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Porous medium equation with absorption and a nonlinear boundary condition

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1. Introduction

We study the behavior of solutions of the following parabolic problem

$$
\begin{aligned}
&u_t = \Delta(|u|^{m-1}u) - \lambda|u|^{p-1}u \quad \text{in } \Omega \times]0,T[= Q_T,\\
&\frac{\partial(|u|^{m-1}u)}{\partial\eta} = |u|^{q-1}u \quad \text{on } \partial\Omega \times]0,T[= S_T,\\
&u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain with smooth boundary, $\partial/\partial\eta$ is the outer normal derivative, $m \geqslant 1, p$, λ and q are positive parameters and u_0 is in $L^\infty(\Omega)$.

Problems of this form arise in mathematical models in a number of areas of science, for instance, in models for gas or fluid flow in porous media [3] and for the spread of certain biological populations [13]. In the semilinear case (that is for $m=1$), there is an extensive literature about global existence and blow-up results for this type of problems, see among others, [5,9,16] and the literature therein. For the degenerate case (that is for $m \neq 1$), with a nonlinear boundary condition, local existence and uniqueness of weak solutions which are limit of solutions of nondegenerate problems has been established in [1]. Also in [2] existence and uniqueness of global weak solutions for a similar

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problem but with opposite sign in the source term and in the boundary condition are studied.

Our aim is to study existence of globally bounded weak solutions or blow-up, depending on the relations between the parameters m, p, q, λ. For this purpose, we construct adequate supersolutions and subsolutions using ideas from [18]. To prove uniqueness we use the technique introduced in [2], for the sake of completeness we give the proofs of some of the technical lemmas we need to establish the uniqueness result. We also prove results about uniqueness and nonuniqueness in the case of null initial data in the line of the ones obtained in [6] (see also [4]).

This paper is structured as follows. In the next section we state our main results on existence, uniqueness and blow-up. In the thrid section we prove the local existence and uniqueness results. In the next section we give the proof of the blow-up results. Finally, the last section is devoted to prove the results about global existence of weak solutions.

2. Statements of main results

In this paper, we use the following definition of weak solution.

Definition 2.1. Given $u_0 \in L^\infty(\Omega)$, by a weak solution of problem (1.1) on Q_T we mean a function $u \in C([0,T]; L^1(\Omega)) \cap L^\infty(Q_T)$ such that $|u|^{m-1}u \in L^2(0,T; H^1(\Omega))$ and satisfies the identity

$$\int_{Q_T} (\nabla(|u|^{m-1}u)\nabla\phi - u\phi_t + \lambda|u|^{p-1}u\phi) - \int_{S_T} |u|^{q-1}u\phi = \int_\Omega u_0(x)\phi(x,0)\,\mathrm{d}x$$

for any test function $\phi \in L^2(0,T; H^1(\Omega)) \cap W^{1,1}(0,T; L^1(\Omega))$ with $\phi(T) = 0$.

We shall say that u is a global weak solution of problem (1.1) if u is a weak solution on Q_T for all positive T.

With respect to local existence and uniqueness our main result is the following.

Theorem 2.1. *Given $u_0 \in L^\infty(\Omega)$ there exists a local weak solution of* (1.1) *defined in* $[0, t_0(u_0))$ *where $t_0(u_0)$ depends on u_0. Moreover, for $q \geqslant m$, if u and v are weak solutions of* (1.1) *with initial data u_0, v_0, respectively, and* $0 < T < \min\{t_0(u_0), t_0(v_0)\}$, *then there exists a constant C such that*

$$\|(u(t) - v(t))^+\|_{L^1(\Omega)} \leqslant \mathrm{e}^{CT}\|(u_0 - v_0)^+\|_{L^1(\Omega)}$$

and

$$\|u(t) - v(t)\|_{L^1(\Omega)} \leqslant \mathrm{e}^{CT}\|u_0 - v_0\|_{L^1(\Omega)}$$

for all $t \in [0,T]$. Therefore, in this case we have uniqueness of weak solutions.

In the case $q < m$ we only know uniqueness or nonuniqueness for initial datum $u_0 \equiv 0$ for some range of parameters. More precisely, we have the following result.

Theorem 2.2. *Let $q < m$.*

(a) *If $2q < m+1$, $m > 1$ and $p > 1$, there exists infinite weak solutions with zero initial datum obtained as limit of nondegenerate problems.*

(b) If $2q \geqslant m+1$, *the weak solution with initial datum $u_0 \equiv 0$ which is obtained as limit of nondegenerate problems is unique.*

Next, we present the global existence and blow-up results depending on the range of the parameters. For blow up of solutions we mean that the solution is defined in $(0,T), 0 < T \leqslant \infty$, and at that time T we have,

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

Theorem 2.3. *Let $q > m$.*

(a) *If $p < 2q-m$, there exist solutions of* (1.1) *that blow up in finite time for initial data large enough.*

(b) *If $p > 2q-m$, for every initial datum in $L^\infty(\Omega)$, there exists a weak solution which is globally bounded.*

(c) *If $p = 2q-m$, the existence of blowing up solutions depends on λ, that is, there exists a critical value $\lambda_0 = q/m$ such that, for every λ small, $\lambda < q/m$, there exist blowing up solutions in finite time for initial data large enough and for every λ large, $\lambda > q/m$, for every initial datum in $L^\infty(\Omega)$ there exists a global weak solution which is globally bounded.*

Theorem 2.4. *Let $q \leqslant m$.*

(a) *If $p > q$, then for every initial datum in $L^\infty(\Omega)$ there exists a global weak solution which is globally bounded.*

(b) *If $p < q$, for $q > 1$ there are solutions with blow-up in finite time for initial data large enough, for $q \leqslant 1$ there exists a global weak solution for every initial datum in $L^\infty(\Omega)$ that it is unbounded if u_0 is large.*

(c) *If $p = q$ and $q \leqslant 1$, there exists a global weak solution for every initial datum in $L^\infty(\Omega)$ and the boundedness depends on λ, for λ small and u_0 large there are unbounded solutions and for λ large there are bounded solutions for any initial datum.*

(d) *If $p=q$ and $q > 1$, the existence of blowing up solutions depends on λ, if λ is small there are solutions with finite time blow-up for initial data large enough, while if λ is large there exists a global weak solution globally bounded for every initial datum in $L^\infty(\Omega)$.*

Remark 2.5. In the case we have uniqueness and $p = 2q-m$ or $p = q$ with $q > 1$, there exists a critical value λ_0 such that for every $\lambda < \lambda_0$ there exist solutions of (1.1) which blow up in finite time. For $\lambda > \lambda_0$, every solution of (1.1) is global. This is a consequence of the fact that positive solutions of (1.1) with λ replaced by $\bar{\lambda}$ are subsolutions of (1.1) for every $\lambda < \bar{\lambda}$.

In the case $p=2q-m$ with $q > m$, we have that λ_0 does not depend on the domain, in fact $\lambda_0 = q/m$.

In the case $p = q$ with $1 < q < m$, if we have uniqueness (see Remark 3.1) the critical value depends on Ω in the following way, consider $\Omega_\mu = \{x/\mu x \in \Omega\}$ then $\lambda_0(\Omega_\mu) = \mu\lambda_0(\Omega)$.

This follows using that if $u(x,t)$ is a solution of (1.1) in Ω with parameter λ then

$$v(x,t) = \mu^{-1/(m-q)}u(\mu x, \mu^{(m-2q+1)/(m-q)}t)$$

is a solution of (1.1) in Ω_μ with parameter $\mu\lambda$.

Finally, in the case $1 < p = q = m$, the critical value λ_0 depends on the domain. For instance, if $\Omega = B(0,R)$ we have that

$$\max\left\{\frac{N}{R}, 1\right\} < \lambda_0(R) < \frac{N}{R} + 1.$$

3. Local existence and uniqueness

In this section, we prove Theorems 2.1 and 2.2.

3.1. Local existence

In proving local existence for degenerate parabolic equations as (1.1) one standard approach is to approximate the problem with a sequence of nondegenerate problems which can be solved in a classical sense. To do that we consider sequences of functions (φ_n), (f_n) and (g_n) with

$$\varphi_n(r) = |r|^{m-1}r, \quad f_n(r) = |r|^{p-1}r, \quad g_n(r) = |r|^{q-1}r \text{ if } \frac{1}{n} < |r| < n,$$

(φ_n), (φ_n'), (f_n) and (g_n) converging uniformly on compact subsets of $\mathbb{R}$ to

$$|r|^{m-1}r, \quad (|r|^{m-1}r)', \quad |r|^{p-1}r, \quad |r|^{q-1}r,$$

respectively, and satisfying the conditions of [14, Theorem 7.4].

Using the same technique as in the proof of [8, Proposition 3], we can find functions $u_{0,n} \in C^3(\bar{\Omega})$, $\|u_{0,n}\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 1$, satisfying the compatibility condition

$$\frac{\partial\varphi_n(u_{0,n})}{\partial\eta} = g_n(u_{0,n}) \quad \text{on } \partial\Omega$$

and

$$\|u_{0,n} - u_0\|_{L^1(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Consider the approximated problems

$$(P_n)\begin{cases} (u_n)_t = \Delta\varphi_n(u_n) - \lambda f_n(u_n) & \text{in } Q_T, \\ \dfrac{\partial\varphi_n(u_n)}{\partial\eta} = g_n(u_n) & \text{on } S_T, \\ u_n(x,0) = u_{0,n}(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

By [14, Theorem 7.4], for any $T > 0$, (P_n) has a unique smooth solution u_n in Q_T.

First, we claim that there exists a time $T > 0$ and a constant C depending on T and $\|u_0\|_\infty$ such that

$$\|u_n(t)\|_{L^\infty(\Omega)} \leqslant C \quad \forall t \in [0,T], \ \forall n \in \mathbb{N}. \tag{3.2}$$

This claim is proved easily using a comparison argument between u_n and $w, \tilde{w}$ solutions with initial data $w(x,0)$ greater than $\|u_0\|_\infty + 1$ and $\tilde{w}(x,0)$ less than $-\|u_0\|_\infty - 1$ verifying the compatibility condition (we can take an extension of $w(x,0) = g(dist(x,\partial\Omega))$ for an appropriate election of g) to obtain $-C \leqslant \tilde{w}(t) \leqslant u_n(t) \leqslant w(t) \leqslant C$ for every $n \geqslant n_0$ and for every t in $[0,T]$.

Next, we prove the following energy estimates.

Proposition 3.1. *Let u_n be the solution of problem (P_n) in $[0,T]$. Then, given $\tau > 0$, there exist constants depending only on $\|u_0\|_{L^\infty(\Omega)}$, T and τ, such that*

$$\|\nabla\varphi_n(u_n(t))\|_{L^2(\Omega)} \leqslant C(\tau, T, \|u_0\|_{L^\infty(\Omega)}), \quad \forall\, T \geqslant t \geqslant \tau, \tag{3.3}$$

$$\int_\tau^T \int_\Omega (u_n)_t \varphi_n(u_n)_t \leqslant K(\tau, T, \|u_0\|_{L^\infty(\Omega)}). \tag{3.4}$$

Proof. Multiplying the equation of (P_n) by $\varphi_n(u_n)_t$ we get

$$\begin{aligned} 0 \leqslant \int_\Omega (u_n)_t \varphi_n(u_n)_t &= -\int_\Omega \nabla\varphi_n(u_n)\nabla(\varphi_n(u_n))_t \\ &\quad + \int_{\partial\Omega} g_n(u_n)(\varphi_n(u_n))_t - \lambda \int_\Omega f_n(u_n)(\varphi_n(u_n))_t. \end{aligned} \tag{3.5}$$

If we set

$$F_n(r) = \lambda \int_0^r f_n(s)\varphi_n'(s)\,\mathrm{d}s, \qquad G_n(r) = \int_0^r g_n(s)\varphi_n'(s)\,\mathrm{d}s,$$

from (3.5) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega |\nabla\varphi_n(u_n)|^2 - \int_{\partial\Omega} G_n(u_n) + \int_\Omega F_n(u_n)\right) \leqslant 0. \tag{3.6}$$

Now, by (3.2), there are constants $M_1, M_2 > 0$, depending only on $\|u_0\|_{L^\infty(\Omega)}$, T and τ, such that

$$-M_1 \leqslant \int_{\partial\Omega} G_n(u_n) \leqslant M_1, \qquad -M_2 \leqslant \int_\Omega F_n(u_n) \leqslant M_2,$$

and consequently

$$M_1 - \int_{\partial\Omega} G_n(u_n) \geqslant 0, \qquad \int_\Omega F_n(u_n) + M_2 \geqslant 0.$$

Moreover, from (3.6), we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega |\nabla\varphi_n(u_n)|^2 + M_1 - \int_{\partial\Omega} G_n(u_n) + \int_\Omega F_n(u_n) + M_2\right) \leqslant 0.$$

On the other hand, multiplying the equation of (P_n) by $\varphi_n(u_n)$, integrating over $]t,t+h[\,\times\,\Omega$ and using the uniform bound for u_n, (3.2), it is easy to see that

$$\int_t^{t+h}\left(\frac{1}{2}\int_\Omega |\nabla\varphi_n(u_n)|^2 + M_1 - \int_{\partial\Omega} G_n(u_n) + \int_\Omega F_n(u_n) + M_2\right)$$
$$\leqslant C(h,T,\|u_0\|_{L^\infty(\Omega)})$$

for $t \geqslant \tau$. Then, by the uniform Gronwall's Lemma ([17]), we obtain (3.3).

Finally, integrating (3.5) over $]\tau,T[$, we get

$$\int_\tau^T \int_\Omega (u_n)_t\varphi_n(u_n)_t + \frac{1}{2}\int_\Omega \left(|\nabla\varphi_n(u_n(T))|^2 - |\nabla\varphi_n(u_n(\tau))|^2\right)$$

$$-\int_{\partial\Omega}(G_n(u_n(T)) - G_n(u_n(\tau))) - \int_\Omega (F_n(u_n(\tau)) - F_n(u_n(T))) = 0.$$

Hence, by (3.3), (3.4) holds. □

Using the uniform estimate (3.2) multiplying the equation in (P_n) by $\varphi_n(u_n)$ and integrating, we get

$$\int_0^T \int_\Omega |\nabla\varphi_n(u_n)|^2 \leqslant C, \quad \forall n \in \mathbb{N}.$$

By compactness arguments, working as in [2], it follows that (up to extraction of a subsequence)

$$\varphi_n(u_n) \to |u|^{m-1}u \quad \text{in } L^2(\Omega\times\,]0,T[) \text{ and a.e. in } \Omega\times\,]0,T[,$$
$$\nabla\varphi_n(u_n) \to \nabla(|u|^{m-1}u) \quad \text{weakly in } (L^2(\Omega\times\,]0,T[))^N,$$
$$g_n(u_n) \to |u|^{q-1}u \quad \text{in } L^2(\partial\Omega\times\,]0,T[),$$
$$(\varphi_n(u_n))_t \to (|u|^{m-1}u)_t \quad \text{weakly in } L^2(\Omega\times\,]\tau,T[).$$

Since u_n is a smooth solution of (P_n), it clearly satisfies

$$\int_0^T \int_\Omega (\nabla\varphi_n(u_n)\nabla\phi - u_n\phi_t + \lambda f_n(u_n)\phi) - \int_0^T \int_{\partial\Omega} g_n(u_n)\phi$$
$$= \int_\Omega u_{0,n}(x)\phi(x,0)\,\mathrm{d}x$$

for any test function ϕ. From here, passing to the limit when $n\to\infty$ we obtain that u satisfies the same equality. Moreover, $u \in C(]0,T];L^1(\Omega))$.

Let us see that $u(t)$ is also continuous at $t=0$. If $u_0 \in C^1(\Omega)$, the above arguments give us the continuity at 0 since we can impose that $|\nabla\varphi_n(u_{0,n})|$ is bounded in $L^2(\Omega)$. In order to get the general case we need the following lemmas.

Lemma 3.1. *Let φ, $\hat{\varphi} : \mathbb{R} \to \mathbb{R}$ be continuous increasing functions, F, $\hat{F} \in L^1(Q_T)$, G, $\hat{G} \in L^1(S_T)$ and v_0, $\hat{v}_0 \in L^\infty(\Omega)$. Suppose v and $\hat{v}$ are smooth solutions of the problem*

$$
\begin{aligned}
&v_t = \Delta\varphi(v) - F \quad \text{in } Q_T,\\
&\frac{\partial\varphi(v)}{\partial\eta} = G \quad \text{on } S_T,\\
&v(x,0) = v_0(x) \quad \text{in } \Omega,
\end{aligned}
$$

and

$$
\hat{v}_t = \Delta\hat{\varphi}(\hat{v}) - \hat{F} \quad \text{in } Q_T, \qquad \frac{\partial\varphi(\hat{v})}{\partial\eta} = \hat{G} \quad \text{on } S_T, \quad \hat{v}(x,0) = \hat{v}_0(x) \quad \text{in } \Omega,
$$

respectively. Let $\psi \in C^2(\bar{\Omega})$, $\psi \geqslant 1$ on $\bar{\Omega}$ such that

$$
\frac{\partial\psi}{\partial\eta} \geqslant L\psi \quad \text{on } \partial\Omega, \tag{3.7}
$$

where $L > 0$ is a given constant. Then

$$
\begin{aligned}
&\int_\Omega (v(t) - \hat{v}(t))^+\psi + L\int_0^t \int_{\partial\Omega} (\varphi(v) - \varphi(\hat{v}))^+\psi\\
&\quad \leqslant \int_\Omega (v_0 - \hat{v}_0)^+\psi + \int_0^t \int_\Omega [(\varphi(v) - \varphi(\hat{v}))^+|\Delta\psi| - (F - \hat{F})\,\mathrm{sign}^+(v - \hat{v})\psi]\\
&\qquad + \int_0^t \int_{\partial\Omega} (G - \hat{G})\,\mathrm{sign}^+(v - \hat{v})\psi
\end{aligned}
$$

for every $0 < t < T$.

Proof. Multiplying the difference of the two equations by $\mathrm{sign}^+(v - \hat{v})\psi$, applying Kato's inequality and integrating over Ω, we get

$$
\begin{aligned}
\int_\Omega (v - \hat{v})_t\,\mathrm{sign}^+(v - \hat{v})\psi \leqslant{}& -\int_\Omega \nabla(\varphi(v) - \varphi(\hat{v}))^+ \cdot \nabla\psi\\
&+ \int_{\partial\Omega} \frac{\partial}{\partial\eta}(\varphi(v) - \varphi(\hat{v}))\,\mathrm{sign}^+(v - \hat{v})\psi\\
&- \int_\Omega (F - \hat{F})\,\mathrm{sign}^+(v - \hat{v})\,\psi\,.
\end{aligned}
$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\Omega}(v-\hat{v})^{+}\psi \leqslant \int_{\Omega}(\varphi(v)-\varphi(\hat{v}))^{+}\Delta\psi - \int_{\partial\Omega}\frac{\partial\psi}{\partial\eta}(\varphi(v)-\varphi(\hat{v}))^{+}$$
$$+\int_{\partial\Omega}(G-\hat{G})\,\mathrm{sign}^{+}(v-\hat{v})\psi - \int_{\Omega}(F-\hat{F})\,\mathrm{sign}^{+}(v-\hat{v})\psi\,.$$

Integrating the above inequality from 0 to t and using (3.7) the proof concludes. □

Lemma 3.2. *Given u_0, $\hat{u}_0 \in L^{\infty}(\Omega)$, let u and $\hat{u}$ be the limit of the smooth solutions of the approximated problem*

$$(Q_n)\quad \begin{cases} (u_n)_t = \Delta\varphi_n(u_n) - \lambda f_n(u_n) & \text{in } Q_T,\\ \dfrac{\partial\varphi_n(u_n)}{\partial\eta} = g_n(u_n) & \text{on } S_T,\\ u_n(x,0) = u_{0,n}(x) & \text{in } \Omega \end{cases}$$

and

$$(\hat{P}_n)\quad \begin{cases} (\hat{u}_n)_t = \Delta\varphi_n(\hat{u}_n) - \lambda f_n(\hat{u}_n) & \text{in } Q_T,\\ \dfrac{\partial\varphi_n(\hat{u}_n)}{\partial\eta} = g_n(\hat{u}_n) & \text{on } S_T,\\ \hat{u}_n(x,0) = \hat{u}_{0,n}(x) & \text{in } \Omega, \end{cases}$$

respectively, where $u_{0,n} \to u_0$, $\hat{u}_{0,n} \to \hat{u}_0$ in $L^1(\Omega)$, $u_{0,n}$ and $\hat{u}_{0,n}$ are bounded in $L^{\infty}(\Omega)$ independently on n, and verify the corresponding compatibility conditions. Then

$$\|u(t)-\hat{u}(t)\|_{L^1(\Omega)} \leqslant K_1(T,\|u_0\|_{\infty},\|\hat{u}_0\|_{\infty},\psi)t + K_2(\psi)\|u_0-\hat{u}_0\|_{L^1(\Omega)}.$$

Proof. Using Lemma 3.1 with $L=0$, the uniform estimate (3.2) and taking $n\to\infty$, we have

$$\int_{\Omega}(u(t)-\hat{u}(t))^{+} \leqslant \int_0^t\int_{\Omega}((|u|^{m-1}u)(s)-(|\hat{u}|^{m-1}\hat{u})(s))^{+}|\Delta\psi|$$
$$+\int_0^t\int_{\partial\Omega}((|u|^{q-1}u)(s)-(|\hat{u}|^{q-1}\hat{u})(s))^{+}\psi + \int_{\Omega}|u_0-\hat{u}_0|\psi$$
$$\leqslant K_1 t + K_2\|u_0-\hat{u}_0\|_{L^1(\Omega)}.$$

Interchanging the role of u and $\hat{u}$ the proof is concluded. □

Let us finish the continuity at zero. If $u_0 \in L^{\infty}(\Omega)$, there exists a sequence of smooth functions $u_{0,n}$ such that $u_{0,n}\to u_0$ in $L^1(\Omega)$. The corresponding u_n, constructed as above, are continuous at 0, then applying Lemma 3.2 we get

$$\|u(t)-u_0\|_1 \leqslant \|u(t)-u_n(t)\|_1 + \|u_n(t)-u_{0,n}\|_1 + \|u_{0,n}-u_0\|_1$$
$$\leqslant K_1 t + K_2\|u_0-u_{0,n}\|_1 + \|u_n(t)-u_{0,n}\|_1 + \|u_{0,n}-u_0\|_1,$$

and the continuity of u at 0 follows. □

3.2. Uniqueness

We start with the following definition of weak solution,

Definition 3.1. Let $\varphi:\mathbb{R}\to\mathbb{R}$ a continuous and increasing function, $F\in L^\infty(Q_T)$, $G\in L^\infty(S_T)$ and $v_0\in L^\infty(\Omega)$, we say that v is a weak solution of the problem

$$S(\varphi,F,G,v_0)\qquad \begin{cases} v_t=\Delta\varphi(v)-F & \text{in } Q_T,\\ \dfrac{\partial\varphi(v)}{\partial\eta}=G & \text{on } S_T,\\ v(x,0)=v_0(x) & \text{in } \Omega,\end{cases}$$

if $v\in C([0,T],L^1(\Omega))\cap L^\infty(Q_T)$, $\varphi(v)\in L^2(0,T;H^1(\Omega))$ and satisfies

$$\int_{Q_T}(\nabla\varphi(v).\nabla\phi - v\phi_t + F\phi) - \int_{S_T}G\phi = \int_\Omega v_0(x)\phi(x,0)\,\mathrm{d}x$$

for any test function $\phi\in L^2(0,T;H^1(\Omega))\cap W^{1,1}(0,T;L^1(\Omega))$ with $\phi(T)=0$.

In order to prove the uniqueness we need the following lemma.

Lemma 3.3. *Let φ, $\hat\varphi:\mathbb{R}\to\mathbb{R}$ be continuous and increasing functions, F, $\hat F\in L^\infty(Q_T)$, G, $\hat G\in L^\infty(S_T)$ and $v_0,\hat v_0\in L^\infty(\Omega)$. If v and $\hat v$ are weak solutions of $S(\varphi,F,G,v_0)$ and $S(\hat\varphi,\hat F,\hat G,\hat v_0)$, respectively, then*

$$\begin{aligned}\int_{Q_T}(v-\hat v)(\varphi(v)-\hat\varphi(\hat v)) \leqslant{}& \int_\Omega(v_0-\hat v_0)\int_0^T(\varphi(v)-\hat\varphi(\hat v))\\ &-\int_{Q_T}(F-\hat F)\int_t^T(\varphi(v)-\hat\varphi(\hat v))\\ &+\int_{S_T}(G-\hat G)\int_t^T(\varphi(v)-\hat\varphi(\hat v)).\end{aligned}$$

Proof. It is enough to take as test function

$$\eta(x,t)=\begin{cases}\displaystyle\int_t^T(\varphi(v(x,s))-\hat\varphi(\hat v(x,s)))\,\mathrm{d}s & \text{if } 0\leqslant t<T,\\ 0 & \text{if } t\geqslant T\end{cases}$$

and the result follows. □

We can begin with the proof of the uniqueness. Suppose that u and $\hat u$ are two weak solutions of problem (1.1) on Q_T with initial data u_0, $\hat u_0\in L^\infty(\Omega)$, respectively. Let F_n, $\hat F_n$, G_n, $\hat G_n$ smooth functions, F_n, $\hat F_n$ bounded in $L^\infty(Q_T)$ and $G_n,\hat G_n$ bounded in $L^\infty(S_T)$ uniformly in n, such that

$$F_n\to\lambda|u|^{p-1}u,\qquad \hat F_n\to\lambda|\hat u|^{p-1}\hat u\quad\text{in } L^2(Q_T),$$

$$G_n\to|u|^{q-1}u,\qquad \hat G_n\to|\hat u|^{q-1}\hat u\quad\text{in } L^2(S_T).$$

Let φ_n as before. Using again the same technique than in Proposition 3 of [8], we can find functions $u_{0,n}, \hat{u}_{0,n} \in C^3(\bar{\Omega})$ bounded in $L^\infty(\Omega)$ uniformly in n satisfying the compatibility conditions

$$\frac{\partial \varphi_n(u_{0,n})}{\partial \eta} = G_n(.,0) \quad \text{on } \partial\Omega,$$

$$\frac{\partial \varphi_n(\hat{u}_{0,n})}{\partial \eta} = \hat{G}_n(.,0) \quad \text{on } \partial\Omega$$

and

$$u_{0,n} \to u_0, \qquad \hat{u}_{0,n} \to u_0 \quad \text{in } L^1(\Omega).$$

By classical results (see Theorem 7.4 of [14]), there exist u_n and $\hat{u}_n$ smooth solutions of the problems $S(\varphi_n, F_n, G_n, u_{0,n})$ and $S(\varphi_n, \hat{F}_n, \hat{G}_n, \hat{u}_{0,n})$, respectively.

First, using the maximum principle, we prove that u_n are bounded in $L^\infty(Q_T)$ uniformly in n. Indeed, there exists $C > 0$ such that $\|F_n\|_{L^\infty(Q_T)} \leqslant C$, $\|G_n\|_{L^\infty(S_T)} \leqslant C$ and $\|\varphi_n(u_{0,n})\|_{L^\infty(\Omega)} \leqslant C$. Consider $\psi \in C^2(\bar{\Omega})$ satisfying

$$\psi \geqslant C \quad \text{in } \Omega \text{ and } \frac{\partial \psi}{\partial \eta} \geqslant \psi \quad \text{on } \partial\Omega.$$

Set $\xi_n = \varphi_n^{-1}(\psi + \gamma t)$, where γ is a positive constant. For C large enough, we have

$$(\xi_n)_t = \frac{\gamma}{\varphi_n'(\varphi_n^{-1}(\psi + \gamma t))} \geqslant \frac{\gamma}{m(\|\psi\|_{L^\infty(\Omega)} + \gamma T)^{(m-1)/m}},$$

which implies

$$(u_n)_t - \Delta\varphi_n(u_n) + F_n = 0 \leqslant (\xi_n)_t - \Delta\varphi_n(\xi_n) + F_n \quad \text{in } Q_T$$

for γ large enough. On the other hand, we have

$$\frac{\partial \varphi_n(\xi_n)}{\partial \eta} = \frac{\partial \psi}{\partial \eta} \geqslant C \geqslant \frac{\partial \varphi_n(u_n)}{\partial \eta} \quad \text{in } S_T.$$

Consequently, $u_n \leqslant \xi_n \leqslant C(\|\psi\|_{L^\infty(\Omega)}, \gamma, T)$ on Q_T. Similarly, a lower bound for u_n can be obtained.

Moreover, multiplying the equation by $\varphi_n(u_n)$ and integrating on Q_T it is easy to see that $\{|\nabla\varphi_n(u_n)|: n \in \mathbb{N}\}$ is bounded in $L^2(Q_T)$.

Using Lemma 3.3 for u_n, the solution of $S(\varphi_n, F_n, G_n, u_{0,n})$, and for u, the weak solution of $S(\varphi, |u|^{p-1}u, |u|^{q-1}u, u_0)$, we have

$$\int_{Q_T} (u_n - u)(\varphi_n(u_n) - |u|^{m-1}u) \leqslant a_n, \quad \text{with } \lim_{n\to\infty} a_n = 0.$$

Then, as

$$\int_{Q_T} (u_n - u)(|u_n|^{m-1}u_n - |u|^{m-1}u)$$

$$= \int_{Q_T} (u_n - u)(|u_n|^{m-1}u_n - \varphi_n(u_n)) + \int_{Q_T} (u_n - u)(\varphi_n(u_n) - |u|^{m-1}u)$$

and $\{u_n : n \in \mathbb{N}\}$ is bounded in $L^\infty(Q_T)$, we obtain

$$\int_{Q_T} (u_n - u)(|u_n|^{m-1}u_n - |u|^{m-1}u) \to 0.$$

By the monotonicity of $|r|^{m-1}r$ we conclude, up to extraction of a subsequence, that u_n converges almost everywhere to u. Consequently, $\varphi_n(u_n) \to |u|^{m-1}u$ in $L^2(Q_T)$. Moreover $\{|\nabla\varphi_n(u_n)| : n \in \mathbb{N}\}$ is bounded in $L^2(Q_T)$. Hence by Theorem 3.4.5 of [15], it follows that

$$\varphi_n(u_n) \to |u|^{m-1}u \quad \text{in } L^2(0,T : L^2(\partial\Omega)).$$

Similarly, we obtain the same for $\hat{u}_n$. Then, applying Lemma 3.1 and passing to the limit we obtain

$$\begin{aligned}
&\int_\Omega (u(t) - \hat{u}(t))^+\psi + L\int_0^t \int_{\partial\Omega} (|u|^{m-1}u - |\hat{u}|^{m-1}\hat{u})^+\psi \\
&\quad\leqslant \int_0^t \int_\Omega (|u|^{m-1}u - |\hat{u}|^{m-1}\hat{u})^+|\Delta\psi| \\
&\quad\quad -\lambda\int_0^t \int_\Omega (|u|^{p-1}u - |\hat{u}|^{p-1}\hat{u})\,\mathrm{sign}^+(u-\hat{u})\psi \\
&\quad\quad +\int_0^t \int_{\partial\Omega} (|u|^{q-1}u - |\hat{u}|^{q-1}\hat{u})\,\mathrm{sign}^+(u-\hat{u})\psi + \int_\Omega (u_0 - \hat{u}_0)^+\psi\,.
\end{aligned}$$

As $q \geqslant m$ and u, $\hat{u}$ are bounded in Q_T, we have that

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} (|u|^{q-1}u - |\hat{u}|^{q-1}\hat{u})\,\mathrm{sign}^+(u-\hat{u})\psi \\
&\quad\leqslant M\int_0^t \int_{\partial\Omega} (|u|^{m-1}u - |\hat{u}|^{m-1}\hat{u})^+\psi\,.
\end{aligned}$$

Choosing $L \geqslant M$, we obtain

$$\int_\Omega (u(t) - \hat{u}(t))^+ \leqslant C_1\int_0^t \int_\Omega (u(s) - \hat{u}(s))^+ + C_2\int_\Omega (u_0 - \hat{u}_0)^+.$$

Finally, applying Gronwall's Lemma we get,

$$\int_\Omega (u(t) - \hat{u}(t))^+ \leqslant \mathrm{e}^{CT}\int_\Omega (u_0 - \hat{u}_0)^+,$$

and the proof of Theorem 2.1 is concluded. □

We now deal with the proof of Theorem 2.2. We begin by case (a).

We consider $u_0 \equiv 0$. The idea of the proof of the nonuniqueness part is, up to some technical arguments, the following, since there is a comparison principle for problem (P_n), the existence of a nontrivial subsolution would imply the existence of a nontrivial solution via a standard monotonicity argument.

Let us first analyze the following ordinary differential equation,

$$(f^m)''(\eta) + \beta\eta f'(\eta) = \alpha f(\eta) \quad \text{in } [0, +\infty), \tag{3.8}$$

where β and α are real numbers.

Solutions of (3.8) provide self-similar solutions of the porous medium equation in the half-line. Eq. (3.8) has been completely studied in [10–12]. We summarize, in the form of a lemma, the part of these results that we will need later.

Lemma 3.4 (Gilding, Peletier [11]). *If $\beta > 0$ and $\alpha > 0$ then for any $U \geqslant 0$ Eq.* (3.8) *has a unique weak solution f such that f is a positive classical solution on an interval* $(0, \xi_0)$, $f(0) = U$, $f(\xi_0) = 0$, $(f^m)'(\xi_0) = 0$ *and* $f(\eta) \equiv 0$ *for* $\eta \in [\xi_0, \infty)$. *Moreover* $(f^m)'(\eta) < 0$ *and* $(f^m)''(\eta) > 0$ *for* $\eta \in [0, \xi_0)$.

Let us define the functions

$$h_1(\eta) = \begin{cases} \dfrac{(f^m)'(\eta)}{(f^m)''(\eta)} & \text{if } \eta \in [0, \xi_0), \\ 0 & \text{if } \eta \in [\xi_0, \infty), \end{cases} \qquad h_2(\eta) = \begin{cases} \dfrac{f^p(\eta)}{(f^m)''(\eta)} & \text{if } \eta \in [0, \xi_0), \\ 0 & \text{if } \eta \in [\xi_0, \infty) \end{cases}$$

and state the following elementary lemma for future reference.

Lemma 3.5. *If $\beta > 0$ and $\alpha > 0$, then the functions h_1 and h_2 are bounded.*

Proof. In order to prove that h_1 is bounded it suffices to show that it is continuous at ξ_0. This is immediate because from the equation it follows that for $\eta \in (0, \xi_0)$ one has

$$-\frac{mf^{m-1}(\eta)}{\beta\eta} \leqslant h_1(\eta) \leqslant 0. \tag{3.9}$$

To see that $f^p/(f^m)''$ is bounded we proceed as follows, first we observe that

$$\alpha^p f^p(\xi) = ((f^m)''(\xi) + \beta f'(\xi)\xi)^p.$$

And hence

$$\frac{\alpha^p f^p}{(f^m)''}(\xi) = \left(\frac{(f^m)''(\xi) + \beta f'(\xi)\xi}{((f^m)''(\xi))^{1/p}}\right)^p = \left(((f^m)''(\xi))^{1-1/p} + \frac{\beta f'(\xi)\xi}{((f^m)''(\xi))^{1/p}}\right)^p.$$

From this and the fact that $((f^m)''(\xi))^{1-1/p}$ is bounded we conclude that it is enough to prove that

$$\frac{f'(\xi)\xi}{((f^m)''(\xi))^{1/p}}$$

is bounded near the point ξ_0 where the function f vanishes. In fact, by (3.9) we have,

$$0 \geqslant \frac{f'(\xi)}{((f^m)''(\xi))^{1/p}} = \frac{1}{m}\frac{(f^m)'(\xi)((f^m)''(\xi))^{(p-1)/p}}{f^{m-1}(\xi)(f^m)''(\xi)}$$

$$= \frac{1}{m} f^{1-m}(\xi)h_1(\xi)((f^m)''(\xi))^{(p-1)/p} \geqslant -\frac{1}{\beta\xi}((f^m)''(\xi))^{(p-1)/p}$$

Hence

$$-\frac{f'(\xi)}{((f^m)''(\xi))^{1/p}}\xi \leqslant \frac{1}{\beta}((f^m)''(\xi))^{(p-1)/p} \leqslant C. \qquad \square$$

Let f be the solution of (3.8) with

$$\alpha = \frac{1}{m+1-2q}$$

and

$$\beta = \frac{m-q}{m+1-2q}.$$

Since we have $2q < m+1$, α and β are positive.

In this case it is easy to see, by rescaling, that it is possible to choose U such that $-(f^m)'(0) = f^q(0)$. Then the function

$$v(s,t) = t^\alpha f\left(\frac{s}{t^\beta}\right)$$

satisfies $v_t = (v^m)_{ss}$ and $-(v^m)_s(0,t) = v^q(0,t)$ in $[0,\infty)\times[0,\infty)$.

Let us consider the following change of variables in a neighborhood of $\partial\Omega$. Let $\bar{x}$ be a point in $\partial\Omega$. We denote by $\hat{n}(\bar{x})$ the inner unit normal to $\partial\Omega$ at the point $\bar{x}$. Since $\partial\Omega$ is smooth it is well known that there exists $\delta > 0$ such that the mapping $\varphi : \partial\Omega \times [0,\delta] \to \mathbb{R}^N$ given by $\varphi(\bar{x},s) = \bar{x} + s\hat{n}(\bar{x})$ defines new coordinates $(\bar{x},s)$ in a neighborhood V of $\partial\Omega$ in $\bar{\Omega}$.

A straightforward computation shows that, in these coordinates, Δ applied to a function $g(\bar{x},s) = g(s)$, which is independent of the variable $\bar{x}$, evaluated at a point $(\bar{x},s)$ is given by

$$\Delta g(\bar{x},s) = \frac{\partial^2 g}{\partial s^2}(\bar{x},s) - \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1-H_j(\bar{x})s)}\frac{\partial g}{\partial s}(\bar{x},s), \tag{3.10}$$

where $H_j(\bar{x})$ for $i = 1,\ldots,N$, denotes the principal curvatures of $\partial\Omega$ at $\bar{x}$.

We proceed now to do the rescaling. Let ε be such that $0 < \varepsilon < 1$ and pick c such that $0 < c < \min\{p-1,(m-1)/2\}$. Choose T_0 such that $\xi_0\varepsilon^{(m-1)/2}((1-\varepsilon^c)T_0)^p \leqslant \delta$. For points in $V\times[0,T_0]$ of coordinates $(\bar{x},s,t)$ such that $0 \leqslant s \leqslant \xi_0\varepsilon^{(m-1)/2}((1-\varepsilon^c)t)^p$ define

$$\underline{u}_\varepsilon(\bar{x},s,t) = \varepsilon v\left(\frac{s}{\varepsilon^{(m-1)/2}},(1-\varepsilon^c)t\right)$$

and extend $\underline{u}_\varepsilon$ as zero to the whole of $\bar{\Omega}\times[0,T_0]$.

We will say that a function $\underline{u}$ is a strict subsolution of (1.1) if $\underline{u}$ is continuous in $\bar{\Omega}\times[0,T[$ and satisfies

$$\underline{u}_t \leqslant \Delta(|\underline{u}|^{m-1}\underline{u}) - \lambda|\underline{u}|^{p-1}\underline{u} \quad \text{in } \Omega\times]0,T[$$

in the weak sense and

$$-\frac{\partial(|\underline{u}|^{m-1}\underline{u})}{\partial\hat{\eta}} < |\underline{u}|^{q-1}\underline{u} \quad \text{on } \partial\Omega\times]0,T[.$$

Proposition 3.2. *There exists* ε_0 *such that for any* ε, $0 < \varepsilon \leqslant \varepsilon_0$, *the function* $\underline{u}_\varepsilon$ *is a strict subsolution of* (1.1) *in* $\bar{\Omega} \times [0, T_0]$.

Proof. As $0 < \varepsilon < 1$ and $q < (m+1)/2$ one has

$$-\frac{\partial \underline{u}_\varepsilon^m}{\partial s}(\bar{x}, 0, t) < \underline{u}_\varepsilon^q(\bar{x}, 0, t) \tag{3.11}$$

and hence the boundary condition is satisfied.

We set

$$\xi = \frac{s}{\varepsilon^{(m-1)/2}((1-\varepsilon^c)t)^p}.$$

A straightforward computation shows that if $(\bar{x}, s, t)$ is such that $0 < t < T_0$ and $0 \leqslant s \leqslant \xi_0 \varepsilon^{(m-1)/2}((1-\varepsilon^c)t)^p$, then

$$\begin{aligned}&(\underline{u}_\varepsilon)_t(\bar{x}, s, t) - \Delta \underline{u}_\varepsilon^m(\bar{x}, s, t) + \lambda \underline{u}_\varepsilon^p(\bar{x}, s, t)\\ &\quad = -\varepsilon^{1+c}((1-\varepsilon^c)t)^{\alpha-1}(f^m)''(\xi)\left[1 - \varepsilon^{((m-1)/2)-c}((1-\varepsilon^c)t)^p\right.\\ &\qquad \left.\times \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1-H_j(\bar{x})s)} h_1(\xi) - \lambda \varepsilon^{p-1-c}((1-\varepsilon^c)t)^{p\alpha-\alpha+1} h_2(\xi)\right].\end{aligned}$$

Now since, by Lemma 3.5, the functions h_1 and h_2 are bounded, $(f^m)''(\xi) > 0$ for $\xi \in [0, \xi_0]$, and we have that $p - 1 - c > 0$ and $p\alpha - \alpha + 1 \geqslant 0$, we obtain that if ε is small enough then

$$(\underline{u}_\varepsilon)_t(\bar{x}, s, t) - \Delta \underline{u}_\varepsilon^m(\bar{x}, s, t) + \lambda \underline{u}^p(\bar{x}, s, t) \leqslant 0 \tag{3.12}$$

if $0 < t < T_0$ and $0 \leqslant s \leqslant \xi_0 \varepsilon^{(m-1)/2}((1-\varepsilon^c)t)^p$. Finally, that $\underline{u}_\varepsilon$ is a subsolution, in the weak sense in the whole of $\bar{\Omega} \times [0, T_0)$, follows from the fact that $\underline{u}_\varepsilon$ is continuous in $\bar{\Omega} \times [0, T_0)$ and, since $(f^m)'(\xi_0) = 0$, one has $\nabla \underline{u}_\varepsilon^m = 0$ on the free boundary as it can be checked by a direct computation. □

We are now in position to give the proof of nonuniqueness in the case $2q < m + 1$. Pick a sequence, v_n, $n = 0, 1, 2, \ldots$, of positive classical solutions of (1.1) with compatible initial data such that $0 < v_n(x, 0) < v_j(x, 0)$ if $n > j$ and $v_n(x, 0) \to 0$ as $n \to \infty$. By a comparison argument, taking T_0 smaller if necessary, we obtain that for a fixed small enough ε one has

$$\underline{u}_\varepsilon \leqslant v_n \leqslant v_j \leqslant v_0$$

in $\bar{\Omega} \times [0, T_0)$ if $n > j$.

We define now $u(x,t) = \lim v_n(x,t)$ as $n \to \infty$. By the monotone convergence theorem we obtain that u is weak solution. Clearly $\underline{u}_\varepsilon \leqslant u$ and hence u is nontrivial and becomes positive for $t > 0$. Hence $u(t - t_0)$ is another nontrivial solution. This proves part (a) of Theorem 2.2.

Respect to the case (b) of Theorem 2.2, the same supersolutions that can be found in [6] provides us with the uniqueness of the zero solution that are constructed as limit of solutions of nondegenerate problems. □

Remark 3.1. In contrast with the case of null initial datum, we observe that the same arguments used in the proof of the uniqueness of weak solutions in the case $q \geqslant m$ work if we consider two weak solutions u, $\hat{u}$ with the same initial datum, such that $u, \hat{u} \geqslant c > 0$ on $\partial\Omega \times [0, t_0]$. Hence, in this case we have uniqueness of weak solutions as long as they are strictly positive over the boundary. For instance, this holds for $t \in [0, t_0]$ if we take a continuous initial datum $u_0 \geqslant c > 0$ on the boundary of Ω, as a consequence of the continuity of the solutions in $\bar{Q}_{t_0}$ given in [7].

4. Blow-up results

In this section, we prove the finite time blow-up results in Theorems 2.3 and 2.4.

Lemma 4.1. *Let $p < m < q$, then there exist solutions of* (1.1) *with finite time blow up.*

Proof. The idea of the proof is to find a subsolution $\underline{u}$ with finite time blow up and to use a comparison argument.

Let $\varphi(s) = (\frac{1}{C-(q-m)s})^{1/(q-m)}$ a solution of the equation $\varphi' = \varphi^{q-m+1}$ and $\underline{u}(x,t) = \varphi(a(x) + b(t))$, where $a(x) = \varepsilon \sum_{i=1}^{N} x_i$ for all $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\varepsilon > 0$, and $b(t) = \delta t$, $\delta > 0$, for all $t \in [0, \infty[$. We choose $C = (q-m)\varepsilon \sup_{x\in\Omega} \sum_{i=1}^{N} x_i + \varepsilon\theta$, with $\theta > 0$. Then $\underline{u}(x,t)$ is well defined for all $x \in \Omega$ if $t \in [0, \varepsilon\theta/((q-m)\delta)[$. Moreover, since

$$\frac{\partial \underline{u}^{\,m}}{\partial \eta} = m\varphi^{m-1}\varphi' \frac{\partial a}{\partial \eta} = m\varphi^{q}\frac{\partial a}{\partial \eta},$$

for ε small enough we get that

$$\frac{\partial \underline{u}^{\,m}}{\partial \eta} \leqslant \underline{u}^{\,q} \quad \text{in } \partial\Omega \times]0, \infty[.$$

On the other hand, since

$$\underline{u}_{\,t} = \varphi' b'$$

and

$$\begin{aligned} \Delta \underline{u}^{\,m} &= m(m-1)\varphi^{m-2}(\varphi')^2|\nabla a|^2 + m\varphi^{m-1}\varphi''|\nabla a|^2 + m\varphi^{m-1}\varphi'\Delta a \\ &= m(m-1)N(\varphi')^2\varepsilon^2\varphi^{m-2} + mN\varphi^{m-1}\varphi''\varepsilon^2, \end{aligned}$$

the inequality

$$\underline{u}_{\,t} \leqslant \Delta\underline{u}^{\,m} - \lambda\underline{u}^{\,p} \quad \text{in } \Omega \times]0, \infty[$$

holds if and only if

$$\delta \leqslant \varphi^{q-1}\left(mqN\varepsilon^2 - \lambda\frac{1}{\varphi^{2q-p-m}}\right).$$

Since φ is increasing, to prove the above inequality it is enough to see that

$$\delta \leqslant \varphi(a(x))^{q-1}\left(mqN\varepsilon^2 - \lambda\frac{1}{\varphi(a(x))^{2q-p-m}}\right)$$

$$= \left(\frac{1}{(q-m)\varepsilon(\sup_{x\in\Omega}\sum_{i=1}^N x_i - \sum_{i=1}^N x_i) + \varepsilon\theta}\right)^{(q-1)/(q-m)}$$

$$\times\varepsilon^2\left(mqN - \lambda\left((q-m)\left(\sup_{x\in\Omega}\sum_{i=1}^N x_i - \sum_{i=1}^N x_i\right) + \theta\right)^{(2q-p-m)/(q-m)} \varepsilon^{(m-p)/(q-m)}\right),$$

which is satisfied for an adequate election of ε and δ.

Finally, taking u_0 such that

$$u_0(x) > \left(\frac{1}{(q-m)\varepsilon\left(\sup_{x\in\Omega}\sum_{i=1}^N x_i - \inf_{x\in\Omega}\sum_{i=1}^N x_i\right) + \varepsilon\theta}\right)^{1/(q-m)},$$

if u is the solution of (1.1) with initial datum u_0, since $u_0(x) > \underline{u}(0,x)$, by using a comparison argument,

$$\underline{u}(x,t) \leqslant u(x,t).$$

Now, let x_0 be such that $a(x_0) = \sup_{x\in\Omega} a(x)$, at the point we have

$$\underline{u}(x_0,t) \geqslant \left(\frac{1}{C - (q-m)(\varepsilon\sup_{x\in\Omega}\sum_{i=1}^N x_i + b(t))}\right)^{1/(q-m)},$$

which goes to infinity as $t \to \varepsilon\theta/((q-m)\delta)$. Therefore, u blows up in finite time. □

Remark that the same proof works if $p = 2q - m$ and λ is small enough. Concretely, we have the following result.

Lemma 4.2. *Let $q > m$, $p = 2q - m$, and λ small enough, $\lambda < \lambda_0 = q/m$, then there exist solutions of* (1.1) *with finite time blow up.*

Now we prove the following result.

Lemma 4.3. *Let $q > m$, $p < 2q - m$ and $p \geqslant m$, then there exist solutions of* (1.1) *with finite time blow up.*

Proof. First we choose $\bar{p} > p$ such that $\bar{p} = 2q - m$. By Lemma 4.2 we can choose $\bar{\lambda}$ small enough such that the solution of (1.1) with $\bar{p}$ and $\bar{\lambda}$ blows up in finite time for every initial data w_0 large enough. Let w be such a solution, we claim that given $K > 0$ there exists M such that $w(x,t) > K$ for every $(x,t) \in \bar{\Omega} \times]0, T_w[$ provided that $w_0 > M$. To see this, fix T such that

$$\left(\frac{1}{(\bar{p}-1)\bar{\lambda}T}\right)^{1/(\bar{p}-1)} > K.$$

As the subsolutions constructed in the proof of Lemma 4.2 blow up at small times, we have that, if M is large, the existence time of w, T_w, satisfies $T_w < T$. Now, let

$$z(t) = \left(\frac{1}{(\bar{p}-1)\bar{\lambda}t + M^{1-\bar{p}}} \right)^{1/(\bar{p}-1)}$$

be the solution of

$$z'(t) = -\bar{\lambda} z^{\bar{p}}(t), \qquad z(0) = M.$$

By a comparison argument, we have that

$$w(x,t) \geqslant z(t) \geqslant \left(\frac{1}{(\bar{p}-1)\bar{\lambda}T + M^{1-\bar{p}}} \right)^{1/(\bar{p}-1)} > K$$

if M is large enough. This proves the claim.

Now we choose K such that $\bar{\lambda} K^{\bar{p}-p} \geqslant \lambda$, by the claim, we have that $w(x,t)$ verifies

$$\lambda w^p \leqslant \bar{\lambda} w^{\bar{p}}$$

and hence it is a subsolution of (1.1) with the original parameters p and λ that blows up in finite time T_w. If we choose $u_0 \geqslant w_0$, by comparison argument, the result follows. □

Lemma 4.4. *If $q \leqslant m$, $p < q$ and $q > 1$, there exist blowing up solutions in finite time.*

Proof. Following the same idea of Lemma 4.1, let $\underline{u}(x,t) = \varphi(a(x) + b(t))$, where φ is the solution of $\varphi' = \varphi^{q-m+1}$, that is, $\varphi(s) = ((m-q)s + C)^{1/(m-q)}$ if $m > q$ and $\varphi(s) = C\mathrm{e}^s$ if $m = q$. In both cases,

$$\lim_{s \to +\infty} \varphi(s) = +\infty. \tag{4.1}$$

We want to show that

$$\frac{\partial \underline{u}^m}{\partial \eta} = m\varphi^{m-1}\varphi' \frac{\partial a}{\partial \eta} = m\varphi^q \frac{\partial a}{\partial \eta} \leqslant \underline{u}^q \quad \text{on } \partial\Omega \times]0,\infty[\tag{4.2}$$

and

$$b' \leqslant m\varphi^{m-1}\Delta a + mq\varphi^{q-1}|\nabla a|^2 - \lambda\varphi^{p-q+m-1} \quad \text{in } \Omega \times]0,\infty[,$$

for which it is enough to see

$$b' \leqslant m\varphi^{m-1}\Delta a - \lambda\varphi^{p-q+m-1} = \varphi^{m-1}(m\Delta a - \lambda\varphi^{p-q}). \tag{4.3}$$

Let $a \in C^2(\Omega)$, $a \geqslant 0$, such that $\Delta a = k > 0$, and $\partial a/\partial\eta \leqslant 1/m$ on $\partial\Omega$. With this election of a (4.2) holds. Let, for $t_0 > 0$ and $A > 0$,

$$b(t) = \left(\frac{1}{A(q-1)/(m-q)(t_0 - t)} \right)^{(m-q)/(q-1)}$$

and for $A < 1/(m-1)t_0$,

$$b(t) = \frac{1}{m-1} \log \frac{1}{A(m-1)(t_0 - t)},$$

the solutions in $[0,t_0[$ of

$$b' = Ab^{(m-1)/(m-q)} \quad \text{if } q < m,$$

$$b' = A\mathrm{e}^{(m-1)b} \quad \text{if } q = m,$$

respectively. In both cases,

$$\lim_{t\to t_0} b(t) = +\infty. \tag{4.4}$$

With these elections of a and b, if we choose A and C such that

$$A \leqslant (m-q)^{(m-1)/(m-q)}\left(mk - \lambda\frac{1}{C^{(q-p)/(m-q)}}\right) \quad \text{if } q < m$$

and

$$A \leqslant C^{m-1}\left(mk - \lambda\frac{1}{C^{q-p}}\right) \quad \text{if } q = m,$$

(4.3) is satisfied

Taking

$$u_0(x) \geqslant \left((m-q)\left(\sup a(x) + \left(\frac{m-q}{A(q-1)t_0}\right)^{(m-q)/(q-1)}\right)\right)^{1/(m-q)}$$

$$\geqslant \underline{u}(x,0) \quad \text{if } q < m,$$

$$u_0(x) \geqslant \mathrm{e}^{\sup a(x) + (1/(m-1))\log(1/(A(m-1)t_0))} \geqslant \underline{u}(0,x) \quad \text{if } q = m,$$

if u is the smooth solution of (1.1) with initial datum u_0, by a comparison argument,

$$u(x,t) \geqslant \underline{u}(x,t).$$

Consequently, by (4.1) and (4.4), u blows up in finite time. □

The same proof works if $p=q$ and λ small enough. We have the following result.

Lemma 4.5. *Let $q \leqslant m$, $p=q$, $q>1$ and λ small enough, $\lambda < \lambda_0$ with λ_0 depending on Ω, then there exist solutions of* (1.1) *with finite time blow up.*

Remark 4.1. Since the blowing up solutions are strictly positive at the boundary in all the above blow-up results, by uniqueness (see Remark 3.1), we have that every weak solution blows up in finite time.

Remark 4.2. In the critical case $1 < m = p = q$, if $\Omega = B(0,R)$ the same arguments used in Lemma 4.4 work if we choose $a(r) = r^2/(2mR)$ if $\lambda < N/R$ or if we choose $a(x) = (1/(mN^{1/2}))\sum_{i=1}^N x_i$ if $\lambda < 1$. This proves that there exist blowing up solutions if $\lambda < \max\{N/R,\ 1\}$.

5. Global weak solutions

In this Section, we prove the global existence results in Theorems 2.3 and 2.4.

In proving the existence of global weak solutions we find a priori estimates for smooth solutions of problem (P_n) and proceed as in Section 3.

Lemma 5.1. *Let $q > m$ and $p > 2q - m$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the solution u_n of (P_n) is globally bounded uniformly on n.*

Proof. We look for supersolutions in the form

$$\overline{u}(x,t) = \varphi(a(x) + b(t)),$$

where $\varphi(s) = (1/(C - (q-m)s))^{1/(q-m)}$ is solution of $\varphi' = \varphi^{q-m+1}$. Let $a \in C^2(\bar{\Omega})$, such that $0 \leqslant a \leqslant \delta$, $\delta a/\delta\eta = 1/m$ on $\partial\Omega$, $|\nabla a| \leqslant 1/m$ in Ω and $|\nabla a|$ is bounded independently of δ by a constant M. Such a function $a(x)$ can be constructed as follows, take an extension of

$$a(x) = g(dist(x, \partial\Omega)),$$

g being an smooth decreasing function such that $g \geqslant 0, g(0) = \delta$, $g(s_0) = 0$, $g'(0) = 1/m$ with g'' bounded independently of δ (see [16]). We choose, $b = 0$, and $C = 2\delta(q-m)$ in order to have φ well defined.

Now, since

$$\left(\frac{1}{2(q-m)\delta}\right)^{1/(q-m)} \leqslant \overline{u}(x,t) \leqslant \left(\frac{1}{(q-m)\delta}\right)^{1/(q-m)},$$

there exist n_δ such that for all $n \geqslant n_\delta$

$$\frac{1}{n} \leqslant \overline{u}(x,t) \leqslant n.$$

Let us see that $\bar{u}$ is a supersolution of (P_n):
Obviously,

$$\frac{\partial \overline{u}^{\,m}}{\partial\eta} = m\varphi^{m-1}\varphi'\frac{\partial a}{\partial\eta} = m\varphi^{q}\frac{\partial a}{\partial\eta} \geqslant \overline{u}^{\,q} \quad \text{on } \partial\Omega \times]0,\infty[.$$

On the other hand,

$$\begin{aligned} b' &\geqslant m\varphi^{m-1}\Delta a + mq\varphi^{q-1}|\nabla a|^2 - \lambda\varphi^{p-q+m-1} \\ &= \varphi^{p-q+m-1}(m\varphi^{q-p}\Delta a + mq\varphi^{2q-p-m}|\nabla a|^2 - \lambda) \quad \text{in } \Omega \times]0,\infty[, \end{aligned}$$

holds if

$$m(C-(q-m)a(x))^{(p-q)/(q-m)}M + \frac{q}{m}(C-(q-m)a(x))^{(p-2q+m)/(q-m)} - \lambda \leqslant 0,$$

which follows taking δ small enough. Consequently,

$$\overline{u}_t \geqslant \Delta\overline{u}^{\,m} - \lambda\overline{u}^{\,p} \quad \text{in } \Omega \times]0,\infty[.$$

Moreover, for δ small,

$$u_0 \leqslant \|u_0\|_\infty \leqslant \left(\frac{1}{C}\right)^{1/(q-m)} \leqslant \overline{u}(x,t).$$

Then, by a comparison argument,

$$u_n(x,t) \leqslant \bar{u}(x,t) \leqslant \left(\frac{1}{(q-m)\delta}\right)^{1/(q-m)} \quad \text{in } \Omega \times [0,\infty[.$$

Taking now $\underline{u} = -\bar{u}$,

$$u_n(x,t) \geqslant \underline{u}(x,t) \geqslant -\left(\frac{1}{(q-m)\delta}\right)^{1/(q-m)} \quad \text{in } \Omega \times [0,\infty[$$

and the proof concludes. □

The same proof gives the following result:

Lemma 5.2. *Let $q > m$ and $p = 2q - m$, with λ large, $\lambda > \lambda_0 = q/m$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the solution u_n of (P_n) is globally bounded uniformly in n.*

Now we prove the following lemma.

Lemma 5.3. *Let $q \leqslant m$ and $p > q$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the solution u_n of (P_n) is globally bounded uniformly on n.*

Proof. Following the same idea as before, let $\bar{u}(x,t) = \varphi(a(x) + b(t))$, where φ is the solution of $\varphi' = \varphi^{q-m+1}$, that is, $\varphi(s) = ((m-q)s + C)^{1/(m-q)}$ if $m > q$, and $\varphi(s) = Ce^s$ if $m = q$. We take $b = 0$ and $a \in C^2(\bar{\Omega})$ such that $a \geqslant 0$, and $\partial a/\partial\eta = 1/m$ on $\partial\Omega$. Then,

$$\frac{\partial \bar{u}^{\,m}}{\partial \eta} = m\varphi^{m-1}\varphi'\frac{\partial a}{\partial \eta} = m\varphi^{q}\frac{\partial a}{\partial \eta} \geqslant \bar{u}^{\,q} \quad \text{on } \partial\Omega \times]0,\infty[.$$

On the other hand, since the inequalities

$$m\Delta a(x) + mq\left(\frac{1}{(m-q)a(x) + C}\right)|\nabla a(x)|^2$$
$$-\lambda((m-q)a(x) + C)^{(p-q)/(m-q)} \leqslant 0 \quad \text{if } m > q$$

and

$$m\Delta a(x) + mq|\nabla a(x)|^2 - \lambda(Ce^{a(x)})^{p-q} \leqslant 0 \quad \text{if } m = q$$

are satisfied for C large enough, we have that

$$\bar{u}_t \geqslant \Delta\bar{u}^{\,m} - \lambda\bar{u}^{\,p} \quad \text{in } \Omega \times]0,\infty[.$$

Moreover, for C large, we have that

$$u_0 \leqslant \|u_0\|_\infty \leqslant C^{1/(q-m)} \leqslant \bar{u}(x,t) \quad \text{if } m > q$$

and

$$u_0 \leqslant \|u_0\|_\infty \leqslant C \leqslant \bar{u}(x,t) \quad \text{if } m = q$$

By a comparison argument, we get

$$u_n(x,t) \leqslant \overline{u}(x,t) \leqslant \left((m-q)\sup_{x\in\Omega} a(x) + C\right)^{1/(m-q)} \quad \text{in } \Omega\times[0,\infty[, \text{ if } m>q$$

and

$$u_n(x,t) \leqslant \overline{u}(x,t) \leqslant C e^{\sup_{x\in\Omega} a(x)} \quad \text{in } \Omega\times[0,\infty[, \text{ if } m=q. \qquad \square$$

Lemma 5.4. *Let $p<q\leqslant 1$, then there exists $n_0\in\mathbb{N}$ such that for all $n\geqslant n_0$, the solution u_n of (P_n) is bounded in $\Omega\times[0,T]$, uniformly in n, for any $T>0$. Moreover, if the initial datum, u_0, is large then u_n goes to infinity as t increases.*

Proof. First we deal with $q<1$. Let $\bar{u}(x,t)=\varphi(a(x)+b(t))$, where $\varphi(s)=((m-q)s)^{1/(m-q)}$ is solution of $\varphi'=\varphi^{q-m+1}$. Let $a\in C^2(\bar{\Omega})$, such that $a\geqslant 0$, and $\partial a/\partial\eta\geqslant 1/m$ on $\partial\Omega$. We take

$$b(t)=\left(A\frac{1-q}{m-q}t+B\right)^{(m-q)/(1-q)}$$

solution of $b'=Ab^{(m-1)/(m-q)}$. Then,

$$\frac{\partial\overline{u}^{\,m}}{\partial\eta}=m\varphi^{m-1}\varphi'\frac{\partial a}{\partial\eta}=m\varphi^{q}\frac{\partial a}{\partial\eta}\geqslant\overline{u}^{\,q} \quad \text{on } \partial\Omega\times]0,T[.$$

On the other hand, since the inequality

$$A\geqslant\left((m-q)\left(1+\frac{a(x)}{B}\right)\right)^{(m-1)/(m-q)}\left(m\Delta a+\frac{mq|\nabla a|^2}{(m-q)B^{(m-q)/(1-q)}}\right)$$

in $\Omega\times]0,\infty[$ is satisfied, given B, for A large enough, for such an election we have that

$$\overline{u}_t\geqslant\Delta\overline{u}^{\,m}-\lambda\overline{u}^{\,p} \quad \text{in } \Omega\times]0,\infty[.$$

Moreover, for B large,

$$u_0\leqslant\|u_0\|_\infty\leqslant((m-q)B^{(m-q)/(1-q)})^{1/(m-q)}\leqslant\overline{u}(x,t),$$

and, by a comparison argument, we obtain

$$u_n(x,t)\leqslant\overline{u}(x,t)$$
$$\leqslant\left((m-q)\left(\sup_{x\in\Omega}a+\left(A\frac{1-q}{m-q}T+B\right)^{(m-q)/(1-q)}\right)\right)^{1/(m-q)} \quad \text{in } \Omega\times[0,T].$$

To deal with the case $q=1$ we only have to follow the same steps as before, but in this case we choose $b(t)=Be^{At}$.

This ends the proof of the first part of the lemma. To see that the solutions go to infinity as t increases we construct a subsolution which is not globally bounded in time. As before, we begin by $q<1$. We choose $\underline{u}(x,t)=\varphi(a(x)+b(t))$, where

$\varphi(s) = ((m-q)s)^{1/(m-q)}$ is solution of $\varphi' = \varphi^{q-m+1}$. Now we choose $a \in C^2(\bar{\Omega})$ such that $a \geqslant 0$, $\Delta a = k > 0$ and $\partial a/\partial \eta \leqslant 1/m$ on $\partial\Omega$. We take

$$b(t) = \left(A\frac{1-q}{m-q}t + B\right)^{(m-q)/(1-q)}$$

solution of $b' = Ab^{(m-1)/(m-q)}$. Then,

$$\frac{\partial \bar{u}^{\,m}}{\partial \eta} = m\varphi^{m-1}\varphi'\frac{\partial a}{\partial \eta} = m\varphi^{q}\frac{\partial a}{\partial \eta} \leqslant \bar{u}^{\,q} \quad \text{on } \partial\Omega \times]0,T[.$$

On the other hand, the inequality

$$A \leqslant (m-q)^{(m-1)/(m-q)}\left(mk - \lambda\left(\frac{1}{(m-q)B^{(m-q)/(1-q)}}\right)^{(q-p)/(m-q)}\right)$$

in $\Omega \times]0,\infty[$ is satisfied for positive small A if B is large enough. For such an election we have that

$$\bar{u}_t \leqslant \Delta\bar{u}^{\,m} - \lambda\bar{u}^{\,p} \quad \text{in } \Omega \times]0,\infty[.$$

Moreover, for u_0 large, $u_0 \geqslant \underline{u}_0$, using a comparison argument, we obtain

$$u_n(x,t) \geqslant \underline{u}(x,t).$$

The case $q=1$ follows as before by choosing $b(t)=Be^{At}$ with an appropriate election of A and B. □

The same proof gives the following result.

Lemma 5.5. *Let $p = q \leqslant 1$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the solution u_n of (P_n) is bounded in $\Omega \times [0,T]$, uniformly in n, for any $T > 0$. Moreover, if λ is small and u_0 is large we obtain an unbounded solution and if λ is large we get globally bounded solutions.*

Finally, working as in the proof of Lemma 5.3 we get.

Lemma 5.6. *If $1 < q = p \leqslant m$, and λ is large, $\lambda \geqslant \lambda_0(\Omega)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the solution u_n of (P_n) is globally bounded uniformly in n.*

Remark 5.1. In the critical case $1 < m = p = q$, if $\Omega = B(0,R)$ the same arguments used in Lemma 5.3 work if we choose $a(r) = r^2/(2mR)$ if $\lambda > N/R + 1$. This proves that there exist global solutions if $\lambda > N/R + 1$.

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