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A Strongly Degenerate Quasilinear Equation: the Parabolic Case

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Abstract

We prove existence and uniqueness of entropy solutions for the Neumann problem for the quasilinear parabolic equation $u_t = \operatorname{div} \mathbf{a}(u, Du)$, where $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$, and f is a convex function of ξ with linear growth as $\|\xi\| \to \infty$, satisfying other additional assumptions. In particular, this class includes the case where $f(z,\xi) = \varphi(z)\psi(\xi)$, $\varphi > 0$, ψ being a convex function with linear growth as $\|\xi\| \to \infty$.

Key words. Nonlinear parabolic equations, nonlinear semigroups functions of bounded variation.

1. Introduction

Let Ω be a bounded set in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1. We are interested in the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, Du) & \text{in} \quad Q_T = (0,T) \times \Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on} \quad S_T = (0,T) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{in} \quad x \in \Omega, \end{cases} \tag{1}$$

where $u_0 \in L^1(\Omega)$, $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$, f being a function with linear growth as $\|\xi\| \to \infty$ and $\frac{\partial}{\partial \eta}$ is the Neumann boundary operator associated to $\mathbf{a}(u, Du)$, i.e.,

$$\frac{\partial u}{\partial \eta} := \mathbf{a}(u, Du) \cdot \nu,$$

with ν the unit outward normal on $\partial\Omega$.

Particular instances of this PDE have been studied in [14],[15],[16] and [25], when $N = 1$. Let us describe their results in some detail. In [14],[15], and [25] the authors considered the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\varphi(u)\mathbf{b}(u_x))_x & \text{in} \quad (0,T) \times I\!R \\ u(0,x) = u_0(x) & \text{in} \quad x \in I\!R \end{cases} \tag{2}$$

corresponding to (1) when $N = 1$ and $\mathbf{a}(u, u_x) = \varphi(u)\mathbf{b}(u_x)$, where $\varphi : I\!R \to I\!R^+$ is smooth and strictly positive, and $\mathbf{b} : I\!R \to I\!R$ is a smooth odd function such that $\mathbf{b}' > 0$ and $\lim_{s\to\infty} \mathbf{b}(s) = \mathbf{b}_\infty$. Such models appear as models for heat and mass transfer in turbulent fluids [10], or in the theory of phase transitions where the corresponding free energy functional has a linear growth rate with respect to the gradient [33]. As the authors observed, in general, there are no classical solutions of (1), indeed, the combination of the dependence on u in $\varphi(u)$ and the constant behavior of $\mathbf{b}(u_x)$ as $u_x \to \infty$ can cause the formation of discontinuities in finite time (see [14], Theorem 2.3). As noticed in [14], the parabolicity of (2) is so weak when $u_x \to \infty$ than solutions become discontinuous and behave like solutions of the first order equation $u_t = \mathbf{b}_\infty(\varphi(u))_x$ (which can be formally obtained differentiating the product in (2) and replacing $\mathbf{b}(u_x)$ by $\mathbf{b}_\infty$). For this reason, they defined the notion of entropy solution and proved existence ([14]) and uniqueness ([25]) of entropy solutions of (2). Existence was proved for bounded strictly increasing initial conditions $u_0 : I\!R \to I\!R$ such that $\mathbf{b}(u_0') \in C(I\!R)$ (where $\mathbf{b}(u_0'(x_0)) = \mathbf{b}_\infty$ if u_0 is discontinuous at x_0), $\mathbf{b}(u_0'(x)) \to 0$ as $x \to \pm\infty$ [14]. The entropy condition was written in Oleinik's form and uniqueness was proved using suitable test functions constructed by regularizing the sign of the difference of two solutions (we shall comment at the end of the paper on the possibility of extending the argument in [25]). Moreover, the authors show that there exist functions φ and initial conditions u_0 for which exist solutions of (2) which do not satisfy the entropy condition ([14], Theorem 2.2). Thus, uniqueness cannot be guaranteed without an additional condition like the entropy condition.

In [16],[17], the author considered the Neumann problem in an interval of $I\!R$

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u, u_x))_x & \text{in} \quad (0,T) \times (0,1) \\ u_x(t,0) = u_x(t,1) = 0 & \text{for} \quad t \in (0,T) \\ u(0,x) = u_0(x) & \text{for} \quad x \in (0,1) \end{cases} \tag{3}$$

for functions $\mathbf{a}(u,v)$ of class $C^{1,\alpha}([0,\infty) \times I\!R)$ such that $\frac{\partial}{\partial v}\mathbf{a}(u,v) < 0$ for any $(u,v) \in [0,\infty) \times I\!R$, $\mathbf{a}(u,0) = 0$ (and some other additional assumptions). After observing that there are, in general, no classical solutions of

(1), the author associated an m-accretive operator to $-(\mathbf{a}(u,u_x))_x$ with Neumann boundary conditions, and proved the existence and uniqueness of a semigroup solution of (3). However, the accretive operator generating the semigroup was not characterized in distributional terms. An example of the equations considered in [16],[17] is the so called *plasma equation* (see [28])

$$\frac{\partial u}{\partial t} = \Big(\frac{u^{5/2}u_x}{1+u|u_x|}\Big)_x \quad \text{in} \quad (0,T)\times(0,1), \tag{4}$$

where the initial condition u_0 is assumed to be positive. In this case u represents the temperature of electrons and the form of the conductivity $\mathbf{a}(u,u_x) = \frac{u^{5/2}u_x}{1+u|u_x|}$ has the effect of limiting heat flux. Thus, existence and uniqueness results for higher dimensional problems were not considered. This will be the purpose of the present paper.

Thus we shall consider Neumann problem (1) for any open bounded set Ω in $\mathbb{R}^N$ whose boundary is of class C^1, and we shall prove existence and uniqueness results for it.

In [5] we have considered the elliptic problem

$$u - \operatorname{div}\, \mathbf{a}(u,Du) = f \quad \text{in} \quad \Omega \tag{5}$$

with Neumann boundary conditions. By introducing a notion of entropy solution for (5), we proved in [5] an existence and uniqueness result when the right hand side f is in $L^1(\Omega)$. This result permitted us to associate to the expression $-\operatorname{div}\, \mathbf{a}(u,Du)$ with Neumann boundary conditions an m-accretive operator $\mathcal{B}$ in $L^1(\Omega)$ with dense domain, which, thus, generates a non-linear contraction semigroup $T(t)$ in $L^1(\Omega)$ ([13],[22], [23]). This result permits us to use Crandall-Ligget's iteration scheme and to define the function

$$u(t) := T(t)u_0 = \lim_{n\to\infty}\Big(I + \frac{t}{n}\mathcal{B}\Big)^{-n} u_0, \qquad u_0 \in L^\infty(\Omega).$$

Then we shall prove that $u(t)$ is an entropy solution of (1) (a notion defined below), and that entropy solutions are unique. As a technical tool we shall use the lower semi-continuity results proved in [24] for energy functionals whose density is a function $g(u,Du)$ convex in Du and with a linear growth rate in Du.

The case of equations of type

$$\frac{\partial u}{\partial t} = \operatorname{div}\, \mathbf{a}(x,Du) \quad \text{in} \quad (0,T)\times\Omega, \tag{6}$$

where $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$, $f(x,\cdot)$ being a convex function of ξ with linear growth as $\|\xi\| \to \infty$ has been considered in [2], [3] and [4] (see also [6]), where existence and uniqueness results of entropy solutions were proved. The present work can be considered as an extension of these works to the case where $\mathbf{a}$ depends on (u,Du) instead of (x,Du). Entropy or renormalized solutions for parabolic problems of types (1) or (6), when $f(u,\xi)$ or $f(x,\xi)$ has a growth of order $p>1$ as $\|\xi\| \to \infty$, or corresponding elliptic problems were considered in [11],[18] and [21] (see also the references therein). The use of Kruzkov's technique [31] of doubling variables to prove uniqueness for parabolic problems was first considered in [20].

Finally, let us explain the plan of the paper. In Section 2 we recall some basic facts about functions of bounded variation, denoted by $BV(\Omega)$, Green's formula, and lower semi-continuity results for energy functionals defined in $BV(\Omega)$. In Section 3 we recall the basic assumptions on the convex function $f(z,\xi)$ and its gradient $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$. In Section 4 we define an m-accretive operator in $L^1(\Omega)$ with dense domain associated to $-\mathrm{div}\, \mathbf{a}(u,Du)$ with Neumann boundary condition, hence, generating a contraction semigroup in $L^1(\Omega)$. In Section 5 we define the notion of entropy solution for (1) and we prove existence and uniqueness of entropy solutions when $u_0 \in L^\infty(\Omega)$, or $u_0 \in L^1(\Omega)$, depending on the set of assumptions . Existence will be proved by means of Crandall-Ligget's scheme and uniqueness by means of Kruzhkov's technique of doubling variables.

2. Preliminaries

We start with some notation. Here $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ are, respectively, the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.

Due to the linear growth condition on the Lagrangian, the natural energy space to study (1) is the space of functions of bounded variation. We recall briefly some facts about functions of bounded variation (for further information we refer to [1], [29] or [37]).

A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, \ldots, \mu_N$ defined in Ω with finite total mass in Ω and

$$\int_\Omega u D_i \varphi dx = -\int_\Omega \varphi d\mu_i \tag{7}$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \ldots, N$. Thus the gradient of u is a vector valued measure with finite total variation

$$|Du|(\Omega) = \sup\left\{ \int_\Omega u \,\mathrm{div}\, \varphi \, dx : \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), |\varphi(x)| \le 1 \text{ for } x \in \Omega \right\}.$$

The space $BV(\Omega)$ is endowed with the norm

$$\| u \|_{BV} = \| u \|_{L^1(\Omega)} + |Du|(\Omega). \tag{8}$$

For $u \in BV(\Omega)$, the gradient Du is a Radon measure that decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \, \mathcal{L}^N$ where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$. Let us denote by $D^s u = \overrightarrow{D^s u} |D^s u|$ the polar decomposition of $D^s u$, where $|D^s u|$ is the total variation measure of $D^s u$. We also split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$. We denote by S_u the set of all $x \in \Omega$ such that x is not a Lebesgue point of u. We say that $x \in \Omega$ is an *approximate jump point of* u if there exist $u^+(x) \neq u^-(x) \in I\!R$ and $\nu_u(x) \in S^{N-1}$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho^+(x, \nu_u(x)))} \int_{B_\rho^+(x,\nu_u(x))} |u(y) - u^+(x)| \, dy = 0$$

$$\lim_{\rho \downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho^-(x, \nu_u(x)))} \int_{B_\rho^-(x,\nu_u(x))} |u(y) - u^-(x)| \, dy = 0,$$

where

$$B_\rho^+(x, \nu_u(x)) = \{ y \in B_\rho(x) \; : \; \langle y - x, \nu_u(x) \rangle > 0 \}$$

and

$$B_\rho^-(x, \nu_u(x)) = \{ y \in B_\rho(x) \; : \; \langle y - x, \nu_u(x) \rangle < 0 \}.$$

We denote by J_u the set of approximate jump points of u. J_u is a Borel subset of S_u and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. We have

$$D^j u = D^s u \llcorner J_u \quad \text{and} \quad D^c u = D^s u \llcorner (\Omega \setminus S_u).$$

It is well known (see for instance [1]) that

$$D^j u = (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

Moreover, if $x \in J_u$, then $\nu_u(x) = \frac{Du}{|Du|}(x)$, $\frac{Du}{|Du|}$ being the Radon-Nikodym derivative of Du with respect to its total variation $|Du|$.

Let Ω be a bounded open subset of $I\!R^N$. Given a Borel function $g : \Omega \times I\!R \times I\!R^N \to I\!R$ such that

$$C\|\xi\| - D \leq g(x, z, \xi) \leq M(1 + \|\xi\|) \qquad \forall \; (x, z, \xi) \in \Omega \times I\!R \times I\!R^N, \quad (9)$$

for some constants $C > 0$, $M \geq 0$, we consider the energy functional

$$G(u) := \int_\Omega g(x, u(x), \nabla u(x)) \, dx$$

defined in the Sobolev space $W^{1,1}(\Omega)$. In order to get an integral representation of the relaxed energy associated with G, i.e.,

$$\mathcal{G}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n \to \infty} G(u_n) \; : \; u_n \in W^{1,1}(\Omega), \; u_n \to u \in L^1(\Omega) \right\},$$

Dal Maso in [24] introduced the following functional for $u \in BV(\Omega)$:

$$\mathcal{R}_g(u) := \int_\Omega g(x,u(x),\nabla u(x))\,dx + \int_\Omega g^0\left(x,u(x),\frac{Du}{|Du|}(x)\right)|D^c u| \\ + \int_{J_u}\left(\int_{u_-(x)}^{u_+(x)} g^0(x,s,\nu_u(x))\,ds\right) d\mathcal{H}^{N-1}(x), \tag{10}$$

where the recession function g^0 of g is defined as

$$g^0(x,z,\xi) = \lim_{t\to 0^+} tg\left(x,z,\frac{\xi}{t}\right), \qquad (x,z)\in\Omega\times\mathbb{R}. \tag{11}$$

It is clear that the function $g^0(x,z,\xi)$ is positively homogeneous of degree one in ξ, i.e.

$$g^0(x,z,s\xi) = sg^0(x,z,\xi) \qquad \text{for all } z,\xi \text{ and } s>0.$$

Let us describe a different way of writing the functional $\mathcal{R}_g(u)$. Let us consider the function $\tilde{g} : \Omega\times\mathbb{R}\times\mathbb{R}^N\times]-\infty,0]\to\mathbb{R}$ defined as

$$\tilde{g}(x,z,\xi,t) := \begin{cases} -g\left(x,z,-\dfrac{\xi}{t}\right)t & \text{if} \quad t<0 \\ g^0(x,z,\xi) & \text{if} \quad t=0. \end{cases} \tag{12}$$

As it is proved in [24], if g is a Borel function satisfying (9) and $g(x,z,\cdot)$ is convex in $\mathbb{R}^N$ for all $(x,z)\in\Omega\times\mathbb{R}$, then one has

$$\mathcal{R}_g(u) = \int_{\Omega\times\mathbb{R}} \tilde{g}\left((x,s),\frac{d\alpha_u}{d|\alpha_u|}(x,s)\right) d|\alpha_u|(x,s) \\ = \int_{\Omega\times\mathbb{R}} \tilde{g}\left((x,s),\nu[(x,s);N(u)]\right) d\mathcal{H}^N(x,s), \tag{13}$$

where $\alpha_u = D\chi_{N(u)}$, with $N(u) := \{(x,s)\in\mathbb{R}\times\Omega \ : \ s<u_+(x)\}$ and $\nu[(x,s);N(u)]$ is the interior normal to $N(u)$ at (s,x) if it exists, and $\nu[(x,s);N(u)]=0$ otherwise.

In [24] Dal Maso proved the following result:

Theorem 1. *Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $g:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ be a lower semi-continuous function satisfying (9) and such that $g(x,z,\cdot)$ is convex in $\mathbb{R}^N$ for any $(x,z)\in\Omega\times\mathbb{R}$. Then, $\mathcal{G}(u)=\mathcal{R}_g(u)$ for all $u\in BV(\Omega)$ and $\mathcal{R}_g(u)$ is lower semi-continuous respect to the $L^1(\Omega)$-convergence.*

We need to consider the following cut-off functions. For $a<b$, let $T_{a,b}(r) := \max(\min(b,r),a)$. It is usual to denote $T_k = T_{-k,k}$. In [5] we have established the following result.

Proposition 1. *Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a lower semi-continuous function satisfying (9) and such that $g(x, z, \cdot)$ is convex in $\mathbb{R}^N$ for any $(x, z) \in \Omega \times \mathbb{R}$. Let us define the functional*

$$R_g^{a,b}(u) = \int_\Omega g(x, u(x), \nabla T_{a,b} u(x))\, dx, \quad u \in W^{1,1}(\Omega).$$

For $u \in BV(\Omega)$, let

$$\mathcal{R}_g^{a,b}(u) := \int_a^b \int_\Omega \tilde{g}\left((x, s), \nu[(s, x); N(u)]\right) d\mathcal{H}^{N-1}(x)\, ds$$

$$+ \int_{[u \le a]} \left(g(x, u(x), 0) - g(x, a, 0)\right) dx + \int_{[u \ge b]} \left(g(x, u(x), 0) - g(x, b, 0)\right) dx.$$

Then $\mathcal{R}_g^{a,b}(u)$ is lower semi-continuous with respect to the $L^1(\Omega)$-convergence, and $\mathcal{R}_g^{a,b}$ coincides with the lower semi-continuous envelope of $R_g^{a,b}$.

Let $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a lower semi-continuous function satisfying

$$C\|\xi\| - D \le g(z, \xi) \le M(1 + \|\xi\|) \qquad \forall\ (z, \xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{14}$$

for some constants $C > 0$, $M \ge 0$. Given $u \in BV(\Omega)$, let us define the measures

$$\mathcal{R}_g(u, \phi) := \int_{\mathbb{R}} \int_\Omega \phi(x) \tilde{g}\left(s, \nu[(s, x); N(u)]\right) d\mathcal{H}^{N-1}(x)\, ds$$

and

$$\mathcal{R}_g^{a,b}(u, \phi) := \int_a^b \int_\Omega \phi(x) \tilde{g}\left(s, \nu[(s, x); N(u)]\right) d\mathcal{H}^{N-1}(x)\, ds$$

$$+ \int_{[u \le a]} \phi(x) \left(g(u(x), 0) - g(a, 0)\right) dx + \int_{[u \ge b]} \phi(x) \left(g(u(x), 0) - g(b, 0)\right) dx$$

for any $\phi \in C(\overline{\Omega})$. For simplicity, we shall write

$$\mathcal{R}_g(u, \phi) = \int_\Omega \phi(x) g(u, Du)$$

and

$$\mathcal{R}_g^{a,b}(u, \phi) = \int_\Omega \phi(x) g(u, DT_{a,b}(u)).$$

The singular parts respect to the Lebesgue measure $\mathcal{L}^N$ of these measures will be denoted by

$$(\mathcal{R}_g)^s(u, \phi) = \int_\Omega \phi(x) g(u, Du)^s$$

and

$$(\mathcal{R}_g^{a,b})^s(u, \phi) = \int_\Omega \phi(x) g(u, DT_{a,b}(u))^s,$$

respectively.

We shall need several results from [9] (see also [30]) in order to give a sense to the integrals of bounded vector fields with divergence in L^p integrated with respect to the gradient of a BV function. Let $p \geq 1$ and $p' \geq 1$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Following [9], let

$$X_p(\Omega) = \{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) : \mathrm{div}(\mathbf{z}) \in L^p(\Omega)\}. \tag{15}$$

If $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$ we define the functional $(\mathbf{z}, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Dw), \varphi \rangle := - \int_\Omega w\, \varphi\, \mathrm{div}(\mathbf{z})\, dx - \int_\Omega w\, \mathbf{z} \cdot \nabla\varphi\, dx. \tag{16}$$

Then $(\mathbf{z}, Dw)$ is a Radon measure in Ω,

$$\int_\Omega (\mathbf{z}, Dw) = \int_\Omega \mathbf{z} \cdot \nabla w\, dx \qquad \forall\ w \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \tag{17}$$

and

$$\left| \int_B (\mathbf{z}, Dw) \right| \leq \int_B |(\mathbf{z}, Dw)| \leq \|\mathbf{z}\|_\infty \int_B |Dw| \tag{18}$$

for any Borel set $B \subseteq \Omega$. Moreover, $(\mathbf{z}, Dw)$ is absolutely continuous with respect to $|Dw|$ with Radon-Nikodym derivative $\theta(\mathbf{z}, Dw, x)$ which is a $|Dw|$ measurable function from Ω to $\mathbb{R}$ such that

$$\int_B (\mathbf{z}, Dw) = \int_B \theta(\mathbf{z}, Dw, x) |Dw| \tag{19}$$

for any Borel set $B \subseteq \Omega$. We also have that

$$\|\theta(\mathbf{z}, Dw, .)\|_{L^\infty(\Omega, |Dw|)} \leq \|\mathbf{z}\|_{L^\infty(\Omega, \mathbb{R}^N)}. \tag{20}$$

Setting

$$\mathbf{z} \cdot D^s u := (\mathbf{z}, Du) - (\mathbf{z} \cdot \nabla u)\ d\mathcal{L}^N,$$

we see that $\mathbf{z} \cdot D^s u$ is a bounded measure. Furthermore, in [30] it is proved that $\mathbf{z} \cdot D^s u$ is absolutely continuous with respect to $|D^s u|$ (and, thus, it is a singular measure with respect to $\mathcal{L}^N$), and

$$|\mathbf{z} \cdot D^s u| \leq \|\mathbf{z}\|_\infty |D^s u|. \tag{21}$$

As a consequence of Theorem 2.4 of [9], we have that

$$\text{if} \quad \mathbf{z} \in X_p(\Omega) \cap C(\Omega, \mathbb{R}^N), \quad \text{then} \quad \mathbf{z} \cdot D^s u = (\mathbf{z} \cdot \overrightarrow{D^s u})\ d|D^s u|. \tag{22}$$

In [9], the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z} \in X_p(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator $\gamma : X_p(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(\mathbf{z})\|_\infty \leq \|\mathbf{z}\|_\infty$$

$$\gamma(\mathbf{z})(x) = \mathbf{z}(x) \cdot \nu(x) \quad \text{ for all } x \in \partial\Omega \text{ if } \mathbf{z} \in C^1(\overline{\Omega}, I\!R^N).$$

We shall denote $\gamma(\mathbf{z})(x)$ by $[\mathbf{z}, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[\mathbf{z}, \nu]$ and the measure $(\mathbf{z}, Dw)$, for $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$, is established:

$$\int_\Omega w \operatorname{div}(\mathbf{z})\, dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w \, d\mathcal{H}^{N-1}. \tag{23}$$

3. Basic assumptions

Here we assume that Ω is an open bounded set in $I\!R^N$, with boundary $\partial\Omega$ of class C^1, and the Lagrangian $f : I\!R \times I\!R^N \to I\!R$ satisfies the following assumptions, which we shall refer collectively as (H):

(H_1) f is continuous on $I\!R \times I\!R^N$ and is a convex differentiable function of ξ such that $\nabla_\xi f(z, \xi) \in C(I\!R \times I\!R^N)$. Further we require f to satisfy the linear growth condition

$$C_0\|\xi\| - D_0 \le f(z, \xi) \le M(\|\xi\| + 1). \tag{24}$$

for any $(z, \xi) \in I\!R \times I\!R^N$, $|z| \le R$ and some positive constants C_0, D_0, M depending on R. Moreover, we assume that f^0 exists.

We consider the function $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$ associated to the Lagrangian f. By the convexity of f

$$\mathbf{a}(z, \xi) \cdot (\eta - \xi) \le f(z, \eta) - f(z, \xi), \tag{25}$$

and the following monotonicity condition is satisfied

$$(\mathbf{a}(z, \eta) - \mathbf{a}(z, \xi)) \cdot (\eta - \xi) \ge 0. \tag{26}$$

Moreover, it is easy to see that

$$|\mathbf{a}(z, \xi)| \le M \qquad \forall\, (z, \xi) \in I\!R \times I\!R^N,\ |z| \le R. \tag{27}$$

We also assume that $\mathbf{a}(z, 0) = 0$ for all $z \in I\!R$. We consider the function $h : I\!R \times I\!R^N \to I\!R$ defined by

$$h(z, \xi) := \mathbf{a}(z, \xi) \cdot \xi.$$

By (26), we have

$$h(z, \xi) \ge 0 \qquad \forall\, \xi \in I\!R^N,\ z \in I\!R. \tag{28}$$

Moreover, from (25) and (24), it follows that

$$C_0\|\xi\| - D_1 \le h(z, \xi) \le M\|\xi\| \tag{29}$$

for any $(z, \xi) \in I\!R \times I\!R^N$, $|z| \le R$, where D_1 is a positive constant depending on R, C_0 and M being as above.

(H$_2$) We assume that $\frac{\partial \mathbf{a}}{\partial \xi_i}(z,\xi) \in C(I\!R \times I\!R^N)$ for any $i = 1, \ldots, N$.

This assumption is not necessary for the case of separated variables described in 1.

We assume that

(H$_3$) $h(z,\xi) = h(z,-\xi)$, for all $z \in I\!R$ and $\xi \in I\!R^N$ and h^0 exists.

Observe that we have

$$C_0\|\xi\| \leq h^0(z,\xi) \leq M\|\xi\| \quad \text{for any } (z,\xi) \in I\!R \times I\!R^N, \, |z| \leq R.$$

(H$_4$) $f^0(z,\xi) = h^0(z,\xi)$, for all $\xi \in I\!R^N$ and all $z \in I\!R$.

(H$_5$) $\mathbf{a}(z,\xi) \cdot \eta \leq h^0(z,\eta)$ for all $\xi, \eta \in I\!R^N$, and all $z \in I\!R$.

(H$_6$) We assume that $h^0(z,\xi)$ can be written in the form $h^0(z,\xi) = \varphi(z)\psi^0(\xi)$ with φ a C^1-function such that for any $R > 0$, we have $\varphi(z) > \alpha_R > 0$ for all $z \in I\!R$, $|z| \leq R$, and ψ^0 being a convex function homogeneous of degree 1.

(H$_7$) For any $R > 0$, there is a constant $C > 0$ such that

$$|(\mathbf{a}(z,\xi) - \mathbf{a}(\hat{z},\xi)) \cdot (\xi - \hat{\xi})| \leq C|z - \hat{z}| \, \|\xi - \hat{\xi}\| \tag{30}$$

for any $(z,\xi), (\hat{z},\hat{\xi}) \in I\!R \times I\!R^N$, $|z|, |\hat{z}| \leq R$.

Observe that, by the monotonicity condition (26) and using (30), it follows that

$$(\mathbf{a}(z,\xi) - \mathbf{a}(\hat{z},\hat{\xi})) \cdot (\xi - \hat{\xi}) \geq -C|z - \hat{z}| \, \|\xi - \hat{\xi}\| \tag{31}$$

for any $(z,\xi), (\hat{z},\xi) \in I\!R \times I\!R^N$, $|z|, |\hat{z}| \leq R$.

Let us observe that under assumptions (H$_4$) and (H$_6$), applying the chain rule for BV-functions (see [1]), we have

$$\mathcal{R}_f(u) = \int_\Omega f(u, \nabla u)dx + \psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(u)|, \tag{32}$$

where $J_\varphi(r) = \int_0^r \varphi(s)\, ds$.

Remark 1. An important particular case of Lagrangian f satisfying all assumptions (H) but (H$_2$), is the one given by $f(z,\xi) = \varphi(z)\psi(\xi)$ with φ a C^1-function such that for any $R > 0$, we have $\varphi(z) > \alpha_R > 0$ for all $z \in I\!R$, $|z| \leq R$, and ψ a convex C^1-function such that

$$C_0\|\xi\| - D_0 \leq \psi(\xi) \leq M(\|\xi\| + 1) \quad \forall \xi \in I\!R^N,$$

and there exists

$$\psi^0(\xi) = \lim_{t \to 0^+} t\psi\left(\frac{\xi}{t}\right).$$

In this case, if $\mathbf{b}(\xi) := \nabla\psi(\xi)$, we have $\mathbf{a}(z,\xi) = \varphi(z)\mathbf{b}(\xi)$, and $h(z,\xi) = \mathbf{a}(z,\xi) \cdot \xi = \varphi(z)\mathbf{b}(\xi) \cdot \xi$. Then, in order to have that (H) holds we need to assume that:

(i) $\mathbf{b}(-\xi)\cdot(-\xi) = \mathbf{b}(\xi)\cdot\xi \geq 0$ for all $\xi \in I\!R^N$ and there exists

$$\lim_{t\to 0^+} \mathbf{b}\left(\frac{\xi}{t}\right)\cdot\xi = \psi^0(\xi).$$

(ii) $\mathbf{b}(\xi)\cdot\eta \leq \psi^0(\xi)\cdot\eta$ for all $\xi, \eta \in I\!R^N$.

We note that in this case (H_2) is not necessary to obtain existence and uniqueness of solutions for problem (1). As we have observed in [5], (H_7) holds.

Remark 2. There are physical models for plasma fusion by inertial confinement in which the temperature evolution of the electrons satisfies an equation of type (1), where $\mathbf{a}(z,\xi) = \frac{|z|^{5/2}\xi}{1+|z||\xi|}$ which corresponds to $f(z,\xi) = |z|^{3/2}|\xi| - |z|^{1/2}\ln(1+|z||\xi|)$ [28], (see also [16] for a mathematical study in the one-dimensional case). It is easy to check that (H_1) (in particular (24) and (29)) holds for any $(z,\xi) \in I\!R \times I\!R^N$ with $z \in [a,R]$, $a > 0$, the constants in (24) and (29) depending on a, R. Note that (H_2) also holds. We also observe that $h^0(z,\xi) = |z|^{3/2}|\xi|$ and (H_3)-(H_6) hold. Finally, as observed in [5], (H_7) for the values of $z, \hat{z} \in [a,R]$ and $\xi \in I\!R^N$. In this case, the results below will prove existence and uniqueness of entropy solutions of (1) for any initial condition $u_0 \in L^\infty(\Omega)$ such that $u_0(x) \geq a > 0$ for some $a > 0$.

4. The operator associated to $-\mathrm{div}\ \mathbf{a}(u, Du)$ with Neumann boundary conditions

We need to consider the function space

$$TBV(\Omega) := \left\{ u \in L^1(\Omega) \ : \ T_k(u) \in BV(\Omega), \ \ \forall\, k > 0 \right\},$$

and to give a sense to the Radon-Nikodym derivative ∇u of a function $u \in TBV(\Omega)$. Notice that the function space $TBV(\Omega)$ is closely related to the space $GBV(\Omega)$ of generalized functions of bounded variation introduced by E. Di Giorgi and L. Ambrosio ([26], see also [1]). As we observed in [3], for every $u \in TBV(\Omega)$ there exists a unique measurable function $v : \Omega \to I\!R^N$ such that

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \qquad \mathcal{L}^N - \text{a.e.} \tag{33}$$

Hence, for a function $u \in TBV(\Omega)$ we can define ∇u as the unique function v which satisfies (33). This notation will be used throughout in the sequel.

Let us recall the following definitions introduced in [5].

Definition 1. Given $v \in L^1(\Omega)$, we say that $u \in L^1(\Omega)$ is an *entropy solution* of

$$\begin{cases} v = -\mathrm{div}\ \mathbf{a}(u, Du) & \text{in} \quad \Omega \\[2mm] \dfrac{\partial u}{\partial \eta} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{34}$$

if $u \in TBV(\Omega)$, $\mathbf{a}(u, \nabla u) \in X_1(\Omega)$ and satisfies:

$$v = -\mathrm{div}\ \mathbf{a}(u, \nabla u) \qquad \text{in } \ \mathcal{D}'(\Omega), \tag{35}$$

$$(\mathbf{a}(u, \nabla u), DT_{a,b}(u)) \geq h(u, DT_{a,b}(u)) \quad \text{as measures } \ \forall\ a < b, \tag{36}$$

$$[\mathbf{a}(u, \nabla u), \nu] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega. \tag{37}$$

Recall that, since $h(z, 0) = 0$ for any $z \in I\!R$, $h(u, DT_{a,b}(u))$ is the density of the measure $\mathcal{R}_h^{a,b}(u) = \mathcal{R}_h(T_{a,b}(u))$. Our assumptions on h permit to apply the results described by 1, and 1. Hence, it has sense that we write $h(u, DT_{a,b}(u))$ which coincides with $h(T_{a,b}(u), DT_{a,b}(u))$.

Observe that (36) is equivalent to

$$\mathbf{a}(u, \nabla u) \cdot D^s T_{a,b}(u) \geq (\mathcal{R}_f^{a,b})^s(u) \quad \text{as measures } \ \forall\ a < b. \tag{38}$$

Definition 2. $(u, v) \in \mathcal{B}_0$ if and only if $u \in TBV(\Omega)$, $v \in L^1(\Omega)$ and u is the entropy solution of problem (34).

If $(u, v) \in \mathcal{B}_0$, we have that

$$\int_\Omega (w - T(u))v\ dx \leq \int_\Omega (\mathbf{a}(u, \nabla u), Dw) - \mathcal{R}_h^{a,b}(u), \tag{39}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$, and any $T = T_{a,b}$, $a < b$. Indeed, this can be easily proved by multiplying (35) by $w - u$, using Green's formula (23) and (36) (see [5]).

We define the operator $B := \mathcal{B}_0 \cap (L^\infty(\Omega) \times L^\infty(\Omega))$, and we denote by $\mathcal{B}$ the closure in $L^1(\Omega)$ of the operator B.

Observe that in the definition of entropy solution we required that $\mathbf{a}(u, \nabla u) \in X_1(\Omega)$, and, thus, $\mathbf{a}(u, \nabla u) \in L^\infty(\Omega, I\!R^N)$. This is reasonable only if we are able to prove that solutions of (40) satisfy it, and this was proved in [5] under some assumptions. The main result of [5] is the following existence and uniqueness result.

Theorem 2. *Assume that assumptions (H) hold. Then,*

(i) for any $v \in L^\infty(\Omega)$ there exists a unique entropy solution $u \in TBV(\Omega) \cap L^\infty(\Omega)$ of the problem

$$\begin{cases} u - \mathrm{div}\ \mathbf{a}(u, Du) = v & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{40}$$

Moreover, if we assume that the bound (27) holds for any $(z, \xi) \in I\!R \times I\!R^N$, then (40) has a unique entropy solution $u \in TBV(\Omega)$ for any $v \in L^1(\Omega)$.

(ii) $\mathcal{B}$ *is an m-accretive operator in* $L^1(\Omega)$ *with dense domain. Moreover, if we assume that the bound (27) holds for any* $(z,\xi) \in I\!R \times I\!R^N$*, then* $\mathcal{B} = \mathcal{B}_0$.

(iii) Given $\lambda > 0$*, and* $u \in L^q(\Omega)$*,*$1 \le q \le \infty$*, if* $v = (I + \lambda\mathcal{B})^{-1}u$*, then* $\|v\|_q \le \|u\|_q$.

Remark 3. As we observed in [5] (H_2) is not used when $f(z,\xi) = \varphi(z)\psi(\xi)$, being φ a bounded smooth function such that $\varphi(z) \ge \alpha_R > 0$ for all $z \in I\!R$, $|z| \le R$ (see 1).

Remark 4. If $\mathbf{a}(z,\xi) = \frac{|z|^{5/2}\xi}{1+|z||\xi|}$ ([28],[16]), then 2 holds for any $v \in L^\infty(\Omega)$ such that $v \ge a$, for some $a > 0$.

By 2, and according to the general theory of nonlinear semigroups (c.f., e.g., [13]), for any $u_0 \in L^1(\Omega)$ there exists a unique mild solution $u \in C([0,T]; L^1(\Omega))$ of the abstract Cauchy problem

$$u'(t) + \mathcal{B}u(t) \ni 0, \qquad u(0) = u_0. \tag{41}$$

Moreover, $u(t) = T(t)u_0$ for all $t \ge 0$, being $(T(t))_{t\ge 0}$ the semigroup in $L^1(\Omega)$ generated by Crandall-Liggett's exponential formula, i.e.,

$$T(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\mathcal{B} \right)^{-n} u_0,$$

and for any $u_0 \in L^q(\Omega)$, $1 \le q \le \infty$, we have

$$\|T(t)u_0\|_q \le \|u_0\|_q \quad \forall\, t \ge 0. \tag{42}$$

5. Existence and uniqueness of solutions of the evolution problem

In this section we give the concept of entropy solution for the Neumann problem (1) and we state the existence and uniqueness result for this type of solution.

To make precise our notion of solution we need to recall the following definitions given in [3].

We define the space

$$Z(\Omega) := \left\{ (\mathbf{z},\xi) \in L^\infty(\Omega, I\!R^N) \times BV(\Omega)^* \ : \ \operatorname{div}(\mathbf{z}) = \xi \ \text{ in } \ \mathcal{D}'(\Omega) \right\}.$$

We denote $R(\Omega) := W^{1,1}(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$. For $(\mathbf{z},\xi) \in Z(\Omega)$ and $w \in R(\Omega)$ we define

$$\langle (\mathbf{z},\xi), w \rangle_{\partial\Omega} := \langle \xi, w \rangle_{BV(\Omega)^*, BV(\Omega)} + \int_\Omega \mathbf{z} \cdot \nabla w \, dx.$$

Then, working as in the proof of Theorem 1.1. of [9], we obtain that if $w, v \in R(\Omega)$ and $w = v$ on $\partial\Omega$ one has

$$\langle(\mathbf{z},\xi), w\rangle_{\partial\Omega} = \langle(\mathbf{z},\xi), v\rangle_{\partial\Omega} \qquad \forall\ (\mathbf{z},\xi) \in Z(\Omega). \tag{43}$$

As a consequence of (43), we can give the following definition: Given $u \in BV(\Omega)\cap L^\infty(\Omega)$ and $(\mathbf{z},\xi) \in Z(\Omega)$, we define $\langle(\mathbf{z},\xi), u\rangle_{\partial\Omega}$ by setting

$$\langle(\mathbf{z},\xi), u\rangle_{\partial\Omega} := \langle(\mathbf{z},\xi), w\rangle_{\partial\Omega},$$

where w is any function in $R(\Omega)$ such that $w = u$ on $\partial\Omega$. In [3] we prove that there exists a linear operator $\gamma : Z(\Omega) \to L^\infty(\partial\Omega)$, with $\gamma(\mathbf{z},\xi) := \gamma_{\mathbf{z},\xi}$, satisfying

$$\langle(\mathbf{z},\xi), w\rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma_{\mathbf{z},\xi}(x) w(x)\ d\mathcal{H}^{N-1} \qquad \forall\ w \in BV(\Omega)\cap L^\infty(\Omega).$$

In case $\mathbf{z} \in C^1(\overline{\Omega}, \mathbb{R}^N)$, we have $\gamma_{\mathbf{z}}(x) = \mathbf{z}(x)\cdot\nu(x)$ for all $x \in \partial\Omega$. Hence, the function $\gamma_{\mathbf{z},\xi}(x)$ is the weak trace of the normal component of $(\mathbf{z},\xi)$. For simplicity of the notation, we shall denote $\gamma_{\mathbf{z},\xi}(x)$ by $[\mathbf{z},\nu](x)$.

We need to consider the space $BV(\Omega)_2$, defined as $BV(\Omega)\cap L^2(\Omega)$ endowed with the norm

$$\|w\|_{BV(\Omega)_2} := \|w\|_{L^2(\Omega)} + |Dw|(\Omega).$$

It is easy to see that $L^2(\Omega) \subset BV(\Omega)_2^*$ and

$$\|w\|_{BV(\Omega)_2^*} \leq \|w\|_{L^2(\Omega)} \qquad \forall\ w \in L^2(\Omega). \tag{44}$$

It is well known (see [35]) that the dual space $\left(L^1(0,T;BV(\Omega)_2)\right)^*$ is isometric to the space $L^\infty(0,T;BV(\Omega)_2^*, BV(\Omega)_2)$ of all weakly* measurable functions $f : [0,T] \to BV(\Omega)_2^*$, such that $v(f) \in L^\infty([0,T])$, where $v(f)$ denotes the supremum of the set $\{|\langle w, f\rangle| \ :\ \|w\|_{BV(\Omega)_2} \leq 1\}$ in the vector lattice of measurable real functions. Moreover, the dual pairing of the isometric is defined by

$$\langle w, f\rangle = \int_0^T \langle w(t), f(t)\rangle\ dt,$$

for $w \in L^1(0,T;BV(\Omega)_2)$ and $f \in L^\infty(0,T;BV(\Omega)_2^*, BV(\Omega)_2)$.

By $L^1_w(0,T,BV(\Omega))$ we denote the space of weakly measurable functions $f : [0,T] \to BV(\Omega)$ (i.e., $t \in [0,T] \to < f(t), \phi >$ is measurable for every $\phi \in BV(\Omega)^*$) such that $\int_0^T \|f(t)\|\, dt < \infty$. Observe that, since $BV(\Omega)$ has a separable predual (see [1]), it follows easily that the map $t \in [0,T] \to \|f(t)\|$ is measurable. By $L^1_{loc,w}(0,T,BV(\Omega))$ we denote the space of weakly measurable functions $f : [0,T] \to BV(\Omega)$ such that the map $t \in [0,T] \to \|f(t)\|$ is in $L^1_{loc}(]0,T[)$

Let us recall the following definitions given in [3].

Definition 3. Let $\Psi \in L^1(0,T,BV(\Omega))$. We say Ψ admits a *weak derivative* in the space $L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$ if there is a function $\Theta \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$ such that $\Psi(t) = \int_0^t \Theta(s)ds$, the integral being taken as a Pettis integral ([27]).

Definition 4. Let $\xi \in \left(L^1(0,T,BV(\Omega)_2)\right)^*$. We say that ξ is *the time derivative* in the space $\left(L^1(0,T,BV(\Omega)_2\right)^*$ of a function $u \in L^1((0,T)\times\Omega)$ if

$$\int_0^T \langle \xi(t), \Psi(t)\rangle dt = -\int_0^T \int_\Omega u(t,x)\Theta(t,x)dxdt$$

for all test functions $\Psi \in L^1(0,T,BV(\Omega))$ with compact support in time, which admit a weak derivative $\Theta \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$.

Note that if $w \in L^1(0,T,BV(\Omega)) \cap L^\infty(Q_T)$ and $\mathbf{z} \in L^\infty(Q_T, I\!R^N)$ is such that there exists $\xi \in \left(L^1(0,T,BV(\Omega))\right)^*$ with $\mathrm{div}(\mathbf{z}) = \xi$ in $\mathcal{D}'(Q_T)$, we can define, associated to the pair $(\mathbf{z},\xi)$, the distribution $(\mathbf{z}, Dw)$ in Q_T by

$$\langle (\mathbf{z}, Dw), \phi\rangle := -\int_0^T \langle \xi(t), w(t)\phi(t)\rangle\, dt - \int_0^T \int_\Omega \mathbf{z}(t,x) w(t,x) \nabla_x \phi(t,x)\, dxdt. \tag{45}$$

for all $\phi \in \mathcal{D}(Q_T)$.

Definition 5. Let $\xi \in \left(L^1(0,T,BV(\Omega)_2)\right)^*$ and $\mathbf{z} \in L^\infty(Q_T, I\!R^N)$. We say that $\xi = \mathrm{div}(\mathbf{z})$ in $\left(L^1(0,T,BV(\Omega)_2\right)^*$ if $(\mathbf{z}, Dw)$ is a Radon measure in Q_T with normal boundary values $[\mathbf{z},\nu] \in L^\infty((0,T)\times\partial\Omega)$, such that

$$\int_{Q_T} (\mathbf{z}, Dw) + \int_0^T \langle \xi(t), w(t)\rangle dt = \int_0^T \int_{\partial\Omega} [\mathbf{z}(t,x),\nu] w(t,x) d\mathcal{H}^{N-1} dt,$$

for all $w \in L^1(0,T,BV(\Omega)) \cap L^\infty(Q_T)$.

Our concept of solution for problem (1) is the following.

Definition 6. A measurable function $u : (0,T)\times\Omega \to I\!R$ is an *entropy solution* of (1) in $Q_T = (0,T)\times\Omega$ if $u \in C([0,T];L^1(\Omega))$, $T_k(u(\cdot)) \in L^1_{loc,w}(0,T,BV(\Omega))$ for all $k>0$, and there exists $\xi \in \left(L^1(0,T,BV(\Omega)_2\right)^*$ such that:

(i) $(\mathbf{a}(u(t),\nabla u(t)), \xi(t)) \in Z(\Omega)$ and $[\mathbf{a}(u(t),\nabla u(t)),\nu] = 0$ a.e. in $t \in [0,T]$,
(ii) ξ is the time derivative of u in $\left(L^1(0,T,BV(\Omega)_2)\right)^*$ in the sense of Definition 4,
(iii) $\xi = \mathrm{div}\,\mathbf{a}(u(t),\nabla u(t))$ in the sense of Definition 5, and

(iv) the following inequality is satisfied

$$-\int_0^T \int_\Omega j_k(u(t)-l)\eta_t \, dxdt + \int_0^T \int_\Omega \eta(t) h(u(t), DT_k(u(t)-l)) \, dt$$

$$+ \int_0^T \int_\Omega \mathbf{a}(u(t), \nabla u(t)) \cdot \nabla \eta(t) T_k(u(t)-l) \, dxdt \leq 0$$

for all $l \in I\!R$ $k > 0$, for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geq 0$, $\eta(t,x) = \phi(t)\rho(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\rho \in C^\infty(\overline{\Omega})$, and $k > 0$, where $j_k(r) = \displaystyle\int_0^r T_k(s) \, ds$.

We have the following existence and uniqueness result.

Theorem 3. *Assume we are under assumptions (H). Then, for any initial datum $u_0 \in L^\infty(\Omega)$ there exists a unique entropy solution u of (1) in $Q_T = (0,T) \times \Omega$ for every $T > 0$ such that $u(0) = u_0$. Moreover, if we assume that the bound (27) holds for any $(z,\xi) \in I\!R \times I\!R^N$, then for any initial datum $u_0 \in L^1(\Omega)$ there exists a unique entropy solution u of (1) in $Q_T = (0,T) \times \Omega$ for every $T > 0$ such that $u(0) = u_0$.*

If $u(t)$, $\overline{u}(t)$ are the entropy solutions corresponding to initial data u_0, $\overline{u}_0 \in L^1(\Omega)$, respectively, then

$$\|u(t) - \overline{u}(t)\|_1 \leq \|u_0 - \overline{u}_0\|_1 \qquad \text{for all } \; t \geq 0. \tag{46}$$

Note that, by 2, if $u_0 \in L^\infty(\Omega)$ we have $\|u(t)\|_\infty \leq \|u_0\|_\infty$.

As in 3 we observe that (H_2) is not required if $f(z,\xi) = \varphi(z)\psi(\xi)$ is as in 1. Using 4 we also observe that 3 holds when $\mathbf{a}(z,\xi) = \frac{|z|^{5/2}\xi}{1+|z||\xi|}$ for any $u_0 \in L^\infty(\Omega)$, $u_0 \geq a$, for some $a > 0$.

Proof. *Existence of entropy solutions.*

Let us point out that both sets of assumptions can be considered within the same proof. In case that assumptions (H) hold, we are considering $u_0 \in L^\infty(\Omega)$, hence the vector fields of type $\mathbf{a}(u, \nabla u)$ are bounded, and this bound is also true when considering that (27) holds for any $(z,\xi) \in I\!R \times I\!R^N$.

Given $u_0 \in L^1(\Omega)$, let $u(t) = T(t)u_0$, being $(T(t))_{t\geq 0}$ the semigroup in $L^1(\Omega)$ generated by the m-accretive operator $\mathcal{B}$. Then, according to the general theory of nonlinear semigroups ([13]), we have that $u(t)$ is a mild-solution of the abstract Cauchy problem

$$u'(t) + \mathcal{B}u(t) \ni 0, \qquad u(0) = u_0. \tag{47}$$

Observe that, by Theorem 2, in any of the considered cases we have

$$T(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\mathcal{B}_0 \right)^{-n} u_0.$$

Let us prove that u is an entropy solution of (1) in Q_T. We divide the proof in several steps.

Step 1. Let $T > 0$, $K \geq 1$, $\Delta t = \frac{T}{K}$, $t_n = n\Delta t$, $n = 0, \ldots, K-1$. We define inductively u^{n+1}, $n = 0, \ldots, K$, to be the unique entropy solution of

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \operatorname{div} \mathbf{a}(u^{n+1}, Du^{n+1}) = 0 & \text{in} \quad \Omega \\[2ex] \dfrac{\partial u^{n+1}}{\partial \eta} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{48}$$

where $u^0 = u_0 \in L^1(\Omega)$. Recall that $\|u^n\|_1 \leq \|u_0\|_1$ for all n. We define

$$u^K(t) = u^0 \chi_{[t_0, t_1]}(t) + \sum_{n=1}^{K-1} u^n \chi_{(t_n, t_{n+1}]}(t).$$

We know that $u^K(t)$ converges uniformly to $u \in C([0,T], L^1(\Omega))$ and $\|u(t)\|_1 \leq \|u_0\|_1$.

If we define

$$\xi^K(t) := \sum_{n=0}^{K-1} \frac{u^{n+1} - u^n}{\Delta t} \chi_{(t_n, t_{n+1}]}(t)$$

and

$$\mathbf{z}^K(t) = \mathbf{a}(u^1, \nabla u^1) \chi_{[t_0, t_1]}(t) + \sum_{n=1}^{K-1} \mathbf{a}(u^{n+1}, \nabla u^{n+1}) \chi_{(t_n, t_{n+1}]}(t),$$

since u^{n+1} is the entropy solution of (48), we have

$$\xi^K(t) = \operatorname{div}(\mathbf{z}^K(t)) \quad \text{in} \ \ \mathcal{D}'(\Omega) \tag{49}$$

$$h(u^K(t + \Delta t), DT_{a,b}(u^K(t + \Delta t)) \leq (\mathbf{z}^K(t), DT_{a,b} u^K(t + \Delta t)) \tag{50}$$

$$[\mathbf{z}^K(t), \nu] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega. \tag{51}$$

Since $\|\mathbf{z}^K(t)\|_\infty \leq M$ for all $K \in \mathbb{N}$ and a.e. $t \in [0,T]$, we may assume that

$$\mathbf{z}^K \to \mathbf{z} \in L^\infty(Q_T, \mathbb{R}^N) \qquad \text{weakly}^*. \tag{52}$$

Step 2. Given $w \in BV(\Omega)_2$, from (49) and (51), it follows that

$$\left| \int_\Omega \xi^K(t) w \, dx \right| = \left| - \int_\Omega (\mathbf{z}^K(t), Dw) \right| \leq M \|w\|_{BV(\Omega)_2}.$$

Thus,

$$\|\xi^K(t)\|_{BV(\Omega)_2^*} \leq M \qquad \forall \ K \in \mathbb{N} \ \text{ and } \ t \in [0,T].$$

Consequently, $\{\xi^K\}$ is a bounded sequence in $L^\infty(0,T;BV(\Omega)_2^*)$. Now, since the space $L^\infty(0,T;BV(\Omega)_2^*)$ is a vector subspace of the dual space $\left(L^1(0,T;BV(\Omega)_2)\right)^*$, we can find a subnet ξ^α of ξ^K such that

$$\xi^\alpha \to \xi \in \left(L^1(0,T;BV(\Omega)_2)\right)^* \quad \text{weakly}^*. \tag{53}$$

Given $\eta \in \mathcal{D}(Q_T)$, since $\eta \in L^1(0,T;BV(\Omega)_2)$, we have

$$\langle \xi, \eta \rangle = \lim_\alpha \langle \xi^\alpha, \eta \rangle = \lim_\alpha \int_0^T \langle \xi^\alpha(t), \eta(t) \rangle \; dt$$

$$= \lim_\alpha \int_0^T \int_\Omega \xi^\alpha(t)\eta(t) \; dxdt = \lim_\alpha \int_0^T \int_\Omega \operatorname{div}(\mathbf{z}^\alpha(t))\eta(t) \; dxdt$$

$$= -\lim_\alpha \int_0^T \int_\Omega \mathbf{z}^\alpha(t) \cdot \nabla\eta(t) \; dxdt = -\int_{Q_T} z \cdot \nabla\eta \; dxdt = \langle \operatorname{div}_x(\mathbf{z}), \eta \rangle.$$

Hence,

$$\xi = \operatorname{div}_x(\mathbf{z}) \qquad \text{in } \mathcal{D}'(Q_T). \tag{54}$$

On the other hand, if we take $\eta(t,x) = \phi(t)\psi(x)$ with $\phi \in \mathcal{D}(]0,T[)$ and $\psi \in \mathcal{D}(\Omega)$, the same calculation as above shows that

$$\xi(t) = \operatorname{div}_x(\mathbf{z}(t)) \qquad \text{in } \mathcal{D}'(\Omega) \;\; \text{a.e.} \;\; t \in [0,T]. \tag{55}$$

Consequently, $(\mathbf{z}(t), \xi(t)) \in Z(\Omega)$ for almost all $t \in [0,T]$, therefore we can consider $[\mathbf{z}(t), \nu]$ defined.

Let us see now that ξ is the time derivative of u in the sense of the 4. Let $\Psi \in L^1(0,T,BV(\Omega))$ be the weak derivative of the function $\Theta \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$, i.e., $\Psi(t) = \int_0^t \Theta(s)ds$, the integral being taken as a Pettis integral, Ψ has compact support in time. We have

$$\int_0^T \int_\Omega \sum_{n=0}^{K-1} \frac{u^{n+1} - u^n}{\Delta t} \chi_{(t_n,t_{n+1}]}(t)\Psi(t) \, dx \, dt$$

$$= \int_0^T \int_\Omega \frac{u^K(t+\Delta t) - u^K(t)}{\Delta t} \Psi(t) \, dx \, dt$$

$$= \frac{1}{\Delta t} \sum_{n=1}^{K} \int_{t_n}^{t_{n+1}} \int_\Omega u^K(s)\Psi(s-\Delta t) \, dxds - \frac{1}{\Delta t} \sum_{n=0}^{K-1} \int_{t_n}^{t_{n+1}} \int_\Omega u^K(t)\Psi(t) \, dxdt$$

$$= \frac{1}{\Delta t} \int_{t_K}^{t_{K+1}} \int_\Omega u^K(s)\Psi(s-\Delta t) \, dxds$$

$$+ \frac{1}{\Delta t} \sum_{n=1}^{K-1} \int_{t_n}^{t_{n+1}} \int_\Omega u^K(t)(\Psi(t-\Delta t) - \Psi(t)) \, dxdt - \frac{1}{\Delta t} \int_{t_0}^{t_1} \int_\Omega u^K(t)\Psi(t) \, dxdt$$

$$= \int_{t_1}^{t_K} \int_\Omega u^K(t) \frac{\Psi(t-\Delta t) - \Psi(t)}{\Delta t} \, dxdt - \frac{1}{\Delta t} \int_{t_0}^{t_1} u^0 \Psi(t) \, dt$$

Letting $K \to \infty$, the first term in the above series of inequalities converges to $\int_0^T \langle \xi(t), \Psi(t) \rangle dt$, while the last term converges to $-\int_0^T \int_\Omega \Theta(t,x) u(t,x) \, dxdt$. We obtain

$$\int_0^T \langle \xi(t), \Psi(t) \rangle dt = -\int_0^T \int_\Omega \Theta(t,x) u(t,x) \; dxds. \tag{56}$$

Step 3. The boundary condition. Let us now prove that

$$[\mathbf{z}(t), \nu] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega, \quad \text{a.e. } t \in [0,T]. \tag{57}$$

Indeed, if $w \in BV(\Omega) \cap L^\infty(\Omega)$, and $v \in R(\Omega)$ such that $v_{|\partial\Omega} = w_{|\partial\Omega}$, we have that

$$\int_0^t \langle \mathbf{z}^\alpha(s), w \rangle_{\partial\Omega} \; ds = \int_0^t \langle \operatorname{div}(\mathbf{z}^\alpha(s)), v \rangle \; ds + \int_0^t \int_\Omega \mathbf{z}^\alpha(s) \cdot \nabla v \; dxds.$$

Hence

$$\begin{aligned} \lim_\alpha \int_0^t \langle \mathbf{z}^\alpha(s), w \rangle_{\partial\Omega} \; ds &= \int_0^t \langle \xi(s), v \rangle \; ds + \int_0^t \int_\Omega \mathbf{z}(s) \cdot \nabla v \; dxds \\ = \int_0^t \langle \mathbf{z}(s), w \rangle_{\partial\Omega} \; ds &= \int_0^t \int_{\partial\Omega} [\mathbf{z}(s), \nu] w \; d\mathcal{H}^{N-1} ds. \end{aligned} \tag{58}$$

On the other hand, since $\mathbf{z}^\alpha(s) \in X_1(\Omega)$, if we apply Green's formula we have that

$$\int_0^t \langle \operatorname{div}(\mathbf{z}^\alpha(s)), v \rangle ds = -\int_0^t \int_\Omega \mathbf{z}^\alpha(s) \cdot \nabla v \; dxds.$$

Consequently,

$$\int_0^t \langle \mathbf{z}_\alpha(s), w \rangle_{\partial\Omega} \; ds = 0.$$

Taking limits in α, we get

$$\int_0^t \int_{\partial\Omega} [\mathbf{z}(s), \nu] w \; d\mathcal{H}^{N-1} ds = 0 \tag{59}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $t \in [0,T]$. Now, if $w \in L^1(\partial\Omega)$, we take $w_k \in BV(\Omega) \cap L^\infty(\Omega)$ such that $w_{k|\partial\Omega} = T_k(w)$, and letting $k \to \infty$, it follows that

$$\int_0^t \int_{\partial\Omega} [\mathbf{z}(s), \nu] w \; d\mathcal{H}^{N-1} ds = 0$$

for all $w \in L^1(\partial\Omega)$ and $t \in [0,T]$. Therefore, (57) holds.

Step 4. Next, we prove that $\xi = \operatorname{div}(\mathbf{z})$ in $\left(L^1(0,T,BV(\Omega)_2)\right)^*$ in the sense of the Definition 5. To do that, let us first observe that $(\mathbf{z}, Dw)$, defined by

(45), is a Radon measure in Q_T for all $w \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$. Let $\phi \in \mathcal{D}(Q_T)$, then

$$\langle (\mathbf{z}, Dw), \phi\rangle = -\int_0^T \langle \xi(t), w(t)\phi(t)\rangle \, dt - \int_0^T \int_\Omega \mathbf{z}(t,x)\cdot\nabla\phi(t,x) w(t,x) \, dxdt$$

$$= -\int_0^T \langle \xi(t) - \xi^\alpha(t), w(t)\phi(t)\rangle \, dt - \int_0^T \langle \xi^\alpha(t), w(t)\phi(t)\rangle \, dt$$

$$- \int_0^T \int_\Omega \mathbf{z}(t,x)\cdot\nabla\phi(t,x) w(t,x) \, dxdt$$

$$= -\int_0^T \langle \xi(t) - \xi^\alpha(t), w(t)\phi(t)\rangle \, dt + \int_0^T \langle (\mathbf{z}^\alpha(t), Dw(t)), \phi(t)\rangle \, dt$$

$$+ \int_0^T \int_\Omega \mathbf{z}^\alpha(t,x)\cdot\nabla\phi(t,x) w(t,x) \, dxdt - \int_0^T \int_\Omega \mathbf{z}(t,x)\cdot\nabla\phi(t,x) w(t,x) \, dxdt$$

Then, taking limits in α, and using (53), we get

$$\langle (\mathbf{z}, Dw), \phi\rangle = \lim_\alpha \int_0^T \langle (\mathbf{z}^\alpha(t), Dw(t)), \phi(t)\rangle \, dt. \tag{60}$$

Therefore, under any set of assumptions of the Theorem, we have

$$|\langle (\mathbf{z}, Dw), \phi\rangle| \le M\|\phi\|_\infty \int_0^T |Dw(t)| \, dt.$$

Hence, $(\mathbf{z}, Dw)$ is a Radon measure in Q_T. Moreover, from (60), applying Green's formula we obtain that

$$\int_{Q_T} (\mathbf{z}, Dw) = \lim_\alpha \int_0^T (\mathbf{z}^\alpha(t), Dw(t)) \, dt$$

$$= -\lim_\alpha \int_0^T \int_\Omega \operatorname{div}(\mathbf{z}_\alpha(t)) w(t) \, dxdt = -\int_0^T \langle \xi(t), w(t)\rangle \, dt,$$

that is,

$$\int_{Q_T} (\mathbf{z}, Dw) + \int_0^T \langle \xi(t), w(t)\rangle \, dt = 0. \tag{61}$$

Step 5. Let $p = T_{a,b}$ be any cut-off function, let j be the primitive of p. Let $0 \le \phi \in \mathcal{D}((0,T)\times\Omega)$. Multiplying (48) by $p(u^{n+1})\phi(t,x)$, $t \in (t_n, t_{n+1}]$ integrating in $(t_n, t_{n+1}] \times \Omega$ and adding from $n = 0$ to $n = K-1$, we have

$$\sum_{n=0}^{K-1} \int_{t_n}^{t_{n+1}} \int_\Omega \frac{u^{n+1} - u^n}{\Delta t} p(u^{n+1})\phi \, dxdt + \int_0^T \int_\Omega (\mathbf{z}^K(t), D(p(u^K(t+\Delta t))\phi) \, dt = 0. \tag{62}$$

Since ϕ has compact support in time in $(0,T)$, for K large enough we have

$$-\sum_{n=0}^{K-1}\int_{t_n}^{t_{n+1}}\int_\Omega \frac{u^{n+1}-u^n}{\Delta t}p(u^{n+1})\phi\,dxdt \le \sum_{n=0}^{K-1}\int_{t_n}^{t_{n+1}}\int_\Omega \frac{j(u^n)-j(u^{n+1})}{\Delta t}\phi\,dxdt$$

$$= \sum_{n=0}^{K-1}\int_{t_n}^{t_{n+1}}\int_\Omega \frac{j(u^K(t))-j(u^K(t+\Delta t))}{\Delta t}\phi\,dxdt = \int_0^T\int_\Omega j(u^K(t))\frac{\phi(t)-\phi(t-\Delta t)}{\Delta t}\,dxdt.$$

Hence, from (62) it follows that

$$\int_0^T\int_\Omega (\mathbf{z}^K(t), D(p(u^K(t+\Delta t))\phi)\,dt \le \int_0^T\int_\Omega j(u^K(t))\frac{\phi(t)-\phi(t-\Delta t)}{\Delta t}\,dxdt. \tag{63}$$

Let $\mathcal{U}$ be an ultrafilter in $\mathbb{N}$ containing the Fréchet filter. We recall that if $\{a_n\}_n$ is a bounded sequence in $\mathbb{R}$, then $\{a_n\}_n$ is convergent along the ultrafilter $\mathcal{U}$ and $\lim_{\mathcal{U}}$ defines a bounded linear operator on $\ell^\infty(\mathbb{N})$ ([34], Chapter IV). Moreover, since $\mathcal{U}$ contains the Fréchet filter any convergent sequence $\{a_n\}_n$ also converges along $\mathcal{U}$ and $\lim_{\mathcal{U}} a_n = \lim_{n\to\infty} a_n$. Using (63), and taking limits in (62) along the ultrafilter $\mathcal{U}$, we obtain

$$\lim_{\mathcal{U}}\int_0^T\int_\Omega (\mathbf{z}^K(t), D(p(u^K(t+\Delta t))\phi)\,dt \le \int_0^T\int_\Omega j(u(t))\phi'(t)\,dxdt, \tag{64}$$

which in turn gives

$$\begin{aligned}&\lim_{\mathcal{U}}\int_0^T\int_\Omega \phi(\mathbf{z}^K(t), Dp(u^K(t+\Delta t))\,dt\\ &\le \int_0^T\int_\Omega j(u(t))\phi'(t)\,dxdt - \int_0^T\int_\Omega \mathbf{z}(t)\cdot\nabla\phi p(u(t))\,dxdt.\end{aligned} \tag{65}$$

Next, let us prove that $p(u(\cdot)) \in L^1_{loc,w}(0,T,BV(\Omega))$. Given $\epsilon > 0$, if we take in (63) $0 \le \phi \in \mathcal{D}((0,T))$ such that $\phi(t) = 1$ for $t \in (\epsilon, T-\epsilon)$, we get

$$\begin{aligned}&\int_\epsilon^{T-\epsilon}\int_\Omega (\mathbf{z}^K(t), D(p(u^K(t+\Delta t)))\,dt\\ &\le \int_0^T\int_\Omega j(u^K(t))\frac{\phi(t)-\phi(t-\Delta t)}{\Delta t}\,dxdt \le M_\epsilon.\end{aligned}$$

This implies that

$$\{(\mathbf{z}^K(t), Dp(u^K(t+\Delta t))\} \quad \text{is a bounded sequence in } L^1_{loc}((0,T),\mathcal{M}_b(\Omega)),$$

where $\mathcal{M}_b(\Omega)$ denotes the space of bounded Radon measures in Ω. Thus, there is $\mu_p \in \mathcal{M}_b(Q_T)$ such that

$$\lim_{\mathcal{U}}\int_0^T\int_\Omega \phi\left(\mathbf{z}^K(t), Dp(u^K(t+\Delta t))\right) = \langle\mu_p,\phi\rangle.$$

On the other hand, having in mind (29) and (50), we obtain that

$$\int_{\epsilon}^{T-\epsilon} \int_{\Omega} |D(p(u^K(t+\Delta t))| \, dt \leq \tilde{M}_{\epsilon}. \tag{66}$$

Moreover, by Lemma 5 of [3], the map $t \mapsto \|p(u^K(t))\|_{BV(\Omega)}$ is measurable, then by the Fatou's Lemma and (66), it follows that

$$\int_{\epsilon}^{T-\epsilon} \liminf_{K\to\infty} \int_{\Omega} |D(p(u^K(t+\Delta t))| \, dt \leq \liminf_{K\to\infty} \int_{\epsilon}^{T-\epsilon} \int_{\Omega} |D(p(u^K(t+\Delta t))| \, dt \leq \tilde{M}_{\epsilon}. \tag{67}$$

Now, since the total variation is lower semi-continuous in $L^1(\Omega)$, we have

$$\int_{\Omega} |Dp(u(t))| \leq \liminf_{K\to\infty} \int_{\Omega} |Dp(u^K(t))|,$$

thus we deduce that $p(u(t)) \in BV(\Omega)$ for almost all $t \in (0,T)$ and consequently $u(t) \in TBV(\Omega)$. Then, by (67), applying again Lemma 5 of [3], we obtain that

$$p(u(\cdot)) \in L^1_{loc,w}(0,T,BV(\Omega)). \tag{68}$$

Step 6. Identification of the vector field. Let us now prove that

$$\mathbf{z}(t) = \mathbf{a}(u(t), \nabla u(t)) \qquad \text{a.e.} \quad t \in (0,T). \tag{69}$$

Let $0 \leq \phi \in C_0^1((0,T)\times\Omega)$ and $g \in C^1([0,T]\times\overline{\Omega})$, and C^2 with respect to x. Let $T = T_{a,b}$, $a < b$ (in case that $v \in L^\infty(\Omega)$ we could dismiss the use of the cut-off functions $T_{a,b}$). For simplicity, we write $T'(r)$ to mean $\chi_{(a,b)}(r)$. As in the proof of Theorem 2 (see [5]), let

$$J_{\mathbf{a}_i}(x,r) := \int_0^r \mathbf{a}_i(s, \nabla g(x)) \, ds, \quad \text{and} \quad J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x,r) := \int_0^r \frac{\partial}{\partial x_j} \mathbf{a}_i(s, \nabla g(x)) \, ds,$$

$i, j \in \{1, \ldots, N\}$. For simplicity, let us write

$$\begin{aligned} &D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) := \\ &\sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i}(x, T(u^K(t+\Delta t))) - J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x, T(u^K(t+\Delta t))) \right]. \end{aligned} \tag{70}$$

Let us make some remarks concerning the measure $D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))$, which using Volpert's averaged superposition

$$\overline{\mathbf{a}}(T(u^K(t+\Delta t)), \nabla g(x))$$

$$= \int_0^1 \mathbf{a}(\tau (T(u^K(t+\Delta t)))^+ + (1-\tau)(T(u^K(t+\Delta t)))^-, \nabla g(x)) \, d\tau$$

and chain's rule for BV functions ([1], Theorem 3.96) as we did in the proof of Theorem 2 (see [5]), can be written as

$$D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))$$

$$= \mathbf{a}(u^K(t+\Delta t), \nabla g)\cdot\nabla T(u^K(t+\Delta t)) + \overline{\mathbf{a}}(T(u^K(t+\Delta t)), \nabla g(x))\cdot D^s T(u^K(t+\Delta t)).$$

In particular, we observe that the absolutely continuous part of

$$D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))$$

is

$$\mathbf{a}(u^K(t+\Delta t), \nabla g)\cdot\nabla T(u^K(t+\Delta t)).$$

We note that using (50) and (26) we have

$$\int_0^T \int_\Omega \phi(\mathbf{z}^K(t), D(T(u^K(t+\Delta t)) - g))\, dt$$

$$-\int_0^T \int_\Omega \phi\left[D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) - \mathbf{a}(u^K(t+\Delta t), \nabla g)\cdot\nabla g\right]\, dt$$

$$\geq \int_0^T \int_\Omega \phi\left[h(u^K(t+\Delta t), DT(u^K(t+\Delta t))) - \mathbf{z}^K(t)\cdot\nabla g\right]\, dt$$

$$+\int_0^T \int_\Omega \phi\mathbf{a}(u^K(t+\Delta t), \nabla g)\cdot(\nabla g - \nabla T(u^K(t+\Delta t)))\, dxdt - \int_0^T \int_\Omega \phi(D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))))^s\, dt$$

$$= \int_0^T \int_\Omega \phi(\mathbf{a}(u^K(t+\Delta t), \nabla u^K(t+\Delta t)))(\nabla T(u^K(t+\Delta t)) - \nabla g)\, dx\, dt$$

$$-\int_0^T \int_\Omega \phi\mathbf{a}(u^K(t+\Delta t), \nabla g)\cdot(\nabla T(u^K(t+\Delta t)) - \nabla g)\, dxdt$$

$$+\int_0^T \int_\Omega \phi\left[h(u^K(t+\Delta t), DT(u^K(t+\Delta t)))^s - (D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))))^s\right]\, dt$$

$$\geq -\int_0^T \int_\Omega \phi(\mathbf{a}(u^K(t+\Delta t), \nabla u^K(t+\Delta t)) - \mathbf{a}(u^K(t+\Delta t), \nabla g))\cdot\nabla g\,(1 - T'(u^K(t+\Delta t)))$$

$$+\int_0^T \int_\Omega \phi\left[h(u^K(t+\Delta t), DT(u^K(t+\Delta t)))^s - (D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))))^s\right]\, dt.$$

On the other hand, by (H_5), (H_6) and using the chain rule for BV-functions,

$$(D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))))^s = \overline{\mathbf{a}}(T(u^K(t+\Delta t)), \nabla g(x)) \cdot D^s T(u^K(t+\Delta t))$$

$$= \overline{\mathbf{a}}(T(u^K(t+\Delta t)), \nabla g(x)) \cdot \frac{D^s T(u^K(t+\Delta t))}{|D^s T(u^K(t+\Delta t))|} |D^s T(u^K(t+\Delta t))|$$

$$= \int_0^1 \mathbf{a}(\tau (T(u^K(t+\Delta t)))^+ + (1-\tau)(T(u^K(t+\Delta t)))^-, \nabla g(x)) \cdot \frac{D^s T(u^K(t+\Delta t))}{|D^s T(u^K(t+\Delta t))|} d\tau \, |D^s T(u^K(t+\Delta t))|$$

$$\leq \int_0^1 h^0 \left(\tau (T(u^K(t+\Delta t)))^+ + (1-\tau)(T(u^K(t+\Delta t)))^-, \frac{D^s T(u^K(t+\Delta t))}{|D^s T(u^K(t+\Delta t))|} \right) d\tau \, |D^s T(u^K(t+\Delta t))|$$

$$= \int_0^1 \varphi(\tau (T(u^K(t+\Delta t)))^+ + (1-\tau)(T(u^K(t+\Delta t)))^-) \, d\tau \, \psi^0 \left(\frac{D^s T(u^K(t+\Delta t))}{|D^s T(u^K(t+\Delta t))|} \right) |D^s T(u^K(t+\Delta t))|$$

$$= \psi^0 \left(\frac{DT(u^K(t+\Delta t))}{|DT(u^K(t+\Delta t))|} \right) |D^s J_\varphi(T(u^K(t+\Delta t)))|$$

$$= h(T(u^K(t+\Delta t)), DT(u^K(t+\Delta t)))^s = h(u^K(t+\Delta t), DT(u^K(t+\Delta t)))^s.$$

Consequently,

$$\int_0^T \int_\Omega \phi \left[h(u^K(t+\Delta t), DT(u^K(t+\Delta t)))^s - (D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))))^s \right] dt \geq 0.$$

Moreover, we have

$$\int_0^T \int_\Omega \phi(\mathbf{a}(u^K(t+\Delta t), \nabla u^K(t+\Delta t)) - \mathbf{a}(u^K(t+\Delta t), \nabla g)) \cdot \nabla g \, (1 - T'(u^K(t+\Delta t)))$$

$$\leq \int_0^T \int_\Omega \phi(\mathbf{a}(u^K(t+\Delta t), \nabla u^K(t+\Delta t)) - \mathbf{a}(u^K(t+\Delta t), \nabla g)) \cdot \nabla g \, (1 - T'(u^K(t+\Delta t))) T'(u(t))$$

$$+ M \|\nabla g\|_\infty \int_0^T \int_\Omega \phi \, (1 - T'(u(t)))$$

Therefore, we obtain that

$$\int_0^T \int_\Omega \phi(\mathbf{z}^K(t), D(T(u^K(t+\Delta t)) - g))\, dt$$
$$- \int_0^T \int_\Omega \phi \left[D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) - \mathbf{a}(u^K(t+\Delta t), \nabla g) \cdot \nabla g\right]\, dt$$
$$+ \int_0^T \int_\Omega \phi(\mathbf{a}(u^K(t+\Delta t), \nabla u^K(t+\Delta t)) - \mathbf{a}(u^K(t+\Delta t), \nabla g)) \cdot \nabla g\,(1 - T'(u^K(t+\Delta t)))T'(u(t))$$
$$+ M\|\nabla g\|_\infty \int_0^T \int_\Omega \phi\,(1 - T'(u(t))) \geq 0. \tag{71}$$

Our purpose is to take limits as $K \to \infty$ in the above inequality. We assume that $\phi(t,x) = \eta(t)\rho(x)$, where $\eta \in \mathcal{D}(0,T)$, $\rho \in \mathcal{D}(\Omega)$, $\eta \geq 0$, $\rho \geq 0$. Let j denote the primitive of T. First, integrating by parts in the first term, for Δt small enough we have

$$\int_0^T \int_\Omega \phi(\mathbf{z}^K(t), D(T(u^K(t+\Delta t)) - g))\, dt = - \int_0^T \int_\Omega (T(u^K(t+\Delta t)) - g)\mathbf{z}^K(t) \cdot \nabla_x \phi(t)\, dxdt$$
$$- \int_0^T \int_\Omega \phi(t)(T(u^K(t+\Delta t)) - g)\mathrm{div}(\mathbf{z}^K(t))\, dxdt = - \int_0^T \int_\Omega (T(u^K(t+\Delta t)) - g)\mathbf{z}^K(t) \cdot \nabla_x \phi(t)\, dxdt$$
$$- \int_0^T \int_\Omega \phi(t)(T(u^K(t+\Delta t)) - g)\xi^K(t)\, dxdt.$$

Now,

$$\int_0^T \int_\Omega \phi(t)(T(u^K(t+\Delta t)) - g)\xi^K(t)\, dxdt$$
$$= \int_0^T \int_\Omega \phi(t) T(u^K(t+\Delta t)) \frac{u^K(t+\Delta t) - u^K(t)}{\Delta t}\, dt - \int_0^T \int_\Omega \phi(t) g \xi^K(t)\, dt$$
$$\geq \int_0^T \int_\Omega \phi(t) \frac{j(u^K(t+\Delta t)) - j(u^K(t))}{\Delta t}\, dt - \int_0^T \int_\Omega \phi(t) g \xi^K(t)\, dt$$
$$= \int_0^T \int_\Omega \frac{\phi(t-\Delta t) - \phi(t)}{\Delta t} j(u^K(t))\, dt - \int_0^T \int_\Omega \phi(t) g \xi^K(t)\, dt.$$

Hence,

$$\int_0^T \int_\Omega \phi(\mathbf{z}^K(t), D(T(u^K(t+\Delta t)) - g)) \leq - \int_0^T \int_\Omega \frac{\phi(t-\Delta t) - \phi(t)}{\Delta t} j(u^K(t))\, dt$$
$$+ \int_0^T \int_\Omega \phi(t) g \xi^K(t)\, dt - \int_0^T \int_\Omega (T(u^K(t+\Delta t)) - g)\mathbf{z}^K(t) \cdot \nabla_x \phi(t)\, dxdt.$$

Then, from (71) it follows that

$$
\begin{aligned}
&-\int_0^T\int_\Omega \frac{\phi(t-\Delta t)-\phi(t)}{\Delta t} j(u^K(t))\,dt + \int_0^T\int_\Omega \phi(t) g \xi^K(t)\,dt \\
&-\int_0^T\int_\Omega (T(u^K(t+\Delta t))-g)\mathbf{z}^K(t)\cdot\nabla_x\phi(t)\,dxdt \\
&+\int_0^T\int_\Omega \phi\left[-D_2 J_{\mathbf{a}}(x,T(u^K(t+\Delta t)))+\mathbf{a}(u^K(t+\Delta t),\nabla g)\cdot\nabla g\right]\,dt \\
&+\int_0^T\int_\Omega \phi(\mathbf{a}(u^K(t+\Delta t),\nabla u^K(t+\Delta t))-\mathbf{a}(u^K(t+\Delta t),\nabla g))\cdot\nabla g\,(1-T'(u^K(t+\Delta t)))T'(u(t)) \\
&+M\|\nabla g\|_\infty\int_0^T\int_\Omega \phi\,(1-T'(u(t)))\ge 0.
\end{aligned}
\tag{72}
$$

Letting $K\to\infty$ in (72), observing that the integral involving the term $(1-T'(u^K(t+\Delta t)))T'(u(t))$ goes to zero, and having in mind that $D_2J_{\mathbf{a}}(x,T(u^K(t+\Delta t)))\to D_2J_{\mathbf{a}}(x,T(u(t+\Delta t)))$ weakly as measures as in the proof of 2 (see [5]), we obtain

$$
\begin{aligned}
&\int_0^T\int_\Omega \phi'(t)j(u(t))\,dt+\int_0^T\langle\phi(t)g,\xi(t)\rangle\,dt-\int_0^T\int_\Omega (T(u(t))-g)\mathbf{z}(t)\cdot\nabla_x\phi(t)\,dxdt \\
&\qquad+\int_0^T\int_\Omega \phi\left[-D_2J_{\mathbf{a}}(x,T(u(t)))+\mathbf{a}(u(t),\nabla g)\cdot\nabla g\right]\,dt \\
&\qquad\qquad+M\|\nabla g\|_\infty\int_0^T\int_\Omega \phi\,(1-T'(u(t)))\ge 0.
\end{aligned}
$$

Now, using (61), we get

$$
\begin{aligned}
&\int_0^T\int_\Omega \phi'(t)j(u(t))\,dt-\int_0^T\int_\Omega \phi(t)\mathbf{z}(t)\cdot\nabla g\,dxdt \\
&-\int_0^T\int_\Omega T(u(t))\mathbf{z}(t)\cdot\nabla_x\phi(t)\,dxdt \\
&+\int_0^T\int_\Omega \phi\left[-D_2J_{\mathbf{a}}(x,T(u(t)))+\mathbf{a}(u(t),\nabla g)\cdot\nabla g\right]\,dt \\
&+M\|\nabla g\|_\infty\int_0^T\int_\Omega \phi\,(1-T'(u(t)))\ge 0.
\end{aligned}
\tag{73}
$$

For any $\tau>0$, we define the function η^τ, as the Dunford integral (see [27])

$$
\eta^\tau(t):=\frac{1}{\tau}\int_{t-\tau}^t \eta(s)T(u(s))\,ds\in BV(\Omega)^{**},
$$

that is,

$$\langle \eta^\tau(t), w\rangle = \frac{1}{\tau}\int_{t-\tau}^{t} \langle \eta(s)T(u(s)), w\rangle \, ds$$

for any $w \in BV(\Omega)^*$. Then $\eta^\tau \in C([0,T]; BV(\Omega))$. Moreover, $\eta^\tau(t) \in L^2(\Omega)$, and, thus, $\eta^\tau(t) \in BV(\Omega)_2$. In [3] we prove that η^τ admits a weak derivative in $L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$.

On the other hand,

$$\begin{aligned}\int_0^T \int_\Omega \phi'(t) j(u(t))\, dt &= \lim_{\tau\to 0}\int_0^T \int_\Omega \frac{\eta(t-\tau)-\eta(t)}{-\tau} j(u(t))\rho(x)\, dxdt \\ &= \lim_{\tau\to 0}\int_0^T \int_\Omega \frac{j(u(t+\tau))-j(u(t))}{-\tau}\eta(t)\rho(x)\, dxdt.\end{aligned} \tag{74}$$

Now, by (56) we have

$$\int_0^T \int_\Omega \frac{j(u(t+\tau))-j(u(t))}{-\tau}\eta(t)\rho(x)\, dxdt \le -\int_0^T \int_\Omega T(u(t))\frac{u(t+\tau)-u(t)}{\tau}\eta(t)\rho(x)\, dxdt$$

$$= \int_0^T \int_\Omega u(t)\rho(x) \frac{T(u(t-\tau))\eta(t-\tau) - T(u(t))\eta(t)}{-\tau}\, dxdt = \int_0^T \int_\Omega u(t)\rho(x)\frac{d}{dt}\eta^\tau(t)\, dxdt$$

$$= -\int_0^T \langle \xi(t), \rho\eta^\tau(t)\rangle\, dt = -\lim_{K\to\infty}\int_0^T \langle \xi^K(t), \rho\eta^\tau(t)\rangle\, dt$$

$$= -\lim_{K\to\infty}\int_0^T \left\langle \operatorname{div}(z^K(t)), \rho\frac{1}{\tau}\int_{t-\tau}^t \eta(s)T(u(s))\, ds \right\rangle dt$$

$$= \lim_{K\to\infty}\int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_\Omega (\mathbf{z}^K(t), D(\rho T(u(s))))\, ds\, dt$$

$$= \lim_{K\to\infty}\int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_\Omega T(u(s))\mathbf{z}^K(t)\cdot\nabla_x\rho\, dx\, ds\, dt$$

$$+ \lim_{K\to\infty}\int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_\Omega \rho(\mathbf{z}^K(t), D(T(u(s))))\, ds\, dt$$

$$= \int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_\Omega T(u(s))\mathbf{z}(t)\cdot\nabla_x\rho\, dx\, ds\, dt$$

$$+ \lim_{K\to\infty}\int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_\Omega \rho\mathbf{z}^K(t)\cdot[\nabla T(u(s)) + D^s T(u(s))]\, dx\, ds\, dt$$

$$\le \int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_\Omega T(u(s))\mathbf{z}(t)\cdot\nabla_x\rho\, dx\, ds\, dt$$

$$+ \int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_\Omega \rho\mathbf{z}(t)\cdot\nabla T(u(s))\, dx\, ds\, dt + \int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(s)\int_\Omega \rho M|D^s(T(u(s)))|\, ds\, dt.$$

Then, from (74) it follows that

$$\int_0^T \int_\Omega \phi'(t) j(u(t))\, dt \leq \int_0^T \eta(t) \int_\Omega T(u(t))\mathbf{z}(t) \cdot \nabla_x \rho \, dx\, dt$$

$$+ \int_0^T \eta(t) \int_\Omega \rho \mathbf{z}(t) \cdot \nabla T(u(t))\, dx\, dt + \int_0^T \eta(t) \int_\Omega \rho M |D^s T(u(t))|\, dt.$$

Hence, taking into account (73), we obtain

$$0 \leq \int_0^T \eta(t) \int_\Omega T(u(t))\mathbf{z}(t) \cdot \nabla_x \rho \, dx\, dt + \int_0^T \eta(t) \int_\Omega \rho \mathbf{z}(t) \cdot \nabla T(u(t))\, dx\, dt$$

$$+ \int_0^T \eta(t) \int_\Omega \rho M |D^s T(u(t))|\, dt - \int_0^T \int_\Omega \phi(t) \mathbf{z}(t) \cdot \nabla g\, dx dt - \int_0^T \int_\Omega T(u(t))\mathbf{z}(t) \cdot \nabla_x \phi(t)\, dx dt$$

$$+ \int_0^T \int_\Omega \phi \left[-D_2 J_{\mathbf{a}}(x, T(u(t))) + \mathbf{a}(u(t), \nabla g) \cdot \nabla g \right]\, dt$$

$$+ M \|\nabla g\|_\infty \int_0^T \int_\Omega \phi \,(1 - T'(u(t))) = \int_0^T \eta(t) \int_\Omega \rho \mathbf{z}(t) \cdot \nabla T(u(t))\, dx\, dt$$

$$+ \int_0^T \eta(t) \int_\Omega \rho M |D^s T(u(t))|\, dt - \int_0^T \eta(t) \int_\Omega \rho(x) \mathbf{z}(t) \cdot \nabla g\, dx dt$$

$$+ \int_0^T \eta(t) \int_\Omega \rho(x) \left[-\mathbf{a}(u, \nabla g) \cdot (\nabla T(u(t)) - \nabla g) \right]\, dx dt - \int_0^T \eta(t) \int_\Omega \rho (D_2 J_{\mathbf{a}}(x, T(u(t))))^s\, dt$$

$$+ M \|\nabla g\|_\infty \int_0^T \int_\Omega \phi \,(1 - T'(u(t)))$$

$$= \int_0^T \eta(t) \int_\Omega \rho(x) (\mathbf{z}(t) - \mathbf{a}(u(t), \nabla g)) \cdot (\nabla T(u(t)) - \nabla g)\, dx dt$$

$$+ \int_0^T \eta(t) \int_\Omega \rho (M |D^s T(u(t))| - (D_2 J_{\mathbf{a}}(x, T(u(t))))^s)\, dt$$

$$+ M \|\nabla g\|_\infty \int_0^T \int_\Omega \phi \,(1 - T'(u(t)))$$

for all $\phi(t,x) = \eta(t)\rho(x)$, $\eta \in \mathcal{D}(0,T)$, $\rho \in \mathcal{D}(\Omega)$, $\eta, \rho \geq 0$. Thus, the measure

$$([\mathbf{z}(t) - \mathbf{a}(u(t), \nabla g)] \cdot \nabla (T(u(t)) - g) + M |D^s T(u(t))| - (D_2 J_{\mathbf{a}}(x, T(u(t))))^s)$$

$$+ M \|\nabla g\|_\infty \,(1 - T'(u(t))) \geq 0.$$

Then its absolutely continuous part

$$[\mathbf{z}(t) - \mathbf{a}(u(t), \nabla g)] \cdot \nabla (u(t) - g) + M \|\nabla g\|_\infty \,(1 - T'(u(t))) \geq 0.$$

In particular, for $x \in [a \leq u(t) \leq b]$ we have

$$[\mathbf{z}(t) - \mathbf{a}(u(t), \nabla g)] \cdot \nabla (u(t) - g) \geq 0 \quad \text{a.e..}$$

Since this holds for any values of a, b with $a < b$, we have that the inequality holds a.e. in Ω. Since we may take a countable set dense in $C^1([0,T]\times\overline{\Omega})$, we have that the above inequality holds for all $(t,x) \in S$ where $S \subseteq (0,T)\times\Omega$ is such that $\mathcal{L}^N((0,T)\times\Omega\setminus S) = 0$, and all $g \in C^1([0,T]\times\overline{\Omega})$. Now, fixed $(t,x)\in S$, and given $y \in I\!\!R^N$, there is $g \in C^1([0,T]\times\overline{\Omega})$ such that $\nabla g(t,x) = y$. Then

$$(\mathbf{z}(t,x) - \mathbf{a}(u(t),y))\cdot(\nabla u(t,x) - y) \geq 0 \quad \forall y \in I\!\!R^N \quad \text{and} \quad \forall (t,x)\in S.$$

and, by an application of the Minty-Browder's method in $I\!\!R^N$, it follows that

$$\mathbf{z}(t,x) = \mathbf{a}(u(t,x), \nabla u(t,x)) \qquad \text{a.e.} \quad (t,x)\in Q_T,$$

and (69) follows.

Step 7. $u(t)$ satisfies the entropy inequalities.

Lemma 1. *As above, let $p = T_{a,b}$, $a < b$. We have*

$$\mathbf{z}\cdot\nabla p(u) + f(p(u), Dp(u))^s \leq \mu_p.$$

In other words,

$$\mu_p \geq h(u, Dp(u)).$$

Proof. By (50) we have

$$\mathbf{z}^K(t)\cdot D^s p(u^K(t+\Delta t))) \geq f(u^K(t+\Delta t), Dp(u^K(t+\Delta t)))^s$$

for all $t \in (0,T)$. Let $0 \leq \phi \in C_0(Q_T)$. Taking the above inequality into account and the convexity of f, we compute

$$\int_0^T\int_\Omega \mathbf{z}^K(t)\cdot\nabla p(u(t))\,\phi\,dxdt \leq \int_0^T\int_\Omega \mathbf{z}^K(t)\cdot\nabla p(u^K(t+\Delta t))\,\phi\,dxdt$$

$$+\int_0^T\int_\Omega f(p(u^K(t+\Delta t)),\nabla p(u(t)))\,\phi\,dxdt - \int_0^T\int_\Omega f(p(u^K(t+\Delta t)),\nabla p(u^K(t+\Delta t)))\,\phi\,dxdt$$

$$\leq \int_0^T\int_\Omega \phi(\mathbf{z}^K(t), Dp(u^K(t+\Delta t))\,dt - \int_0^T\int_\Omega \phi f(u^K(t+\Delta t), Dp(u^K(t+\Delta t)))^s\,dt$$

$$+\int_0^T\int_\Omega f(p(u^K(t+\Delta t)),\nabla p(u(t)))\,\phi\,dxdt - \int_0^T\int_\Omega f(p(u^K(t+\Delta t)),\nabla p(u^K(t+\Delta t)))\,\phi\,dxdt$$

$$= \int_0^T\int_\Omega \phi(\mathbf{z}^K(t), Dp(u^K(t+\Delta t))\,dt + \int_0^T\int_\Omega f(p(u^K(t+\Delta t)),\nabla p(u(t)))\,\phi\,dxdt$$

$$-\int_0^T\int_\Omega \phi f(p(u^K(t+\Delta t)), Dp(u^K(t+\Delta t)))\,dt.$$

Letting $K\to\infty$ and using that

$$\int_0^T\int_\Omega \phi f(p(u(t)), Dp(u(t)))\,dt \leq \liminf_K \int_0^T\int_\Omega \phi f(p(u^K(t+\Delta t)), Dp(u^K(t+\Delta t)))\,dt,$$

we obtain

$$\int_0^T \int_\Omega \mathbf{z}(t) \cdot \nabla p(u(t))\, \phi \, dxdt$$

$$\leq \langle \mu_p, \phi\rangle + \int_0^T \int_\Omega f(p(u(t)), \nabla p(u(t)))\phi \, dxdt - \int_0^T \int_\Omega \phi f(p(u(t)), Dp(u(t)))\, dt$$

$$= \langle \mu_p, \phi\rangle - \int_0^T \int_\Omega \phi f(p(u(t)), Dp(u(t)))^s \, dt,$$

and the proof of the lemma is complete. □

From the above lemma, using (65) we obtain that the mild-solution u satisfies the entropy inequalities

$$\begin{aligned} &\int_0^T \int_\Omega \phi h(u, Dp(u))\, dt \leq \\ &\int_0^T \int_\Omega j(u(t))\phi'(t)\, dxdt - \int_0^T \int_\Omega \mathbf{z}(t) \cdot \nabla \phi(t) p(u(t))\, dxdt. \end{aligned} \tag{75}$$

for any cut-off function p and any ϕ a smooth function of compact support in time, in particular of the form $\phi(t,x) = \phi_1(t)\rho(x)$, $\phi_1 \in \mathcal{D}((0,T))$, $\rho \in C^1(\overline{\Omega})$. Now, given $l \in I\!\!R$ and $k > 0$, if we take in (75) $p(r) := T_k(r-l) = T_{l-k,l+k}(r) - l$ and $j(r) = \int_l^r p(s)\, ds$, we obtain that

$$-\int_0^T \int_\Omega j_k(u(t)-l)\eta_t \, dxdt + \int_0^T \int_\Omega \eta(t) h(u(t), DT_k(u(t)-l))\, dt$$

$$+ \int_0^T \int_\Omega \mathbf{a}(u(t), \nabla u(t)) \cdot \nabla \eta(t) T_k(u(t)-l)\, dxdt \leq 0,$$

and this concludes the proof of existence of solutions of (1).

Uniqueness of entropy solutions.

Let u be an entropy solution of (1) corresponding to the initial conditions $u_0 \in L^1(\Omega)$, and let $\overline{u}$ be the semigroup solution of (1) corresponding to the initial condition $\overline{u}_0 \in L^\infty(\Omega)$. Notice that, by the existence proof, $\overline{u}$ is also an entropy solution of (1). Then, there exist $\xi, \overline{\xi} \in \left(L^1(0,T,BV(\Omega)_2\right)^*$ such that if $\mathbf{z}(t) := \mathbf{a}(u(t), \nabla u(t))$ and $\overline{\mathbf{z}}(t) := \mathbf{a}(\overline{u}(t), \nabla \overline{u}(t))$, we have $(\mathbf{z}(t), \xi(t)), (\overline{\mathbf{z}}(t), \overline{\xi}(t)) \in Z(\Omega)$ for almost all $t \in [0,T]$ and

$$[\mathbf{z}(t), \nu] = [\overline{\mathbf{z}}(t), \nu] = 0 \quad \text{a.e. in } \; t \in [0,T], \tag{76}$$

$$\xi, \overline{\xi} \;\text{ are the time derivatives of } u, \overline{u} \;\text{ in } \left(L^1(0,T,BV(\Omega)_2)\right)^*, \text{resp.,} \tag{77}$$

$$\xi = \operatorname{div} \mathbf{z}(t) \text{ and } \overline{\xi} = \operatorname{div} \overline{\mathbf{z}}(t) \text{ in the sense of 5,} \tag{78}$$

and, if $l_1, l_2 \in I\!R$, then

$$
\begin{aligned}
&-\int_0^T \int_\Omega j_\epsilon(u(t)-l_1)\eta_t + \int_0^T \int_\Omega \eta(t) h(u(t), DT_\epsilon(u(t)-l_1)) \\
&+\int_0^T \int_\Omega \mathbf{z}(t)\cdot\nabla\eta(t)\; T_\epsilon(u(t)-l_1) \le 0,
\end{aligned}
\tag{79}
$$

and

$$
\begin{aligned}
&-\int_0^T \int_\Omega j_\epsilon(\overline{u}(t)-l_2)\eta_t + \int_0^T \int_\Omega \eta(t) h(\overline{u}(t), DT_\epsilon(\overline{u}(t))-l_2) \\
&+\int_0^T \int_\Omega \overline{\mathbf{z}}(t)\cdot\nabla\eta(t)\; T_\epsilon(\overline{u}(t)-l_2) \le 0,
\end{aligned}
\tag{80}
$$

for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \ge 0$, $\eta(t,x) = \phi(t)\rho(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\rho \in C^\infty(\overline{\Omega})$, and $j_\epsilon(r) = \displaystyle\int_0^r T_\epsilon(s)\; ds$.

We choose two different pairs of variables (t,x), (s,y) and consider u, $\mathbf{z}$ as functions in (t,x), $\overline{u}$, $\overline{\mathbf{z}}$ in (s,y). Let $0 \le \phi \in \mathcal{D}(]0,T[)$, ρ_n a classical sequence of mollifiers in $I\!R^N$ and $\tilde{\rho}_n$ a sequence of mollifiers in $I\!R$. Define

$$
\eta_n(t,x,s,y) := \rho_n(x-y)\tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right).
$$

For (s,y) fixed, if we take in (79) $l_1 = \overline{u}(s,y)$ we get

$$
\begin{aligned}
&-\int_0^T \int_\Omega j_\epsilon(u(t,x)-\overline{u}(s,y))(\eta_n)_t\; dxdt \\
&+\int_0^T \int_\Omega \eta_n h(u(t,x), D_x T_\epsilon(u(t,x)-\overline{u}(s,y)))\; dt \\
&+\int_0^T \int_\Omega \mathbf{z}(t,x)\cdot\nabla_x\eta_n\; T_\epsilon(u(t,x)-\overline{u}(s,y))\; dxdt \le 0.
\end{aligned}
\tag{81}
$$

Similarly, for (t,x) fixed, if we take in (80) $l_2 = u(t,x)$ we get

$$
\begin{aligned}
&-\int_0^T \int_\Omega j_\epsilon(\overline{u}(s,y)-u(t,x))(\eta_n)_s\; dyds \\
&+\int_0^T \int_\Omega \eta_n h(\overline{u}(s,y), D_y T_\epsilon(\overline{u}(s,y)-u(t,x)))\, ds \\
&+\int_0^T \int_\Omega \overline{\mathbf{z}}(s,y)\cdot\nabla_y\eta_n\; T_\epsilon(\overline{u}(s,y)-u(t,x))\, dyds \le 0.
\end{aligned}
\tag{82}
$$

Now, since $T_\epsilon(-r) = -T_\epsilon(r)$, $j_\epsilon(-r) = j_\epsilon(r)$ and $h(x,\xi) = h(x,-\xi)$, we can rewrite the last inequality as

$$\begin{aligned}
&-\int_0^T \int_\Omega j_\epsilon(u(t,x) - \overline{u}(s,y))(\eta_n)_s \; dyds \\
&+\int_0^T \int_\Omega \eta_n h(\overline{u}(s,y), D_y T_\epsilon(u(t,x) - \overline{u}(s,y)))\, ds \\
&-\int_0^T \int_\Omega \overline{\mathbf{z}}(s,y)\cdot \nabla_y \eta_n \; T_\epsilon(u(t,x) - \overline{u}(s,y))\, dyds \leq 0.
\end{aligned} \tag{83}$$

Integrating (81) in (s,y), (83) in (t,x), adding the two inequalities and taking into account that $\nabla_x \eta_n + \nabla_y \eta_n = 0$, we have

$$\begin{aligned}
&-\int_{Q_T\times Q_T} j_\epsilon(u(t,x) - \overline{u}(s,y))\big((\eta_n)_t + (\eta_n)_s\big) \\
&+\int_{Q_T\times Q_T} \eta_n h(u(t,x), D_x T_\epsilon(u(t,x) - \overline{u}(s,y))) \\
&+\int_{Q_T\times Q_T} \eta_n h(\overline{u}(s,y), D_y T_\epsilon(u(t,x) - \overline{u}(s,y))) \\
&+\int_{Q_T\times Q_T} \overline{\mathbf{z}}(s,y)\cdot \nabla_x \eta_n T_\epsilon(u(t,x) - \overline{u}(s,y)) \\
&-\int_{Q_T\times Q_T} \mathbf{z}(t,x)\cdot \nabla_y \eta_n T_\epsilon(u(t,x) - \overline{u}(s,y)) \leq 0.
\end{aligned} \tag{84}$$

Now, by Green's formula and the identities $\mathbf{z}(t,x) = a(u(t,x), \nabla u(t,x))$, $\overline{\mathbf{z}}(s,y) = a(\overline{u}(s,y), \nabla \overline{u}(s,y))$, we have

$$\begin{aligned}
J_n :=& \int_{Q_T\times Q_T} \overline{\mathbf{z}}(s,y)\cdot\nabla_x\eta_n \; T_\epsilon(u(t,x)-\overline{u}(s,y)) + \int_{Q_T\times Q_T} \eta_n h(u(t,x), D_x T_\epsilon(u(t,x)-\overline{u}(s,y))) \\
&-\int_{Q_T\times Q_T} \mathbf{z}(t,x)\cdot\nabla_y\eta_n \; T_\epsilon(u(t,x)-\overline{u}(s,y)) + \int_{Q_T\times Q_T} \eta_n h(\overline{u}(s,y), D_y T_\epsilon(u(t,x)-\overline{u}(s,y))) \\
=& -\int_{Q_T\times Q_T} \eta_n \big(\overline{\mathbf{z}}(s,y), D_x T_\epsilon(u(t,x) - \overline{u}(s,y))\big) \\
&+\int_{Q_T\times Q_T} \eta_n h(u(t,x), D_x T_\epsilon(u(t,x)-\overline{u}(s,y))) + \int_{Q_T\times Q_T} \eta_n \big(\mathbf{z}(t,x), D_y T_\epsilon(u(t,x)-\overline{u}(s,y))\big) \\
&+\int_{Q_T\times Q_T} \eta_n h(\overline{u}(s,y), D_y T_\epsilon(u(t,x) - \overline{u}(s,y))) \\
=& \int_{Q_T\times Q_T} \eta_n (T_\epsilon)'(u(t,x)-\overline{u}(s,y))[\mathbf{z}(t,x)-\overline{\mathbf{z}}(s,y)]\cdot(\nabla_x u(t,x)-\nabla_y \overline{u}(s,y))
\end{aligned}$$

$$-\int_{Q_T\times Q_T}\eta_n\overline{\mathbf{z}}(s,y)\cdot D^s_x T_\epsilon(u(t,x)-\overline{u}(s,y))+\int_{Q_T\times Q_T}\eta_n h(u(t,x),D_xT_\epsilon(u(t,x)-\overline{u}(s,y)))^s$$

$$+\int_{Q_T\times Q_T}\eta_n\mathbf{z}(t,x)\cdot D^s_y T_\epsilon(u(t,x)-\overline{u}(s,y))+\int_{Q_T\times Q_T}\eta_n h(\overline{u}(s,y),D_yT_\epsilon(u(t,x)-\overline{u}(s,y)))^s$$

$$=J_n^1+J_n^2+J_n^3.$$

Let us analyze the term

$$J_n^2:=-\int_{Q_T\times Q_T}\eta_n\overline{\mathbf{z}}(s,y)\cdot D^s_x T_\epsilon(u(t,x)-\overline{u}(s,y))+\int_{Q_T\times Q_T}\eta_n h(u(t,x),D_xT_\epsilon(u(t,x)-\overline{u}(s,y)))^s.$$

Let us define $u_\epsilon(t,x,s,y):=T_{-\epsilon+\overline{u}(s,y),\epsilon+\overline{u}(s,y)}(u(t,x))$. Observe that, since $\overline{u}\in L^\infty(Q_T)$, we have that $u_\epsilon(t,\cdot,s,y)\in BV(\Omega)$. Since

$$T_\epsilon(u(t,x)-\overline{u}(s,y))=T_{-\epsilon+\overline{u}(s,y),\epsilon+\overline{u}(s,y)}(u(t,x))-\overline{u}(s,y)),$$

we have

$$D^s_x(T_\epsilon(u(t,x)-\overline{u}(s,y))=D^s_x u_\epsilon(t,x,s,y)$$

For simplicity, we shall not always write all arguments of these functions. On the other hand, by (H$_5$) and (H$_6$), we have

$$\overline{\mathbf{z}}(s,y)\cdot D^s_x u_\epsilon(t,x,s,y)=\left(\overline{\mathbf{z}}(s,y)\cdot\overrightarrow{D^s_x u_\epsilon(t,x,s,y)}\right)|D^s_x u_\epsilon(t,x,s,y)|$$

$$\le\varphi(\overline{u}(s,y))\psi^0\left(\overrightarrow{D^s_x u_\epsilon(t,x,s,y)}\right)|D^s_x u_\epsilon(t,x,s,y)|.$$

Hence,

$$J_n^2\ge-\int_{Q_T\times Q_T}\eta_n\varphi(\overline{u}(s,y))\psi^0\left(\overrightarrow{D^s_x u_\epsilon(t,x,s,y)}\right)|D^s_x u_\epsilon(t,x,s,y)|$$

$$+\int_{Q_T\times Q_T}\eta_n h(u(t,x),D_x u_\epsilon(t,x,s,y)))^s$$

$$=\int_{Q_T\times Q_T}\eta_n\varphi(u(t,x))\psi^0\left(\overrightarrow{D^s_x u_\epsilon(t,x,s,y)}\right)|D^c_x u_\epsilon(t,x,s,y)|$$

$$-\int_{Q_T\times Q_T}\eta_n\varphi(\overline{u}(s,y))\psi^0\left(\overrightarrow{D^s_x u_\epsilon(t,x,s,y)}\right)|D^c_x u_\epsilon(t,x,s,y)|$$

$$+\int_{Q_T}\int_0^T\left[\int_{J_{u_\epsilon}}\eta_n\frac{1}{(u_\epsilon)^+(x)-(u_\epsilon)^-(x)}\left(\int_{(u_\epsilon)^-(x)}^{(u_\epsilon)^+(x)}\varphi(\tau)\,d\tau\right)\psi^0\left(\overrightarrow{D^s_x u_\epsilon(t,x,s,y)}\right)|D^j_x u_\epsilon(t,x,s,y)|\right]$$

$$-\int_{Q_T}\int_0^T\left(\int_\Omega\eta_n\varphi(\overline{u}(s,y))\psi^0\left(\overrightarrow{D^s_x u_\epsilon(t,x,s,y)}\right)|D^j_x u_\epsilon(t,x,s,y)|\right)$$

$$=J_n^{2,1}+J_n^{2,2},$$

where $J_n^{2,1}$ denotes the first and second terms of the above expression, and $J_n^{2,2}$ the third and fourth terms. Let us write $S = T_{-\|\overline{u}\|-\epsilon,\|\overline{u}\|+\epsilon}$. Now, since φ is Lipschitz continuous, we have

$$|J_n^{2,1}| \leq \int_{Q_T\times Q_T} \eta_n |\varphi(u(t,x)) - \varphi(\overline{u}(s,y))| \psi^0 \left(\overrightarrow{D_x^s u_\epsilon(t,x,s,y)}\right) |D_x^c u_\epsilon(t,x,s,y)|$$

$$\leq R_n \int_{Q_T\times Q_T} |u(t,x) - \overline{u}(s,y)| \, |D_x^c u_\epsilon(t,x)|$$

$$= R_n \int_{Q_T} \int_{\{-\epsilon+\overline{u}(s,y)<u(t,x)<\epsilon+\overline{u}(s,y)\}} |u(t,x) - \overline{u}(s,y)| \, |D_x^c S(u(t,x))|$$

$$\leq \epsilon R_n \int_{Q_T} \int_{\{-\epsilon+\overline{u}(s,y)<u(t,x)<\epsilon+\overline{u}(s,y)\}} |D_x^c S(u(t,x))|$$

$$= \epsilon R_n \int_{Q_T} \left(\int_0^T \left(\int_{-\epsilon+\overline{u}(s,y)}^{\epsilon+\overline{u}(s,y)} \mathrm{Per}(\{S(u)(t,x) \geq \lambda\}) \, d\lambda \right) dt \right) dyds,$$

(for some constant $R_n > 0$) which yields

$$\frac{1}{\epsilon} J_n^{2,1} \geq o(\epsilon) \quad \forall\, n \in I\!N, \tag{85}$$

where $o(\epsilon)$ is an expression that tends to 0 as $\epsilon \to 0^+$.

On the other hand, working in a similar way as before,

$$\frac{1}{\epsilon} J_n^{2,2} \geq o(\epsilon) \quad \forall\, n \in I\!N. \tag{86}$$

Then, by (85) and (86), we get

$$\frac{1}{\epsilon} J_n^{2} \geq o(\epsilon) \quad \forall\, n \in I\!N. \tag{87}$$

In a similar way, we obtain

$$\frac{1}{\epsilon} J_n^{3} \geq o(\epsilon) \quad \forall\, n \in I\!N, \tag{88}$$

where

$$J_n^3 := \int_{Q_T\times Q_T} \eta_n \mathbf{z}(t,x) \cdot D_y^s T_\epsilon(u(t,x) - \overline{u}(s,y)) + \int_{Q_T\times Q_T} \eta_n h(\overline{u}(s,y), D_y T_\epsilon(u(t,x) - \overline{u}(s,y)))^s.$$

Now let us compute J_n^1. By (31) it follows that

$$J_n^1 := \int_{Q_T\times Q_T} \eta_n (T_\epsilon)'(u(t,x) - \overline{u}(s,y)) [\mathbf{z}(t,x) - \overline{\mathbf{z}}(s,y)] \cdot (\nabla_x S(u(t,x)) - \nabla_y \overline{u}(s,y))$$

$$\geq -C \int_{Q_T\times Q_T} \eta_n (T_\epsilon)'(u(t,x) - \overline{u}(s,y)) |u(t,x) - \overline{u}(s,y)| \, \|\nabla_x S(u(t,x)) - \nabla_y \overline{u}(s,y)\|$$

$$\geq -C\epsilon \int_{Q_T \times Q_T} \eta_n (T_\epsilon)'(u(t,x) - \overline{u}(s,y)) \, \|\nabla_x S(u(t,x)) - \nabla_y \overline{u}(s,y)\|.$$

Since $\nabla_x S(u), \nabla_y \overline{u}$ are both integrable, we have that

$$\frac{1}{\epsilon} J_n^1 \geq o(\epsilon) \quad \forall\, n \in I\!N. \tag{89}$$

From (87), (88) and (89), it follows that

$$\frac{1}{\epsilon} J_n \geq o(\epsilon) \quad \forall\, n \in I\!N,$$

which yields, taking into account (84),

$$-\frac{1}{\epsilon} \int_{Q_T \times Q_T} j_\epsilon(u(t,x) - \overline{u}(s,y)) \big((\eta_n)_t + (\eta_n)_s\big) + o(\epsilon) \leq 0. \tag{90}$$

Letting $\epsilon \to 0^+$ in (90), it yields

$$\int_{Q_T \times Q_T} |u(t,x) - \overline{u}(s,y)| \big((\eta_n)_t + (\eta_n)_s\big) \geq 0. \tag{91}$$

Then, since

$$(\eta_n)_t + (\eta_n)_s = \rho_n(x-y) \tilde{\rho}_n(t-s) \phi' \left(\frac{t+s}{2} \right),$$

passing to the limit in (91) as $n \to +\infty$, we obtain

$$\int_{Q_T} |u(t,x) - \overline{u}(t,x)| \phi'(t) \, dxdt \geq 0.$$

Since this is true for all $0 \leq \phi \in \mathcal{D}(]0,T[)$, we have

$$\frac{d}{dt} \int_\Omega |(u(t,x) - \overline{u}(t,x)| \leq 0.$$

Hence

$$\int_\Omega |u(t,x) - \overline{u}(t,x)| \, dx \leq \int_\Omega |u_0(x) - \overline{u}_0(x)| \, dx \qquad \text{for all} \;\; t \geq 0.$$

In particular, let $u_n(t) := T(t)u_0$, where $u_{0,n} \in L^\infty(\Omega)$ and $u_{0,n} \to u_0$ in $L^1(\Omega)$. By the above estimate, we have

$$\int_\Omega |u(t,x) - u_n(t)| \, dx \leq \int_\Omega |u_0 - u_{0,n}| \, dx \qquad \text{for all} \;\; t \geq 0.$$

Consequently, letting $n \to \infty$, we obtain that $u(t) = T(t)u_0$. We have proved that entropy solutions coincide with semigroup solutions. Estimate (46) follows. □

Remark 5. When applying Kruzhkov's method, if instead of multiplying by $T_\epsilon(u(t,x) - \overline{u}(s,y))$ we multiply by $T_\epsilon(u(t,x) - \overline{u}(s,y))^+$, we obtain the estimate

$$\int_\Omega (u(t,x) - \overline{u}(t,x))^+ \, dx \leq \int_\Omega (u_0(x) - \overline{u}_0(x))^+ \, dx, \quad \forall t \geq 0. \tag{92}$$

Remark 6. Note that the proof of 3 shows that under assumptions (H) we have uniqueness of entropy solutions of (1). Moreover, given $u_0 \in L^1(\Omega)$, if there exists an entropy solution $u(t)$ of (1) such that $u(0) = u_0$, then $u(t) = T(t)u_0$ for all $t \geq 0$. Now, we only know that $u(t)$ is an entropy solution (in which case $u(t) = T(t)u_0$) if we assume that the bound (27) holds for any $(z,\xi) \in I\!R \times I\!R^N$, or $u_0 \in L^\infty(\Omega)$. It is an open problem if $u(t) = T(t)u_0$ is an entropy solution for all $u_0 \in L^1(\Omega)$ under assumptions (H).

Remark 7. The following question was raised by one of the referees: can we adapt the uniqueness proof in [25] (for (2)) to the case of several space variables? Indeed, it is difficult to give a complete answer, but we can comment on the main difficulties raised by this extension.

1) The uniqueness proof in [25] is based on the fact that solutions u are BV in (t,x). This implies that the set of jump points of u is countably rectifiable and the proof exploits (as we also do) the structure of BV functions. We know that there exist solutions of (1) such that $T_k(u(t)) \in BV(\Omega)$ for almost any $t \in (0,T)$ and any $k > 0$. If $u \in L^\infty((0,T) \times \Omega)$, then we know that $u(t) \in BV(\Omega)$ for almost any $t \in (0,T)$. But we do not know if u_t is a Radon measure, i.e., we do not know if $u \in BV((0,T) \times \Omega)$ (or $T_k(u) \in BV((0,T) \times \Omega)$ for any $k > 0$). This seems plausible (and it is further supported by the results in [7,8]) but we do not have a proof of it.

2) Even in the case $\mathbf{a}(z,\xi) = \varphi(z)\psi(\xi)$, $z \in I\!R$, $\xi \in I\!R^N$, we do not know if $\psi(\nabla u)$ is continuous. This result was needed to prove a Rankine-Hugoniot formula on the discontinuities of the solution u (Lemma 3.2 in [25]). The continuity of $\psi(\nabla u)$ was used to compute averages of products like $\varphi(u)\psi(\nabla u)$. The only thing we know is that $\operatorname{div} \mathbf{a}(u(t), \nabla u(t)) \in BV(\Omega)^*$. Even if we know that $\operatorname{div} \mathbf{a}(u(t), \nabla u(t))$, or equivalently u_t, is a Radon measure (which in view of the examples in [7,8] seems plausible), or even a function in $L^1(\Omega)$, we would need to make possible the computations in [25], Lemma 3.2. The computations involved would require to prove some normal trace theorems for vector fields whose divergence is a Radon measure (even in the simpler case where the divergence is in L^1 the trace theorems should be proved).

3) The uniqueness proof in [25] is based on a suitable choice of test functions constructed by regularizing the sign of the difference of two solutions. The entropy conditions are imposed in Oleinik's form. The extension of Oleinik's entropy conditions to the case of scalar conservation laws in several space variables was done by Volpert [36] who proved uniqueness of such entropy solutions in case that they are in BV with respect to space and time variables.

In the formulation of Kruzhkov [31] those entropy conditions are written as a family of integral inequalities which coincide with Volpert's conditions in case that our solutions are in BV with respect to (t,x), but extend them since they are valid for functions which are only in L^1_{loc} in (t,x). In this general context, Kruzhkov's formulation of entropy conditions [31] seems more adapted and permits to prove uniqueness by means of the doubling variables technique introduced in [31]. This approach has been pursued for degenerate parabolic and elliptic equations in [20,21,11], to quote some works. The extension to the case of equations of type (6) where $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$, $f(x,\cdot)$ being a convex function of ξ with linear growth as $\|\xi\| \to \infty$ has been considered in [2,3] (see also [6]) where solutions are considered in a sense analogous to 6 and uniqueness is proved using the doubling variables technique. In general, in this case, as in the present paper, we do not know if the solution u is BV in (t,x). Thus, the remaining question is if we can prove in some cases that solutions of (1) are BV in (t,x) and the uniqueness proof given by P. Dal Passo in [25] can be extended to this case. This is an interesting question and deserves further exploration.

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References

1. L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, 2000.
2. F. Andreu, V. Caselles, and J.M. Mazón, *A parabolic quasilinear problem for linear growth functionals.* Rev. Mat. Iberoamericana **18** (2002), 135-185.
3. F. Andreu, V. Caselles, and J.M. Mazón, *Existence and uniqueness of solution for a parabolic quasilinear problem for linear growth functionals with L^1 data.* Math. Ann. **322** (2002), 139-206.
4. F. Andreu, V. Caselles, and J.M. Mazón, *The Cauchy Problem for Linear Growth Functionals with.* J. Evol. Equat. **3** (2003), 39-65.
5. F. Andreu, V. Caselles, and J.M. Mazón, *A Strongly Degenerate Quasilinear Equations: the Elliptic Case.* To appear at the Annali della Scuola Normale Superiore di Pisa, 2004.
6. F. Andreu, V. Caselles, and J.M. Mazón, *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals.* Progress in Mathematics 223, Birkhauser Verlag, 2004.
7. F. Andreu, V. Caselles, J.M. Mazón, and S. Moll, *On the relativistic heat equation and an asymptotic regime of it.* Preprint 2004.
8. F. Andreu, V. Caselles, G. Haro, and J.M. Mazón, *On free boundaries of the relativistic heat equation.* In preparation.
9. G. Anzellotti, *Pairings Between Measures and Bounded Functions and Compensated Compactness*, Ann. di Matematica Pura ed Appl. IV (135) (1983), 293-318.

10. G.I. Barenblatt, M. Bertsch, R. Dal Passo, V.M. Prostokishin, and M. Ughi, *A mathematical Model of Turbulent Heat and Mass Transfer in Stable Stratified Shear Flow*, J. Fluid Mechanics **253** (1993), 341-358.
11. Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre and J.L Vazquez, *An L^1-Theory of Existence and Uniqueness of Solutions of Nonlinear Elliptic Equations*, Ann. Scuola Normale Superiore di Pisa, IV, Vol. XXII (1995), 241-273.
12. Ph. Bénilan and M.G. Crandall, *Completely Accretive Operators*, in Semigroups Theory and Evolution Equations, Ph. Clement et al. editors, Marcel Dekker, 1991, pp. 41-76.
13. Ph. Bénilan, M.G. Crandall and A. Pazy, *Evolution Equations Governed by Accretive Operators*, Book in preparation.
14. M. Bertsch and R. Dal Passo, *Hyperbolic Phenomena in a Strongly Degenerate Parabolic Equation*, Arch Rational Mech. Anal. **117** (1992), 349-387.
15. M. Bertsch and R. Dal Passo, *A Parabolic Equation with Mean-Curvature Type Operator*, In Progres Nonlinear Differential Equation Appl. **7** (F. Browder, ed.) Birkhäuser 1992, 89-97.
16. Ph. Blanc, *On the regularity of the solutions of some degenerate parabolic equations*, Comm. in Partial Diff. Equat. **18** (1993), 821-846.
17. Ph. Blanc, *Sur une classe d'equations paraboliques degeneréesa une dimension d'espace possedant des solutions discontinues*, Ph.D. Thesis, number 798, Ecole Polytechnique Federale de Lausanne, 1989.
18. D. Blanchard and F. Murat, *Renormalized solutions on nonlinear parabolic problems with L^1 data: existence and uniqueness*, Proc. Royal Soc. Edinburgh Sect. A **127** (1997), 1137-1152.
19. H. Brezis, *Operateurs Maximaux Monotones*, North Holland, Amsterdam, 1973.
20. J. Carrillo, *On the uniqueness of solution of the evolution dam problem*, Nonlinear Analysis **22** (1994), 573-607.
21. J. Carrillo and P. Wittbold, *Uniqueness of Renormalized Solutions of Degenerate Elliptic-Parabolic problems*, Jour. Diff. Equat. **156** (1999), 93-121.
22. M.G. Crandall, *Nonlinear Semigroups and Evolution Governed by Accretive Operators*, In Proceeding of Symposium in Pure Mat., Part I (F. Browder, ed.) A.M.S., Providence 1986, 305-338.
23. M.G. Crandall and T.M. Liggett, *Generation of Semigroups of Nonlinear Transformations on General Banach Spaces*, Amer. J. Math. **93** (1971), 265-298.
24. G. Dal Maso *Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals*, manuscripta math. **30** (1980), 387-416.
25. R. Dal Passo, *Uniqueness of the entropy solution of a strongly degenerate parabolic equation*, Comm. in Partial Diff. Equat. **18** (1993), 265-279.
26. E. De Giorgi, L. Ambrosio, *Un nuovo tipo di funzionale del calcolo delle variazioni*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat Fis. Natur. s. 8 **82** (1988), 199-210.
27. J. Diestel and J.J. Uhl, Jr., *Vector Measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, 1977.
28. J.J. Duderstadt and G.A. Moses, *Inertial Confinement Fusion*, John Wiley and Sons, 1982.
29. L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Math., CRC Press, 1992.
30. R. Kohn and R. Temam, *Dual space of stress and strains with application to Hencky plasticity*, Appl. Math. Optim. **10** (1983), 1-35.
31. S.N. Kruzhkov, *First order quasilinear equations in several independent variables*, Math. USSR-Sb. **10** (1970), 217-243.

32. J. Leray and J.L. Lions, *Quelques resultats de Visik sur les problemes elliptiques semi-lineaires par les metode de Minty et Browder*, Bull. Soc. math. France **93** (1965), 97-107.
33. P. Rosenau, *Free Energy Functionals at the High Gradient Limit*, Phys. Review A **41** (1990), 2227-2230.
34. H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer Verlag, 1974.
35. L. Schwartz, *Fonctions mesurables et *-scalairement mesurables, mesures banachiques majorées, martingales banachiques, et propiété de Radon-Nikodým*, Sém. Maurey-Schawartz, 19674-75, Ecole Polytech., Centre de Math.
36. A.I. Volpert, *The spaces BV and quasilinear equations*, Math. USSR Sbornik **2** (1967), 225-267.
37. W.P. Ziemer, *Weakly Differentiable Functions*, GTM 120, Springer Verlag, 1989.