Tema 2

Edgeworth's Exchange Theory

The exchange Theory of Edgeworth. A simple exchange model 2X2.

- 2 agents A y B and 2 goods: x_1 y x_2
- No production
- Initial endowments are given by:

$$w^A = (w_1^A, w_2^A) \text{ y } w^B = (w_1^B, w_2^B) \text{ con}$$

$$w_1^A + w_1^B = w_1 \quad \text{y} \quad w_2^A + w_2^B = w_2$$

- Each agent has well-defined preferences over baskets of goods and can consume either her initial endowment or exchange it with the other agents.

The exchange Theory of Edgeworth. A simple exchange model 2X2.

- Let a consumption basket of A and B be:

$$x^A = (x_1^A, x_2^A) \text{ y } x^B = (x_1^B, x_2^B)$$

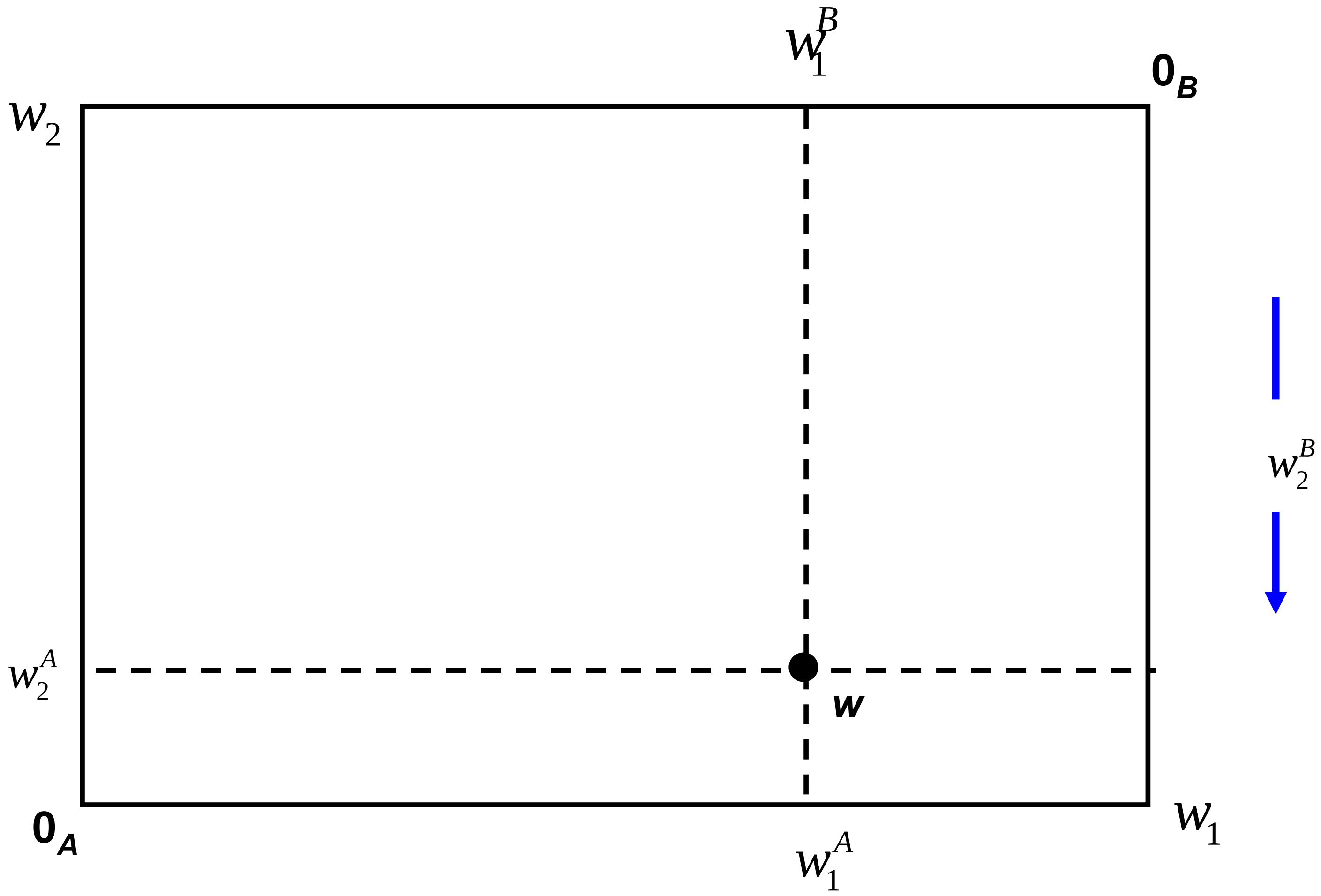
- An *allocation* is a pair of consumption baskets :

$$x = (x^A, x^B)$$

- An allocation is feasible if:

$$x_1^A + x_1^B = w_1^A + w_1^B = w_1 \quad \text{y} \quad x_2^A + x_2^B = w_2^A + w_2^B = w_2$$

A simple model of Pure exchange. The Edgeworth-Bowley box summarizes the set of all feasible allocations.



The Exchange Theory of Edgeworth. Pareto Rationality.

PARETO RATIONALITY: An allocation of goods is **Pareto efficient** if no one can be made better off without making someone else worse off. Formally:

Definition: A feasible allocation ***x*** *is Pareto optimal* (or Pareto Efficient) if there is no other feasible allocation ***y*** such that:

1) $u^i(y^i) \geq u^i(x^i)$ for all i, and

2) $u^j(y^j) > u^j(x^j)$ for at least some j

- The **contract curve** shows all the efficient allocations of goods between two consumers.
- To calculate the contract curve, the utility of an agent is maximized subject to both the feasibility constraint and to the utility level of the other agent´s constraint: *Max* $u^1(x^1)$, *subject to* $u^2(x^2) \geq \underline{u}^2$ and subjetc to feasibility (the two FOC of the assoc. Lagrangian $\rightarrow RMS^1 = RMS^2$ and feasibiity).

The Exchange Theory of Edgeworth. The core of an economy

- INDIVIDUAL RATIONALITY: An allocation x^i satisfies individual rationality (IR) with respect to w^i *if*: $u^i(x^i) \geq u^i(w^i)$

- The **core** of an exchange economy is *the set of feasible allocations which cannot be improved upon (or blocked) by any coalition of agents*.

- For 2 agent-exchange economies core allocations are those satisfying ***individual rationality and Pareto efficiency*.**

- For n agent-economies: We need to define "coalitions" of agentes and how they can block a given allocation.

The Exchange Theory of Edgeworth. The core of an economy

Coalition: A coalition *S* is any subset of agents with mandatory agreements.

- Any coalition *S* can block a proposed allocation x whenever the agents in *S* can reallocate their initial endowments among themselves and be better than under x.

- **Core**: RI, Pareto Rationality and rationality of all the remaining coalitions.

- Example: three agents {A,B,C}
- Coalitions: {A},{B},{C}; {A,B,C,}; {A,B}, {B,C} y
- {A,C}

Core of an exchange economy

- Let n be the number of agents of the economy,
- $w=(w^1,w^2,\ldots,w^n)$ the vector of initial endowments,
- $x=(x^1,x^2,\ldots,x^n)$ an allocation of the economy, and
- $F(w)=\{x: \sum_i x^i = \sum_i w^i\}$ the set of feasible allocations.

- ***Blocking coalition:*** Let S be a coalition. *S blocks* allocation x in F(W), through y in F(W) if:
- 1) $u^i(y^i) \geq u^i(x^i)$ for all i in S, and
- 2) $u^j(y^j) > u^j(x^j)$ for at least some j in S
- 3) $\sum_{i \text{ en } S} y^i \leqslant \sum_{i \text{ en } S} w^i$ (feasibility in S)

- The **core** of an exchange economy ***C(w):***
- ***C(w)={x: there is no y satisfying 1),2) y 3) with x and y* in *F(w)}***

Existence of the core of an exchange economy. Is the core empty? No, whenever there exists a WE, since it belongs to the core.

<u>Definition</u> (alternative): A pair allocation-price (x^*, p^*) is a WE:

- 1) $\sum_i x^{*i} = \sum_i w^i$ (x^* is feasible), and
- 2) If $u^i(x^i) > u^i(x^{*i})$, then $p^* x^i > p^* w^i$ (x is not affordable).

Proposition: If (x^*, p^*) is a WE for the initial endowment w, then x^* belongs to $C(w)$.

<u>Proof:</u> Suppose on the contrary that x^* does not belong to $C(w)$. Then there is a coalition *S and an allocation x* such that for all i in S, $u^i(x^i) > u^i(x^{*i})$, and

$\sum_{i \text{ en } S} x^i = \sum_{i \text{ en } S} w^i$ (x is feasible for S), $\rightarrow p^* \sum_{i \text{ en } S} x^i = p^* \sum_{i \text{ en } S} w^i$ *(1)*

As x^* is a WE, then by definition, for all i in S

$p^* x^i > p^* w^i$ and adding over all i's in *S:*

$$p^* \sum_{i \text{ en } S} x^i > p^* \sum_{i \text{ en } S} w^i \text{, which contradicts } (1) \; (= \sum_{i \text{ en } S} x^i)$$

Then x* belongs to C(w).

Contraction of the core and replica-economies

- The core has ore allocations than the WE.
  - We show that if the economy increases its size, then new coalitions will appear and more opportunities to block or to improve upon:
  - ***The core shrinks (contracts)***
- We use a very simple type of growth.
- ***Definition***: 2 agents are of the same type if both their preferences and their initial endowment are identical.
- ***Definition***: An economy is a ***replica of size r*** of another economy, if there are r-times as many agents of each type in the former economy as in the later.
- Thus, if a large economy replicates a smaller economy, then it will just be as a "scale up" version of the small one.
- We only consider 2 types of agents: type A and type B.

Equal Treatment in the Core

The ***r-core*** (r-C) of an economy is the core of its replica of size r.

Lemma:*Equal treatment in the Core.*

Suppose that agents' preferences are stricty convex, continuous and strongly monotone. If *x belongs to the r-core of a given economy*, then any two agents of the same type will receive the same bundle in *x.*

Proof: Let

$A_1, A_2, \ldots. A_r$ and

$B_1, B_2, \ldots\ldots B_r,$

2 types of agents in the r-replica.

If all agents of the same type do no get the same allocation, there will be one agent of each type who is the most poorly treated.

Call these two agents: type A underdog (marginated): A_M and type B underdog: $B_{M.}$

Equal Treatment in the Core.

- (cont. proof.) Let the mean (average) allocations be:

$$\underline{x}_A = \frac{1}{r}\sum_{j=1}^{r} x_{A_j} \quad \text{y} \quad \underline{x}_B = \frac{1}{r}\sum_{j=1}^{r} x_{B_j}$$

- and we have that: $x_{A_M} < \underline{x}_A, \quad x_{B_M} \leq \underline{x}_B$
- (Note that if all agents receive the same, then they will get the average allocations). By convexity of preferences A_M and B_M prefer the mean allocations to their allocation in *x*:

$$u_A(\underline{x}_A) > u_A(x_{A_M}), \quad u_B(\underline{x}_B) \geq u_B(x_{B_M})$$

- *Can A_M and B_M block core-allocation x through the average allocations?*
- They could whenever average allocations are **feasible** for the coalition of them: $\underline{x}_A + \underline{x}_B = w_A + w_B$

Equal treatment in the Core.

- (cont. proof.) We check the feasibility of average allocations for the underdog-coalition: by feasibility of *x* and given that all agents of the same type have the same initial endowment.

$$\underline{x}_A + \underline{x}_B = \frac{1}{r}\sum_{j=1}^{r} x_{A_j} + \frac{1}{r}\sum_{j=1}^{r} x_{B_j} = x_{A_1} + x_{A_2} + \ldots + x_{A_r} + x_{B_1} + \ldots. + x_{B_r} =$$

$$w_{A_1} + w_{A_2} + \ldots + w_{A_r} + w_{B_1} + \ldots. + w_{B_r} = \frac{1}{r}\sum_{j=1}^{r} w_{A_j} + \frac{1}{r}\sum_{j=1}^{r} w_{B_j} =$$

$$\frac{1}{r} r w_A + \frac{1}{r} r w_B = w_A + w_B.$$

- Then, average allocations are feasible for the coalition of A_M and B_M.

Equal Treatment in the Core.

- (Cont). The underdog A_M strictly prefers its type average allocation to x_{AM} and the underdog B_M considers its type average allocation at least as good as x_{BM}.

- Strong monotonicity allows A_M to remove a little quantity from its average allocation $\underline{x}_A - \varepsilon$
- and to brive B_M by offering him: $\underline{x}_B + \varepsilon$

  thus forming a coalition that can improve upon allocation *x.*

- **Then agents cannot receive a different treatment in the *r-core* of an exchange economy. In the core, all agents of the same type have to receive the same bundle.**

Contraction of the Core

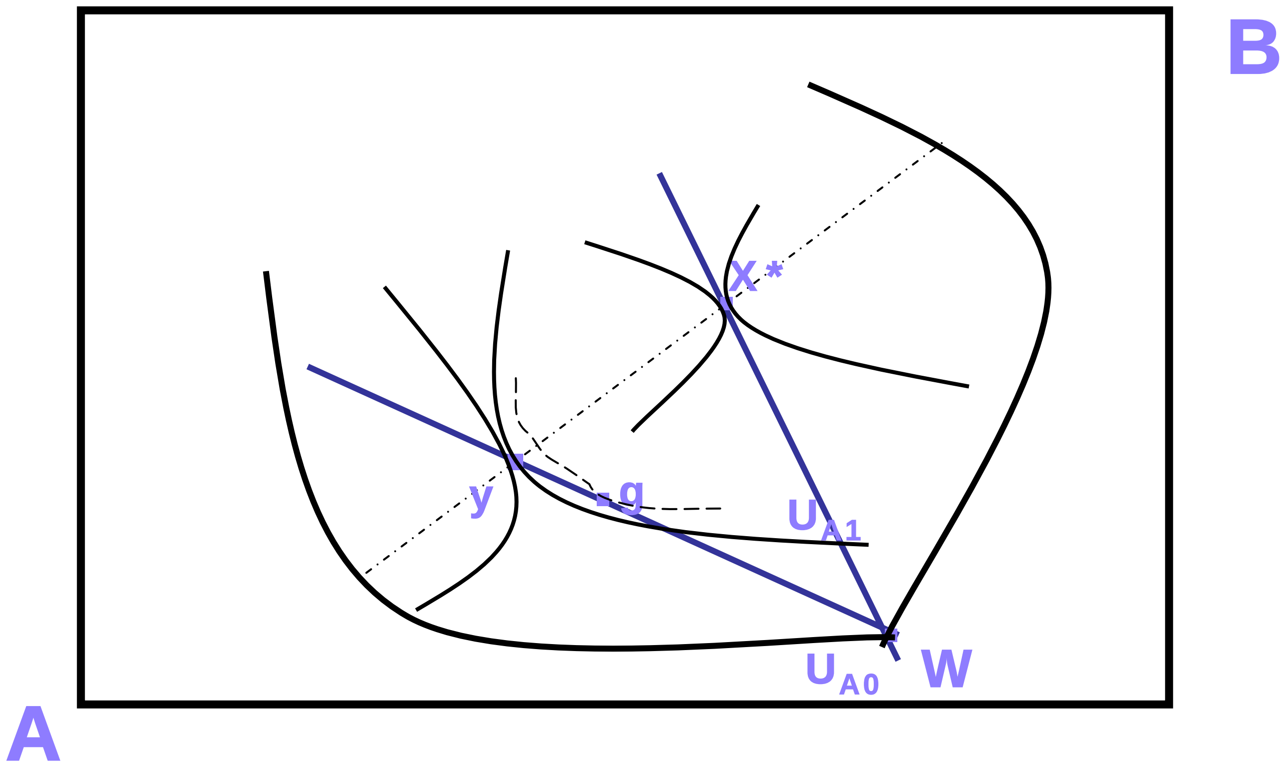
- ***Lemma implications***: To simplify the analysis of the core in replica-economies: an allocation *x* in *C* tell us what each agent type A and type B obtain and then we can keep on representing core-allocations in two-dimensions (in the Edgeworth's box).

- Any allocation ***x*** that is not a WE must eventually not be in the r-core of the economy. Hence, core allocations of large economies look like market equilibria.

- ***Proposition***: *Contraction of the core*:
- *Suppose that preferences are strictly convex and strongly monotone and that there is a unique WE:* ***x****, *for initial endowments* ***w****. Then if* ***y*** *is not a WE allocation, then, there will exit some* ***r****-replication of the economy, such that* ***y*** *is not in the* ***r-core****.*

Contraction of the Core

- Proof: Observe the following drawing:



Contraction of the Core

Since ***y*** *is not a* ***EW***, the line trhough ***w*** and ***y*** must cut at least one agent's indifference, say $\boldsymbol{u_{A1}}$ through ***y***. Then, it is possible to choose a point such as ***g*** which ***A*** prefers to ***y***.

We look for a replica-economy and a coalition that blocks allocation y.

By continuity of the preferences: $g=\theta w+(1-\theta)y$

Let $\theta=T/V<1$, con T *y* V integer numbers.

Then: $g_A=(T/V)w_A+(1-T/V)y_A$

Take the ***V-replica of the economy.***

Form the **coalition**: ***V agents of type A and V-T agents of type B,***

And consider the allocation **asignación *z*** giving $\boldsymbol{g_A}$ to type A agents and $\boldsymbol{y_B}$ to those of type B:

Z: $\boldsymbol{g_A}$ to *type* ***A*** with $\boldsymbol{u_A(z_A)> u_A(y_A)}$

$\boldsymbol{y_B}$ to *type* ***B,*** with $\boldsymbol{u_B(z_B)\geq u_B(y_B)}$,

Then, ***z*** is strictly preferred to ***y*** since agents' type A can always brive agents' type B giving them some epsilon.

Contraction of the Core

- In order this coalition can block allocation ***y*** through ***z***, ***z*** has to be feasible for the coalition. Let us check that this is the case:

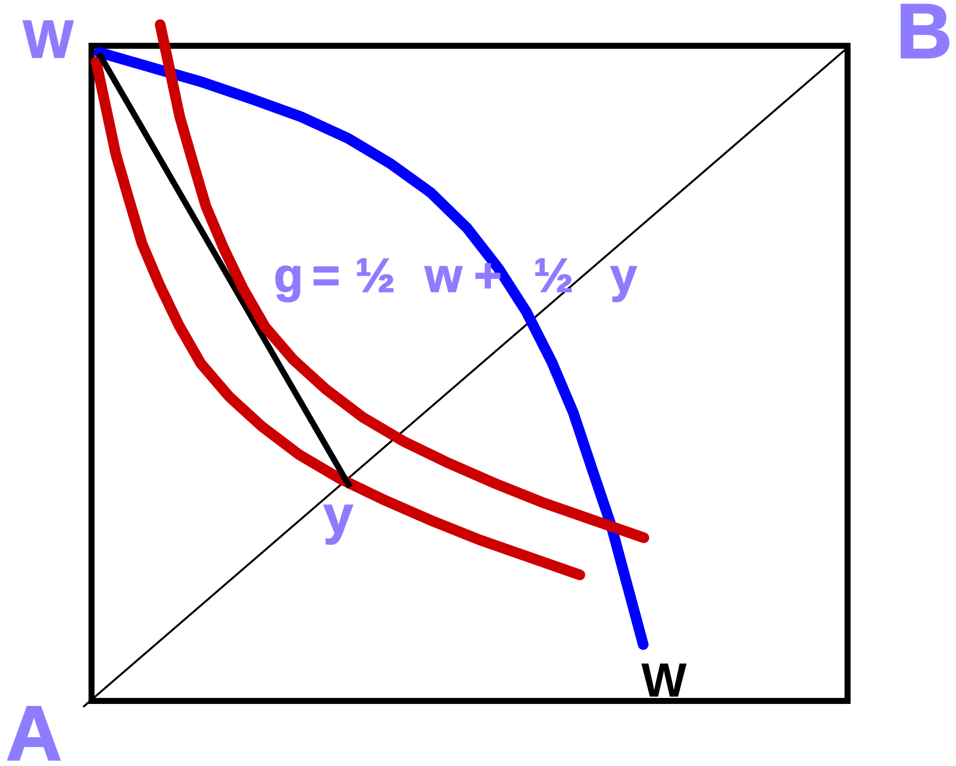
- $Vz_A+(V-T)z_B=Vg_A+(V-T)y_B=V[(T/V)w_A+(1-T/V)y_A]+ (V-T)y_B=$
- $Tw_A+(V-T)y_A+ (V-T)y_B= Tw_A+ +(V-T) (y_A+y_B)=$ *(by feasbt y)*
- $Tw_A+ +(V-T) (w_A+w_B)= V w_A+ +(V-T) w_B,$ which is the initial endowment of the proposed coalition.

- Then, the proposed coalition can block allocation ***y*** through z, in the ***V***-replica of the economy.

- In this way, all allocations of the core not being WE will disappear in some replicas of the economy.
- All core-allocations in huge economies are WE.

Contraction of the Core. Example

- 2 agents: A y B



Contraction of the Core. Example

- What **replica** of the economy and what **coalition** can block ***y*** through ***z*,** with $\boldsymbol{g_A}$ for type A and ***y*** for type B?
- $g=1/2\ w+1/2\ y$ then, $\theta=T/V=1/2$,
- That is: $V=2$ y $V-T=2-1=1$
- Replicate the economy at escale $V=2$ (duplicate the economy: 4 agents) and form the coalition:
- $V=2$ agents of type A and $(V-T)=1$ agent of type B (coalition of 3 agents).
- This coalition can block ***y*** through ***z*** whenever ***z*** is feasible for the coalition. Checking feasibility:
- $Vz_A+(V-T)z_B=Vg_A+ +(V-T)y_B=2(1/2\ w_A+1/2\ y_A)+y_B=$
- $w_A+y_A+y_B=w_A+w_A+w_B=2w_A+w_B$,
- *which are the initial endowments of the coalition. Then, y can be blocked in the duplication of the economy by the proposed coalition of three agents and through allocation z.*

Contraction of the Core. Example

In general:

If the allocation is $\boldsymbol{g=1/n\ w+(1-1/n)y}$

Then $\theta=T/V=1/n$, that is: $V=n$ y $V-T=n-1$

With $g_A=1/n\ w_A+(n-1)/n\ y_A$

- Take the replica of the economy at escale $\boldsymbol{V=n}$
- and the coalition:
- $V=\boldsymbol{n}$ agents type A and $V-T=\boldsymbol{n-1}$ agents type B