ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF QUASI-LINEAR PARABOLIC EQUATIONS WITH NON-LINEAR FLUX

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Abstract. In this paper we study the large time behaviour of solutions of the quasi-linear parabolic equation with nonlinear boundary conditions

$$u_t = \operatorname{div}\, \mathbf{a}(x, Du) \quad \text{in} \quad (0,\infty)\times\Omega$$
$$-\frac{\partial u}{\partial \eta_a} \in \beta(u) \quad \text{on} \quad (0,\infty)\times\partial\Omega$$
$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega.$$

We show that the solutions stabilize as $t \to \infty$ by converging to a constant function. For some particular boundary conditions we also obtain a decay rate.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $1 < p < \infty$. Consider a vector valued function $\mathbf{a}$ mapping $\Omega \times \mathbb{R}^N$ into $\mathbb{R}^N$ and satisfying

(H_1) $\mathbf{a}$ is a Carathéodory function (i.e., the map $\xi \to \mathbf{a}(x,\xi)$ is continuous for almost all x and the map $x \to \mathbf{a}(x,\xi)$ is measurable for every ξ) and there exists $\lambda > 0$ such that

$$\langle \mathbf{a}(x,\xi), \xi\rangle \geq \lambda |\xi|^p \qquad (\ 1 < p < \infty \ )$$

holds for every ξ and a.e. $x \in \Omega$, where $\langle , \rangle$ means scalar product in $\mathbb{R}^N$. There is no restriction in assuming that $\lambda = 1$.

(H_2) For every ξ and $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and a.e. $x \in \Omega$ it holds

$$\langle \mathbf{a}(x,\xi) - \mathbf{a}(x,\eta), \xi - \eta\rangle > 0.$$

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(H_3) There exists $\Lambda \in \mathbb{R}$ such that

$$|\mathbf{a}(x,\xi)| \leq \Lambda(j(x) + |\xi|^{p-1})$$

holds for every $\xi \in \mathbb{R}^N$ with $j \in L^{p'}$, $p' = p/(p-1)$.

The hypotheses (H_1), (H_2) and (H_3) are classical in the study of nonlinear operators in divergence form (see [16]). The model example of a function $\mathbf{a}$ satisfying these hypothesis is $\mathbf{a}(x,\xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \operatorname{div}\big(|Du|^{p-2}\, Du\big)$.

The aim of this paper is to study the large time behaviour of solutions for equations of the form

$$(I) \qquad \begin{cases} u_t = \operatorname{div} \mathbf{a}(x, Du) & \text{in } \Omega \times (0,\infty) \\[2mm] -\dfrac{\partial u}{\partial \eta_a} \in \beta(u) & \text{on } \partial\Omega \times (0,\infty) \\[2mm] u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

where $\partial/\partial\eta_a$ is the Neumann boundary operator associated to $\mathbf{a}$, i.e.,

$$\frac{\partial u}{\partial \eta_a} := \langle \mathbf{a}(x, Du), \eta\rangle$$

with η the unit outward normal on $\partial\Omega$, Du the gradient of u and β a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta(0)$. These nonlinear fluxes on the boundary occur in heat transfer between solids and gases ([14]) and in some problems in Mechanics and Physics [13] (see also [9]). Observe also that the classical Neumann and Dirichlet boundary conditions correspond to $\beta = \mathbb{R} \times \{0\}$ and $\beta = \{0\} \times \mathbb{R}$, respectively.

In order to discuss the asymptotic behaviour of solutions of problem (I) we must be sure such solutions exist. In general, problem (I) is not solvable in the classical sense and it is necessary to introduce a suitable class of generalized solutions. In [1], following the idea of entropy solution introduced in [7], we study problem (I) in the context of Nonlinear Semigroup Theory. We associate to problem (I) an m-T-accretive operator in $L^1(\Omega)$. So, for us a solution of problem (I) will be the mild-solution obtained via the Crandall-Liggett exponential formula. These mild-solutions have been characterized in [2] by introducing a new class of weak solutions, namely *entropy solutions*. We show that any solution of problem (I) converges to a spatially constant function K, $K \in \beta^{-1}(0)$, i.e., $\|u(.,t) - K\|_1 \to 0$ as $t \to \infty$. For some particular boundary conditions we also obtain a decay rate. Our main tool is the Lyapunov method for semigroups of nonlinear contractions introduced by A. Pazy [18].

1. Preliminaries

In this section we give the results about existence and uniqueness of mild-solutions of problem (I) we need. We start with some notation and definitions used later. If $\Omega \subset \mathbb{R}^N$ is a Lebesgue measurable set then $\lambda_N(\Omega)$ denotes its measure. The norm in $L^p(\Omega)$ is denoted by $\|.\|_p$, $1 \leq p \leq \infty$. If $k \geq 0$ is an integer and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ is the Sobolev space of functions u on the open set $\Omega \subset \mathbb{R}^N$ for which $D^\alpha u$ belongs to $L^p(\Omega)$ when $|\alpha| \leq k$, with its usual norm $\|.\|_{k,p}$. $W_0^{k,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. If $v \in L^1(\Omega)$ and $\lambda_N(\Omega) < \infty$, we denote by $\overline{v}$ the average of v, i.e.,

$$\overline{v} := \frac{1}{\lambda_N(\Omega)} \int_\Omega v(x)\, dx.$$

We use some terminology and notations from classical topological dynamics. For a continuous semigroup $(T(t))_{t\geq 0}$ on a metric space X, the *orbit or trajectory* of $u \in X$ is the set

$$\gamma(u) = \{T(t)u \ : t \geq 0\}$$

and the *ω-limit set* of u is

$$\omega(u) = \{v \in X \ : \ v = \lim_{n\to\infty} T(t_n)u \ \text{ for some sequence } t_n \to \infty\}.$$

This set is possibly empty. Now, it is well-known that if $\gamma(u)$ is relatively compact, then $\omega(u)$ is a non empty, compact and connected subset of X. Furthermore, $\omega(u)$ is positive invariant under $T(t)$, i.e., $T(t)\omega(u) = \omega(u)$ for any $t \geq 0$. An *equilibrium or stationary point* $u \in X$ is a point such that $\gamma(u) = \omega(u) = \{u\}$, or equivalently, $T(t)u = u$ for all $t \geq 0$.

As we said in the introduction, our abstract framework is the Theory of Nonlinear Semigroups. We refer the reader to [2], [4], [6] and [11] for background material on nonlinear contraction semigroups.

Ph. Bénilan and M. Crandall introduce in [5] the concept of completely accretive operator, whose precedents are the results of Brézis- Strauss [10] on semilinear elliptic equations (see also [4]). This type of operators, in the particular case of $L^1(\Omega)$ with Ω bounded, can be defined in the following way: An operator A in $L^1(\Omega)$, possibly multivalued (i.e., $A \subset L^1(\Omega) \times L^1(\Omega)$), is said to be *completely accretive* if one of the following conditions is satisfied:

1. For $\lambda > 0$, $(u, v), (\hat{u}, \hat{v}) \in A$ and $j \in J_0$,

$$\int_\Omega j(u - \hat{u}) \leq \int_\Omega j(u - \hat{u} + \lambda(v - \hat{v})), \tag{1.1}$$

where

$$J_0 = \{\text{convex lower-semicontinuous maps } j : \mathbb{R} \to [0, \infty] \text{ satisfying } j(0) = 0\}.$$

2. For $(u, v), (\hat{u}, \hat{v}) \in A$ and $p \in P_0$,

$$\int_\Omega p(u - \hat{u})(v - \hat{v}) \geq 0, \tag{1.2}$$

where

$$P_0 = \{p \in C^\infty(\mathbb{R}) \ : \ 0 \leq p' \leq 1, \ \text{supp}(p') \text{ is compact, and } 0 \notin \text{supp}(p)\}.$$

Remark that if A is a completely accretive operator in $L^1(\Omega)$ and $1 \leq q \leq \infty$, then the restriction A_q of A to $L^q(\Omega)$ is T-accretive in $L^q(\Omega)$. Consequently, the corresponding resolvent $J_\lambda = (I + \lambda A_q)^{-1}$ is an order preserving contraction in $L^q(\Omega)$. If a completely accretive operator A in $L^1(\Omega)$ satisfies the *range condition*: "there exists $\lambda > 0$ such that $R(I + \lambda A)$ is dense in $L^1(\Omega)$", then the closure $\overline{A}$ of A is an m-T-accretive operator in $L^1(\Omega)$. So, by Crandall-Liggett's Theorem, the operator A generates, on the closure of its domain, a semigroup of order-preserving contractions given by the exponential formula

$$e^{t\overline{A}}u = \lim_{n\to\infty}(I + \frac{t}{n}\overline{A})^{-n}u \quad \text{for} \quad u \in \overline{D(\overline{A})}.$$

This semigroup solves the corresponding initial value problem for the operator $\overline{A}$

$$u' + \overline{A}u \ni 0, \quad u(0) = u_0. \tag{1.3}$$

The function $u(t) := e^{-t\overline{A}}u_0$ is called the *mild-solution* of problem (1.3).

From now on, Ω will be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ of class C^1, $1 < p < N$, $\mathbf{a}$ is a vector valued map from $\Omega \times \mathbb{R}^N$ into $\mathbb{R}^N$ satisfying (H_1) - (H_3) and β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$.

In order to study problem (I) from the point of view of Nonlinear Semigroup Theory we introduce a nonlinear completely accretive operator A_β in $L^1(\Omega)$ associated with the formal expression

$$-\mathrm{div}\ \mathbf{a}(x, Du) \quad + \quad \text{nonlinear boundary conditions.}$$

Since β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, there exists a convex lower semicontinuous (l.s.c.) function j on $\mathbb{R}$, $j(0) = 0$, such that $\beta = \partial j$. Consider $\Phi : W^{1,p}(\Omega) \to [0, +\infty]$, defined by

$$\varPhi(u) := \begin{cases} \int_{\partial\Omega} j(u) & \text{if} \quad j(u) \in L^1(\partial\Omega) \\ +\infty & \text{if } j(u) \notin L^1(\partial\Omega). \end{cases}$$

It is well-known (cf. [9]) that $\varPhi$ is a convex l.s.c. function in $W^{1,p}(\Omega)$. Moreover $\varPhi \geq 0$ because $j \geq 0$. Now let us define the operator A_β in $L^1(\Omega)$ by:

$(u, v) \in A_\beta$ if and only if $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v \in L^1(\Omega)$ and

$$\varPhi(w) \geq \varPhi(u) + \int_\Omega v(w-u) - \int_\Omega \langle \mathbf{a}(x, Du), D(w-u)\rangle \quad \text{for every } w \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Here and below the integrals over Ω are with respect to Lebesgue measure λ_N and the integrals over $\partial\Omega$ are with respect to the area measure μ on $\partial\Omega$.

In the following theorem we summarize all the results we need about A_β given in [1].

Theorem 1.1. *The operator A_β satisfies the following statements:*
(i) A_β is univalued, i.e., if $(u, v) \in A_\beta$, then

$$v = -\mathit{div}\ \mathbf{a}(x, Du) \quad \text{in the sense of distributions.}$$

(ii) A_β *is completely accretive.*
(iii) $L^\infty(\Omega) \subset R(I + A_\beta)$.
(iv) The domain of the operator A_β *is dense in* $L^1(\Omega)$.
(v) If $(u, v) \in A_\beta$, *then*

$$\int_\Omega |Du|^p \leq \|u\|_\infty \|v\|_1.$$

We associate with problem (I) the operator

$$\mathcal{A}_\beta := \overline{A_\beta}^{L^1(\Omega)}, \quad \text{i.e., the closure of } A_\beta \text{ in } L^1(\Omega)$$

which is m-completely accretive in $L^1(\Omega)$. Thus the abstract Cauchy problem in $L^1(\Omega)$ corresponding to (I) reads as follows:

$$(II) \quad \begin{cases} u'(t) + \mathcal{A}_\beta u(t) \ni 0 & t \geq 0, \\ u(0) = u_0. \end{cases}$$

Since $\mathcal{A}_\beta$ is m-completely accretive in $L^1(\Omega)$ a unique mild-solution $u \in C(\mathbb{R}^+; L^1(\Omega))$ of (II) is known to exist in the sense of Nonlinear Semigroup Theory for any $u_0 \in \overline{D(\mathcal{A}_\beta)} = L^1(\Omega)$.

In [1], following the idea of entropy solutions introduced in [7], we characterize the closure $\mathcal{A}_\beta$ of the operator A_β in some cases.

2. The stabilization results

In this section we establish that the mild-solutions of problem (II) stabilize as $t \to 0$ by converging to a constant function. We use the Lyapunov method for semigroups of nonlinear contractions introduced by A. Pazy [18].

In order to prove the stabilization theorem we need the orbits to be relatively compact.

Theorem 2.1. *Let* $(S(t))_{t \geq 0}$ *be the semigroup generated by* $\mathcal{A}_\beta$ *and* J_λ *its resolvent. Then,*

(i) $J_\lambda(B)$ *is a relatively compact subset of* $L^1(\Omega)$ *if* B *is a bounded subset of* $L^\infty(\Omega)$.

(ii) For every $u_0 \in L^1(\Omega)$ *the orbit* $\gamma(u_0) = \{S(t)u_0 \ : \ t \geq 0\}$ *is a relatively compact subset of* $L^1(\Omega)$.

Proof. (i): Let B a bounded subset of $L^\infty(\Omega)$. Take $(f_n) \subset B$ and let $u_n := J_\lambda f_n$. Set $M := \sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$. By Theorem 1.1, $\|u_n\|_\infty \leq M$ for every $n \in \mathbb{N}$ and

$$\int_\Omega |Du_n|^p \leq \frac{2M^2 \lambda_N(\Omega)}{\lambda} \qquad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

Thus, $\{u_n \ : \ n \in \mathbb{N}\}$ is a bounded sequence in $W^{1,p}(\Omega)$, and by the Rellich-Kondrachov Theorem we have that $\{u_n \ : \ n \in \mathbb{N}\}$ is a relatively compact subset of $L^1(\Omega)$.

(ii): Consider first $u_0 \in \mathcal{D}(A_\beta) \cap L^\infty(\Omega)$. Then, since

$$\|S(t)u_0\|_\infty \leq \|u_0\|_\infty \quad \text{for all } t \geq 0,$$

as a consequence of (i), we have that $J_\lambda(\gamma(u_0))$ is a relatively compact subset of $L^1(\Omega)$ for all $\lambda > 0$. Moreover,

$$\|S(t)u_0 - J_\lambda S(t)u_0\|_1 \leq \lambda \inf\{\|v\|_1 : v \in \mathcal{A}_\beta(u_0)\}.$$

Hence, $\gamma(u_0)$ is relatively compact in $L^1(\Omega)$.

Finally, since $\mathcal{D}(\mathcal{A}_\beta) \cap L^\infty(\Omega)$ is dense in $L^1(\Omega)$, given $u_0 \in L^1(\Omega)$ and $\epsilon > 0$, there exists $v_0 \in \mathcal{D}(\mathcal{A}_\beta) \cap L^\infty(\Omega)$ such that $\|u_0 - v_0\|_1 < \epsilon$. So we have,

$$\sup_{t \geq 0} \inf_{s \geq 0} \|S(t)u_0 - S(s)v_0\|_1 \leq \sup_{t \geq 0} \|S(t)u_0 - S(t)v_0\|_1 \leq \|u_0 - v_0\|_1 < \epsilon.$$

From where it follows that $\gamma(u_0)$ is relatively compact in $L^1(\Omega)$.

Now we come to the main result.

Theorem 2.2. *Let $u_0 \in L^1(\Omega)$ and $u(x,t)$ be the mild-solution of problem (I). Then, there exists a constant K, $K \in \beta^{-1}\{0\}$ such that*

$$\|u(.,t) - K\|_1 \to 0 \quad as \quad t \to \infty.$$

Proof. Suppose first that $u_0 \in L^\infty(\Omega)$. Let $(S(t))_{t \geq 0}$ be the semigroup generated by $\mathcal{A}_\beta$ and J_λ its resolvent. Let $\mathcal{V} : L^1(\Omega) \to [0, +\infty]$ be defined by

$$\mathcal{V}(u) = \begin{cases} \frac{1}{2}\int_\Omega u^2, & \text{if } u \in L^2(\Omega) \\ \\ +\infty, & \text{if } u \notin L^2(\Omega) \end{cases}$$

It is well-known that $\mathcal{V}$ is lower semicontinuous (see [9, pag. 160]). On the other hand, since $\mathcal{A}_\beta$ is completely accretive, we have

$$\frac{1}{2}\int_\Omega (J^n_{t/n} f)^2 \leq \frac{1}{2}\int_\Omega f^2 \quad \text{for } f \in L^2(\Omega),\ t > 0 \ \text{ and } \ n \in \mathbb{N}.$$

Now, by the Crandall-Liggett Theorem, since $\mathcal{V}$ is lower semicontinuous, we have

$$\mathcal{V}\big(S(t)f\big) \leq \liminf_{n \to \infty} \mathcal{V}\big(J^n_{t/n} f\big) \leq \mathcal{V}(f), \quad \text{for } t \geq 0.$$

Therefore, $\mathcal{V}$ is a Lyapunov functional for the semigroup $\big(S(t)\big)_{t \geq 0}$.

Let $\mathcal{W} : L^1(\Omega) \to]-\infty, +\infty]$ be defined by

$$\mathcal{W}(u) = \begin{cases} \int_\Omega |Du|^p, & \text{if } |Du| \in L^p(\Omega) \\ \\ +\infty, & \text{if } |Du| \notin L^p(\Omega) \end{cases}$$

It is easy to see that $\mathcal{W}$ is lower semicontinuous in $L^p(\Omega)$. Since $u_0 \in L^\infty(\Omega)$, by Theorem 1.1, $J_\lambda u_0 = (I + \lambda A_\beta)^{-1} u_0 \in \mathcal{D}(\mathcal{A}_\beta) \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then,

$(J_\lambda u_0, \frac{1}{\lambda}(u_0 - J_\lambda u_0)) \in A_\beta$. Thus, taking $w = 0$ as a test function in the definition of the operator A_β, we have

$$\int_\Omega \langle \mathbf{a}(x, DJ_\lambda u_0), DJ_\lambda u_0 \rangle \leq \frac{1}{\lambda} \int_\Omega (u_0 - J_\lambda u_0) J_\lambda u_0 - \Phi(J_\lambda u_0).$$

Now, using (H_1) and $\Phi(J_\lambda u_0) \geq 0$, we obtain

$$\mathcal{W}(J_\lambda u_0) \leq \frac{1}{\lambda} \int_\omega (u_0 - J_\lambda u_0) J_\lambda u_0.$$

Then, since

$$\mathcal{V}(J_\lambda u_0) - \mathcal{V}(u_0) = \frac{1}{2} \int_\Omega (J_\lambda u_0)^2 - \frac{1}{2} \int_\Omega u_0^2 \leq - \int_\Omega (u_0 - J_\lambda u_0) J_\lambda u_0,$$

we get

$$\mathcal{V}(J_\lambda u_0) + \lambda \mathcal{W}(J_\lambda u_0) - \mathcal{V}(u_0) \leq 0. \tag{2.3}$$

Replacing u_0 by $J_\lambda^{k-1} u_0$ in (2.3) we find

$$\mathcal{V}(J_\lambda^k u_0) + \lambda \mathcal{W}(J_\lambda^k u_0) - \mathcal{V}\big(J_\lambda^{k-1} u_0\big) \leq 0.$$

Summing these inequalities from $k = 1$ to $k = n$ and choosing $\lambda = t/n$, it yields

$$\mathcal{V}(J_{\frac{t}{n}}^n u_0) + \sum_{k=1}^n \frac{t}{n} \mathcal{W}(J_{\frac{t}{n}}^k u_0) - \mathcal{V}(u_0) \leq 0. \tag{2.4}$$

Next we define a piecewise constant function

$$F_n(\tau) = \mathcal{W}(J_{\frac{t}{n}}^k u_0) \qquad \text{for } (k-1)t/n < \tau \leq kt/n.$$

Then

$$\sum_{k=1}^n \frac{t}{n} \mathcal{W}(J_{\frac{t}{n}}^k u_0) = \int_0^t F_n(\tau)\, d\tau.$$

On the other hand, by the Crandall-Liggett Theorem,

$$\lim_{n \to \infty} J_{\frac{t}{n}}^k u_0 = S(\tau) u_0 \quad \text{in } L^1(\Omega)$$

where $k = k_n(\tau) = [n\tau/t] + 1$. By the Dominated Convergence Theorem, taking a subsequence if necessary, it follows that

$$\lim_{n \to \infty} J_{\frac{t}{n}}^k u_0 = S(\tau) u_0 \quad \text{in } L^p(\Omega).$$

Since $\mathcal{W}$ is lower semicontinuous in $L^p(\Omega)$, we have

$$\mathcal{W}(S(t) u_0) \leq \liminf_{n \to \infty} \mathcal{W}(J_{\frac{t}{n}}^k u_0) = \liminf_{n \to \infty} F_n(\tau).$$

Thus, by Fatou's lemma, we obtain

$$\int_0^t \mathcal{W}\big(S(\tau)u_0\big)\, d\tau \leq \liminf_{n\to\infty} \int_0^t F_n(\tau)\, d\tau = \liminf_{n\to\infty} \sum_{k=1}^n \frac{t}{n} \mathcal{W}(J^k_{\frac{t}{n}} u_0). \tag{2.5}$$

Passing to the limit as $n \to \infty$ in (2.4) and taking into account (2.5) and the lower semicontinuity of $\mathcal{V}$, we get

$$\mathcal{V}\big(S(t)u_0\big) + \int_0^t \mathcal{W}\big(S(\tau)u_0\big)\, d\tau - \mathcal{V}(u_0) \leq 0.$$

Consequently

$$\int_0^\infty \mathcal{W}\big(S(\tau)u_0\big)\, d\tau \leq \mathcal{V}(u_0). \tag{2.6}$$

Thus, there exists a sequence $t_n \to \infty$, such that $\mathcal{W}\big(S(t_n)u_0\big) \to 0$ as $n \to \infty$. Now by Theorem 2.1, there exists a subsequence (t_{n_k}) such that

$$\lim_{k\to\infty} S(t_{n_k})u_0 = v \in \omega(u_0).$$

Hence, by the Dominated Convergence Theorem,

$$\lim_{k\to\infty} S(t_{n_k})u_0 = v \quad \text{in } L^p(\Omega)$$

and by the lower semicontinuity of $\mathcal{W}$, it follows that

$$\mathcal{W}(v) \leq \liminf_{k\to\infty} \mathcal{W}\big(S(t_{n_k})u_0\big) = 0.$$

Therefore, v is a constant K. If $K = 0$, since 0 is an equilibrium, $\omega(u_0) = \{0\}$. Suppose $K > 0$. Then, since $\|S(t)K\|_\infty \leq \|K\|_\infty = K$,

$$0 \leq S(t)K \leq K. \tag{2.7}$$

Since $S(t)K$, $K \in \omega(u_0)$ and $\mathcal{V}$ is a Lyapunov functional, it follows from the invariance principle of Dafermos [12, Proposition 4.1] that $\mathcal{V}\big(S(t)K\big) = \mathcal{V}(K)$. Consequently, by (2.7) and the definition of $\mathcal{V}$, $S(t)K = K$ for all $t \geq 0$, so as $S(t)$ are contractions, we get $\omega(u_0) = \{K\}$ and the proof for the case $u_0 \in L^\infty(\Omega)$ concludes. Now, since $L^\infty(\Omega)$ is dense in $\overline{\mathcal{D}(A_\beta)} = L^1(\Omega)$ and $S(t)$ is a T-contraction, from the above we obtain easily the conclusion in the general case $u_0 \in L^1(\Omega)$. Finally, as K is an equilibrium, it follows that $K \in \beta^{-1}\{0\}$.

3. Some particular boundary conditions

For some particular boundary conditions we can say more about the constant K of Theorem 2.2. For instance, when $\beta = \{0\} \times \mathbb{R}$, then $K = 0$, i.e., for Dirichlet boundary conditions the solutions of problem (I) stabilize to 0. Now, in this section we are going to see that for Dirichlet and Neumann boundary conditions we also can estimate a decay rate of the solutions when $t \to \infty$. We start with the Neumann boundary valued problem. First we need the following result about the conservation of mass.

Proposition 3.1. *Let* $((S(t))_{t\geq 0}$ *be the semigroup in* $L^1(\Omega)$ *generated by* $\mathcal{A}_\beta$*, with* β *corresponding to the Neumann boundary condition, and* $u_0 \in L^1(\Omega)$*. Then, we have consevation of mass, that is,*

$$\int_\Omega S(t)u_0 = \int_\Omega u_0, \quad \textit{for all} \ \ t \geq 0.$$

Proof. Without loss of generality we can assume that $u_0 \in L^\infty(\Omega)$. For $\lambda > 0$, let $J_\lambda = (I + A_\beta)^{-1}$ be the resolvent of A_β and define v_i by $v_0 = u_0$, $v_{i+1} = J_\lambda v_i$, $i = 1, 2, \cdots$. Letting $v_\lambda(t) = v_i$ for $i\lambda \leq t < (i+1)\lambda$, we have by the Crandall-Ligett Theorem that

$$\lim_{\lambda \to 0} v_\lambda(t) = S(t)u_0 \quad \text{in} \ \ L^1(\Omega),$$

and the limit is uniform for t in compact subsets of $[0, \infty[$.

Since $(v_{i+1}, \frac{1}{\lambda}(v_i - v_{i+1})) \in A_\beta$, we have

$$\int_\Omega \langle \mathbf{a}(x, Dv_{i+1}), D(v_{i+1} - \phi)\rangle \leq \int_\Omega \frac{1}{\lambda}(v_i - v_{i+1})(v_{i+1} - \phi), \tag{3.1}$$

for every $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Now, given $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, taking $v_{i+1} - \phi$ and $v_{i+1} + \phi$ as a test functions in (3.1) we get

$$\int_\Omega \langle \mathbf{a}(x, Dv_{i+1}), D\phi\rangle = \int_\Omega \frac{1}{\lambda}(v_i - v_{i+1})\phi, \tag{3.2}$$

for every $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. In particular, taking $\phi(x) = 1$ for all $x \in \Omega$ in (3.2), it follows that

$$\int_\Omega v_{i+1} = \int_\Omega v_i = \int_\Omega u_0, \quad i = 1, 2, \cdots$$

Therefore,

$$\int_\Omega S(t)u_0 = \lim_{\lambda \to 0} \int_\Omega v_\lambda(t) = \int_\Omega u_0.$$

Concerning Neumann boundary problem, we have the following result

Theorem 3.2. *Let* $u_0 \in L^1(\Omega)$*. Then, if* $u(x,t)$ *is the mild-solution of problem (I) with Neumann boundary conditions, we have*

$$\|u(.,t) - \overline{u_0}\|_1 \to 0 \quad \textit{as} \ \ t \to \infty.$$

Moreover, if $u_0 \in L^\infty(\Omega)$ *there exists a constant* C*, independent of* u_0*, such that*

$$\|u(.,t) - \overline{u_0}\|_p \leq \left(\frac{C\|u_0\|_2^2}{t}\right)^{1/p} \quad \textit{for all} \ \ t > 0.$$

Proof. Let $u_0 \in L^1(\Omega)$ and $u(x,t)$ be the mild-solution of problem (I) with $\beta = \mathbb{R} \times \{0\}$. By Theorem 2.2, there exists a constant $K \in \beta^{-1}\{0\}$ such that

$$\lim_{t\to\infty} \|u(.,t) - K\|_1 = 0.$$

Then, since the average of any solution is preserved (Proposition 3.1), it follows that $K = \overline{u_0}$.

Now suppose $u_0 \in L^\infty(\Omega)$. Then, by (2.6) we have that $u(.,t) \in W^{1,p}(\Omega)$ for almost all $t > 0$ and, moreover,

$$\int_0^t \int_\Omega |Du(.,s)|^p \, ds \le \frac{1}{2}\|u_0\|_2^2 \quad \text{for any } t > 0. \tag{3.3}$$

On the other hand, since $\overline{u(.,s)} = \overline{u_0}$, by the Poincaré-Wirtinger inequality, it follows that

$$\|u(.,t) - \overline{u_0}\|_p^p = \|u(.,t) - \overline{u(.,s)}\|_p^p \le M\|Du(.,s)\|_p^p. \tag{3.4}$$

Then, (3.3) and (3.4) imply that

$$\int_0^t \|u(.,t) - \overline{u_0}\|_p^p \, ds \le \frac{1}{2}M\|u_0\|_2^2 \quad \text{for any } t > 0. \tag{3.5}$$

Now, since A_β is completely accretive and $j(r) = |r|^p$ is an element of J_0, $\mathcal{U}(u) = \int_\Omega j(u)$ is a Lyapunov functional for the semigroup generated A_β. Then, by (3.5) we get

$$t\|u(.,t) - \overline{u_0}\|_p^p \le \int_0^t \mathcal{U}(u(.,s) - \overline{u_0}) \, ds \le \frac{1}{2}M\|u_0\|_2^2.$$

Therefore

$$\|u(.,t) - \overline{u_0}\|_p \le \left(\frac{C\|u_0\|_2^2}{t}\right)^{1/p} \quad \text{for all } t > 0.$$

with $C = \frac{M}{2}$.

Unilateral boundary conditions of type

$$u > 0 \;\Rightarrow\; \frac{\partial u}{\partial \eta} = 0,$$

$$u = 0 \;\Rightarrow\; \frac{\partial u}{\partial \eta} \ge 0.$$

which correspond to variational inequalities introduced by J. L. Lions and G. Stampachia [17] (see also [8] and [9]), appear in elasticity (Signorini's problem) and in problems of heat control. In the above notation this boundary condition is

$$\frac{\partial u}{\partial \eta_a} \in \beta(u)$$

with β being the maximal monotone graph

$$\beta(r) = \begin{cases} 0, & \text{for } r > 0 \\ ]-\infty, 0], & \text{for } r = 0 \\ \emptyset, & \text{for } r < 0. \end{cases}$$

For this type of boundary conditions and for similar ones which appear in problems of temperature control through the boundary (see [14]), we have the following result.

Theorem 3.3. *Let* β *be a maximal monotone graph in* $\mathbb{R}\times\mathbb{R}$ *with* $0\in\beta(0)$. *Let* $0\leq u_0\in L^1(\Omega)$ *and* $u(x,t)$ *the mild-solution of problem (I). Then, there exists a constant* $K\geq 0$ *such that*

$$\|u(.,t)-K\|_1\to 0 \quad as \quad t\to\infty.$$

Moreover, if $d:=\sup\{r\geq 0 \ : \ 0\in\beta(r)\}$. *Then,*

$$\overline{\inf\{d,u_0\}}\leq K\leq\inf\{d,\overline{u_0}\}.$$

Proof. We suppose first that $u_0\geq\alpha>0$. Let

$$\tilde{\beta}(r)=\begin{cases} 0, & \text{for } \ r<0\\ \beta(r)\cap[0,+\infty[, & \text{for } \ r\geq 0.\end{cases}$$

It is easy to see that

$$e^{-t\mathcal{A}_\beta}u_0=e^{-t\mathcal{A}_{\tilde{\beta}}}u_0 \quad \text{ for } \ t\geq 0.$$

By Theorem 2.2, there exists $0\leq K\in\beta^{-1}\{0\}$ such that $\omega(u_0)=\{K\}$. By the definition of d, $K\leq d$. Moreover, having in mind the proof of Proposition 3.1, we have

$$\int_\Omega e^{-t\mathcal{A}_{\tilde{\beta}}}u_0\leq\int_\Omega u_0$$

and consequently $K\leq\overline{u_0}$. Consider $\hat{u}_0:=\inf\{d,u_0\}$. Then, $\omega(\hat{u}_0)\leq\omega(u_0)=\{K\}$ since $\hat{u}_0\leq u_0$. Now, if $\alpha=\mathbb{R}\times\{0\}$ it is not difficult to see that

$$e^{-t\mathcal{A}_\beta}\hat{u}_0=e^{-t\mathcal{A}_\alpha}\hat{u}_0 \quad \text{ for } \ t\geq 0.$$

Hence, by Theorem 3.2 we have

$$\omega(\hat{u}_0)=\{\frac{1}{\mu(\Omega)}\int_\Omega\hat{u}_0\}.$$

Then

$$K\geq\frac{1}{\mu(\Omega)}\int_\Omega\hat{u}_0=\overline{\inf\{d,u_0\}}$$

and the proof finishes in the case $u_0\geq\alpha>0$.

Finally, suppose that $u_0\geq 0$. Then, given $\epsilon>0$, if $u_\epsilon=u_0(x)+\epsilon$, the above implies the existence of a constant K_ϵ such that $\omega(u_\epsilon)=\{K_\epsilon\}$. Moreover we have

$$P_\epsilon:=\overline{\inf\{d,u_\epsilon\}}\leq K_\epsilon\leq\inf\{d,\overline{u_\epsilon}\}=:Q_\epsilon$$

Now,

$$P_\epsilon\to\overline{\inf\{d,u_0\}} \quad \text{as} \ \ \epsilon\to 0$$

and

$$Q_\epsilon\to\inf\{d,\overline{u_0}\} \quad \text{as} \ \ \epsilon\to 0.$$

Therefore, there exists a sequence $\{\epsilon_n\}$, $\epsilon_n \to 0$, such that $K_{\epsilon_n} \to K$ and

$$\overline{\inf\{d, u_0\}} \leq K \leq \inf\{d, \overline{u}_0\}.$$

Let us prove $\omega(u_0) = \{K\}$. In fact:

$$\|e^{-t\mathcal{A}_\beta} u_0 - K\|_1 \leq \|e^{-t\mathcal{A}_\beta} u_0 - e^{-t\mathcal{A}_\beta} u_{\epsilon_n}\|_1 + \|e^{-t\mathcal{A}_\beta} u_{\epsilon_n} - K_{\epsilon_n}\|_1 + \|K_{\epsilon_n} - K\|_1$$

$$\leq \|u_0 - u_{\epsilon_n}\|_1 + \|e^{-t\mathcal{A}_\beta} u_{\epsilon_n} - K_{\epsilon_n}\|_1 + \|K_{\epsilon_n} - K\|_1.$$

Now, since $\omega(u_{\epsilon_n}) = \{K_{\epsilon_n}\}$, we have that

$$\lim_{t\to\infty} \|e^{-t\mathcal{A}_\beta} u_0 - K\|_1 \leq \|u_0 - u_{\epsilon_n}\|_1 + \|K_{\epsilon_n} - K\|_1.$$

But $\|u_0 - u_{\epsilon_n}\|_1 \to 0$ and $\|K_{\epsilon_n} - K\|_1 \to 0$ as $n \to \infty$, hence $\omega(u_0) = \{K\}$.

Remark that in the above theorem, $K = \overline{u_0}$ if $\beta(r) = 0$ for all $r > 0$, and $K = 0$ if $\beta(r) > 0$ for all $r > 0$.

To finish, let us see that for the Dirichlet boundary value problem we also obtain a decay rate. To be more concrete, we have

Theorem 3.4. *Let* $u_0 \in L^\infty(\Omega)$ *the mild-solution of the problem*

$$u_t = \text{div } \mathbf{a}(x, Du) \quad \text{in} \quad \Omega \times (0, \infty)$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty)$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega.$$

Then, there exists a constant C*, independent of* u_0*, such that*

$$\|u(.,t)\|_p \leq \left(\frac{C\|u_0\|_2^2}{t} \right)^{1/p} \quad \textit{for all} \quad t > 0.$$

Proof. By the definiton of A_β, $J_\lambda u_0 \in W_0^{1,p}(\Omega)$. So, by the Poincaré inequality, there exists a constant M such that

$$\int_\Omega |J_\lambda u_0|^p \leq M \int_\Omega |D(J_\lambda u_0)|^p.$$

Then, if we set

$$\mathcal{U}(v) := \frac{1}{M} \|v\|_p^p,$$

(2.3) implies that

$$\mathcal{V}(J_\lambda u_0) + \lambda \mathcal{U}(J_\lambda u_0) - \mathcal{V}(u_0) \leq 0$$

Thus, proceeding as in the proof of Theorem 2.2, we get

$$\int_0^\infty \mathcal{U}(u(.,s)) \, ds \leq \mathcal{V}(u_0). \tag{3.6}$$

Now, by the complete accretiveness of A_β, $\mathcal{U}$ is a Lyapunov functional for the semigroup generated by A_β. Then, by (3.6) we get

$$t\|u(.,t) - \|_p^p \leq \int_0^t \mathcal{U}(u(.,s))\ ds \leq \frac{1}{2}M\|u_0\|_2^2.$$

Therefore

$$\|u(.,t)\|_p \leq \left(\frac{C\|u_0\|_2^2}{t}\right)^{1/p} \quad \text{for all} \quad t > 0,$$

with $C = \frac{M}{2}$.

Remark 3.5.. The asymptotic behaviour of solutions of the Dirichlet problem for more general **a** is studied by P. Wittbold [19]. Also in [15] precise decay estimates are given for the particular case of the p-Laplacian with $p > 2$.

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