Nonlinear Analysis 56 (2004) 1175–1209

Nonlinear Analysis

www.elsevier.com/locate/na

Quasi-linear diffusion equations with gradient terms and L^1 data

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Received 3 August 2003; accepted 20 November 2003

Abstract

In this article we study the following quasi-linear parabolic problem:

$$\begin{cases} u_t - \Delta u + |u|^{\beta-2}u|\nabla u|^q = |u|^{\alpha-2}u|\nabla u|^p & \text{in } \Omega \times]0,T[, \\ u(x,t) = 0 & \text{on } \partial\Omega \times]0,T[, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ and $T > 0$. We prove that if $\alpha, \beta > 1$, $0 \leqslant p < q$, $1 \leqslant q \leqslant 2$, and $\alpha + p < \beta + q$, then there exists a generalized solution for all $u_0 \in L^1(\Omega)$.

Keywords: Nonlinear diffusion equations; Global existence; Generalized solutions

1. Introduction

Given $T > 0$, consider the following quasi-linear parabolic problem:

$$\begin{cases} u_t - \Delta u + |u|^{\beta-2}u|\nabla u|^q = |u|^{\alpha-2}u|\nabla u|^p & \text{in } Q_T := \Omega \times]0,T[, \\ u(x,t) = 0 & \text{on } S_T := \partial\Omega \times]0,T[, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, whose boundary is denoted by $\partial\Omega$, $1 \leqslant q \leqslant 2$, $0 \leqslant p < q$ and $\alpha, \beta > 1$. (We denote $|\nabla u|^0 = 1$.)

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0362-546X/$ - see front matter
doi:10.1016/j.na.2003.11.010

For the concrete case $p=0$ and $\beta=1$, and for positive initial data, problem (1) was introduced by Chipot and Weissler in [8] in order to investigate the effect of a damping term on existence or nonexistence of classical solutions. Several authors have studied the existence of nonglobal positive classical solutions, giving conditions for blow-up under certain assumptions on α, q, N and Ω; see [4] and the references there in. Global existence for nonnegative initial data has been proved in the case $q+1 \geqslant \alpha > 2$ (see [9,16]). On the other hand, it is observed in [19] that problem (1), with $q=2$ and $p=0$, does not admit global classical solution in the case $\alpha > 2$, $\beta \geqslant 1$ and $\beta+2 < \alpha$.

For positive initial data and $p=0$, the degenerate case (the term Δu is replaced by Δu^m in problem (1)) has been studied in [4], where the existence of global weak solutions for nonnegative initial data in $L^{m+1}(\Omega)$ is proved under the following assumptions: Ω a smooth bounded domain, $m \geqslant 1$, $(\beta+q-1)/q > m/2$, $1 \leqslant q < 2$ and $2 \leqslant \alpha < \beta+q$. We remark that the methods used in our paper are different of that of [4] which does not work in the limit case $q=2$; moreover, we obtain an existence result for, not necessarily positive, initial data in $L^1(\Omega)$.

We point out that in [4,18] a model in population dynamics is described by this type of equations.

Problem (1), with $p=0$ and $q=2$, has been dealt with in [2] to obtain existence for L^1-initial data. We point out that the technique we use here is different from that employed in [2], which, moreover, does not work when $q<2$.

Related problems are also studied in [1] in the degenerate case with measure initial data. In contrast with the above references, in [1] it is considered an equation with right-hand side depending on the gradient.

The aim of this paper is to prove the existence of a generalized solution of problem (1) for initial data $u_0 \in L^1(\Omega)$ under the following hypotheses on the parameters: α, $\beta > 1$, $0 \leqslant p < q$, $1 \leqslant q \leqslant 2$, and $\alpha + p < \beta + q$. The existence result lies on a stability theorem with respect to the initial datum (Theorem 3.1). We point out that the techniques employed in this paper also work for more general evolution problems, as, for example, those involving a general Lions type operator of linear growth (see [12]) instead of the Laplacian.

This article is organized as follows. In Section 2, we define the concept of generalized solution and we prove that these solutions are solutions in the sense of distributions. Section 3 is devoted to prove the existence of generalized solutions of problem (1) for initial datum $u_0 \in L^1(\Omega)$ by proving our stability result. Finally, in the appendix we give an example which shows that the hypothesis $\alpha+p<\beta+q$ in our stability result cannot be avoided.

2. Generalized solutions

In this section we define and analyze our concept of solution of problem (1). This kind of solutions was introduced in [5] for stationary problems, and in [3,15] for evolution ones, as entropy solutions.

We use the following notation, for each $k>0$, we denote $T_k(r) = (r \wedge k) \vee (-k)$ and J_k the primitive of T_k such that $J_k(0)=0$.

Definition 2.1. Let $u_0 \in L^1(\Omega)$. By a generalized solution of problem (1) in Q_T we mean a function $u \in C([0,T]; L^1(\Omega))$, such that $T_k(u) \in L^2(0,T; H_0^1(\Omega))$ for all $k > 0$, $u|u|^{\alpha-2}|\nabla u|^p \in L^1(Q_T)$, $u|u|^{\beta-2}|\nabla u|^q \in L^1(Q_T)$ and

$$\int_\Omega J_k(u(t)-\phi(t)) + \int_0^t\int_\Omega \nabla u\cdot\nabla T_k(u-\phi) + \int_0^t\int_\Omega |u|^{\beta-2}u|\nabla u|^q T_k(u-\phi)$$
$$= -\int_0^t \langle T_k(u-\phi),\phi_s\rangle + \int_0^t\int_\Omega |u|^{\alpha-2}u|\nabla u|^p T_k(u-\phi) + \int_\Omega J_k(u_0-\phi(0))$$

for all $k > 0$, all $t \in [0,T]$ and all test function $\phi \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q_T)$ such that its derivative in time in the sense of distributions, ϕ_t, belongs to $L^2(0,T; H^{-1}(\Omega)) + L^1(Q_T)$.

Remark 2.1. (1) If ϕ belongs to $L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q_T)$ and its distributional derivative in time is such that $\phi_t \in L^2(0,T; H^{-1}(\Omega)) + L^1(Q_T)$, it is well known that $\phi \in C([0,T]; L^2(\Omega))$. As a consequence, the functions $\phi(0)$ and $\phi(t)$ in the above definition have sense.

(2) Since $T_k(u) \in L^2(0,T; H_0^1(\Omega))$ and $\phi \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q_T)$, it follows that $T_k(u-\phi) \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q_T)$ (see [5]).

(3) It follows from $\nabla T_k(u-\phi) = 0$ when $|u-\phi| > k$, that $\nabla T_k(u-\phi) = 0$ when $|u| > M := k + \|\phi\|_\infty$. Thus, $\nabla u \cdot \nabla T_k(u-\phi) = \nabla T_M u \cdot \nabla T_k(u-\phi) \in L^1(Q_T)$ and the second term is well defined.

(4) Since $\phi_t \in L^2(0,T; H^{-1}(\Omega)) + L^1(Q_T)$, we have $\phi_t = \beta_1 + \beta_2$ where $\beta_1 \in L^2(0,T; H^{-1}(\Omega))$ and $\beta_2 \in L^1(Q_T)$. We use the notation

$$\int_0^t \langle T_k(u-\phi),\phi_s\rangle = \int_0^t \langle T_k(u-\phi),\beta_1\rangle_{H_0^1,H^{-1}} + \int_{Q_t} T_k(u-\phi)\beta_2$$

in the above definition.

(5) Taking $\phi = 0$ and $k = 1$ in the generalized formulation, it yields

$$\int_{Q_T} |\nabla T_1(u)|^2 \leqslant \int_{Q_T} |u|^{\alpha-1}\|T_1(u)\|\nabla u|^p + \int_\Omega J_1(u_0) < \infty.$$

Moreover, we also have

$$\int_{Q_T} |\nabla(u - T_1(u))|^q \leqslant \int_{Q_T} |u|^{\beta-1}\|T_1(u)\|\nabla u|^q < \infty.$$

Hence, these estimates imply $\int_{Q_T} |\nabla u|^q < \infty$ and so $u \in L^q(0,T; W_0^{1q}(\Omega))$.

(6) Actually, the condition $u|u|^{\beta-2}|\nabla u|^q \in L^1(Q_T)$ in the above definition is redundant. Indeed, on the one hand, if $q < 2$,

$$\int_{\{|u|\leqslant k\}} |u|^{\beta-1}|\nabla u|^q \leqslant \frac{2-q}{2}\int_{\{|u|\leqslant k\}} k^{2(\beta-1)/(2-q)} + \frac{q}{2}\int_{\{|u|\leqslant k\}} |\nabla u|^2 < +\infty$$

and if $q = 2$,

$$\int_{\{|u|\leqslant k\}} |u|^{\beta-1}|\nabla u|^2 \leqslant \int_{\{|u|\leqslant k\}} k^{\beta-1}|\nabla u|^2 < +\infty.$$

On the other hand, taking $\phi=0$ and $t=T$ in the generalized formulation and disregarding nonnegative terms, it yields

$$\int_{\{|u|>k\}} |u|^{\beta-1}|\nabla u|^q \leqslant \int_{Q_T} |u|^{\alpha-1}|\nabla u|^p + \int_\Omega |u_0| < +\infty.$$

Thus, $|u|^{\beta-1}|\nabla u|^q \in L^1(Q)$.

Next, we are going to see that generalized solutions satisfy our equation in the sense of distributions. We will first prove that every generalized solution is a kind of "weak solution". (We point out that this is possible since $q \geqslant 1$; in another case this formulation has no sense, although the generalized formulation still has it. Nevertheless our methods do not work to obtain existence of solutions when $0 < q < 1$.) In order to see it, we have to regularize our initial datum and apply the time-regularization procedure introduced in [10] (see also [11] and, for non-zero initial datum, [13] and [14]): for a fixed $v \in \mathbb{N}$ and a given function $w \in L^2(0,T;H_0^1(\Omega))$, we set

$$w_v(x,t) = v\int_0^t w(x,s)\mathrm{e}^{v(s-t)}\,\mathrm{d}s \tag{2}$$

for $t \in [0,T]$. This regularization function has the following properties:

$$\begin{cases} w_v \in C([0,T];H_0^1(\Omega)), \\ (w_v)_t = v(w - w_v) \quad \text{in the sense of distributions,} \\ w_v \to w \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ as } v \to \infty. \end{cases} \tag{3}$$

Moreover, $\|w_v\|_\infty \leqslant \|w\|_\infty$ if $w \in L^\infty(Q_T)$ and, when $w \in C([0,T];L^1(\Omega))$, $w_v(.,t) \to w(.,t)$ in $L^1(\Omega)$ for $0 < t \leqslant T$.

Proposition 2.1. *Let $T > 0$. If u is a generalized solution of (1) and $\phi \in L^{q'}(0,T;W_0^{1,q'}(\Omega)) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$, then the following equality holds*:

$$\int_\Omega u(T)\phi(T) + \int_{Q_T} \nabla u\cdot\nabla\phi + \int_{Q_T} |u|^{\beta-2}u|\nabla u|^q\phi$$

$$= \int_{Q_T} u\phi_t + \int_{Q_T} |u|^{\alpha-2}u|\nabla u|^p\phi + \int_\Omega u_0\phi(0).$$

Proof. Fix $k > 0$ such that $k > \|\phi\|_\infty$ and let $h > k$. Consider a sequence $(\psi_j)_{j=1}^\infty$ in $\mathscr{D}(\Omega)$ such that $\psi_j \to u_0$ in $L^1(\Omega)$.

Now define $\eta_{v,j}(u)=(T_h(u))_v+\mathrm{e}^{-vt}T_h(\psi_j)$. By (3), $\eta_{v,j}(u) \in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^1(\Omega))\cap L^\infty(Q_T)$ and, in a distributional sense, $(\eta_{v,j}(u))_t = v(T_h(u)-\eta_{v,j}(u)) \in L^\infty(Q_T)$. Thus, if $\phi \in L^{q'}(0,T;W_0^{1,q'}(\Omega)) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$, then $\eta_{v,j}(u) - \phi$ may be taken as test function in the generalized formulation of problem (1) which yields

$$\int_\Omega J_k(u(T)-\eta_{v,j}(u)(T)+\phi(T))+\int_{Q_T}\nabla u\cdot\nabla T_k(u-\eta_{v,j}(u)+\phi)$$
$$+\int_{Q_T}u|u|^{\beta-2}|\nabla u|^q T_k(u-\eta_{v,j}(u)+\phi)$$
$$=-\int_{Q_T}(\eta_{v,j}(u))_t\,T_k(u-\eta_{v,j}(u)+\phi)+\int_{Q_T}\phi_t T_k(u-\eta_{v,j}(u)+\phi)$$
$$+\int_{Q_T}u|u|^{\alpha-2}|\nabla u|^p T_k(u-\eta_{v,j}(u)+\phi)$$
$$+\int_\Omega J_k(u_0-T_h(\psi_j)+\phi(0)). \tag{4}$$

We now analyze the following term:

$$\int_{Q_T}(\eta_{v,j}(u))_t T_k(u-\eta_{v,j}(u)+\phi)=v\int_{Q_T}(T_h(u)-\eta_{v,j}(u))T_k(u-\eta_{v,j}(u)+\phi).$$

Observe that the functions $T_h(u)-\eta_{v,j}(u)$ and $u-\eta_{v,j}(u)$ have the same sign. Indeed, when $|u|\leqslant h$ both functions coincide and when $|u|>h$, taking into account that $|\eta_{v,j}(u)|\leqslant h$, we have that

$$\operatorname{sgn}(T_h(u)-\eta_{v,j}(u))=\operatorname{sgn}(u-\eta_{v,j}(u)).$$

On the other hand, since T_k is an increasing function, sgn $a=$ sgn $\bar a$ implies $a(T_k(\bar a+b)-T_k(b))\geqslant 0$; that is, $aT_k(\bar a+b)\geqslant aT_k(b)$. Hence,

$$\int_{Q_T}(T_h(u)-\eta_{v,j}(u))T_k(u-\eta_{v,j}(u)+\phi)$$
$$\geqslant\int_{Q_T}(T_h(u)-\eta_{v,j}(u))T_k(\phi)$$
$$=\int_{Q_T}(T_h(u)-\eta_{v,j}(u))\phi=\frac{1}{v}\int_{Q_T}(\eta_{v,j}(u))_t\phi$$

so that

$$\int_{Q_T}(\eta_{v,j}(u))_t T_k(u-\eta_{v,j}(u)+\phi)$$
$$\geqslant\int_\Omega\eta_{v,j}(u)(T)\phi(T)-\int_\Omega T_h(\psi_j)\phi(0)-\int_{Q_T}\eta_{v,j}(u)\phi_t.$$

Thus, (4) becomes

$$\int_\Omega J_k(u(T)-\eta_{v,j}(u)(T)+\phi(T))+\int_{Q_T}\nabla u\cdot\nabla T_k(u-\eta_{v,j}(u)+\phi)$$
$$+\int_{Q_T}u|u|^{\beta-2}|\nabla u|^q T_k(u-\eta_{v,j}(u)+\phi)\leqslant-\int_\Omega\eta_{v,j}(u)(T)\phi(T)$$

$$+ \int_\Omega T_h(\psi_j)\phi(0) + \int_{Q_T} \eta_{v,j}(u)\phi_t + \int_{Q_T} \phi_t T_k(u - \eta_{v,j}(u) + \phi)$$

$$+ \int_{Q_T} u|u|^{\alpha-2}|\nabla u|^p T_k(u - \eta_{v,j}(u) + \phi) + \int_\Omega J_k(u_0 - T_h(\psi_j) + \phi(0)). \qquad (5)$$

In order to take limit as v goes to ∞ we have to study the term

$$\int_{Q_T} \nabla u \cdot \nabla T_k(u - \eta_{v,j}(u) + \phi) = \int_{\{|u-\eta_{v,j}(u)+\phi|<k\}} \nabla u \cdot \nabla(u - \eta_{v,j}(u) + \phi),$$

which can be split up as

$$\int_{Q_T} \nabla u \cdot \nabla T_k(u - \eta_{v,j}(u) + \phi) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\{|u-\eta_{v,j}(u)+\phi|<k\}} \nabla u \cdot \nabla(u - \eta_{v,j}(u) + \phi)\chi_{\{|u-T_h(u)+\phi|>k\}},$$

$$I_2 = \int_{\{|u-\eta_{v,j}(u)+\phi|<k\}} \nabla u \cdot \nabla(T_h(u) - \eta_{v,j}(u))\chi_{\{|u-T_h(u)+\phi|\leqslant k\}}$$

and

$$I_3 = \int_{\{|u-\eta_{v,j}(u)+\phi|<k\}} \nabla u \cdot \nabla(u - T_h(u) + \phi)\chi_{\{|u-T_h(u)+\phi|\leqslant k\}}.$$

Since $\lim_{v\to\infty} \eta_{v,j}(u) = T_h(u)$, it is easy to see that $\lim_{v\to\infty} I_1 = 0 = \lim_{v\to\infty} I_2$ and $\lim_{v\to\infty} I_3 = \int_{\{|u-T_h(u)+\phi|<k\}} \nabla u \cdot \nabla(u - T_h(u) + \phi)$. So that

$$\lim_{v\to\infty} \int_{Q_T} \nabla u \cdot \nabla T_k(u - \eta_{v,j}(u) + \phi) = \int_{Q_T} \nabla u \cdot \nabla T_k(u - T_h(u) + \phi).$$

Thus, by this convergence and Lebesgue's Theorem, we may take limit in (5) first when v tends to ∞ and then when j goes to ∞, and it follows that:

$$\int_\Omega J_k(u(T) - T_h(u)(T) + \phi(T)) + \int_{Q_T} \nabla u \cdot \nabla T_k(u - T_h(u) + \phi)$$

$$+ \int_{Q_T} u|u|^{\beta-2}|\nabla u|^q T_k(u - T_h(u) + \phi) \leqslant - \int_\Omega T_h(u)(T)\phi(T)$$

$$+ \int_\Omega T_h(u_0)\phi(0) + \int_{Q_T} T_h(u)\phi_t + \int_{Q_T} \phi_t T_k(u - T_h(u) + \phi)$$

$$+ \int_{Q_T} u|u|^{\alpha-2}|\nabla u|^p T_k(u - T_h(u) + \phi) + \int_\Omega J_k(u_0 - T_h(u_0) + \phi(0)). \qquad (6)$$

Note that

$$\int_{Q_T} \nabla u \cdot \nabla T_k(u - T_h(u) + \phi)$$

$$= \int_{\{|u| \leqslant h\}} \nabla u \cdot \nabla \phi + \int_{\{h<|u|<k+h+\|\phi\|_\infty\} \cap \{|u-T_h(u)+\phi|<k\}} \nabla u \cdot \nabla (u + \phi)$$

$$\geqslant \int_{Q_T} \nabla u \cdot \nabla \phi (\chi_{\{|u| \leqslant h\}} + \chi_{\{h<|u|<k+h+\|\phi\|_\infty\} \cap \{|u-T_h(u)+\phi|<k\}})$$

and the last term in the above inequality converges to $\int_{Q_T} \nabla u \cdot \nabla \phi$ when h tends to ∞. As a consequence, we obtain from (6) that

$$\int_\Omega u(T)\phi(T) + \int_\Omega J_k(\phi(T)) + \int_{Q_T} \nabla u \cdot \nabla \phi$$
$$+ \int_{Q_T} u|u|^{\beta-2}|\nabla u|^q T_k(\phi) \leqslant \int_{Q_T} u\phi_t + \int_{Q_T} \phi_t T_k(\phi)$$
$$+ \int_{Q_T} u|u|^{\alpha-2}|\nabla u|^p T_k(\phi) + \int_\Omega J_k(\phi(0)) + \int_\Omega u_0 \phi(0). \tag{7}$$

Taking now into account that

$$\int_{Q_T} \phi_t T_k(\phi) = \int_\Omega J_k(\phi(T)) - \int_\Omega J_k(\phi(0)),$$

we deduce from (7) that

$$\int_\Omega u(T)\phi(T) + \int_{Q_T} \nabla u \cdot \nabla \phi + \int_{Q_T} u|u|^{\beta-2}|\nabla u|^q \phi$$
$$\leqslant \int_{Q_T} u\phi_t + \int_{Q_T} u|u|^{\alpha-2}|\nabla u|^p \phi + \int_\Omega u_0 \phi(0).$$

Finally, the desired equality follows by considering $\pm\phi$.

Corollary 2.1. *Every generalized solution of* (1) *in* Q_T *satisfies the equation in the sense of distributions.*

Remark 2.2. The above result implies the following fact for bounded solutions. Since each bounded generalized solution u satisfies the equation in the sense of distributions, it follows that $u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$ and so, by Remark (2.1) (1), we deduce that $u \in C([0,T];L^2(\Omega))$.

Generalized solutions and distributional solutions are different in general; nevertheless they coincide for bounded solutions.

Proposition 2.2. *Let u belong to* $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q_T)$ *satisfying*

$$|u|^{\beta-1}|\nabla u|^q,\ |u|^{\alpha-1}|\nabla u|^p\in L^1(Q_T).$$

Then u is a solution of (1) *in the sense of distributions if and only if u is a generalized solution.*

Proof. Having in mind Corollary 2.1, we only have to see that distributional solutions are generalized solutions.

We begin by observing that if u is a distributional solution of (1) then, by Remark 2.2, $u_t\in L^2(0,T;H^{-1}(\Omega))+L^1(Q_T)$ and $u\in C([0,T];L^2(\Omega))$. Next fix $\phi\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q_T)$ such that $\phi_t\in L^2(0,T;H^{-1}(\Omega))+L^1(Q_T)$, and consider a sequence $(\varphi_n)_{n=1}^\infty$ in $\mathscr{D}(Q_T)$ such that $\varphi_n\to u-\phi$ in $L^2(0,T;H_0^1(\Omega))$ and a.e.

Now, let $S:\mathbb{R}\to\mathbb{R}$ be a bounded C^∞-function satisfying $S(0)=0$, $0\leqslant S'\leqslant 1$, $S'(s)=0$ for all s big enough, $S(-s)=-S(s)$ for all $s\in\mathbb{R}$, and $S''(s)\leqslant 0$ for all $s\geqslant 0$. Taking $S(\varphi_n)$ as test function in the distributional formulation and passing to the limit when n goes to infinity, it yields

$$\int_0^t\langle S(u-\phi),u_s\rangle+\int_0^t\int_\Omega\nabla u\cdot\nabla S(u-\phi)+\int_0^t\int_\Omega|u|^{\beta-2}u|\nabla u|^qS(u-\phi)$$
$$=\int_0^t\int_\Omega|u|^{\alpha-2}u|\nabla u|^pS(u-\phi)$$

for all $t\in[0,T]$.

From here, denoting by $J_S(s)=\int_0^s S(r)\,\mathrm{d}r$, we get

$$\int_\Omega J_S(u(t)-\phi(t))+\int_0^t\int_\Omega\nabla u\cdot\nabla S(u-\phi)+\int_0^t\int_\Omega|u|^{\beta-2}u|\nabla u|^qS(u-\phi)$$
$$=-\int_0^t\langle S(u-\phi),\phi_s\rangle+\int_0^t\int_\Omega|u|^{\alpha-2}u|\nabla u|^pS(u-\phi)$$
$$+\int_\Omega J_S(u(0)-\phi(0))$$

for all $t\in[0,T]$.

Finally, approximating the truncature T_k by an increasing sequence of functions $(S_m)_{m=1}^\infty$ as in [5, Lemma 3.2] and letting m tend to infinity, we obtain that u satisfies the generalized formulation.

3. Existence of generalized solutions

In this section we prove a stability result from which the existence of generalized solutions follows.

Theorem 3.1. *Assume that u_n is a bounded generalized solution of*

$$\begin{cases} (u_n)_t - \Delta u_n + u_n|u_n|^{\beta-2}|\nabla u_n|^q = |u_n|^{\alpha-2}u_n|\nabla u_n|^p & \text{in } Q_T, \\ u_n = 0 & \text{on } S_T, \\ u_n(x,0) = u_{0n}(x) & \text{in } \Omega, \end{cases} \tag{8}$$

where $\alpha, \beta > 1$, $1 \leqslant q \leqslant 2$, $0 \leqslant p < q$, $p + \alpha < q + \beta$, and $u_{0n} \in L^\infty(\Omega)$ for all $n \in \mathbb{N}$. If

$$u_{0n} \to u_0 \quad \text{in } L^1(\Omega), \tag{9}$$

then there exists a subsequence (still denoted by u_n) and a function $u : Q_T \to \mathbb{R}$ satisfying

$$u_n \to u \quad \text{in } L^q(0,T; W_0^{1,q}(\Omega)), \tag{10}$$

$$T_k(u_n) \to T_k(u) \quad \text{in } L^2(0,T; H_0^1(\Omega)) \text{ for all } k > 0, \tag{11}$$

$$|u_n|^{\beta-1}|\nabla u_n|^q \to |u|^{\beta-1}|\nabla u|^q \quad \text{in } L^1(Q_T), \tag{12}$$

$$|u_n|^{\alpha-1}|\nabla u_n|^p \to |u|^{\alpha-1}|\nabla u|^p \quad \text{in } L^1(Q_T), \tag{13}$$

$$u_n \to u \quad \text{in } C([0,T]; L^1(\Omega)). \tag{14}$$

Moreover, this function u is a generalized solution of problem (1).

Proof. In this proof C will denote a positive constant that only depends on Ω, T, a bound of $\|u_{0n}\|_1$ and on the parameters α, β, p and q. The value of C may vary from line to line.

The following equality will be used several times in what follows:

$$\int_\Omega J_k(u_n(t)) + \int_{Q_t} |\nabla T_k(u_n)|^2 + \int_{Q_t} |u_n|^{\beta-2}u_n T_k(u_n)|\nabla u_n|^q$$

$$= \int_{Q_t} |u_n|^{\alpha-2}u_n T_k(u_n)|\nabla u_n|^p + \int_\Omega J_k(u_{0n}). \tag{15}$$

To obtain (15) it is enough to fix $t \in [0,T]$ and take $\phi = 0$ as test function in the generalized formulation of (8). Moreover, dividing by k, dropping a nonnegative term and letting $k \to 0^+$, it follows that

$$\int_\Omega |u_n(t)| + \int_{Q_t} |u_n|^{\beta-1}|\nabla u_n|^q \leqslant \int_{Q_t} |u_n|^{\alpha-1}|\nabla u_n|^p + \int_\Omega |u_{0n}|. \tag{16}$$

3.1. A priori estimates

We will prove that

$$\int_{Q_T} |u_n|^{\alpha-1}|\nabla u_n|^p \leqslant C \quad \text{for all } n \in \mathbb{N}. \tag{17}$$

Assume first that $(\alpha-1)q>(\beta-1)p$. Applying Young's inequality it follows that

$$\int_{Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p \leqslant \frac{p}{q}\int_{Q_T}|u_n|^{\beta-1}|\nabla u_n|^q + \frac{q-p}{q}\int_{Q_T}|u_n|^{(\alpha q-\beta p)/(q-p)-1}. \tag{18}$$

Taking $t=T$ in (16), the above inequality implies

$$\int_{Q_T}|u_n|^{\beta-1}|\nabla u_n|^q \leqslant \int_{Q_T}|u_n|^{(\alpha q-\beta p)/(q-p)-1} + \frac{q}{q-p}\int_{\Omega}|u_{0n}|. \tag{19}$$

Taking into account (9) and applying Poincaré's inequality we get

$$\int_{Q_T}|u_n|^{\beta-1+q} \leqslant C\int_{Q_T}|\nabla(|u_n|^{(\beta-1)/q}u_n)|^q = C\left(\frac{\beta-1}{q}+1\right)^q\int_{Q_T}|u_n|^{\beta-1}|\nabla u_n|^q$$
$$\leqslant C\left(\int_{Q_T}|u_n|^{(\alpha q-\beta p)/(q-p)-1}+1\right). \tag{20}$$

Since $p+\alpha<q+\beta$ and $p<q$ imply $(\alpha q-\beta p)/(q-p)<q+\beta$, it follows from (20) that

$$\int_{Q_T}|u_n|^{\beta-1+q} \leqslant C \quad \text{for all } n\in\mathbb{N}, \tag{21}$$

and so

$$\int_{Q_T}|u_n|^{(\alpha q-\beta p)/(q-p)-1} \leqslant C \quad \text{for all } n\in\mathbb{N}. \tag{22}$$

Going back to (19), we deduce that

$$\int_{Q_T}|u_n|^{\beta-1}|\nabla u_n|^q \leqslant C \quad \text{for all } n\in\mathbb{N}. \tag{23}$$

Now, (18), (22) and (23) imply that (17) holds when $(\alpha-1)q>(\beta-1)p$.

The case $(\alpha-1)q=(\beta-1)p$ is proved in a similar way. Consider finally the case $(\alpha-1)q<(\beta-1)p$. Then we deduce from (15), with $t=T$ and $k=1$, that

$$\int_{Q_T}|\nabla T_1(u_n)|^2 + \int_{\{|u_n|<1\}\cap Q_T}|u_n|^{\beta}|\nabla u_n|^q + \int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q$$
$$= \int_{Q_T}|\nabla T_1(u_n)|^2 + \int_{Q_T}|u_n|^{\beta-2}u_nT_1(u_n)|\nabla u_n|^q$$
$$\leqslant \int_{Q_T}|u_n|^{\alpha-2}u_nT_1(u_n)|\nabla u_n|^p + \int_{\Omega}|u_{0n}|$$
$$= \int_{\{|u_n|<1\}\cap Q_T}|u_n|^{\alpha}|\nabla u_n|^p + \int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p + \int_{\Omega}|u_{0n}|. \tag{24}$$

Thus,

$$\int_{Q_T} |\nabla T_1(u_n)|^2 + \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\beta-1}|\nabla u_n|^q$$
$$\leqslant \int_{Q_T} |\nabla T_1(u_n)|^p + \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\alpha-1}|\nabla u_n|^p + \int_{\Omega} |u_{0n}|.$$

Since $p < 2$, using Young's inequality,

$$\int_{Q_T} |\nabla T_1(u_n)|^p \leqslant \frac{p}{2} \int_{Q_T} |\nabla T_1(u_n)|^2 + C$$

and

$$\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\alpha-1}|\nabla u_n|^p$$
$$\leqslant \frac{p}{q} \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\beta-1}|\nabla u_n|^q + \frac{q-p}{q}$$
$$\times \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{(\alpha q-\beta p)/(q-p)-1}. \tag{25}$$

Consequently,

$$\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\beta-1}|\nabla u_n|^q \leqslant \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{(\alpha q-\beta p)/(q-p)-1}$$
$$+ \frac{q}{q-p} \int_{\Omega} |u_{0n}| + C,$$

where the right-hand side in the above inequality is bounded, since $(\alpha q - \beta p)/(q - p) - 1 < 0$. Hence,

$$\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\beta-1}|\nabla u_n|^q \leqslant C.$$

Therefore, the above inequality and (25) imply

$$\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\alpha-1}|\nabla u_n|^p \leqslant C. \tag{26}$$

On the other hand, dropping nonnegative terms in inequality (24) we obtain

$$\int_{Q_T} |\nabla T_1(u_n)|^2 \leqslant \int_{Q_T} |\nabla T_1(u_n)|^p + C$$

and, using Young's inequality, it follows:

$$\int_{Q_T} |\nabla T_1(u_n)|^p \leqslant C.$$

From here and (26), we get that (17) holds in every case. As a consequence, the right-hand side in equality (16) is bounded, which implies

$$\int_{Q_T} |u_n|^{\beta-1}|\nabla u_n|^q \leqslant C \quad \text{for all } n \in \mathbb{N}$$

and

$$\sup_{t\in[0,T]} \int_{\Omega} |u_n(t)| \leqslant C \quad \text{for all } n\in\mathbb{N}. \tag{27}$$

Furthermore, from equality (15) the following estimates also hold:

$$\int_{Q_T} |\nabla T_k(u_n)|^2 \leqslant Ck \quad \text{for all } n\in\mathbb{N} \tag{28}$$

and, for $k=1$,

$$\int_{Q_T} T_1(u_n)u_n|u_n|^{\beta-2}|\nabla u_n|^q \leqslant C \quad \text{for all } n\in\mathbb{N}.$$

Denoting $G_k(r)=r-T_k(r)$, this last estimate implies

$$\int_{Q_T} |\nabla G_1(u_n)|^q \leqslant \int_{Q_T} T_1(u_n)u_n|u_n|^{\beta-2}|\nabla(u_n)|^q \leqslant C \quad \text{for all } n\in\mathbb{N}.$$

From this fact and (28) we obtain

$$\int_{Q_T} |\nabla u_n|^q \leqslant C \quad \text{for all } n\in\mathbb{N}. \tag{29}$$

Moreover, for q close to 1 a better estimate can be obtained; indeed, from (27) and (28) we may follow the procedure used in [3] (see also [6]) and deduce that, for $1\leqslant r<(N+2)/(N+1)$,

$$\int_{Q_T} |\nabla u_n|^r \leqslant C \tag{30}$$

for all $n\in\mathbb{N}$.

Going back again to (8), we get that the sequence $((u_n)_t)_{n=1}^{\infty}$ is bounded in the spaces $L^q(0,T;W^{-1,q}(\Omega))+L^1(Q_T)$ and $L^r(0,T;W^{-1,r}(\Omega))+L^1(Q_T)$ for $1\leqslant r<(N+2)/(N+1)$. Using this fact, (29) and (30), we obtain from [17, Corrollary 4] that $(u_n)_{n=1}^{\infty}$ is relatively compact in $L^q(Q_T)$.

Summing up, there exists a function $u\in L^q(0,T;W_0^{1,q}(\Omega))$ and a subsequence, still denoted by $(u_n)_{n=1}^{\infty}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^q(0,T;W_0^{1,q}(\Omega))$$

and

$$u_n \to u \quad \text{in } L^q(Q_T) \text{ and a.e. in } Q_T. \tag{31}$$

Moreover, by (28), we may assume that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)).$$

Finally, assuming $(\alpha-1)q>(\beta-1)p$, we also deduce that

$$|u_n|^{(\alpha q-\beta p)/(q-p)-1} \to |u|^{(\alpha q-\beta p)/(q-p)-1} \quad \text{in } L^1(Q_T). \tag{32}$$

Indeed, because of (31), we just have to show that the sequence $(|u_n|^{(\alpha q-\beta p)/(q-p)-1})_{n=1}^{\infty}$ is equi-integrable, but it is straightforward taking (21) and Hölder's inequality into account.

3.2. Convergence of truncations in $L^2(0,T;H_0^1(\Omega))$

Our aim is to prove that (11) holds; that is,

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{in } L^2(Q_T) \text{ for all } k\in\mathbb{N}. \tag{33}$$

From here, applying a diagonal procedure, we deduce that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } Q_T. \tag{34}$$

To prove (33), we have to regularize the initial datum u_0 and to use the time-regularization function given in (2). Let $\psi_j \in \mathscr{D}(\Omega)$ be such that

$$\psi_j \to u_0 \quad \text{in } L^1(\Omega)$$

and let

$$\eta_{\nu j}(u^+) = (T_k(u^+))_\nu + \mathrm{e}^{-\nu t} T_k(\psi_j^+),$$

which has the following properties (see (3)):

$$(\eta_{\nu j}(u^+))_t = \nu(T_k(u^+) - \eta_{\nu j}(u^+)),$$

$$\eta_{\nu j}(u^+)(0) = T_k(\psi_j^+),$$

$$|\eta_{\nu j}(u^+)| \leqslant k,$$

$$\eta_{\nu j}(u^+) \to T_k(u^+) \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ as } \nu\to\infty.$$

By denoting $\omega(n,\nu,j,h)$ any quantity such that

$$\lim_{h\to\infty}\lim_{j\to\infty}\lim_{\nu\to\infty}\lim_{n\to\infty}\omega(n,\nu,j,h) = 0,$$

all we have to prove is that

$$\int_{Q_T} |\nabla(T_k(u_n) - \eta_{\nu j}(u))|^2 \leqslant \omega(n,\nu,j,h),$$

where h is a parameter we will consider later. The proof of this fact will be split up into several stages. We begin by showing that

Claim 1.

$$\int_{Q_T} |\nabla(T_k(u_n^+) - \eta_{\nu j}(u^+))^+|^2 \leqslant \omega(n,\nu,j,h)$$

Proof. Consider

$$w_n = T_{2k}(u_n^+ - T_h(u_n^+) + (T_k(u_n) - \eta_{\nu j}(u^+))^+)$$

with $h > k$, and observe that $u_n w_n \geqslant 0$. Using Remark 2.2 we can multiply problem (8) by w_n and integrate to obtain

$$\int_0^T \langle w_n, (u_n)_t \rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n + \int_{Q_T} |u_n|^{\beta-1} |\nabla u_n|^q |w_n|$$
$$= \int_{Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n|. \tag{35}$$

Let us prove that

$$\int_0^T \langle w_n, (u_n)_t \rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n \leqslant \omega(n, v, h). \tag{36}$$

We have to consider three cases; assume first $(\alpha - 1)q > (\beta - 1)p$. Then using Young's inequality in the right-hand side of (35) we obtain

$$\int_0^T \langle w_n, (u_n)_t \rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n + \int_{Q_T} |u_n|^{\beta-1} |\nabla u_n|^q |w_n|$$
$$\leqslant \frac{p}{q} \int_{Q_T} |u_n|^{\beta-1} |\nabla u_n|^q |w_n| + \frac{q-p}{q} \int_{Q_T} |u_n|^{(\alpha q - \beta p)/(q-p)-1} |w_n|$$

and consequently, there exists a constant $C > 0$ such that

$$\int_0^T \langle w_n, (u_n)_t \rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n \leqslant C \int_{Q_T} |u_n|^{(\alpha q - \beta p)/(q-p)-1} |w_n|.$$

Having in mind (32), the properties of $\eta_{vj}(u^+)$ and Lebesgue's Theorem, it is easy to see that

$$\lim_{h\to\infty} \lim_{v\to\infty} \lim_{n\to\infty} \int_{Q_T} |u_n|^{(\alpha q - \beta p)/(q-p)-1} |w_n| = 0,$$

thus, (36) is proved in this case. The case $(\alpha - 1)q = (\beta - 1)p$ is similar. Consider next $(\alpha - 1)q < (\beta - 1)p$, then

$$\int_{Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n| = \int_{\{|u_n|<1\} \cap Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n|$$
$$+ \int_{\{|u_n|\geqslant 1\} \cap Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n|.$$

The first integral can be manipulated as follows:

$$\int_{\{|u_n|<1\} \cap Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n| \leqslant \int_{\{|u_n|<1\} \cap Q_T} |\nabla u_n|^p |w_n|$$
$$\leqslant \left(\int_{Q_T} |\nabla T_1(u_n)|^2 \right)^{p/2} \left(\int_{Q_T} |w_n|^{2/(2-p)} \right)^{(2-p)/2}.$$

With respect to the second integral, we use Young's inequality to get

$$\int_{\{|u_n|\geqslant 1\}\cap Q_T} |u_n|^{\alpha-1}|\nabla u_n|^p|w_n|$$

$$\leqslant \frac{p}{q}\int_{\{|u_n|\geqslant 1\}\cap Q_T} |u_n|^{\beta-1}|\nabla u_n|^q|w_n| + \frac{q-p}{q}$$

$$\times \int_{\{|u_n|\geqslant 1\}\cap Q_T} |u_n|^{(\alpha q-\beta p)/(q-p)-1}|w_n|$$

$$\leqslant \frac{p}{q}\int_{Q_T} |u_n|^{\beta-1}|\nabla u_n|^q|w_n| + \frac{q-p}{q}\int_{Q_T}|w_n|.$$

On account of (35), we have

$$\int_0^T \langle w_n,(u_n)_t\rangle + \int_{Q_T} \nabla u_n\cdot\nabla w_n$$

$$\leqslant \left(\int_{Q_T}|\nabla T_1(u_n)|^2\right)^{p/2}\left(\int_{Q_T}|w_n|^{2/(2-p)}\right)^{(2-p)/2} + \frac{q-p}{q}\int_{Q_T}|w_n|$$

$$\leqslant \omega(n,v,h).$$

Therefore, (36) is proved.

In order to analyze the left-hand side terms in (36), we will follow the procedure introduced in [14, Theorem 3.1, Step 3], which is included here for the sake of completeness.

Let us begin with the term $\int_0^T \langle w_n,(u_n)_t\rangle$ in (36). Note that $w_n = w_n\chi_{\{u_n\geqslant 0\}}$ and, if $u_n\geqslant 0$, then

$$w_n = T_{h+k}(u_n-\eta_{vj}(u^+))^+ - T_{h-k}(u_n^+ - T_k u_n^+)$$

and so

$$\int_0^T \langle w_n,(u_n)_t\rangle$$

$$= \int_0^T \langle T_{h+k}(u_n-\eta_{vj}(u^+))^+,(u_n)_t\rangle - \int_0^T \langle T_{h-k}(u_n^+ - T_k(u_n^+)),(u_n)_t\rangle. \tag{37}$$

On the one hand,

$$\int_0^T \langle T_{h+k}(u_n-\eta_{vj}(u^+))^+,(u_n)_t\rangle$$

$$= \int_{Q_T} (\eta_{vj}(u^+))_t T_{h+k}(u_n-\eta_{vj}(u^+))^+ + \int_\Omega J_{h+k}((u_n-\eta_{vj}(u^+))^+(T))$$

$$- \int_\Omega J_{h+k}((u_{0n}-T_k(\psi_j^+))^+)$$

$$
\begin{aligned}
&= \nu \int_{Q_T} (T_k(u^+) - \eta_{\nu j}(u^+)) T_{h+k}(u_n - \eta_{\nu j}(u^+))^+ \\
&\quad + \int_\Omega J_{h+k}((u_n - \eta_{\nu j}(u^+))^+(T)) - \int_\Omega J_{h+k}((u_{0n} - T_k(\psi_j^+))^+) \\
&\geqslant \omega(n) + \int_\Omega J_{h+k}((u_n - \eta_{\nu j}(u^+))^+(T)) - \int_\Omega J_{h+k}((u_{0n} - T_k(\psi_j^+))^+),
\end{aligned}
$$

having in mind $|\eta_{\nu j}(u^+)| \leqslant k$ and $(T_k(u^+) - \eta_{\nu j}(u^+))T_{h+k}(u - \eta_{\nu j}(u^+))^+ \geqslant 0$.

In order to estimate the last term in (37), we have to approximate the functions u_n. We begin by splitting up $(u_n)_t = \beta_{1n} + \beta_{2n}$ where $\beta_{1n} \in L^2(0,T;H^{-1}(\Omega))$ and $\beta_{2n} \in L^1(Q_T)$. Applying [7, Lemma 2.2] to each $u_n - u_{0n}$ and then adding u_{0n} to the obtained sequence, we may consider a sequence $(z_{n\sigma})_{\sigma=1}^{\infty}$ in $L^2([0,T];H_0^1(\Omega))$ such that $z_{n\sigma}(0) = u_{0n}$, and $z_{n\sigma} \to u_n$ in $L^2(0,T;H_0^1(\Omega))$ when σ tends to infinity. Moreover, $(z_{n\sigma})_t \in L^2(Q_T)$ and $\lim_{\sigma\to\infty}(z_{n\sigma})_t = (u_n)_t$ in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$.

$$
\begin{aligned}
&\int_0^T \langle T_{h-k}(u_n^+ - T_k(u_n^+)), (u_n)_t\rangle \\
&\quad = \lim_{\sigma\to\infty} \int_{Q_T} T_{h-k}(z_{n\sigma}^+ - T_k(z_{n\sigma}^+))(z_{n\sigma})_t \\
&\quad = \lim_{\sigma\to\infty} \int_{Q_T} T_{h-k}(G_k(z_{n\sigma}^+))G_k(z_{n\sigma}^+)_t \\
&\quad = \lim_{\sigma\to\infty} \int_\Omega J_{h-k}(G_k(z_{n\sigma}^+(T)) - \int_\Omega J_{h-k}(G_k(z_{n\sigma}^+(0)) \\
&\quad = \int_\Omega J_{h-k}(G_k(u_n^+(T)) - \int_\Omega J_{h-k}(G_k(u_{0n}^+)).
\end{aligned}
$$

Therefore, (37) becomes

$$
\begin{aligned}
\int_0^T \langle w_n, (u_n)_t\rangle \geqslant \omega(n) &+ \int_\Omega J_{h+k}((u_n - \eta_{\nu j}(u^+))^+(T)) \\
&- \int_\Omega J_{h+k}((u_{0n} - T_k(\psi_j^+))^+) - \int_\Omega J_{h-k}(G_k(u_n^+(T)) \\
&+ \int_\Omega J_{h-k}(G_k(u_{0n}^+)).
\end{aligned}
$$

As $|\eta_{\nu j}(u^+)| \leqslant k$ it follows that

$$
J_{h+k}((u_n - \eta_{\nu j}(u^+))^+(T)) - J_{h-k}(u_n(T) - k)\chi_{\{u_n(T)\geqslant k\}} \geqslant 0,
$$

so

$$
\int_0^T \langle w_n, (u_n)_t\rangle \geqslant \omega(n) - \int_\Omega J_{h+k}((u_{0n} - T_k(\psi_j^+))^+) + \int_{\{u_{0n}\geqslant k\}} J_{h-k}(u_{0n} - k).
$$

Taking limits as n goes to infinity, it follows that

$$\int_0^T \langle w_n, (u_n)_t \rangle \geqslant \omega(n) - \int_\Omega J_{h+k}((u_0 - T_k(\psi_j^+))^+) + \int_{\{u_0 \geqslant k\}} J_{h-k}(u_0 - k).$$

Taking now limits as v and j tend to infinity,

$$\begin{aligned}\int_0^T \langle w_n, (u_n)_t \rangle &\geqslant \omega(n,v,j) - \int_\Omega J_{h+k}((u_0 - T_k(u_0^+))^+) + \int_{\{u_0 \geqslant k\}} J_{h-k}(u_0 - k) \\ &= \omega(n,v,j) + \int_{\{u_0 \geqslant k\}} (J_{h-k}(u_0 - k) - J_{h+k}((u_0 - k)^+)).\end{aligned}$$

Since

$$-2k \int_{\{u_0 \geqslant h\}} u_0 \leqslant \int_{\{u_0 \geqslant k\}} (J_{h-k}(u_0 - k) - J_{h+k}((u_0 - k)^+)),$$

it yields

$$\int_0^T \langle w_n, (u_n)_t \rangle \geqslant \omega(n,v,j,h).$$

Then, by (36) and the above inequality,

$$\int_{Q_T} \nabla u_n \cdot \nabla w_n \leqslant \omega(n,v,j,h). \tag{38}$$

Now, since $\nabla w_n = 0$ when $u_n \geqslant h + 4k$ and $w_n = w_n \chi_{\{u_n \geqslant 0\}}$, it follows that

$$\begin{aligned}\int_{Q_T} \nabla u_n \cdot \nabla w_n &= \int_{Q_T} \nabla T_{h+4k}(u_n^+) \cdot \nabla w_n \\ &= \int_{Q_T} \nabla T_k(u_n^+) \cdot \nabla (T_k(u_n^+) - \eta_{vj}(u^+))^+ \\ &\quad + \int_{\{u_n \geqslant k\}} \nabla T_{h+4k}(u_n^+) \cdot \nabla w_n,\end{aligned} \tag{39}$$

where

$$\begin{aligned}&\int_{\{u_n \geqslant k\}} \nabla T_{h+4k}(u_n^+) \cdot \nabla w_n \\ &\quad = \int_{\{u_n \geqslant k\} \cap \{u_n - T_h(u_n) - \eta_{vj}(u^+) \leqslant k\}} \nabla T_{h+4k}(u_n^+) \cdot \nabla (u_n - T_h(u_n) + k - \eta_{vj}(u^+)) \\ &\quad \geqslant - \int_{\{u_n \geqslant k\} \cap \{u_n - T_h(u_n) - \eta_{vj}(u^+) \leqslant k\}} \nabla T_{h+4k}(u_n^+) \cdot \nabla \eta_{vj}(u^+) \\ &\quad \geqslant - \int_{\{u_n \geqslant k\}} |\nabla T_{h+4k}(u_n^+)| |\nabla \eta_{vj}(u^+)|\end{aligned}$$

$$\geqslant -\int_{\{u_n\geqslant k\}} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+)|$$

$$-\int_{\{u_n\geqslant k\}} |\nabla T_{h+4k}(u_n^+)|| \times \nabla T_k(u^+) - \nabla\eta_{vj}(u^+)|$$

$$\geqslant -\int_{\{u_n\geqslant k\}} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+)| - \int_{Q_T} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+) - \nabla\eta_{vj}(u^+)|$$

with

$$\lim_{n\to\infty}\int_{\{u_n\geqslant k\}} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+)| = 0$$

and

$$\lim_{v\to\infty}\lim_{n\to\infty}\int_{Q_T} |\nabla T_{h+4k}(u_n^+)\,\nabla T_k(u^+) - \nabla\eta_{vj}(u^+)| = 0,$$

so that (38) and (39) imply

$$\int_{Q_T} \nabla T_k(u_n^+)\cdot\nabla(T_k(u_n^+) - \eta_{vj}(u^+))^+ \leqslant \omega(n,v,j,h).$$

On the other hand, it is easy to see that

$$\lim_{v\to\infty}\lim_{n\to\infty}\int_{Q_T} \nabla\eta_{vj}(u^+)\cdot\nabla(T_k(u_n^+) - \eta_{vj}(u^+))^+ = 0.$$

Therefore,

$$\int_{Q_T} |\nabla(T_k(u_n^+) - \eta_{vj}(u^+))^+|^2 \leqslant \omega(n,v,j,h)$$

and Claim 1 is proved.

Claim 2.

$$\int_{Q_T} |\nabla T_k(u_n^-)|^2\eta_{vj}(u^+) \leqslant \omega(n,v,j,h).$$

Proof. We multiply problem (8) by

$$\theta_n = T_{k^2}(-u_n^- + T_h(u_n^-) - T_k(u_n^-)\eta_{vj}(u^+))$$

with $h > k$, integrate and work as in the above claim to deduce that

$$\int_0^T \langle\theta_n,(u_n)_t\rangle + \int_{Q_T} \nabla u_n\cdot\nabla\theta_n \leqslant \omega(n,v,h). \tag{40}$$

Let us next study the term $\int_0^T \langle\theta_n,(u_n)_t\rangle$. From the above claim, we know there exists a sequence $(z_{n\sigma})_{\sigma=1}^{\infty}$ in $L^2([0,T];H_0^1(\Omega))$, with $(z_{n\sigma})_t \in L^2(Q_T)$, such that $z_{n\sigma}(0) = u_{0n}$, $\lim_{\sigma\to\infty} z_{n\sigma} = u_n$ in $L^2(0,T;H_0^1(\Omega))$ and $\lim_{\sigma\to\infty}(z_{n\sigma})_t = (u_n)_t$ in

$L^2(0,T;H^{-1}(\Omega))+L^1(Q_T)$. Now, denote

$$\theta_{n\sigma}=T_{k^2}(-z_{n\sigma}^-+T_h(z_{n\sigma}^-)-T_k(z_{n\sigma}^-)\eta_{\nu j}(u^+)),$$

which belongs to $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q_T)$. Then we have

$$\int_0^T\langle\theta_n,(u_n)_t\rangle=\lim_{\sigma\to\infty}\int_{Q_T}(z_{n\sigma})_t\theta_{n\sigma}=\lim_{\sigma\to\infty}(A_{n\sigma}+B_{n\sigma}), \tag{41}$$

where $A_{n\sigma}=\int_{\{-h<z_{n\sigma}<0\}}(z_{n\sigma})_t\theta_{n\sigma}$ and $B_{n\sigma}=\int_{\{z_{n\sigma}<-h\}}(z_{n\sigma})_t\theta_{n\sigma}$.

On the one hand, note that

$$\begin{aligned}A_{n\sigma}&=\int_{Q_T}(-T_h(z_{n\sigma}^-))_t\,T_k(-T_h(z_{n\sigma}^-))\eta_{\nu j}(u^+)\\&=\int_{Q_T}(J_k(-T_h(z_{n\sigma}^-)))_t\eta_{\nu j}(u^+)=-\int_{Q_T}J_k(-T_h(z_{n\sigma}^-))(\eta_{\nu j}(u^+))_t\\&\quad+\int_\Omega J_k(-T_h(z_{n\sigma}^-(T)))\eta_{\nu j}(u^+)(T)-\int_\Omega J_k(-T_h(u_{0n}^-))T_k(\psi_j^+)\\&\geqslant-\int_{Q_T}J_k(-T_h(z_{n\sigma}^-))(\eta_{\nu j}(u^+))_t-\int_\Omega J_k(-T_h(u_{0n}^-))T_k(\psi_j^+),\end{aligned}$$

consequently,

$$\lim_{\sigma\to\infty}A_{n\sigma}\geqslant-\int_{Q_T}J_k(-T_h(u^-))(\eta_{\nu j}(u^+))_t-\int_\Omega J_k(-T_h(u_0^-))T_k(\psi_j^+)+\omega(n).$$

Since

$$\int_{Q_T}J_k(-T_h(u^-))(\eta_{\nu j}(u^+))_t=\nu\int_{Q_T}J_k(-T_h(u^-))(-\eta_{\nu j}(u^+))\leqslant 0,$$

it follows that

$$\lim_{\sigma\to\infty}A_{n\sigma}\geqslant-\int_\Omega J_k(-T_h(u_0^-))T_k(\psi_j^+)+\omega(n). \tag{42}$$

On the other hand,

$$\begin{aligned}B_{n\sigma}&=\int_{\{z_{n\sigma}<-h\}}(z_{n\sigma})_tT_{k^2}(-z_{n\sigma}^-+T_h(z_{n\sigma}^-)-T_k(z_{n\sigma}^-)\eta_{\nu j}(u^+))\\&=\int_{Q_T}(-G_h(z_{n\sigma}^-))_tT_{k^2}(-G_h(z_{n\sigma}^-)-k\eta_{\nu j}(u^+))\\&=\int_\Omega J_{k^2}(-G_h(z_{n\sigma}^-(T))-k\eta_{\nu j}(u^+)(T))-\int_\Omega J_{k^2}(-G_h(u_{0n}^-)-kT_k(\psi_j^+))\\&\quad+k\int_{Q_T}(\eta_{\nu j}(u^+))_tT_{k^2}(-G_h(z_{n\sigma}^-)-k\eta_{\nu j}(u^+))\end{aligned}$$

$$\geqslant \int_\Omega J_{k^2}(-k\eta_{vj}(u^+)(T)) - \int_\Omega J_{k^2}(-G_h(u_{0n}^-) - kT_k(\psi_j^+))$$

$$+ k \int_{Q_T} (\eta_{vj}(u^+))_t T_{k^2}(-G_h(z_{n\sigma}^-) - k\eta_{vj}(u^+)).$$

Next, we analyze the last term in the above expression. Observe first that it can be split up as

$$k \int_{Q_T} (\eta_{vj}(u^+))_t T_{k^2}(-G_h(z_{n\sigma}^-) - k\eta_{vj}(u^+)) = B_{n\sigma}^1 - B_{n\sigma}^2$$

with

$$B_{n\sigma}^1 = kv \int_{Q_T} T_k(u^+) T_{k^2}(-G_h(z_{n\sigma}^-) - k\eta_{vj}(u^+))$$

and

$$B_{n\sigma}^2 = kv \int_{Q_T} \eta_{vj}(u^+) T_{k^2}(-G_h(z_{n\sigma}^-) - k\eta_{vj}(u^+)).$$

Thus, taking limits, it yields

$$\lim_{n\to\infty} \lim_{\sigma\to\infty} B_{n\sigma}^1 = kv \int_{Q_T} T_k(u^+) T_{k^2}(-G_h(u^-) - k\eta_{vj}(u^+))$$

$$= -k^2 v \int_{Q_T} T_k(u^+)\eta_{vj}(u^+)$$

and

$$\lim_{n\to\infty} \lim_{\sigma\to\infty} B_{n\sigma}^2 = kv \int_{Q_T} \eta_{vj}(u^+) T_{k^2}(-G_h(u^-) - k\eta_{vj}(u^+))$$

$$\leqslant kv \int_{Q_T} \eta_{vj}(u^+) T_{k^2}(-k\eta_{vj}(u^+))$$

$$= -k^2 v \int_{Q_T} \eta_{vj}(u^+)^2.$$

Hence, we obtain,

$$\lim_{n\to\infty} \lim_{\sigma\to\infty} k \int_{Q_T} (\eta_{vj}(u^+))_t T_{k^2}(-G_h(z_{n\sigma}^-) - k\eta_{vj}(u^+))$$

$$\geqslant -k^2 \int_{Q_T} v(T_k(u^+) - \eta_{vj}(u^+))\eta_{vj}(u^+) = -\frac{k^2}{2} \int_{Q_T} (\eta_{vj}(u^+)^2)_t.$$

Consequently,

$$\lim_{\sigma\to\infty} B_{n\sigma} \geqslant \int_\Omega J_{k^2}(-k\eta_{vj}(u^+)(T)) - \int_\Omega J_{k^2}(-G_h(u_0^-) - kT_k(\psi_j^+))$$

$$-\frac{k^2}{2} \int_\Omega \eta_{vj}(u^+)^2(T) + \frac{k^2}{2} \int_\Omega T_k(\psi_j^+)^2 + \omega(n). \tag{43}$$

Having in mind (42) and (43), it follows from (41) that

$$\int_0^T \langle \theta_n, (u_n)_t \rangle \geqslant \omega(n) - \int_\Omega J_k(-T_h(u_0^-))T_k(\psi_j^+) + \int_\Omega J_{k^2}(-k\eta_{vj}(u^+)(T)) - \int_\Omega J_{k^2}(-G_h(u_0^-) - kT_k(\psi_j^+)) - \frac{k^2}{2}\int_\Omega \eta_{vj}(u^+)^2(T) + \frac{k^2}{2}\int_\Omega T_k(\psi_j^+)^2.$$

Taking now $v \to \infty$ and then $j \to \infty$, we deduce that

$$\int_0^T \langle \theta_n, (u_n)_t \rangle \geqslant \omega(n,v,j) + \int_\Omega J_{k^2}(-kT_k(u^+(T)) - \int_\Omega J_{k^2}(-G_h(u_0^-) - kT_k(u_0^+)) - \frac{k^2}{2}\int_\Omega T_k(u^+(T))^2 + \frac{k^2}{2}\int_\Omega T_k(u_0^+)^2.$$

Letting h go to infinity, we have that

$$\int_0^T \langle \theta_n, (u_n)_t \rangle \geqslant \omega(n,v,j,h) + \int_\Omega J_{k^2}(-kT_k(u^+(T))) - \int_\Omega J_{k^2}(-kT_k(u_0^+)) - \frac{k^2}{2}\int_\Omega T_k(u^+(T))^2 + \frac{k^2}{2}\int_\Omega T_k(u_0^+)^2 = \omega(n,v,j,h),$$

since $J_{k^2}(-kT_k(u^+(T))) = (k^2/2)T_k(u^+(T))^2$ and $J_{k^2}(-kT_k(u_0^+)) = (k^2/2)T_k(u_0^+)^2$.

Hence, from (40), we conclude that

$$\int_{Q_T} \nabla u_n \cdot \nabla \theta_n \leqslant \omega(n,v,j,h). \tag{44}$$

We next turn to study this term. It is straightforward that

$$\int_{Q_T} \nabla u_n \cdot \nabla \theta_n = \int_{\{-k<u_n<0\}} |\nabla u_n|^2 \eta_{vj}(u^+) + \int_{\{-k<u_n<0\}} u_n \nabla u_n \cdot \nabla \eta_{vj}(u^+) - k\int_{\{-h<u_n<-k\}} \nabla u_n \cdot \nabla \eta_{vj}(u^+) - \int_{\{u_n<-h\}} \nabla u_n^- \cdot \nabla T_{k^2}(-u_n^- + h - k\eta_{vj}(u^+)). \tag{45}$$

Observing that the first term in the right-hand side is equal to $\int_{Q_T} |\nabla T_k(u_n^-)|^2 \eta_{\nu j}(u^+)$, to prove our claim all we have to see is that the other terms in (45) tend to 0.

First we analyze the second and the third term. Since $T_k(u_n)^- \rightharpoonup T_k(u)^-$ weakly in $L^2(0,T;H_0^1(\Omega))$, we have that

$$\int_{\{-k<u_n<0\}} u_n \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) = \int_{Q_T} T_k(u_n^-) \nabla T_k(u_n^-) \cdot \nabla \eta_{\nu j}(u^+)$$

tends, as n and ν go to ∞, to $\int_{Q_T} T_k(u^-) \nabla T_k(u^-) \cdot \nabla T_k(u^+) = 0$. Similarly,

$$-\int_{\{-h<u_n<-k\}} \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) = \int_{Q_T} (\nabla T_h(u_n^-) - \nabla T_k(u_n^-)) \cdot \nabla \eta_{\nu j}(u^+)$$

tends to $\int_{Q_T} (\nabla T_h(u^-) - \nabla T_k(u^-)) \cdot \nabla T_k(u^+) = 0$. Consequently,

$$\int_{\{-k<u_n<0\}} u_n \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) - k \int_{\{-h<u_n<-k\}} \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) = \omega(n,\nu). \quad (46)$$

Next, in order to analyze the last term in (45), we use the following notation; we set $M = k^2 + h$,

$$E_n^+ = \{-M + k\eta_{\nu j}(u^+) < u_n < -h\} \cap \{u \geqslant 0\}$$

and

$$E_n^- = \{-M + k\eta_{\nu j}(u^+) < u_n < -h\} \cap \{u < 0\}.$$

Then

$$\begin{aligned}
&\int_{\{u_n<-h\}} \nabla u_n^- \cdot \nabla T_{k^2}(-u_n^- + h - k\eta_{\nu j}(u^+)) \\
&\quad = \int_{\{-k^2-h+k\eta_{\nu j}(u^+)<u_n<-h\}} \nabla u_n^- (-\nabla u_n^- - k \nabla \eta_{\nu j}(u^+)) \\
&\quad \leqslant -k \int_{\{-k^2-h+k\eta_{\nu j}(u^+)<u_n<-h\}} \nabla u_n^- \cdot \nabla \eta_{\nu j}(u^+) \\
&\quad = -k \int_{Q_T} \chi_{E_n^+} \nabla T_M(u_n)^- \cdot \nabla \eta_{\nu j}(u^+) - k \int_{Q_T} \chi_{E_n^-} \nabla T_M(u_n)^- \cdot \nabla \eta_{\nu j}(u^+). \quad (47)
\end{aligned}$$

Since $\chi_{E_n^+}(x,t) \to 0$ a.e. and $T_M(u_n)^- \rightharpoonup T_M(u)^-$ weakly in $L^2(0,T;H_0^1(\Omega))$, it follows that

$$\lim_{n\to\infty} \int_{Q_T} \chi_{E_n^+} \nabla T_M(u_n)^- \cdot \nabla \eta_{\nu j}(u^+) = 0.$$

With respect to the last term in (47), we apply Cauchy–Schwarz's inequality to get

$$\begin{aligned}
&\left| \int_{Q_T} \chi_{E_n^-} \nabla T_M(u_n)^- \cdot \nabla \eta_{\nu j}(u^+) \right| \\
&\quad \leqslant \left(\int_{Q_T} |\nabla T_M(u_n)^-|^2 \right)^{1/2} \left(\int_{E_n^-} |\nabla \eta_{\nu j}(u^+)|^2 \right)^{1/2},
\end{aligned}$$

which is equal to $\omega(n,v)$ just noting the integrals $\int_{Q_T} |\nabla T_M(u_n)^-|^2$ are bounded by a constant only depending on M and

$$\lim_{v\to\infty}\lim_{n\to\infty}\int_{E_n^-}|\nabla\eta_{vj}(u^+)|^2=\int_{\{-M+k\eta_{vj}(u^+)<u<-h\}}|\nabla T_k(u^+)|^2=0.$$

Going back to (47), it yields

$$\int_{\{u_n<-h\}}\nabla u_n^-\cdot\nabla T_{k^2}(-u_n^-+h-k\eta_{vj}(u^+))\leqslant\omega(n,v).$$

From that last inequality, taking into account (45) and (46), we obtain

$$\int_{Q_T}\nabla u_n\cdot\nabla\theta_n\geqslant\int_{Q_T}|\nabla T_k(u_n^-)|^2\eta_{vj}(u^+)+\omega(n,v).$$

Therefore, Claim 2 follows from (44).

Claim 3.

$$\int_{Q_T}|\nabla(T_k(u_n^+)-\eta_{vj}(u^+))^-|^2\leqslant\omega(n,v,j,h).$$

Proof. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be an increasing C^1 function such that $\varphi(0)=0$, $\varphi(-s)=-\varphi(s)$ for all s, and $\varphi'(s)\leqslant\varphi'(r)$ if $0\leqslant s\leqslant r$. Consider the following functions: $\Phi(s)=\int_0^s\varphi(\tau^-)\,d\tau$ and $S(s)=\frac{1}{k}\int_0^s(k-\tau^-)^+\,d\tau$, that is,

$$S(s)=\begin{cases} s & \text{if } s\geqslant 0,\\ s+\dfrac{s^2}{2k} & \text{if } -k\leqslant s\leqslant 0,\\ -\dfrac{k}{2} & \text{if } s\leqslant -k.\end{cases}$$

Observe that we may multiply problem (8) by the function

$$\xi_n=(\varphi((S(u_n)-\eta_{vj}(u^+)))^- -\varphi(S(u_n)^-))S'(u_n)$$

and integrate to get

$$\int_0^T\langle\xi_n,(u_n)_t\rangle+\int_{Q_T}\nabla u_n\cdot\nabla\xi_n+\int_{Q_T}|u_n|^{\beta-2}u_n|\nabla u_n|^q\xi_n$$
$$=\int_{Q_T}|u_n|^{\alpha-2}u_n|\nabla u_n|^p\xi_n.$$

Performing obvious manipulations, from the above equality and the following computations:

$$\int_{Q_T} \nabla u_n \cdot \nabla \xi_n$$

$$= \int_{Q_T} \nabla u_n \cdot \nabla (S(u_n) - \eta_{vj}(u^+))^- \varphi'((S(u_n) - \eta_{vj}(u^+))^-) S'(u_n)$$

$$- \int_{Q_T} \nabla u_n \cdot \nabla S(u_n)^- \varphi'(S(u_n)^-) S'(u_n)$$

$$+ \int_{Q_T} |\nabla u_n|^2 (\varphi((S(u_n) - \eta_{vj}(u^+))^-) - \varphi(S(u_n)^-)) S''(u_n)$$

$$= \int_{\{0 \leqslant u_n \leqslant \eta_{vj}(u^+)\}} \nabla u_n \cdot \nabla (u_n - \eta_{vj}(u^+))^- \varphi'((S(u_n) - \eta_{vj}(u^+))^-) S'(u_n)$$

$$+ \int_{\{-k \leqslant u_n \leqslant 0\}} \nabla u_n \cdot \nabla \eta_{vj}(u^+) \varphi'((S(u_n) - \eta_{vj}(u^+))^-) S'(u_n)$$

$$+ \int_{\{-k \leqslant u_n \leqslant 0\}} \nabla u_n \cdot \nabla (S(u_n))(\varphi'(S(u_n)^-) - \varphi'((S(u_n) - \eta_{vj}(u^+))^-)) S'(u_n)$$

$$+ \frac{1}{k} \int_{\{-k \leqslant u_n \leqslant 0\}} |\nabla u_n|^2 (\varphi((S(u_n) - \eta_{vj}(u^+))^-) - \varphi(S(u_n)^-)),$$

we deduce that

$$\int_{Q_T} |\nabla (T_k(u_n^+) - \eta_{vj}(u^+))^-|^2 \varphi'((u_n^+ - \eta_{vj}(u^+))^-)$$

$$= - \int_{\{0 \leqslant u_n \leqslant \eta_{vj}(u^+)\}} \nabla (u_n - \eta_{vj}(u^+)) \cdot \nabla (u_n^+ - \eta_{vj}(u^+))^- \times \varphi'((u_n^+ - \eta_{vj}(u^+))^-)$$

$$+ \int_{\{u_n \leqslant 0\}} |\nabla \eta_{vj}(u^+)|^2 \varphi'(\eta_{vj}(u^+))$$

$$= - \int_{Q_T} \nabla u_n \cdot \nabla \xi_n$$

$$+ \int_{\{-k \leqslant u_n \leqslant 0\}} \nabla u_n \cdot \nabla \eta_{vj}(u^+) \varphi'((S(u_n) - \eta_{vj}(u^+))^-) S'(u_n)$$

$$+ \int_{\{-k \leqslant u_n \leqslant 0\}} \nabla u_n \cdot \nabla S(u_n)(\varphi'(S(u_n)^-) - \varphi'((S(u_n) - \eta_{vj}(u^+))^-)) S'(u_n)$$

$$+ \frac{1}{k} \int_{\{-k \leqslant u_n \leqslant 0\}} |\nabla u_n|^2 (\varphi((S(u_n) - \eta_{vj}(u^+))^-) - \varphi(S(u_n)^-))$$

$$+\int_{Q_T} \nabla\eta_{vj}(u^+)\cdot\nabla(T_k(u_n^+)-\eta_{vj}(u^+))^-\varphi'((u_n^+-\eta_{vj}(u^+))^-)$$

$$=I_1+I_2+I_3+I_4+I_5+I_6+I_7, \tag{48}$$

where

$$I_1=\int_0^T\langle\xi_n,(u_n)_t\rangle,$$
$$I_2=\int_{Q_T} u_n|u_n|^{\beta-2}|\nabla u_n|^q\xi_n,$$
$$I_3=-\int_{Q_T} u_n|u_n|^{\alpha-2}|\nabla u_n|^p\xi_n,$$
$$I_4=\int_{\{-k\leqslant u_n\leqslant 0\}}\nabla u_n\cdot\nabla\eta_{vj}(u^+)\varphi'((S(u_n)-\eta_{vj}(u^+))^-)S'(u_n),$$
$$I_5=\int_{\{-k\leqslant u_n\leqslant 0\}}\nabla u_n\cdot\nabla S(u_n)(\varphi'(S(u_n)^-)-\varphi'((S(u_n)-\eta_{vj}(u^+))^-))S'(u_n),$$
$$I_6=\frac{1}{k}\int_{\{-k\leqslant u_n\leqslant 0\}}|\nabla u_n|^2(\varphi((S(u_n)-\eta_{vj}(u^+))^-)-\varphi(S(u_n)^-)),$$
$$I_7=\int_{Q_T}\nabla\eta_{vj}(u^+)\cdot\nabla(T_k(u_n^+)-\eta_{vj}(u^+))^-\varphi'((u_n^+-\eta_{vj}(u^+))^-).$$

We are going to study each of these terms. Firstly, observe that, as the function φ is Lipschitz-continuous on $[-k,0]$,

$$|\varphi((S(u_n)-\eta_{vj}(u^+))^-)-\varphi(S(u_n)^-)|\leqslant M\eta_{vj}(u^+)\quad\text{in }\{-k\leqslant u_n\leqslant 0\}. \tag{49}$$

Let us begin with the first term,

$$I_1=\int_0^T\langle\xi_n,(u_n)_t\rangle=\int_0^T\langle\varphi((S(u_n)-\eta_{vj}(u^+))^-)-\varphi(S(u_n)^-),(S(u_n))_t\rangle$$
$$=\int_\Omega\Phi(S(u_n(T))-\eta_{vj}(u^+)(T))-\int_\Omega\Phi(S(u_{0n})-T_k(\psi_j^+))$$
$$+\int_{Q_T}(\eta_{vj}(u^+))_t\varphi((S(u_n)-\eta_{vj}(u^+))^-)-\int_\Omega\Phi(S(u_n(T)))$$
$$+\int_\Omega\Phi(S(u_{0n})).$$

Since Φ is increasing and $\eta_{vj}(u^+)\geqslant 0$, we get

$$\Phi(S(u_n(T))-\eta_{vj}(u^+)(T))-\Phi(S(u_n(T)))\leqslant 0$$

and consequently

$$\int_0^T\langle\xi_n,(u_n)_t\rangle\leqslant\int_\Omega\Phi(S(u_0))-\int_\Omega\Phi(S(u_0)-T_k(\psi_j^+))$$
$$+\int_{Q_T}(\eta_{vj}(u^+))_t\varphi((S(u)-\eta_{vj}(u^+))^-)+\omega(n). \tag{50}$$

On the other hand,

$$\lim_{j\to\infty}\int_\Omega \Phi(S(u_0)) - \int_\Omega \Phi(S(u_0) - T_k(\psi_j^+))$$

$$= \int_\Omega (\Phi(S(u_0)) - \Phi(S(u_0) - T_k(u_0^+)))$$

$$= \int_{\{u_0\geqslant 0\}} (\Phi(S(u_0)) - \Phi(S(u_0) - T_k(u_0^+)))$$

$$+ \int_{\{u_0\leqslant 0\}} (\Phi(S(u_0)) - \Phi(S(u_0) - T_k(u_0^+))) = 0.$$

Finally, since $S(u) \leqslant \eta_{vj}(u^+)$ implies $T_k(u^+) \leqslant \eta_{vj}(u^+)$, we get

$$\int_{Q_T} (\eta_{vj}(u^+))_t \varphi((S(u) - \eta_{vj}(u^+))^-)$$

$$= v\int_{Q_T} (T_k(u^+) - \eta_{vj}(u^+))\varphi((S(u) - \eta_{vj}(u^+))^-) \leqslant 0.$$

Hence, we deduce

$$I_1 \leqslant \omega(n, v, j). \tag{51}$$

Let us turn to analyze I_2 and I_3:

$$I_2 = \int_{\{0\leqslant u_n\leqslant \eta_{vj}(u^+)\}} |u_n|^{\beta-2}u_n|\nabla u_n|^q \varphi((u_n - \eta_{vj}(u^+))^-)$$

$$+ \int_{\{-k\leqslant u_n\leqslant 0\}} |u_n|^{\beta-2}u_n|\nabla u_n|^q(\varphi((S(u_n) - \eta_{vj}(u^+))^-) - \varphi(S(u_n)^-))S'(u_n)$$

$$\leqslant \int_{Q_T} |T_k(u_n^+)|^{\beta-1}|\nabla T_k(u_n^+)|^q \varphi((u_n - \eta_{vj}(u^+))^-).$$

From here, using Young's inequality if $q < 2$,

$$I_2 \leqslant C\int_{Q_T} |\nabla T_k(u_n^+)|^2 \varphi((u_n - \eta_{vj}(u^+))^-) + \omega(n, v)$$

$$= C\int_{Q_T} |\nabla(T_k(u_n^+) - \eta_{vj}(u^+))|^2 \varphi((u_n - \eta_{vj}(u^+))^-)$$

$$+ C\int_{Q_T} \nabla\eta_{vj}(u^+)\cdot\nabla(T_k(u_n^+) - \eta_{vj}(u^+))\varphi((u_n - \eta_{vj}(u^+))^-)$$

$$+ C\int_{Q_T} \nabla T_k(u_n^+)\cdot\nabla\eta_{vj}(u^+)\varphi((u_n - \eta_{vj}(u^+))^-) + \omega(n, v)$$

$$= C\int_{Q_T} |\nabla(T_k(u_n^+) - \eta_{vj}(u^+))|^2 \varphi((u_n - \eta_{vj}(u^+))^-) + \omega(n, v). \tag{52}$$

Respect to I_3,

$$I_3 = -\int_{\{0\leqslant u_n\leqslant \eta_{\nu j}(u^+)\}} |u_n|^{\alpha-2}u_n|\nabla u_n|^p\varphi((u_n-\eta_{\nu j}(u^+))^-)$$

$$-\int_{\{-k\leqslant u_n\leqslant 0\}} |u_n|^{\alpha-2}u_n|\nabla u_n|^p(\varphi((S(u_n)-\eta_{\nu j}(u^+))^-)-\varphi(S(u_n)^-))S'(u_n),$$

and then, using (49), Young and Hölder's inequalities:

$$I_3 \leqslant \int_{Q_T} |T_k(u_n^+)|^{\alpha-1}|\nabla T_k(u_n^+)|^p\varphi((u_n-\eta_{\nu j}(u^+))^-)$$

$$+M\int_{Q_T} |T_k(u_n^-)|^{\alpha-1}|\nabla T_k(u_n^-)|^p\eta_{\nu j}(u^+)$$

$$\leqslant C\int_{Q_T} |\nabla T_k(u_n^+)|^2\varphi((u_n-\eta_{\nu j}(u^+))^-)+M\left(\int_{Q_T}|\nabla T_k(u_n^-)|^2\right)^{p/2}$$

$$\times\left(\int_{Q_T}(|T_k(u_n^-)|^{\alpha-1}\eta_{\nu j}(u^+))^{2/(2-p)}\right)^{(2-p)/2}+\omega(n,\nu)$$

$$= C\int_{Q_T} |\nabla T_k(u_n^+)|^2\varphi((u_n-\eta_{\nu j}(u^+))^-)+\omega(n,\nu).$$

Thus, proceeding as in the above term,

$$I_3 \leqslant C\int_{Q_T} |\nabla(T_k(u_n^+)-\eta_{\nu j}(u^+))|^2\varphi((u_n-\eta_{\nu j}(u^+))^-)+\omega(n,\nu). \tag{53}$$

The term I_4 verifies

$$\lim_{\nu\to\infty}\lim_{n\to\infty} I_4 = -\int_{Q_T} \nabla T_k(u^-)\cdot\nabla T_k(u^+)\varphi'((S(u)-T_k(u^+))^-)S'(u)=0. \tag{54}$$

With respect to the following term, since

$$\varphi'(S(u_n)^-)\leqslant \varphi'((S(u_n)-\eta_{\nu j}(u^+))^-)$$

on the set $\{-k<u_n<0\}$, we have

$$I_5 = \int_{\{-k<u_n<0\}} |\nabla u_n|^2[\varphi'(S(u_n)^-)-\varphi'((S(u_n)-\eta_{\nu j}(u^+))^-)]S'(u_n)^2 \leqslant 0. \tag{55}$$

We now analyze I_6. Having in mind (49) and Claim 2,

$$I_6 = \frac{1}{k}\int_{\{-k\leqslant u_n\leqslant 0\}} |\nabla u_n|^2(\varphi((S(u_n)-\eta_{\nu j}(u^+))^-)-\varphi(S(u_n)^-))$$

$$\leqslant \frac{M}{k}\int_{Q_T} |\nabla T_k(u_n^-)|^2\eta_{\nu j}(u^+)\leqslant \omega(n,\nu,j,h). \tag{56}$$

Finally, it is straightforward that

$$\lim_{v\to\infty}\lim_{n\to\infty} I_7 = \int_{Q_T} \nabla T_k(u^+)\cdot\nabla(T_k(u^+)-T_k(u^+))^-\varphi'((u^+-T_k(u^+))^-)$$
$$= 0. \tag{57}$$

Therefore, going back to (48), estimates (51), (52), (53), (54), (55), (56) and (57) imply

$$\int_{Q_T} |\nabla(T_k(u_n^+)-\eta_{vj}(u^+))^-|^2[\varphi'((u_n^+-\eta_{vj}(u^+))^-)-C\varphi((u_n^+-\eta_{vj}(u^+))^-)]$$
$$\leqslant \omega(n,v,j,h).$$

Choosing $\varphi(s)=s e^{\lambda s^2}$ with λ large such that $\varphi'(s)-C\varphi(s)\geqslant \frac{1}{2}$, it follows that

$$\int_{Q_T} |\nabla(T_k(u_n^+)-\nabla\eta_{vj}(u^+))^-|^2 \leqslant \omega(n,v,j,h)$$

and so Claim 3 is proved.

Now, it follows from Claims 1 and 3 that

$$\int_{Q_T} |\nabla(T_k(u_n^+)-\eta_{vj}(u^+))|^2 \leqslant \omega(n,v,j,h),$$

consequently, since $\eta_{vj}(u^+)\to T_k(u^+)$ in $L^2(0,T;H_0^1(\Omega))$, we obtain

$$\lim_{n\to\infty}\nabla T_k(u_n^+)=\nabla T_k(u^+) \quad \text{in } L^2(Q_T). \tag{58}$$

The corresponding result for the negative part of truncations may be obtained by similar arguments, or by using the fact that $-u_n$ is a solution of

$$\begin{cases} v_t-\Delta v+v|v|^{\beta-2}|\nabla v|^q=|v|^{\alpha-2}v|\nabla v|^p & \text{in } Q_T,\\ v=0 \quad \text{on } S_T, \\ v(x,0)=-u_{0n}(x) & \text{in } \Omega \end{cases}$$

and so we deduce

$$\lim_{n\to\infty}\nabla T_k(-u_n)^+=\nabla T_k(-u)^+ \quad \text{in } L^2(Q_T),$$

that is,

$$\lim_{n\to\infty}\nabla T_k(u_n^-)=\nabla T_k(u^-) \quad \text{in } L^2(Q_T). \tag{59}$$

Therefore, from (58) and (59), we conclude that (33) holds true.

3.3. Convergence of gradient terms in $L^1(Q_T)$

Our aim in this step is to show (12) and (13); as a consequence, we also prove (10).

We begin with the proof of (12); since almost everywhere convergence is guaranteed by (31) and (34), by Vitali's convergence theorem, we only need to show that the sequence $(|u_n|^{\beta-1}|\nabla u_n|^q)_{n=1}^{\infty}$ is equi-integrable. This fact is a consequence of

$$\lim_{h\to\infty}\int_{\{|u_n|\geqslant h\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q=0 \quad \text{uniformly on } n\in\mathbb{N}. \tag{60}$$

To see (60), we multiply problem (8) by $T_k(u_n-T_h(u_n))$ to obtain

$$\begin{aligned}
&\int_{\{|u_n(T)|\geqslant h\}\cap\Omega}J_k(|u_n(T)|-h)+\int_{\{h<|u_n|<k+h\}\cap Q_T}|\nabla u_n|^2\\
&\quad+\int_{Q_T}|u_n|^{\beta-2}u_nT_k(u_n-T_h(u_n))|\nabla u_n|^q\\
&\qquad\leqslant\int_{Q_T}|u_n|^{\alpha-2}u_nT_k(u_n-T_h(u_n))|\nabla u_n|^p+\int_{\{|u_{0n}|\geqslant h\}\cap\Omega}J_k(|u_{0n}|-h).
\end{aligned}$$

Disregarding nonnegative terms, dividing by k and letting k go to 0, it yields

$$\begin{aligned}
\int_{\{|u_n|\geqslant h\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q&\leqslant\int_{\{|u_n|\geqslant h\}\cap Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p\\
&\quad+\int_{\{|u_{0n}|\geqslant h\}\cap\Omega}(|u_{0n}|-h)\\
&\leqslant\int_{\{|u_n|\geqslant h\}\cap Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p+\int_{\{|u_{0n}|\geqslant h\}\cap\Omega}|u_{0n}|.
\end{aligned}$$

Applying Young's inequality we get

$$\begin{aligned}
\int_{\{|u_n|\geqslant h\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q&\leqslant\int_{\{|u_n|\geqslant h\}\cap Q_T}|u_n|^{(\alpha q-\beta p)/(q-p)-1}\\
&\quad+C\int_{\{|u_{0n}|\geqslant h\}\cap\Omega}|u_{0n}|.
\end{aligned}$$

When $(\alpha-1)q>(\beta-1)p$, (60) holds from (9) and (32). In the other cases, $(\alpha-1)q\leqslant(\beta-1)p$, (60) follows in a straightforward way.

Now we are ready to see that the sequence $(|u_n|^{\beta-1}|\nabla u_n|^q)_{n=1}^{\infty}$ is equi-integrable. Indeed, if E is a measurable subset of Q_T, then

$$\begin{aligned}
\int_E|u_n|^{\beta-1}|\nabla u_n|^q&=\int_{E\cap\{|u_n|<k\}}|u_n|^{\beta-1}|\nabla u_n|^q+\int_{E\cap\{|u_n|\geqslant k\}}|u_n|^{\beta-1}|\nabla u_n|^q\\
&\leqslant k^{\beta-1}\int_E|\nabla T_k(u_n)|^q+\int_{\{|u_n|\geqslant k\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q.
\end{aligned}\tag{61}$$

Let $\varepsilon > 0$. By (60), we may choose $k > 0$ such that

$$\int_{\{|u_n|\geqslant k\}\cap Q_T} |u_n|^{\beta-1}|\nabla u_n|^q < \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$. Fixed $k > 0$, as a consequence of (33), we have that the sequence $(|\nabla T_k u_n|^q)_{n=1}^{\infty}$ is equi-integrable. So we may find $\delta > 0$ such that $|E| < \delta$ implies

$$\int_E |\nabla T_k(u_n)|^q < \frac{\varepsilon}{2k^{\beta-1}}$$

for all $n \in \mathbb{N}$. Hence, it follows from (61) that $|E| < \delta$ implies $\int_E |u_n|^{\beta-1}|\nabla u_n|^q < \varepsilon$ for all $n \in \mathbb{N}$.

In order to see (13), we apply Young's inequality to obtain

$$|u_n|^{\alpha-1}|\nabla u_n|^p \leqslant |\nabla T_1(u_n)|^p + \frac{q-p}{q}\,|u_n|^{(\alpha q-\beta p)/(q-p)-1}\chi_{\{|u_n|>1\}} + \frac{p}{q}\,|u_n|^{\beta-1}|\nabla u_n|^q.$$

From here, distinguishing the cases $(\alpha-1)q > (\beta-1)p$ and $(\alpha-1)q \leqslant (\beta-1)p$, and using (31), (32), (33), (34) and (12), (13) follows.

Finally, we prove (10) by showing that

$$|\nabla u_n| \to |\nabla u| \quad \text{in } L^q(Q_T).$$

To do that we only need to apply Vitali's Theorem again. The pointwise convergence follows from (34), while the equi-integrability is a consequence of (11), (12) and the following inequality:

$$\int_E |\nabla u_n|^q = \int_E |\nabla T_1(u_n)|^q + \int_E |\nabla(u_n - T_1(u_n))|^q$$

$$\leqslant |E|^{(2-q)/2}\left(\int_E |\nabla T_1(u_n)|^2\right)^{q/2} + \int_E |u_n|^{\beta-1}|\nabla(u_n)|^q.$$

3.4. Convergence in $C([0,T];L^1(\Omega))$

In this step we prove (14). To do this fix $t \in [0,T]$, and $m, n \in \mathbb{N}$. Take u_m as test function in the generalized formulation of (8) corresponding to u_n, and u_n in that of u_m; adding up both identities we deduce that

$$\int_\Omega J_k(u_n(t)-u_m(t)) + \int_{Q_t} \nabla(u_n-u_m)\cdot\nabla T_k(u_n-u_m)$$

$$+\int_{Q_t} (|u_n|^{\beta-2}u_n|\nabla u_n|^q - |u_m|^{\beta-2}u_m|\nabla u_m|^q)T_k(u_n-u_m)$$

$$= \int_{Q_t} (|u_n|^{\alpha-2}u_n|\nabla u_n|^p - |u_m|^{\alpha-2}u_m|\nabla u_m|^p)T_k(u_n-u_m)$$

$$+\int_\Omega J_k(u_{0n}-u_{0m}).$$

From here, disregarding the nonnegative second term, we obtain that

$$\begin{aligned}\int_\Omega J_k(u_n(t)-u_m(t)) \leqslant k\int_{Q_T} &\left||u_n|^{\beta-2}u_n|\nabla u_n|^q - |u_m|^{\beta-2}u_m|\nabla u_m|^q\right| \\ &+ k\int_{Q_T}\left||u_n|^{\alpha-2}u_n|\nabla u_n|^p - |u_m|^{\alpha-2}u_m|\nabla u_m|^p\right| \\ &+ k\int_\Omega |u_{0n}-u_{0m}|.\end{aligned}$$

Next, dividing this inequality by k and letting k go to 0 we have

$$\begin{aligned}\int_\Omega |u_n(t)-u_m(t)| \leqslant \int_{Q_T} &\left||u_n|^{\beta-2}u_n|\nabla u_n|^q - |u_m|^{\beta-2}u_m|\nabla u_m|^q\right| \\ &+ \int_{Q_T}\left||u_n|^{\alpha-2}u_n|\nabla u_n|^p - |u_m|^{\alpha-2}u_m|\nabla u_m|^p\right| \\ &+ \int_\Omega |u_{0n}-u_{0m}|.\end{aligned}$$

Hence,

$$\begin{aligned}\sup_{t\in[0,T]}\int_\Omega |u_n(t)-u_m(t)| \leqslant \int_{Q_T} &\left||u_n|^{\beta-2}u_n|\nabla u_n|^q - |u_m|^{\beta-2}u_m|\nabla u_m|^q\right| \\ &+ \int_{Q_T}\left||u_n|^{\alpha-2}u_n|\nabla u_n|^p - |u_m|^{\alpha-2}u_m|\nabla u_m|^p\right| \\ &+ \int_\Omega |u_{0n}-u_{0m}|.\end{aligned}$$

Thus, it follows from (9), (12) and (13), that $(u_n)_{n=1}^\infty$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$ and consequently (14) holds.

3.5. *u is a generalized solution*

To finish the proof, we consider $\phi\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q_T)$ such that $\phi_t\in L^2(0,T;H^{-1}(\Omega))+L^1(Q_T)$. Taking ϕ as test function in the approximating problem (8) and letting n go to ∞, having in mind (11), (12), (13) and (14), we deduce the generalized formulation of problem (1) and so the proof of Theorem 3.1 is concluded.

Theorem 3.2. *Assume that* $\alpha,\beta>1$, $1\leqslant q\leqslant 2$, $p+\alpha<q+\beta$, $0\leqslant p<q$. *Then, for every* $u_0\in L^1(\Omega)$, *there exists a generalized solution of problem* (1).

Proof. We will prove this result using the previous theorem. To this end, take an approximating sequence $u_{0n}\in L^\infty(\Omega)$ which converges to u_0 in $L^1(\Omega)$ and consider the corresponding problems with these initial data. Next, we will apply [19] to solve these approximating problems; so that, we need supersolutions and subsolutions of them.

Since Ω is bounded, we have $R>0$ such that $|x_1| \leqslant R-1$ for all $x \in \Omega$. Thus, fixed $n \in \mathbb{N}$, there is $K>0$ such that the function defined by $u^*(x)=K(x_1+R)$ is a supersolution of our approximating problem; indeed,

$$(u^*)_t - \Delta u^* + |u^*|^{\beta-2}u^*|\nabla u^*|^q - |u^*|^{\alpha-2}u^*|\nabla u^*|^p$$

$$= K^{p+\alpha-1}(x_1+R)(K^{(q+\beta)-(p+\alpha)}(x_1+R)^{\beta-2} - (x_1+R)^{\alpha-2}) \geqslant 0 \quad \text{in } Q,$$

$$(u^*)(x,t) \geqslant K > 0 \quad \text{on } S,$$

$$u^*(x,0) \geqslant K \geqslant \|u_{0n}\|_\infty \quad \text{in } \Omega$$

for K big enough. Likewise, the function defined by $u_*(x)=-K(x_1+R)$ is a subsolution. Hence, as a consequence of [7], we get a bounded distributional solution u_n of each approximating problem such that $u_n \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$, and $|u_n|^{\beta-1}|\nabla u_n|^q$ and $|u_n|^{\alpha-1}|\nabla u_n|^p$ belong to $L^1(Q_T)$, and so, by Proposition 2.2, they are generalized solutions. Now, by Theorem 3.1, we obtain a generalized solution of our problem.

Remark 3.1. Observe that assumptions $\alpha, \beta > 1$ in the above theorem are only needed for changing sign initial data, for positive initial data the result obtained is true for $\alpha, \beta \geqslant 1$.

Acknowledgements

We wish to thank Alessio Porretta for his useful suggestions, and also the referee for his comments and remarks.

This article has partially been supported by the Spanish PNPGC, Proyecto BFM2002-01145. The first author has also been supported by EC through the RTN Programme *Nonlinear Partial Differential Equations describing Front Propagation and other Singular Phenomena*, HPRN-CT-2002-00274.

Appendix A

In the case $p=0$, $\alpha+p>\beta+q$, it is known that solutions of problem (1), for initial data $u_0 \in L^\infty(\Omega)$, may blow up in finite time. Nevertheless, in the case $p>0$ a solution of problem (1) for initial datum $u_0 \in L^\infty(\Omega)$ is a global solution since $u^*(x,t)=\|u_0\|_\infty$ is a supersolution and $u_*(x,t)=-\|u_0\|_\infty$ is a subsolution of our problem (see [7]).

In this appendix, we will show that our condition $\alpha+p<\beta+q$ is not arbitrary. In fact, we are going to construct an example in one dimension, for the parameters $p=q=1$ and $\alpha>\beta \geqslant 2$, where our stability result does not work. So, in this case, we are not able to deduce an existence result.

Let us consider the following problem:

$$\begin{cases} u_t - u_{xx} + |u|^{\beta-2}u|u_x| = |u|^{\alpha-2}u|u_x| & \text{in }]-1,1[\times]0,T[, \\ u(x,t)=0 & \text{on } \{-1,1\}\times]0,T[\\ u(x,0)=|x|^{-\gamma}-1 & \text{in }]-1,1[, \end{cases} \tag{A.1}$$

where $0 < \gamma < 1$. We are going to see that, if $\alpha > \beta \geqslant 2$, for a suitable γ, then there exists a sequence of approximate solutions for which our stability result does not apply.

Let

$$L(u) = u_t - u_{xx} + u^{\beta-1}|u_x| - u^{\alpha-1}|u_x|$$

and let

$$u(x,t) = \mathrm{e}^{-k^{\delta}t} h(|x|)$$

where $k > 1$, $\delta > 0$ and

$$h(x) = \begin{cases} -\dfrac{\gamma}{2} k^{(\gamma+2)/\gamma} x^2 + \dfrac{\gamma+2}{2} k - 1 & \text{if } 0 \leqslant x < k^{-1/\gamma}, \\ x^{-\gamma} - 1 & \text{if } k^{-1/\gamma} \leqslant x \leqslant 1. \end{cases}$$

We remark that $h(|x|)$ belongs to C^1 in $[-1, 1]$.

Let us see that choosing $\delta > \max\{\frac{2}{\gamma}, \beta - 1 + \frac{1}{\gamma}\}$ and k large enough,

$$L(u) \leqslant 0 \quad \text{pointwise in }]0,1[\times [0,T].$$

On the one hand, in $]0, k^{-1/\gamma}] \times [0,T]$,

$$L(u) \leqslant -k^{\delta}\mathrm{e}^{-k^{\delta}t} h(x) + \mathrm{e}^{-k^{\delta}t}\gamma k^{(\gamma+2)/\gamma} + \mathrm{e}^{-k^{\delta}t\beta} h^{\beta-1}(x)\gamma k^{(\gamma+2)/\gamma} x \leqslant 0.$$

Indeed, $-\frac{1}{2}k^{\delta}\mathrm{e}^{-k^{\delta}t}h(x) + \mathrm{e}^{-k^{\delta}t}\gamma k^{(\gamma+2)/\gamma} \leqslant 0$ if $\delta > 2/\gamma$ and k is big enough, and $-\frac{1}{2}k^{\delta}\mathrm{e}^{-k^{\delta}t}h(x) + \mathrm{e}^{-k^{\delta}t\beta}h^{\beta-1}(x)\gamma k^{(\gamma+2)/\gamma}x \leqslant 0$ if $\delta > \beta - 1 + 1/\gamma$ and k is large enough. On the other hand, in $[k^{-1/\gamma}, 1[\times [0,T]$,

$$L(u) \leqslant -k^{\delta}\mathrm{e}^{-k^{\delta}t}h(x) + \mathrm{e}^{-k^{\delta}t\beta}h^{\beta-1}(x)\gamma x^{-\gamma-1} \leqslant 0$$

if $\delta > \beta - 1 + 1/\gamma$ and k is large enough. Finally, by a symmetric argument, we conclude that u is a nonnegative subsolution of problem:

$$\begin{cases} u_t - u_{xx} + |u|^{\beta-2}u|u_x| = |u|^{\alpha-2}u|u_x|, & \text{in }]-1,1[\times]0,T[, \\ u(x,t) = 0, \quad \text{on } \{-1,1\} \times]0,T[, \\ u(x,0) = h(|x|), \quad \text{in }]-1,1[. \end{cases}$$

A supersolution of this problem is the constant function $\|h(|x|)\|_{\infty}$. Then, by Boccardo et al. [7] and Proposition 2.2, a bounded generalized solution v of the above problem exists, with $u \leqslant v \leqslant \|h(|x|)\|_{\infty}$.

Since $h(|x|) \to |x|^{-\gamma} - 1$ in L^1 as $k \to \infty$, if Theorem 3.1 holds for these parameters, it would exist a solution w of (A.1) such that

$$\int_0^T \int_{-1}^1 v^{\alpha-1}|v_x| \to \int_0^T \int_{-1}^1 w^{\alpha-1}|w_x| \quad \text{as } k \to \infty. \tag{A.2}$$

Now, it follows from $0 \leqslant u \leqslant v$ that

$$\int_0^T \int_0^1 v^{\alpha-1}|v_x| = \frac{1}{\alpha}\int_0^T \int_0^1 |(v^\alpha)_x| \geqslant \frac{1}{\alpha}\int_0^T \int_0^1 -(v^\alpha)_x$$
$$= \frac{1}{\alpha}\int_0^T v^\alpha(0) \geqslant \frac{1}{\alpha}\int_0^T u^\alpha(0) = \frac{1}{\alpha}\int_0^T (\mathrm{e}^{-k^\delta t}h(0))^\alpha$$
$$= \frac{1}{\alpha}\int_0^T \mathrm{e}^{-k^\delta t\alpha}h^\alpha(0) = \frac{1}{\alpha^2}\left(\frac{\gamma+2}{2}k-1\right)^\alpha k^{-\delta}(1-\mathrm{e}^{-k^\delta T\alpha}).$$

So that, if we may take $\delta < \alpha$, then the last term in the above inequality goes to infinity as k goes to infinity, which contradicts (A.2).

Hence, to get the contradiction we need to find $\delta > 0$ such that

$$\max\left\{\frac{2}{\gamma}, \beta - 1 + \frac{1}{\gamma}\right\} < \delta < \alpha \tag{A.3}$$

for some $0 < \gamma < 1$. Since $\alpha > \beta \geqslant 2$, there exists $0 < \gamma < 1$ such that

$$\max\left\{\frac{2}{\gamma}, \beta - 1 + \frac{1}{\gamma}\right\} < \alpha$$

and consequently there is $\delta > 0$ satisfying (A.3).

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