assume that $h^0(z,\xi)$ can be written in the form $h^0(z,\xi) = \varphi(z)\psi^0(\xi)$ with φ a Lipschitz continuous function such that $\varphi(z) > 0$ for any $z \neq 0$, and ψ^0 being a convex function homogeneous of degree 1.

(H_7) For any $R > 0$, there is a constant $C > 0$ such that

$$|(\mathbf{a}(z,\xi) - \mathbf{a}(\hat{z},\xi)) \cdot (\xi - \hat{\xi})| \le C|z - \hat{z}|\,\|\xi - \hat{\xi}\| \tag{27}$$

for any $z, \hat{z} \in \mathbb{R}$, $\xi, \hat{\xi} \in \mathbb{R}^N$, with $|z|, |\hat{z}| \le R$.

Observe that, by the monotonicity condition (20) and using (27), it follows that

$$(\mathbf{a}(z,\xi) - \mathbf{a}(\hat{z},\hat{\xi})) \cdot (\xi - \hat{\xi}) \ge -C|z - \hat{z}|\,\|\xi - \hat{\xi}\| \tag{28}$$

for any $(z,\xi), (\hat{z},\hat{\xi}) \in \mathbb{R} \times \mathbb{R}^N$, $|z|, |\hat{z}| \le R$.

Let us observe that under assumptions (H_4) and (H_6), applying (12), we have

$$f(u,Du)^s = h(u,Du)^s = \psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(u)|. \tag{29}$$

Remark 1. The function $f(z,\xi) = \frac{c^2}{\nu}|z|\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}$ satisfies the assumptions (H_1)-(H_7), with $\mathbf{a}(z,\xi) = \nu\frac{|z|\xi}{\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}}$. This particular case corresponds to the *relativistic heat equation* (3). The Lagrangian

$$f(z,\xi) := cz\left(|\xi| - \frac{cz}{\nu}\log\left(1 + \frac{\nu}{cz}|\xi|\right)\right)$$

is associated with the flux limited diffusion equation (2) and satisfies also the assumptions (H_1)-(H_7).

3.2. A functional calculus

Let $g : I\!R \times I\!R^N \to [0, \infty[$ be a Borel function satisfying (9). Observe that both functions f, h satisfy (9).

Let $T \in \mathcal{T}^+ \cup \mathcal{T}^-$. Then there is some $T_{a,b} \in \mathcal{T}_r$ and a constant $c \in I\!R$ such that $T = T_{a,b} - c$. For each $u \in TBV^+(I\!R^N) \cap L^\infty(I\!R^N)$ and $\phi \in C_c(I\!R^N)$, $\phi \geq 0$, we define

$$\mathcal{R}(\phi g, T)(u) := \mathcal{R}_{\phi g}(T_{a,b}(u)) + \int_{[u \leq a]} \phi(x)\,(g(u(x), 0) - g(a, 0))\; dx \\ + \int_{[u \geq b]} \phi(x)\,(g(u(x), 0) - g(b, 0))\; dx. \tag{30}$$

If $\phi \in C_c(I\!R^N)$, we define $\mathcal{R}(\phi g, T)(u) := \mathcal{R}(\phi^+ g, T)(u) - \mathcal{R}(\phi^- g, T)(u)$.

Observe that, with this notation, we have $\mathcal{R}(\phi g, T)(u) = \mathcal{R}(\phi g, T_{a,b})(u)$. Moreover, if $u \in W^{1,1}(I\!R^N)$, we get

$$\mathcal{R}(\phi g, T)(u) = \int_{I\!R^N} \phi(x) g(u(x), \nabla T(u(x)))\, dx. \tag{31}$$

We recall that, if $g(z, \xi)$ is continuous in (z, ξ), convex in ξ for any $z \in I\!R$, and $\phi \in C^1(I\!R^N)$ has compact support, then we have that $\mathcal{R}(\phi g, T)$ is lower semi-continuous in $TBV^+(I\!R^N)$ with respect to $L^1(I\!R^N)$-convergence [18]. We shall not need this here, but this property is used to prove the existence part of Theorem 1.

For $u \in TBV^+(I\!R^N) \cap L^\infty(I\!R^N)$ and $T \in \mathcal{T}^+ \cup \mathcal{T}^-$, we define the Radon measure $g(u, DT(u))$ in $I\!R^N$ by

$$\langle g(u, DT(u)), \phi\rangle := \mathcal{R}(\phi g, T)(u) \qquad \forall \phi \in C_c(I\!R^N). \tag{32}$$

Let $u \in TBV^+(I\!R^N) \cap L^\infty(I\!R^N)$, $S \in \mathcal{P}^+$ and $T \in \mathcal{T}^+ \cup \mathcal{T}^-$. We denote by $h_S(u, DT(u))$, the Radon measure defined by (32) with $g(z, \xi) := S(z) h(z, \xi)$. If $-S \in \mathcal{P}^+$ and $T \in \mathcal{T}^+ \cup \mathcal{T}^-$, by definition we set $h_S(u, DT(u)) := -h_{(-S)}(u, DT(u))$.

If $h(z, 0) = 0$ for all $z \in I\!R$, and $S, T \in \mathcal{T}^+ \cup \mathcal{T}^-$ with $T = T_{a,b} - c$, we have

$$h_S(u, DT(u)) = h_S(T_{a,b}(u), DT(u)) = h_S(T_{a,b}(u), DT_{a,b}(u)). \tag{33}$$

Moreover, if $h^0(z, \xi) = \varphi(z)\psi^0(\xi)$, with φ being Lipschitz continuous and ψ^0 an homogeneous function of degree 1, then, by (12), we have

$$(h_S(u, DT(u)))^s = (h_S(u, DT_{a,b}(u)))^s \\ = \psi^0\left(\frac{DT_{a,b}(u)}{|DT_{a,b}(u)|}\right) |D^s J_{S\varphi}(T_{a,b}(u))| \quad \text{if} \quad S \in \mathcal{T}^+\,, \tag{34}$$

and

$$
\begin{aligned}
&(h_S(u, DT(u)))^s = (h_S(u, DT_{a,b}(u)))^s \\
&= -\psi^0\left(\tfrac{DT_{a,b}(u)}{|DT_{a,b}(u)|}\right)|D^s J_{(-S)\varphi}(T_{a,b}(u))| \quad \text{ if } \; S \in \mathcal{T}^-.
\end{aligned} \tag{35}
$$

By the representation formulas in Subsection 2.2, the absolutely continuous part of $h_S(u, DT(u))$ is $S(u)h(u, \nabla T(u))$. Similar identities are true when $S = 1$.

3.3. The notion of entropy solution. Existence and uniqueness

The notion of entropy solution is certainly complex and requires some explanation. Those explanations will be given in Remark 3. If the reader wants to have a first heuristic explanation he can go directly to it, but it will be helpful to have the notation introduced here.

To make precise our notion of solution we need to recall several definitions given in [2] (see also [3]). We define the space

$$Z(I\!R^N) := \left\{(\mathbf{z}, \xi) \in L^\infty(I\!R^N, I\!R^N) \times BV(I\!R^N)^* \; : \; \operatorname{div}(\mathbf{z}) = \xi \;\text{ in }\; \mathcal{D}'(I\!R^N)\right\}.$$

We need to consider the space $BV(I\!R^N)_2$, defined as $BV(I\!R^N) \cap L^2(I\!R^N)$ endowed with the norm

$$\|w\|_{BV(I\!R^N)_2} := \|w\|_{L^2(I\!R^N)} + |Dw|(I\!R^N).$$

It is easy to see that $L^2(I\!R^N) \subset BV(I\!R^N)_2^*$ and

$$\|w\|_{BV(I\!R^N)_2^*} \leq \|w\|_{L^2(I\!R^N)} \qquad \forall \, w \in L^2(I\!R^N). \tag{36}$$

It is well known (see [24]) that the dual space $\left(L^1(0, T; BV(I\!R^N)_2)\right)^*$ is isometric to the space of all weakly* measurable functions $f : [0, T] \to BV(I\!R^N)_2^*$, such that $v(f) \in L^\infty([0, T])$, where $v(f)$ denotes the supremum of the set $\{|\langle w, f\rangle| \; : \; \|w\|_{BV(I\!R^N)_2} \leq 1\}$ in the vector lattice of measurable real functions. Moreover, the duality pair is

$$\langle w, f\rangle = \int_0^T \langle w(t), f(t)\rangle \, dt,$$

for $w \in L^1(0, T; BV(I\!R^N)_2)$ and $f \in \left(L^1(0, T; BV(I\!R^N)_2)\right)^*$.

By $L^1_w(0, T; BV(I\!R^N))$ we denote the space of weakly measurable functions $w : [0, T] \to BV(I\!R^N)$ (i.e., $t \in [0, T] \to \langle w(t), \phi\rangle$ is measurable for every $\phi \in BV(I\!R^N)^*$) such that $\int_0^T \|w(t)\| \, dt < \infty$. Observe that, since $BV(I\!R^N)$ has a separable predual (see [1]), it follows easily that the map $t \in [0, T] \to \|w(t)\|$ is measurable. By $L^1_{loc,w}(0, T; BV(I\!R^N))$ we denote the space of weakly measurable functions $w : [0, T] \to BV(I\!R^N)$ such that the map $t \in [0, T] \to \|w(t)\|$ is in $L^1_{loc}(]0, T[)$.

Let us recall the following definitions given in [2].

Definition 1. Let $\Psi \in L^1(0,T;BV(I\!R^N))$. We say Ψ admits a *weak derivative* in the space $L^1_w(0,T;BV(I\!R^N)) \cap L^\infty(Q_T)$ if there is a function $\Theta \in L^1_w(0,T;BV(I\!R^N)) \cap L^\infty(Q_T)$ such that $\Psi(t) = \int_0^t \Theta(s)ds$, the integral being taken as a Pettis integral ([19]).

Definition 2. Let $\xi \in \left(L^1(0,T;BV(I\!R^N)_2)\right)^*$. We say that ξ is *the time derivative* in the space $\left(L^1(0,T;BV(I\!R^N)_2)\right)^*$ of a function $u \in L^1((0,T)\times I\!R^N)$ if

$$\int_0^T \langle \xi(t), \Psi(t)\rangle dt = -\int_0^T \int_{I\!R^N} u(t,x)\Theta(t,x)dxdt$$

for all test functions $\Psi \in L^1(0,T;BV(I\!R^N))$ with compact support in time, which admit a weak derivative $\Theta \in L^1_w(0,T;BV(I\!R^N)) \cap L^\infty(Q_T)$.

Note that if $w \in L^1(0,T;BV(I\!R^N)) \cap L^\infty(Q_T)$ and $\mathbf{z} \in L^\infty(Q_T, I\!R^N)$ is such that there exists $\xi \in \left(L^1(0,T;BV(I\!R^N))\right)^*$ with $\operatorname{div}(\mathbf{z}) = \xi$ in $\mathcal{D}'(Q_T)$, we can define, associated to the pair $(\mathbf{z},\xi)$, the distribution $(\mathbf{z}, Dw)$ in Q_T by

$$\begin{aligned} \langle (\mathbf{z}, Dw), \phi\rangle := -\int_0^T \langle \xi(t), w(t)\phi(t)\rangle\, dt \\ -\int_0^T \int_{I\!R^N} \mathbf{z}(t,x) w(t,x) \nabla_x \phi(t,x)\, dxdt. \end{aligned} \tag{37}$$

for all $\phi \in \mathcal{D}(Q_T)$.

Definition 3. Let $\xi \in \left(L^1(0,T;BV(I\!R^N)_2)\right)^*$ and $\mathbf{z} \in L^\infty(Q_T, I\!R^N)$. We say that $\xi = \operatorname{div}(\mathbf{z})$ in $\left(L^1(0,T;BV(I\!R^N)_2)\right)^*$ if $(\mathbf{z}, Dw)$ is a Radon measure in Q_T such that

$$\int_{Q_T} (\mathbf{z}, Dw) + \int_0^T \langle \xi(t), w(t)\rangle dt = 0,$$

for all $w \in L^1(0,T;BV(I\!R^N)) \cap L^\infty(Q_T)$.

Our concept of solution for problem (17) is the following one.

Definition 4. A measurable function $u : (0,T)\times I\!R^N \to I\!R$ is an *entropy solution* of (17) in $Q_T = (0,T)\times I\!R^N$ if $u \in C([0,T]; L^1(I\!R^N))$, $T_{a,b}(u(\cdot)) \in L^1_{loc,w}(0,T;BV(I\!R^N))$ for all $0 < a < b$, and there exists $\xi \in \left(L^1(0,T;BV(I\!R^N)_2)\right)^*$ such that

(i) $(\mathbf{a}(u(t),\nabla u(t)), \xi(t)) \in Z(I\!R^N)$ a.e. in $t \in [0,T]$,
(ii) ξ is the time derivative of u in $\left(L^1(0,T;BV(I\!R^N)_2)\right)^*$ in the sense of Definition 2,
(iii) $\xi = \operatorname{div} \mathbf{a}(u(t),\nabla u(t))$ in the sense of Definition 3, and

(iv) the following inequality is satisfied

$$\int_0^T \int_{I\!R^N} \phi h_S(u, DT(u))\, dt + \int_0^T \int_{I\!R^N} \phi h_T(u, DS(u))\, dt$$

$$\leq \int_0^T \int_{I\!R^N} J_{TS}(u(t))\phi'(t)\, dxdt$$

$$- \int_0^T \int_{I\!R^N} \mathbf{a}(u(t), \nabla u(t)) \cdot \nabla\phi\ T(u(t))S(u(t))\, dxdt$$

for truncatures $S, T \in \mathcal{T}^+$ and any nonnegative smooth function ϕ of compact support, in particular of the form $\phi(t,x) = \phi_1(t)\rho(x)$, $\phi_1 \in \mathcal{D}((0,T))$, $\rho \in \mathcal{D}(I\!R^N)$.

In [7] we give the following existence and uniqueness result.

Theorem 1. *Assume we are under assumptions (H). Then, for any initial datum* $0 \leq u_0 \in L^\infty(I\!R^N) \cap L^1(I\!R^N)$ *there exists a unique entropy solution* u *of (17) in* $Q_T = (0,T) \times I\!R^N$ *for every* $T > 0$ *such that* $u(0) = u_0$. *Moreover, if* $u(t)$, $\overline{u}(t)$ *are the entropy solutions corresponding to initial data* u_0, $\overline{u}_0 \in \left(L^\infty(I\!R^N) \cap L^1(I\!R^N)\right)^+$, *respectively, then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{for all} \quad t \geq 0. \tag{38}$$

Remark 2. If $u(t)$ is the entropy solution corresponding to the initial datum $u_0 \in \left(L^\infty(I\!R^N) \cap L^1(I\!R^N)\right)^+$, then ([7])

$$\int_{I\!R^N} j(u(t))\, dx \leq \int_{I\!R^N} j(u_0)\, dx \tag{39}$$

for any convex function $j : I\!R \to [0,\infty)$. This implies that $u(t) \in L^p(I\!R^N)$ for any $p \in [1,\infty)$ if $u_0 \in \left(L^\infty(I\!R^N) \cap L^1(I\!R^N)\right)^+$. Moreover, since entropy solutions coincide with semigroup solutions – obtained using Crandall-Liggett's discretization scheme – for which the conservation of the mass is immediate to prove by integrating the resolvent equations ([7]), we have

$$\int_{I\!R^N} u(t,x)\, dx = \int_{I\!R^N} u_0(x)\, dx.$$

In order to prove that the Boltzmann entropy is a decreasing function of time, i.e.,

$$\frac{d}{dt} \int_{I\!R^N} u(t)(\log u(t) - 1)\, dx \leq 0, \tag{40}$$

we need to observe that (39) also holds for any convex function $j : [0,\infty) \to I\!R$ continuous at $r = 0$. Let us sketch the proof of it. Indeed, the entropy solution $u(t)$ is obtained (see [7]) as limit as $K \to \infty$ of $u^K(t) := u^0\chi_{[0,t_1]} + \sum_{n=1}^K u^n \chi_{(t_n,t_{n+1}]}$ where $u^0 = u_0$, $t_n = n\Delta t$, $\Delta t = T/K$, and u^{n+1} are the solutions of

$$u^{n+1} - \Delta t\, \mathrm{div}\, \mathbf{a}(u^{n+1}, Du^{n+1}) = u^n. \tag{41}$$

Let us first assume j is such that $j'' \in \mathcal{P}$ and $p(r) := j'(r)$ is constant for $r \in [0,a]$ for some $a > 0$. Now we multiply (41) by $p(u^{n+1})$ and integrate by parts to obtain

$$\int_{I\!R^N} (u^{n+1} - u^n) p(u^{n+1})\, dx = -\Delta t \int_{I\!R^N} (\mathbf{a}(u^{n+1}, Du^{n+1}), Dp(u^{n+1}))$$
$$= -\Delta t \int_{I\!R^N} (\mathbf{a}(u^{n+1}, Du^{n+1}), DJ_{T'S}(u^{n+1})).$$

Now we consider $T(r) = T_{a,b}(r)$ with $0 < a < b$, and $b \geq \|u_0\|_\infty$, $S(r) = p'(r) = j''(r) \in \mathcal{P}^+$. By the definition of entropy solution of (41) (see [6]) we have

$$(\mathbf{a}(u^{n+1}, Du^{n+1}), DJ_{T'S}(u^{n+1})) \geq h_S(u^{n+1}, DT(u^{n+1})) \geq 0.$$

Hence, we obtain

$$\int_{I\!R^N} (u^{n+1} - u^n) p(u^{n+1})\, dx \leq 0. \tag{42}$$

Since $j(r)$ is convex, we deduce that $j(u^{n+1}) - j(u^n) \leq (u^{n+1} - u^n) p(u^{n+1})$ a.e., hence, from (42) we have

$$\int_{I\!R^N} (j(u^{n+1}) - j(u^n))\, dx \leq 0. \tag{43}$$

The case of a general convex function j follows by approximation. Letting $K \to \infty$ we obtain that (39) holds for any convex function $j : [0,\infty) \to I\!R$ continuous at $r = 0$.

Remark 3. To explain the notion of entropy solution, let us collect several observations:

a) As it is well known, equations of type (17) where $\mathbf{a}(u, Du)$ has a explicit dependence of u generate a contraction semigroup in L^1 and have estimates like (39). The contractivity in L^1 guarantees uniqueness and is usually proved using Kruzkov's doubling variables technique. This has been the approach followed in [14] and in many other papers. One can formally get the estimate (39) multiplying (17) by test functions $T(u)$, where $T = T^l_{a,b}$, and integrating by parts since $\mathbf{a}(u, Du) \cdot DT(u) \geq 0$. This estimate implies that $u(t) \in L^p(I\!R^N)$ for any $p \in [1,\infty)$ if $u_0 \in \left(L^\infty(I\!R^N) \cap L^1(I\!R^N)\right)^+$. Finally, the linear growth of $f(u, Du)$ in $|Du|$ permits to obtain a BV estimate on u. To fix ideas, let us concentrate on the relativistic heat equation (3). In this case, h satisfies (25) with $C_0(z) = c|z|$, $D_1(z) = \frac{c^2}{\nu}|z|^2$ and formal computations which involve integration by parts give

$$\frac{d}{dt}\int_{I\!R^N} j(u(t))\, dx + c\int_{I\!R^N} T'(u)|u||Du| \leq \frac{c^2}{\nu}\int_{I\!R^N} |u|^2\, dx.$$

In particular we obtain that $\int_0^t \int_{\mathbb{R}^N} |D \max(a,u)| < \infty$ for any $t > 0$, and any $a > 0$. These are all natural estimates for (3) and, more generally, for (17).

b) The notion of entropy solution is partly justified by the results of this paper. As we shall prove in Theorem 4 there are solutions which are discontinuous on a front which moves at the speed of light. In that case u_t is not a function and the best regularity we can expect is that u_t is a Radon measure. Indeed, it can be proved that this is indeed the case for certain radially symmetric initial conditions ([8]), but we do not know if this is true in general.

c) Admitting that we were able to prove that u_t is a Radon measure, we would obtain that $\operatorname{div} \mathbf{a}(u, Du)$ is a Radon measure. In the formal computations in a) we required the use of test functions of the form $T(u)$ for some Lipschitz function T. Observe that $T(u)$ is at most in $BV(\mathbb{R}^N)$, hence we need that the Radon measure $\operatorname{div} \mathbf{a}(u, Du)$ can be integrated against BV functions. Those Radon measures can be characterized as being absolutely continuous with respect to $\mathcal{H}^{N-1}$ (see [25]). Again, we know that this is the case for some radial initial conditions ([8]) but nothing seems to be known in general. To be able to circumvent this difficulty we observe that, being the divergence of a bounded measurable vector field, the expression $\operatorname{div} \mathbf{a}(u, Du)$ defines an element of $BV(\mathbb{R}^N)^*$, i.e, the dual of $BV(\mathbb{R}^N)$, and we can use test functions in $BV(\mathbb{R}^N)$. To be more precise, the time dependence has to be included and we have that $\operatorname{div} \mathbf{a}(u, Du) \in \left(L^1(0,T; BV(\mathbb{R}^N)_2)\right)^*$ and we can use test functions in $L^1(0,T; BV(\mathbb{R}^N)_2)$. To integrate by parts we have to extend Anzellotti's integration by parts formula to the time dependent case. This is what we did in [2] for the Dirichlet problem for the minimizing total variation flow (see also [3]).

d) Since $u_t = \operatorname{div} \mathbf{a}(u, Du) \in \left(L^1(0,T; BV(\mathbb{R}^N)_2)\right)^*$, the formal computations of a) require that we are able to integrate by parts with respect to time when the test functions are in $L^1(0,T; BV(\mathbb{R}^N)_2)$.

e) Remarks a), c), and d) explain the requirements in the definition of entropy solution, in particular, (i), (ii), (iii). Condition (iv) is a formulation of Kruzkov's entropy inequalities for elliptic PDEs of type (17) and permits to adapt Kruzkov's technique of doubling variables to prove uniqueness of entropy solutions and the contractivity estimate (38) (see [6,7]).

f) As we mentioned in a) and e) we need a family of Kruzkov's inequalities to be able to prove uniqueness of entropy solutions. They are derived by multiplying (17) by test functions $T(u)S(u)\phi$ where T, S are truncature functions in $\mathcal{T}^+$ and ϕ is a smooth test function of compact support in $\mathbb{R}^N$. These computations require the use of the functional calculus for BV functions summarized in Section 3.1. To give some hint about it is convenient to consider the elliptic equation

$$u - \lambda \operatorname{div} \mathbf{a}(u, Du) = f \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+. \tag{44}$$

Following the approach in [6,7] we studied first (44) and proved existence and uniqueness of entropy solutions for it. Since this is the resolvent equation for the Cauchy problem (17), Crandall-Liggett's iteration scheme permitted to prove existence of entropy solutions for (17). To derive the entropy inequalities for (44) we multiply it by $T(u)S(u)\phi$ and integrate by parts in $I\!R^N$. We have to give sense to expressions of the form

$$\mathbf{a}(u,Du)\cdot D(S(u)T(u)) = S(u)\mathbf{a}(u,Du)\cdot DT(u) + T(u)\mathbf{a}(u,Du)\cdot DS(u).$$

This is possible if we observe that

$$S(u)\mathbf{a}(u,Du)\cdot DT(u) = \mathbf{a}(u,Du)\cdot DJ_{T'S}(u)$$

(and the similar identity with S and T interchanged) and use Anzellotti's results [9] to give sense to the pairings between gradients of BV functions and bounded measurable vector fields with divergence in $L^\infty(I\!R^N)$. Finally, we observe that formally we have $\mathbf{a}(u,Du)\cdot DJ_{T'S}(u) = h_S(u,DT(u))$. When it comes to a rigorous proof, we have been able to prove only that $\mathbf{a}(u,Du)\cdot DJ_{T'S}(u) \geq h_S(u,DT(u))$, but this is sufficient to derive Kruzkov's inequalities and prove uniqueness of entropy solutions with Kruzkov's technique [6, 7]. The respective roles of S and T in the proof can be seen in those references.

4. Sub and super-solutions. Comparison principles

Definition 5. Given $0 \leq u_0 \in L^\infty(I\!R^N)\cap L^1(I\!R^N)$, we say that a measurable function $u:(0,T)\times I\!R^N \to I\!R$ is an *entropy super-solution* (respectively, *entropy sub-solution*) of the Cauchy problem (17) in $Q_T=(0,T)\times I\!R^N$ if $u\in C([0,T];L^1(I\!R^N))$, $u(0)\geq u_0$ (resp. $u(0)\leq u_0$), $T_{a,b}(u(\cdot))\in L^1_{loc,w}(0,T,BV(I\!R^N))$ for all $0<a<b$, $\mathbf{a}(u(\cdot),\nabla u(\cdot))\in L^\infty(Q_T)$, and the following inequality is satisfied:

$$\begin{aligned}&\int_{Q_T} h_S(u,DT(u))\phi + \int_{Q_T} h_T(u,DS(u))\phi \qquad (45)\\ &\leq \int_{Q_T} J_{TS}(u)\phi' - \int_0^T\int_{I\!R^N} \mathbf{a}(u(t),\nabla u(t))\cdot\nabla\phi T(u(t))S(u(t))dxdt,\end{aligned}$$

(resp. with $\geq$ sign instead of $\leq$) for any $\phi\in\mathcal{D}((0,T)\times I\!R^N)$, $\phi\geq 0$, and any $T\in\mathcal{T}^+$, $S\in\mathcal{T}^-$.

Note that taking $T(r)=1$ and $S(r)=-1$, for all $r\in I\!R$, from (45), we get

$$\frac{\partial u}{\partial t} \geq \operatorname{div}\mathbf{a}(u(\cdot),\nabla u(\cdot)) \quad \text{in } \mathcal{D}'(Q_T). \qquad (46)$$

We can not use these truncation functions directly, instead we can use $T=T_{\frac{1}{n},\frac{2}{n}}+1$ and $S=T_{\frac{1}{n},\frac{2}{n}}-1$, and so obtain (46) by a limit process.

We have the following comparison principle between entropy super-solutions and entropy solutions.

Theorem 2. *Assume that there is some constant $C > 0$ such that the function $M(z)$ in (24) satisfies $M(z) \le Cz$ for $z \ge 0$ small enough. Assume that u is an entropy solution of (17) corresponding to initial datum $u_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$, and $\overline{u}$ is an entropy super-solution of (17) corresponding to initial datum $\overline{u}_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$ such that $\overline{u}(t) \in BV(\mathbb{R}^N)$ for almost all $0 < t < T$. Then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \le \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{for all } \ t \ge 0. \tag{47}$$

Proof. Let $b > a > 2\epsilon > 0$, $T(r) := T_{a,b}(r) - a$. Let us denote

$$R_{\epsilon,l}(r) := T_\epsilon(r-l)^+ = T_{l,l+\epsilon}(r) - l \quad \text{if} \quad l > 0,$$

$$S_{\epsilon,l}(r) := T_\epsilon(r-l)^- = -T_\epsilon(l-r)^+ = T_{l-\epsilon,l}(r) - l \quad \text{if} \quad l > \epsilon.$$

Observe that $T, R_{\epsilon,l} \in \mathcal{T}^+$, $S_{\epsilon,l} \in \mathcal{T}^-$. Let us denote

$$J^+_{T,\epsilon,l}(r) = \int_l^r T(s) T_\epsilon(s-l)^+ \, ds,$$

$$J^-_{T,\epsilon,l}(r) = \int_l^r T(s) T_\epsilon(s-l)^- \, ds = -\int_l^r T(s) T_\epsilon(l-s)^+ \, ds.$$

Since u is an entropy solution of (17) and $\overline{u}$ is an entropy super-solution of (17), if $\mathbf{z}(t) := \mathbf{a}(u(t), \nabla u(t))$, $\overline{z}(t) := \mathbf{a}(\overline{u}(t), \nabla \overline{u}(t))$, and $l_1, l_2 > \epsilon$, we have

$$\int_0^T \int_{\mathbb{R}^N} \eta(t)(h_T(u(t), DR_{\epsilon,l_1}(u(t))) + h_{R_{\epsilon,l_1}}(u(t), DT(u(t))))$$
$$- \int_0^T \int_{\mathbb{R}^N} J^+_{T,\epsilon,l_1}(u(t))\eta_t + \int_0^T \int_{\mathbb{R}^N} \mathbf{z}(t) \cdot \nabla \eta(t) \; T(u(t)) R_{\epsilon,l_1}(u(t)) \le 0, \tag{48}$$

and

$$\int_0^T \int_{\mathbb{R}^N} \eta(t)(h_T(\overline{u}(t), DS_{\epsilon,l_2}(\overline{u}(t))) + h_{S_{\epsilon,l_2}}(\overline{u}(t), DT(\overline{u}(t))))$$
$$- \int_0^T \int_{\mathbb{R}^N} J^-_{T,\epsilon,l_2}(\overline{u}(t))\eta_t + \int_0^T \int_{\mathbb{R}^N} \overline{\mathbf{z}}(t) \cdot \nabla \eta(t) \; T(\overline{u}(t)) S_{\epsilon,l_2}(\overline{u}(t)) \le 0, \tag{49}$$

for all $\eta \in C^\infty(Q_T)$, with $\eta \ge 0$, $\eta(t,x) = \phi(t)\rho(x)$, being $\phi \in \mathcal{D}((0,T))$, $\rho \in \mathcal{D}(\mathbb{R}^N)$.

We choose two different pairs of variables (t,x), (s,y), and consider u, $\mathbf{z}$ as functions of (t,x), and $\overline{u}$, $\overline{\mathbf{z}}$ as functions of (s,y). Let $0 \le \phi \in \mathcal{D}(]0,T[)$, ρ_m be a classical sequence of mollifiers in $\mathbb{R}^N$ and $\tilde{\rho}_n$ a sequence of mollifiers in $\mathbb{R}$. We define

$$\eta_{m,n}(t,x,s,y) := \rho_m(x-y)\tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right).$$

For (s,y) fixed, if we take $l_1 = \overline{u}(s,y)$ in (48), we have

$$\begin{aligned} &-\int_0^T \int_{I\!R^N} J^+_{T,\epsilon,\overline{u}(s,y)}(u(t,x))(\eta_{m,n})_t \; dxdt \\ &+\int_0^T \int_{I\!R^N} \eta_{m,n}(h_T(u(t,x), D_x R_{\epsilon,\overline{u}(s,y)}(u(t,x))) + h_{R_{\epsilon,\overline{u}(s,y)}}(u(t), D_x T(u(t)))) \; dt \\ &+\int_0^T \int_{I\!R^N} \mathbf{z}(t,x) \cdot \nabla_x \eta_{m,n} \; T(u(t,x)) \; R_{\epsilon,\overline{u}(s,y)}(u(t,x)) \; dxdt \le 0. \end{aligned} \tag{50}$$

Similarly, for (t,x) fixed, if we take $l_2 = u(t,x)$ in (49), we have

$$\begin{aligned} &-\int_0^T \int_{I\!R^N} J^-_{T,\epsilon,u(t,x)}(\overline{u}(s,y))(\eta_{m,n})_s \; dyds \\ &+\int_0^T \int_{I\!R^N} \eta_{m,n}(h_T(\overline{u}(s,y), D_y S_{\epsilon,u(t,x)}(\overline{u}(s,y))) + h_{S_{\epsilon,u(t,x)}}(\overline{u}(s), D_y T(\overline{u}(s)))) \, ds \\ &+\int_0^T \int_{I\!R^N} \overline{\mathbf{z}}(s,y) \cdot \nabla_y \eta_{m,n} \; T(\overline{u}(s,y)) \; S_{\epsilon,u(t,x)}(\overline{u}(s,y)) \, dyds \le 0. \end{aligned} \tag{51}$$

Observe that, since $a > 2\epsilon$, if $\overline{u}(s,y) \le \epsilon$ or $u(t,x) \le \epsilon$, then the integrals in (50) and (51)are zero.

Integrating (50) in (s,y), (51) in (t,x), adding both inequalities and taking into account that $\nabla_x \eta_{m,n} + \nabla_y \eta_{m,n} = 0$, we have

$$\begin{aligned} &-\int_{Q_T \times Q_T} \left(J^+_{T,\epsilon,\overline{u}(s,y)}(u(t,x))(\eta_{m,n})_t + J^-_{T,\epsilon,u(t,x)}(\overline{u}(s,y))(\eta_{m,n})_s \right) \\ &\quad + \int_{Q_T \times Q_T} \eta_{m,n} h_T(u(t,x), D_x R_{\epsilon,\overline{u}(s,y)}(u(t,x)) \\ &\quad + \int_{Q_T \times Q_T} \eta_{m,n} h_T(\overline{u}(s,y), D_y S_{\epsilon,u(t,x)}(\overline{u}(s,y))) \\ &\quad + \int_{Q_T \times Q_T} \eta_{m,n} h_{R_{\epsilon,\overline{u}(s,y)}}(u(t), D_x T(u(t))) \\ &\quad + \int_{Q_T \times Q_T} \eta_{m,n} h_{S_{\epsilon,u(t,x)}}(\overline{u}(s), D_y T(\overline{u}(s))) \\ &\quad - \int_{Q_T \times Q_T} \overline{\mathbf{z}}(s,y) \cdot \nabla_x \eta_{m,n} T(\overline{u}(s,y)) S_{\epsilon,u(t,x)}(\overline{u}(s,y)) \\ &\quad - \int_{Q_T \times Q_T} \mathbf{z}(t,x) \cdot \nabla_y \eta_{m,n} T(u(t,x)) R_{\epsilon,\overline{u}(s,y)}(u(t,x)) \le 0. \end{aligned}$$

Then, since

$$\int_{Q_T\times Q_T}\eta_{m,n}h_{R_{\epsilon,\overline{u}(s,y)}}(u(t),D_xT(u(t)))\geq 0,$$

we get

$$\begin{aligned}
&-\int_{Q_T\times Q_T}\left(J^+_{T,\epsilon,\overline{u}(s,y)}(u(t,x))(\eta_{m,n})_t+J^-_{T,\epsilon,u(t,x)}(\overline{u}(s,y))(\eta_{m,n})_s\right)\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_T(u(t,x),D_xR_{\epsilon,\overline{u}(s,y)}(u(t,x))\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_T(\overline{u}(s,y),D_yS_{\epsilon,u(t,x)}(\overline{u}(s,y)))\\
&-\int_{Q_T\times Q_T}\overline{\mathbf{z}}(s,y)\cdot\nabla_x\eta_{m,n}T(\overline{u}(s,y))S_{\epsilon,u(t,x)}(\overline{u}(s,y))\\
&-\int_{Q_T\times Q_T}\mathbf{z}(t,x)\cdot\nabla_y\eta_{m,n}T(u(t,x))R_{\epsilon,\overline{u}(s,y)}(u(t,x))\\
&\leq-\int_{Q_T}\eta_{m,n}h_{S_{\epsilon,u(t,x)}}(\overline{u}(s),D_yT(\overline{u}(s))).
\end{aligned}\tag{52}$$

Let I_1, I_2 be, respectively, the first term and the sum of the rest of terms at the left hand side of the above inequality. Arguing as in the proof of uniqueness in [6] (see also [7]) we prove that

$$\frac{1}{\epsilon}I_2\geq\|\phi\|_\infty o(\epsilon)$$

where $o(\epsilon)$ denotes an expression converging to 0 as $\epsilon\to 0+$. Thus, by (52), it follows that

$$\begin{aligned}
&-\frac{1}{\epsilon}\int_{Q_T\times Q_T}\left(J^+_{T,\epsilon,\overline{u}(s,y)}(u)(\eta_{m,n})_t+J^-_{T,\epsilon,u(t,x)}(\overline{u})(\eta_{m,n})_s\right)\\
&\leq\|\phi\|_\infty o(\epsilon)-\frac{1}{\epsilon}\int_{Q_T\times Q_T}\eta_{m,n}h_{S_{\epsilon,u(t,x)}}(\overline{u}(s),D_yT(\overline{u}(s)))\\
&\leq\|\phi\|_\infty o(\epsilon)+\int_{Q_T\times Q_T}\eta_{m,n}h(\overline{u}(s),D_yT(\overline{u}(s))).
\end{aligned}\tag{53}$$

Therefore, letting $\epsilon\to 0$ in (53) we obtain

$$\begin{aligned}
&-\int_{Q_T\times Q_T}\left(J_{T,sign^+,\overline{u}(s,y)}(u)(\eta_{m,n})_t+J_{T,sign^-,u(t,x)}(\overline{u})(\eta_{m,n})_s\right)\\
&\leq\int_{Q_T\times Q_T}\eta_{m,n}h(\overline{u}(s),D_yT(\overline{u}(s))),
\end{aligned}\tag{54}$$

where

$$J_{T,sign^{\pm},l}(r) = \int_l^r T(s)\mathrm{sign}_0^{\pm}(s-l)ds \quad l \in I\!R,\, r \geq 0.$$

Now, letting $m \to \infty$ we obtain

$$-\int_{(0,T)\times(0,T)\times I\!R^N} \left(J_{T,sign^+,\overline{u}(s,x)}(u(t,x))(\chi_n)_t + J_{T,sign^-,u(t,x)}(\overline{u}(s,x))(\chi_n)_s\right)$$

$$\leq \int_{(0,T)\times(0,T)\times I\!R^N} \chi_n h(\overline{u}(s), D_y T(\overline{u}(s))), \tag{55}$$

where

$$\chi_n = \tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right).$$

Letting $a \to 0^+$ in (55) we obtain

$$\begin{aligned} &-\int_{(0,T)\times(0,T)\times I\!R^N} J_{T_{0,b},sign^+,\overline{u}(s,x)}(u(t,x))(\chi_n)_t \\ &-\int_{(0,T)\times(0,T)\times I\!R^N} J_{T_{0,b},sign^-,u(t,x)}(\overline{u}(s,x))(\chi_n)_s \\ &\leq \int_{(0,T)\times(0,T)\times I\!R^N} \chi_n h(\overline{u}(s), D_y T_{0,b}(\overline{u}(s))). \end{aligned} \tag{56}$$

Now, using (24), our assumption on $M(z)$ and the coarea formula we have

$$\int_{I\!R^N} h(\overline{u}(s), D_y T_{0,b}(\overline{u}(s))) \leq Cb \int_0^b P([\overline{u} \geq \lambda])\, d\lambda,$$

where $P(X)$ denotes the perimeter of X for any rectifiable subset $X \subseteq I\!R^N$. Since $P([\overline{u} \geq \lambda])$ is integrable as a function of λ, we deduce

$$\lim_{b\to 0^+} \frac{1}{b} \int_{(0,T)\times(0,T)\times I\!R^N} \chi_n h(\overline{u}(s), D_y T_{0,b}(\overline{u}(s))) = 0.$$

Hence, dividing (56) by b and letting $b \to 0^+$, we obtain

$$-\int_{(0,T)\times(0,T)\times I\!R^N} (u(t,x) - \overline{u}(s,x))^+ \left((\chi_n)_t + (\chi_n)_s\right) \leq 0. \tag{57}$$

Since

$$(\chi_n)_t + (\chi_n)_s = \tilde{\rho}_n(t-s)\phi'\left(\frac{t+s}{2}\right),$$

we may write (57) as

$$-\int_{(0,T)\times(0,T)\times I\!R^N} (u(t,x) - \overline{u}(s,x))^+ \, \tilde{\rho}_n(t-s)\phi'\left(\frac{t+s}{2}\right) \leq 0. \tag{58}$$

Now, letting $n \to \infty$, we obtain

$$-\int_{(0,T)\times \mathbb{R}^N} (u(t,x) - \overline{u}(t,x))^+ \, \phi'(t) \, dt \, dx \leq 0. \tag{59}$$

Since this is true for all $0 \leq \phi \in \mathcal{D}(]0,T[)$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^N} (u(t,x) - \overline{u}(t,x))^+ \, dx \leq 0.$$

Hence

$$\int_{\mathbb{R}^N} (u(t,x) - \overline{u}(t,x))^+ \, dx \leq \int_{\mathbb{R}^N} (u_0(x) - \overline{u}_0(x))^+ \, dx \qquad \text{for all} \quad t \geq 0.$$

□

Working as in the proof of Theorem 2, we can prove the following comparison principle between sub-solutions and entropy solutions of the Cauchy problem (17).

Theorem 3. *Assume that there is some $C > 0$ such that the function $M(z)$ in (24) satisfies $M(z) \leq Cz$ for $z \geq 0$ small enough. Assume that u is an entropy solution of (17) corresponding to initial datum $u_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$, and $\overline{u}$ is an entropy sub-solution of (17) corresponding to initial datum $\overline{u}_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$ such that $\overline{u}(t) \in BV(\mathbb{R}^N)$ for almost all $0 < t < T$. Then*

$$\|(\overline{u}(t) - u(t))^+\|_1 \leq \|(\overline{u}_0 - u_0)^+\|_1 \qquad \text{for all} \quad t \geq 0. \tag{60}$$

5. The evolution of the support of the solutions of the relativistic heat equation

To study the evolution of the support of entropy solutions of the relativistic heat equation, we need to compute some explicit entropy super and sub-solutions.

5.1. Some entropy super-solutions of the relativistic heat equation

Proposition 1. *Let $C \subset \mathbb{R}^N$ a compact set, $0 < \alpha \leq \beta$. For $s > 0$, let $C(s) := \{x \in \mathbb{R}^N : d(x,C) \leq s\}$. Then $u(t,x) := \beta \chi_{C(ct)}(x)$ is an entropy super-solution of the Cauchy problem for the relativistic heat equation (3) with $u_0 = \alpha \chi_C$ as initial datum.*

Proof. Since

$$\frac{\partial u}{\partial t} = c\beta \mathcal{H}^{N-1} \llcorner \partial C(ct) \quad \text{in} \quad \mathcal{D}'(Q_T)$$

and $\mathbf{a}(u(\cdot), \nabla u(\cdot)) \equiv 0$, we have

$$\frac{\partial u}{\partial t} \geq \operatorname{div} \mathbf{a}(u(\cdot), \nabla u(\cdot)) \quad \text{in} \;\; \mathcal{D}'(Q_T).$$

Let $\varphi(r) = cr$. Let us prove that if $T \in \mathcal{T}^+$, $S \in \mathcal{T}^-$, then

$$(h_S(u(t), DT(u(t))))^j = J_{S\varphi T'}(\beta)\mathcal{H}^{N-1} \llcorner \partial C(ct) \tag{61}$$

and

$$(h_T(u(t), DS(u(t))))^j = J_{T\varphi S'}(\beta)\mathcal{H}^{N-1} \llcorner \partial C(t). \tag{62}$$

For that, let $T = T_{a,b} + d$, where $0 < a < b$ and $d \in I\!R$. By (35), we have

$$(h_S(u(t), DT(u(t))))^j = -|D^j J_{(-S)\varphi}(T_{a,b}(u(t)))|. \tag{63}$$

Now, applying chain's rule in BV, we have

$$\begin{aligned}
&D^j J_{(-S)\varphi}(T_{a,b}(u(t))) = \\
&\frac{J_{(-S)\varphi}(T_{a,b}(u(t))_+) - J_{(-S)\varphi}(T_{a,b}(u(t))_-)}{T_{a,b}(u(t))_+ - T_{a,b}(u(t))_-} D^j(T_{a,b}(u(t))) \\
&= \left(\frac{J_{(-S)\varphi}(T_{a,b}(u(t))_+) - J_{(-S)\varphi}(T_{a,b}(u(t))_-)}{T_{a,b}(u(t))_+ - T_{a,b}(u(t))_-} \right) \\
&\quad \times \left(\frac{T_{a,b}(u(t))_+ - T_{a,b}(u(t))_-}{u(t)_+ - u(t)_-} \right) D^j(u(t)) \\
&= \frac{J_{(-S)\varphi}(T_{a,b}(u(t))_+) - J_{(-S)\varphi}(T_{a,b}(u(t))_-)}{u(t)_+ - u(t)_-} D^j(u(t)) \\
&= \frac{J_{(-S)\varphi T'}(u(t)_+) - J_{(-S)\varphi T'}(u(t)_-)}{u(t)_+ - u(t)_-} D^j(u(t)) = D^j J_{(-S)\varphi T'}(u(t)).
\end{aligned}$$

Recall that for any Lipschitz nondecreasing function g, we have

$$Dg(u(t)) = -(g(\beta) - g(0))\nu^{C(ct)}\mathcal{H}^{N-1} \llcorner \partial C(ct).$$

Using the previous computations we obtain

$$D^j J_{(-S)\varphi}(T_{a,b}(u(t))) = J_{S\varphi T'}(\beta)\mathcal{H}^{N-1} \llcorner \partial C(ct). \tag{64}$$

Combining (63) and (64) we obtain (61). The proof of (62) is similar.

Thus, using (61) and (62), we get

$$\begin{aligned}
&h_S(u(t), DT(u(t))) + h_T(u(t), DS(u(t))) \\
&= J_{(TS)'\varphi}(\beta)\mathcal{H}^{N-1} \llcorner \partial C(ct) = (TS\varphi(\beta) - cJ_{TS}(\beta))\, \mathcal{H}^{N-1} \llcorner \partial C(ct).
\end{aligned}$$

Hence, for any $0 \leq \phi \in \mathcal{D}((0,T) \times I\!R^N)$,

$$\begin{aligned}
&\int_{Q_T} \phi(t) h_S(u(t), DT(u(t)))\, dt + \int_{Q_T} \phi(t) h_T(u(t), DS(u(t)))\, dt \\
&= \int_0^T (TS\varphi(\beta) - cJ_{TS}(\beta)) \int_{\partial C(ct)} \phi \, d\mathcal{H}^{N-1}\, dt.
\end{aligned} \tag{65}$$

On the other hand, since $J_{TS}(u(t)) = J_{TS}(\beta)\chi_{C(ct)}$, we have

$$\frac{\partial}{\partial t} J_{TS}(u(t)) = cJ_{TS}(\beta)\mathcal{H}^{N-1} \llcorner C(ct).$$

Therefore,

$$\begin{aligned} \int_{Q_T} J_{TS}(u(t))\phi'(t)\,dt &= -\int_{Q_T} \phi(t)\frac{\partial}{\partial t} J_{TS}(u(t))\,dt \\ &= -c\int_0^T J_{TS}(\beta)\int_{\partial C(ct)} \phi(t)\,d\mathcal{H}^{N-1}\,dt. \end{aligned} \tag{66}$$

Finally, observe that $(TS\varphi)(\beta) \leq 0$ and $\mathbf{a}(u(t), \nabla(u(t))) = 0$ as $\nabla u(t) \equiv 0$. Thus, using (65) and (66), we obtain

$$\int_{Q_T} \phi(t)h_S(u(t), DT(u(t)))\,dt + \int_{Q_T} \phi(t)h_T(u(t), DS(u(t)))\,dt$$

$$\leq \int_{Q_T} J_{TS}(u(t))\phi'(t)dxdt - \int_0^T \int_{I\!R^N} \mathbf{a}(u(t), \nabla u(t))\cdot\nabla\phi(t)T(u(t))S(u(t))dxdt.$$

Therefore, if $\alpha \leq \beta$, then $u(0) \geq u_0$ and u is a super-solution of (3). □

5.2. Some entropy sub-solutions of the relativistic heat equation

Proposition 2. *Given $R_0, \alpha_0 > 0$ and $\gamma_0 \geq 0$, there are values $\beta_1, \beta_2 > 0$ large enough such that*

$$u(t,x) = \begin{cases} e^{-\beta_1 t - \beta_2 t^2}\left(\alpha_0 \frac{c}{\nu}\sqrt{R(t)^2 - |x|^2} + \gamma_0\right) & \text{if } |x| < R(t) \\ 0 & \text{if } |x| \geq R(t), \end{cases}$$

where $R(t) = R_0 + ct$, is an entropy sub-solution of (3).

Proof. Observe that $v(t,x)$ is an entropy solution of (3) if and only if $u(t,x) = v(\frac{\nu}{c^2}t, \frac{\nu}{c}x)$ is an entropy solution of

$$u_t = \operatorname{div}\left(\frac{|u|Du}{\sqrt{u^2 + |Du|^2}}\right). \tag{67}$$

Thus, without loss of generality we may assume that $\nu = c = 1$. In this case,

$$\mathbf{a}(z,\xi) = \frac{|z|\ \xi}{\sqrt{z^2 + \|\xi\|^2}}, \qquad h(z,\xi) = \frac{|z|\ \|\xi\|^2}{\sqrt{z^2 + \|\xi\|^2}}$$

and

$$h^0(z,\xi) = |z|\,\|\xi\| = \varphi(z)\psi^0(\xi),$$

with

$$\varphi(z) = |z| \quad \text{and} \quad \psi^0(\xi) = \|\xi\|.$$

We denote $\alpha(t) = \alpha_0 e^{-\beta_1 t - \beta_2 t^2}$, $\gamma(t) = \gamma_0 e^{-\beta_1 t - \beta_2 t^2}$ and

$$\chi(t,x) = \alpha(t)\sqrt{R(t)^2 - |x|^2} + \gamma(t).$$

Since

$$Du(t) = -\frac{\alpha(t)x}{\sqrt{R(t)^2 - |x|^2}} \chi_{C(t)} \mathcal{L}^N + \gamma(t)\mathcal{H}^{N-1} \llcorner \partial C(t),$$

with $C(t) = B_{R(t)}(0)$, we have

$$\nabla u(t,x) = -\frac{\alpha(t)x}{\sqrt{R(t)^2 - |x|^2}} \chi_{C(t)}(x).$$

Let $\mathbf{z}$ be the vector field

$$\mathbf{z}(t,x) = \mathbf{a}(u(t,x), \nabla u(t,x))$$

$$= -\frac{\alpha(t)x[\alpha(t)\sqrt{R(t)^2 - |x|^2} + \gamma(t)]}{\sqrt{(R(t)^2 - |x|^2)[\alpha(t)\sqrt{R(t)^2 - |x|^2} + \gamma(t)]^2 + \alpha(t)^2|x|^2}} \chi_{C(t)}(x)$$

$$= -\frac{x[\alpha(t)\sqrt{R(t)^2 - |x|^2} + \gamma(t)]}{\sqrt{(R(t)^2 - |x|^2)[\sqrt{R(t)^2 - |x|^2} + \tilde{\gamma}]^2 + |x|^2}} \chi_{C(t)}(x) =: \eta(t,x)\chi_{C(t)}(x),$$

where $\tilde{\gamma} = \frac{\gamma(t)}{\alpha(t)} = \frac{\gamma_0}{\alpha_0}$ and

$$\eta(t,x) = -\frac{x[\alpha(t)\sqrt{R(t)^2 - |x|^2} + \gamma(t)]}{\sqrt{(R(t)^2 - |x|^2)[\sqrt{R(t)^2 - |x|^2} + \tilde{\gamma}]^2 + |x|^2}}.$$

To prove that u is a entropy sub-solution of (67), we have to prove that the following inequality is satisfied:

$$\int_{Q_T} h_S(u, DT(u))\phi + \int_{Q_T} h_T(u, DS(u))\phi \tag{68}$$
$$\geq \int_{Q_T} J_{TS}(u)\phi_t - \int_0^T \int_{I\!R^N} \mathbf{z}(t,x) \cdot \nabla\phi T(u(t))S(u(t))dxdt,$$

for any $\phi \in \mathcal{D}((0,T) \times I\!R^N)$, $\phi \geq 0$, and any $T \in \mathcal{T}^+$, $S \in \mathcal{T}^-$.

We divide the proof of (68) in several steps.

Step 1. Working as in the proof of Proposition 1, we get

$$(h_S(u(t), DT(u(t))))^j = J_{S\varphi T'}(\gamma(t))\mathcal{H}^{N-1} \llcorner \partial C(t) \tag{69}$$

and

$$(h_T(u(t), DS(u(t))))^j = J_{T\varphi S'}(\gamma(t))\mathcal{H}^{N-1} \llcorner \partial C(t). \tag{70}$$

Thus, by (69) and (70), we get

$$\begin{aligned} &(h_S(u(t), DT(u(t))))^j + (h_T(u(t), DS(u(t))))^j \\ &= (TS\varphi(\gamma(t)) - J_{TS}(\gamma(t))\,\mathcal{H}^{N-1} \llcorner \partial C(t). \end{aligned} \tag{71}$$

Using (71) and

$$\begin{aligned} &(h_S(u, DT(u)))^{ac} + (h_T(u, DS(u)))^{ac} \\ &= \eta(t,x) \cdot \nabla(T(\chi(t,x))S(\chi(t,x)))\chi_{C(t)}, \end{aligned}$$

it follows that

$$\begin{aligned} &\int_{Q_T} \phi h_S(u(t), DT(u(t)))\,dt + \int_{Q_T} \phi h_T(u(t), DS(u(t)))\,dt \\ &= \int_0^T (TS\varphi(\gamma(t)) - J_{TS}(\gamma(t)) \int_{\partial C(t)} \phi\, d\mathcal{H}^{N-1}\,dt \\ &+ \int_0^T \int_{C(t)} \eta(t,x) \cdot \nabla(T(\chi(t,x))S(\chi(t,x)))\phi(t)\,dxdt. \end{aligned} \tag{72}$$

Let us compute

$$\begin{aligned} &\int_{Q_T} \mathbf{z}(t,x) \cdot \nabla\phi(t)\,T(u(t))S(u(t))\,dxdt \\ &= -\int_0^T \int_{C(t)} \operatorname{div}(\mathbf{z}(t)T(u(t))S(u(t))\phi(t)\,dxdt \\ &+ \int_0^T \left(\int_{\partial C(t)} [\mathbf{z}(t)T(u(t))S(u(t)), \nu^{C(t)}]\,\phi(t)\,d\mathcal{H}^{N-1} \right) dt \\ &= -\int_0^T \int_{C(t)} \operatorname{div}(\eta(t,x))\,T(\chi(t,x))S(\chi(t,x))\phi(t)\,dxdt \\ &- \int_0^T \int_{C(t)} \eta(t,x) \cdot \nabla(T(\chi(t,x))S(\chi(t,x)))\phi(t)\,dxdt \\ &- \int_0^T \left(\int_{\partial C(t)} \gamma(t)T(\gamma(t))S(\gamma(t))\,\phi(t)\,d\mathcal{H}^{N-1} \right) dt. \end{aligned}$$

On the other hand, since $J_{TS}(u(t)) = J_{TS}(\chi(t))\chi_{C(t)}$, we have

$$\frac{\partial}{\partial t} J_{TS}(u(t)) = \frac{\partial}{\partial t} J_{TS}(\chi(t))\chi_{C(t)}\mathcal{L}^N + J_{TS}(\gamma(t))\mathcal{H}^{N-1} \llcorner C(t).$$

Hence

$$\int_{Q_T} J_{TS}(u(t))\phi_t \, dxdt = -\int_{Q_T} \phi(t) \frac{\partial}{\partial t} J_{TS}(u(t)) \, dt$$

$$= -\int_0^T \int_{C(t)} \phi(t) \frac{\partial}{\partial t} J_{TS}(\chi(t)) \, dxdt - \int_0^T J_{TS}(\gamma(t)) \int_{\partial C(t)} \phi(t) \, d\mathcal{H}^{N-1} \, dt$$

$$= -\int_0^T \int_{C(t)} \phi(t)\chi_t(t,x) T(\chi(t,x)) S(\chi(t,x)) \, dxdt$$

$$- \int_0^T J_{TS}(\gamma(t)) \int_{\partial C(t)} \phi(t) \, d\mathcal{H}^{N-1} \, dt.$$

Therefore,

$$\int_{Q_T} J_{TS}(u)\phi_t - \int_0^T \int_{\mathbb{R}^N} \mathbf{z}(t,x) \cdot \nabla\phi T(u(t)) S(u(t)) dxdt$$

$$= -\int_0^T \int_{C(t)} \phi(t)\chi_t(t,x) T(\chi(t,x)) S(\chi(t,x)) \, dxdt$$

$$- \int_0^T J_{TS}(\gamma(t)) \int_{\partial C(t)} \phi(t) \, d\mathcal{H}^{N-1} \, dt$$

$$+ \int_0^T \int_{C(t)} \operatorname{div}(\eta(t,x)) \, T(\chi(t,x)) S(\chi(t,x)) \phi(t) \, dxdt$$

$$+ \int_0^T \int_{C(t)} \eta(t,x) \cdot \nabla(T(\chi(t,x)) S(\chi(t,x))) \chi(t) \, dxdt$$

$$+ \int_0^T \left(\int_{\partial C(t)} \gamma(t) T(\gamma(t)) S(\gamma(t)) \, \phi(t) \, d\mathcal{H}^{N-1} \right) dt.$$

Then, by (72), to prove (68), it will be sufficient to prove that

$$\begin{aligned} &\int_0^T \int_{C(t)} \phi(t)\chi_t(t,x) T(\chi(t,x)) S(\chi(t,x)) \, dxdt \\ &\geq \int_0^T \int_{C(t)} \operatorname{div}(\eta(t,x)) \, T(\chi(t,x)) S(\chi(t,x)) \phi(t) \, dxdt. \end{aligned} \tag{73}$$

Step 2. We have

$$\chi_t(t,x) \leq \operatorname{div}(\eta(t,x)) \quad \text{a.e. } \ x \in C(t), \ t \in (0,T). \tag{74}$$

Since $T \in \mathcal{T}^+$ and $S \in \mathcal{T}^-$, (74) implies (73).

Working in radial coordinates, $r = |x|$, and after some tedious computations, we obtain that (74) holds if and only if for almost all $0 \le r \le R(t)$, $t \in (0,T)$,

$$\begin{aligned} &\alpha'(t)(R(t)^2 - r^2) + R(t)\alpha(t) + \gamma'(t)(R(t)^2 - r^2)^{1/2} \le \\ &-\frac{\alpha(t)N[\tilde\gamma + (R(t)^2 - r^2)^{1/2}](R(t)^2 - r^2)^{1/2}}{D(t,r)} + \frac{\alpha(t)r^2}{D(t,r)} \\ &+\frac{\alpha(t)r^2[\tilde\gamma + (R(t)^2 - r^2)^{1/2}](R(t)^2 - r^2)^{1/2}}{D(t,r)^3}B(t,r), \end{aligned} \tag{75}$$

where

$$D(t,r) = \sqrt{[\tilde\gamma + (R(t)^2 - r^2)^{1/2}]^2(R(t)^2 - r^2) + r^2}$$

and

$$B(t,r) =$$

$$\left(1 - \left[\tilde\gamma + (R(t)^2 - r^2)^{1/2}\right]^2 - (R(t)^2 - r^2)^{1/2}\left[\tilde\gamma + (R(t)^2 - r^2)^{1/2}\right]\right)$$

For notation simplicity we shall write $A = R(t)^2 - r^2$. Introducing $\alpha(t) = \alpha_0 e^{-\beta_1 t - \beta_2 t^2}$, $\gamma(t) = \gamma_0 e^{-\beta_1 t - \beta_2 t^2}$ in (75) and dividing by $\alpha(t)$, we deduce that (74) holds if and only if for almost all $0 \le r \le R(t)$, $t \in (0,T)$,

$$\begin{aligned} &-(\beta_1 + 2\beta_2 t)A + R(t) - \tilde\gamma(\beta_1 + 2\beta_2 t)A^{1/2} \le \\ &-\frac{N\left[\tilde\gamma + A^{1/2}\right]A^{1/2}}{D(t,r)} + \frac{r^2}{D(t,r)} + \frac{r^2\left[\tilde\gamma + A^{1/2}\right]A^{1/2}}{D(t,r)^3}B(t,r). \end{aligned} \tag{76}$$

We shall prove that there exist values of $\lambda > 0$ and $\beta_1, \beta_2 > 0$ such that

$$R(t) - 2\beta_2 tA - 2\tilde\gamma\beta_2 t\sqrt{A} \le \frac{r^2}{D(t,r)} + \lambda A \quad \text{and} \tag{77}$$

$$-\beta_1 A - \beta_1\tilde\gamma\sqrt{A} \le -\frac{N(\tilde\gamma + \sqrt{A})\sqrt{A}}{D(t,r)} + \frac{r^2(\tilde\gamma + \sqrt{A})\sqrt{A}}{D(t,r)^3}B - \lambda A, \tag{78}$$

in $0 \le r \le R(t)$. Then (76) follows by adding the inequalities in (77) and (78).

To prove (77) we shall prove that there exist values of $\lambda_0, \beta_2 > 0$ and $0 < \Delta_0 < \frac{R_0}{2}$ such that for any $\lambda \ge \lambda_0$ we have

$$R(t) \le \frac{r^2}{D(t,r)} + \lambda A \quad \forall r \in [0, \tfrac{R(t)}{2}] \cup [R(t) - \Delta_0, R(t)], \tag{79}$$

$$R(t) \le \lambda A + 2\beta_2 tA \quad \forall r \in [\tfrac{R(t)}{2}, R(t) - \Delta_0]. \tag{80}$$

Choosing

$$\lambda_0 \ge \frac{4}{3R_0} \tag{81}$$

we have that $\lambda A \geq \frac{3}{4}\lambda R(t)^2 \geq R(t)$ for any $\lambda \geq \lambda_0$ and any $0 \leq r \leq \frac{R(t)}{2}$. Hence, (79) holds for any $0 \leq r \leq \frac{R(t)}{2}$. To prove (79) for r near $R(t)$ observe that, after multiplying both sides by $D(t,r)$, we may write it as

$$R(t)r\left(1+\frac{1}{r^2}\left[\tilde{\gamma}+\sqrt{A}\right]^2 A\right)^{1/2} \leq r^2+\lambda Ar\left(1+\frac{1}{r^2}\left[\tilde{\gamma}+\sqrt{A}\right]^2 A\right)^{1/2}. \tag{82}$$

Since $\left(\tilde{\gamma}+\sqrt{A}\right)^2 \leq 2\tilde{\gamma}^2+2A$ the inequality (82) will be implied by the inequality

$$R(t)r\left(1+\frac{2}{r^2}\left[\tilde{\gamma}^2+A\right]A\right)^{1/2} \leq r^2+\lambda Ar\left(1+\frac{A^2}{r^2}\right)^{1/2}. \tag{83}$$

Let us write $\Delta = R(t)-r$, $X = \frac{(\tilde{\gamma}+\sqrt{A})^2 A}{r^2}$. Using that $1+\frac{X}{2}-\frac{X^2}{4} \leq \sqrt{1+X} \leq 1+\frac{X}{2}$ we observe that (83) will be in turn implied by the inequality

$$\begin{aligned} R(t)(R(t)-\Delta)+\tilde{\gamma}^2\frac{R(t)}{R(t)-\Delta}\Delta(2R(t)-\Delta)+R(t)\frac{\Delta^2(2R(t)-\Delta)^2}{(R(t)-\Delta)} \\ \leq (R(t)-\Delta)^2+\lambda\Delta(2R(t)-\Delta)(R(t)-\Delta) \\ +\lambda\Delta^3\frac{(2R(t)-\Delta)^3}{2(R(t)-\Delta)}-\frac{1}{4}\lambda\Delta^5\frac{(2R(t)-\Delta)^5}{(R(t)-\Delta)^3}. \end{aligned}$$

To simplify the above inequality we observe that the terms of order Δ^0 cancel and, after dividing by $R(t)\Delta$, we can find $\Delta_0 > 0$ such that for any $\Delta \leq \Delta_0$ we may write it as

$$\frac{1+2\tilde{\gamma}^2}{R(t)}+C'\Delta \leq 2\lambda \tag{84}$$

for some constant $C' > 0$ which depends on R_0. If we take

$$\lambda_0 \geq \frac{1+2\tilde{\gamma}^2}{2R_0}+C'\Delta_0 \tag{85}$$

then (84) holds for any $\lambda \geq \lambda_0$. We have proved that (79) holds.

To prove (80) we choose $\lambda_0 > 0$ such that

$$\lambda_0(2R_0\Delta_0-\Delta_0^2) \geq R_0 \tag{86}$$

and $\beta_2 > 0$ such that

$$2\beta_2(2R_0\Delta_0-\Delta_0^2) \geq 1. \tag{87}$$

Our choice (86) implies that $\lambda(R(t)^2-r^2) \geq R_0$. Our choice (87) implies that $2\beta_2 t(R(t)^2-r^2) \geq t$. Both things together imply that (80) holds when $\frac{R(t)}{2} \leq r \leq R(t)-\Delta_0$. This concludes the proof of (77).

Let us prove (78). After division by $\sqrt{A}$ we may write it as

$$\beta_1\sqrt{A} + \beta_1\tilde{\gamma} \geq \frac{N(\tilde{\gamma}+\sqrt{A})}{D(t,r)} + \lambda\sqrt{A} - \frac{r^2(\tilde{\gamma}+\sqrt{A})}{D(t,r)^3}B(t,r). \tag{88}$$

Now, we separate those terms which contain $\sqrt{A}$ of those which do not contain it. By doing this, we observe that (88) is implied by the following two inequalities

$$\begin{aligned}\beta_1\sqrt{A} \geq \frac{N\sqrt{A}}{D(t,r)} + \lambda\sqrt{A} - \frac{r^2\sqrt{A}}{D(t,r)^3}B(t,r) + \frac{r^2\tilde{\gamma}}{D(t,r)^3}\sqrt{A}(\tilde{\gamma}+\sqrt{A}) \\ + \frac{r^2\tilde{\gamma}}{D(t,r)^3}(A + 2\tilde{\gamma}\sqrt{A}),\end{aligned} \tag{89}$$

$$\beta_1 \geq \frac{N}{D(t,r)} + \frac{R(t)^2}{D(t,r)^3}(\tilde{\gamma}^2 - 1). \tag{90}$$

After division by $\sqrt{A}$, (89) becomes

$$\beta_1 \geq \frac{N}{D(t,r)} + \lambda - \frac{r^2}{D(t,r)^3}B(t,r) + \frac{r^2\tilde{\gamma}}{D(t,r)^3}\left[3\tilde{\gamma} + 2\sqrt{A}\right]. \tag{91}$$

Since

$$-B(t,r) + 3\tilde{\gamma}^2 + 2\tilde{\gamma}\sqrt{A} \geq 4\tilde{\gamma}^2 + 5\tilde{\gamma}\sqrt{A} + 2\tilde{\gamma}A,$$

then (91) holds if

$$\beta_1 \geq \frac{N}{D(t,r)} + \lambda + \frac{r^2}{D(t,r)^3}\left[4\tilde{\gamma}^2 + 5\tilde{\gamma}\sqrt{A} + 2\tilde{\gamma}\sqrt{A}\right]. \tag{92}$$

After observing that

$$D(t,r) \geq \sqrt{(R(t)^2 - r^2)^2 + r^2} \geq \begin{cases} \frac{R(t)^2}{2} & \text{if } \ r^2 \leq \frac{R(t)^2}{2} \\ \frac{R(t)}{\sqrt{2}} & \text{if } \ r^2 \geq \frac{R(t)^2}{2}, \end{cases} \tag{93}$$

it is easy to see that the right hand sides of (92) and (90), respectively, are bounded independent of t and r, and consequently a proper choice of β_1 permits to satisfy (92) and (90). This proves (88). This conclude the proof of (74). □

5.3. The evolution of the support

Theorem 4. *Let C be an open bounded set in $\mathbb{R}^N$. Let $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$ with support equal to $\overline{C}$. Let $u(t)$ be the entropy solution of the Cauchy problem for the relativistic heat equation (3) with u_0 as initial datum. Then*

$$\mathrm{supp}(u(t)) \subset C(ct) \qquad \text{for all } t \geq 0. \tag{94}$$

Moreover, if we assume that

(*) for any closed set $F \subseteq C$, there is a constant $\alpha_F > 0$ such that $u_0 \geq \alpha_F$ in F,

then

$$\mathrm{supp}(u(t)) = C(ct) \qquad \text{for all } \ t \geq 0.$$

Proof. Let $\overline{u}_0 := \|u_0\|_\infty \chi_C$. By Proposition 1, $\overline{u}(t,x) := \|u_0\|_\infty \chi_{C(ct)}(x)$ is an entropy super-solution of (3) with $\overline{u}_0$ as initial datum. Then, by Theorem 2, we have that $u(t) \leq \overline{u}(t)$ for all $t \geq 0$, and consequently

$$\mathrm{supp}(u(t)) \subset \mathrm{supp}(\overline{u}(t)) \subset C(ct) \qquad \text{for all } t \geq 0.$$

To prove the second assertion, assume that $(*)$ holds. As in Proposition 2 we may assume that $\nu = c = 1$. Let $y \in C$ and let $R_y := \frac{d(y,\partial C)}{2}$ and $\alpha_y > 0$ such that $u_0(x) \geq u_{0,y}(x)$ for all $x \in \mathbb{R}^N$, where

$$u_{0,y}(x) := \begin{cases} \alpha_y \sqrt{R_y^2 - |x-y|^2} & \text{if } \ |x-y| < R_y \\ 0 & \text{if } \ |x-y| \geq R_y, \end{cases}$$

Then, by the comparison principle with sub-solutions (Theorem 3), we have that there exist positive constants β_1, β_2 such that

$$u(t,x) \geq u_y(t,x),$$

where

$$u_y(t,x) := \begin{cases} \alpha_y e^{-\beta_1 t - \beta_2 t^2} \sqrt{(R_y+t)^2 - |x-y|^2} & \text{if } \ |x-y| < R_y + t \\ 0 & \text{if } \ |x-y| \geq R_y + t. \end{cases}$$

Hence $\overline{C} \subset \mathrm{supp}(u(t))$. Let $x \in C(t) \setminus \overline{C}$. Then $d(x,\partial C) \leq t$ and there exists $y \in \partial C$ such that $d(y,x) \leq t$. Let $y_n \in C$ be such that $y_n \to y$. Observe that $R_{y_n} \leq d(y_n, y) \to 0$. By our previous argument, we know that $u(t,\cdot) > 0$ in $B_{R_{y_n}+t}(y_n)$. Since we may approximate x by points in $\cup_n B_{R_{y_n}+t}(y_n)$, we deduce that $x \in \mathrm{supp}(u(t))$. This proves that $C(t) \subseteq \mathrm{supp}(u(t))$. $\square$

Finally, the following result can be derived from Proposition 2 and the comparison principle with sub-solutions.

Proposition 3. *Let* $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$ *and let* u *be the entropy solution of the Cauchy problem for the equation (3) with* u_0 *as initial datum. Assume that* $u_0(y) \geq \alpha > 0$ *for any* $y \in B_R(x)$, $R > 0$. *Then* $u(t,y) \geq \alpha(t)$ *for any* $y \in B_{R+ct}(x)$ *and any* $t > 0$, *for some function* $\alpha(t) > 0$. *In particular, if* u_0 *is continuous at* $x \in \mathbb{R}^N$ *and* $u_0(x) > 0$, *then* $u(t,x) > 0$ *for any* $t > 0$.

This result implies the propagation of discontinuity fronts for any $t > 0$.

Acknowledgements. We would like to thank the reviewer for his useful comments which contributed to improve the paper. The first, third and fourth authors have been partially supported by EC through the RTN Programme *Nonlinear Partial Differential Equations Describing Front propagation and other Singular Phenomena*, HPRN-CT-2002-00274, and by PNPGC project, reference BFM2002-01145. The second author acknowledges partial support by the Departament d'Universitats, Recerca i Societat de la Informació de la Generalitat de Catalunya and by PNPGC project, reference BFM2003-02125.

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