PRAGMATIC LANGUAGES WITH UNIVERSAL GRAMMARS

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Supplementary material of Pragmatic Languages with Universal Grammars

1. DERIVATION OF THE RECEIVER'S BEST RESPONSE FOR ANY FEASIBLE BINARY CORPUS OR ENCODING RULE.

Notations:

To make easier the comparison between any observed output sequence y and any pair of standar prototypes $\widehat{\sigma}_i^S = (x_i^1, \ldots, x_i^n)$ and $\widehat{\sigma}_j^S = (x_j^1, \ldots, x_j^n)$ (under any binary encoding rule), let us split the sequences $\widehat{\sigma}_i^S$, $\widehat{\sigma}_j^S$ in four different blocks or subsequences. Each block is formed by the elements of the sequences with subindexes in the following sets:

$$\begin{aligned}
T_{00}^{ij} &= \{l \mid 1 \leq l \leq n \text{ and } x_i^l = 0 \text{ and } x_j^l = 0\} \\
T_{01}^{ij} &= \{l \mid 1 \leq l \leq n \text{ and } x_i^l = 0 \text{ and } x_j^l = 1\} \\
T_{10}^{ij} &= \{l \mid 1 \leq l \leq n \text{ and } x_i^l = 1 \text{ and } x_j^l = 0\} \\
T_{11}^{ij} &= \{l \mid 1 \leq l \leq n \text{ and } x_i^l = 1 \text{ and } x_j^l = 1\}
\end{aligned}$$

For instance, $l \in T_{10}^{ij}$ means that the element of $\widehat{\sigma}_i^S$ placed in position l is a 1, while that in the $l-$position in $\widehat{\sigma}_j^S$ is a 0. Moreover, some of the sets $T_{\alpha\beta}^{ij}$ may be empty, for $\alpha, \beta = 0, 1$. Notice that this subdivision of any two sequences in four blocks may be differente for each pair of prototypes to be compared (the use of superindexes ij is then needed to distinguish among each pair of them).

To further proceed, we need to introduce some additional notation representing the cardinality of these subsets and the Hamming distance among the four subsequences of the output and the prototypes to be compared. Formally, for $\alpha, \beta = 0, 1$, define:

$$n_{\alpha\beta}^{ij} = |T_{\alpha\beta}^{ij}|$$

$$h_{\alpha\beta}^{ij}(y, \widehat{\sigma}_i^S) = \sum\nolimits_{l \in T_{\alpha\beta}^{ij}} I_{y^l \neq x_i^l} \text{ and } h_{\alpha\beta}^{ij}(y, \widehat{\sigma}_j^S) = \sum\nolimits_{l \in T_{\alpha\beta}^{ij}} I_{y^l \neq x_j^l}$$

where I stands for the indicator function.

Example. For instance, T_{01}^{ij} is the subset of indexes $l = 1, \ldots, n$ corresponding to those position with zeros in $\widehat{\sigma}_i^S$ and by ones in $\widehat{\sigma}_j^S$ (i. e., $x_i^l = 0$ and $x_j^l = 1$). If $n = 6$ and $\widehat{\sigma}_i^S = (1,1,0,0,1,0)$ and $\widehat{\sigma}_j^S = (1,0,0,1,0,1)$ we have that $T_{01}^{ij} = \{4,6\}$ and $n_{01}^{ij} = 2$. Additionally $h_{01}^{ij}(\widehat{\sigma}_i^S, y) = h((0,0),(y^4,y^6))$ is the number of digits in y located in positions 4 and 6 that are not zeros and $h_{01}^{ij}(\widehat{\sigma}_j^S, y) = h((1,1),(y^4,y^6))$ is the number of those located in the position in y that are not ones. Notice that each sequence y is split if four separate blocks whose elements are placed at positions $T_{00}^{ij} = \{3\}$, $T_{01}^{ij} = \{4,6\}$, $T_{10}^{ij} = \{2,5\}$ and $T_{11}^{ij} = \{1\}$.

It is straightforward to check that:

Lemma 1. *For all $i, j = 1, 2, \ldots, |\Omega|, i \neq j$ and for all $y \in \{0,1\}^n$ we have that*

(1) $n_{00}^{ij} + n_{01}^{ij} + n_{10}^{ij} + n_{11}^{ij} = n$
(2) $n_{00}^{ij} = n_{00}^{ji}; n_{11}^{ij} = n_{11}^{ji}; n_{01}^{ij} = n_{10}^{ji}$
(3) $h_{00}^{ij}(y, \widehat{\sigma}_i^S) = h_{00}^{ij}(y, \widehat{\sigma}_j^S)$
(4) $h_{11}^{ij}(y, \widehat{\sigma}_i^S) = h_{11}^{ij}(y, \widehat{\sigma}_j^S)$
(5) $h_{01}^{ij}(y, \widehat{\sigma}_i^S) = n_{01}^{ij} - h_{01}^{ij}(y, \widehat{\sigma}_j^S)$
(6) $h_{10}^{ij}(y, \widehat{\sigma}_i^S) = n_{10}^{ij} - h_{10}^{ij}(y, \widehat{\sigma}_j^S)$

Proposition 1. *For all $l, k = 1, \ldots, |\Omega|$, $l \neq k$ and for all $y \in Y$,*

$$\frac{p(\widehat{\sigma}_l^S|y)}{p(\widehat{\sigma}_k^S|y)} = \frac{q_l}{q_k}\left(\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}\right)^{h_{10}^{lk}(\widehat{\sigma}_l^S,y)+h_{01}^{lk}(\widehat{\sigma}_l^S,y)}\left(\frac{1-\varepsilon_1}{\varepsilon_0}\right)^{n_{10}^{lk}}\left(\frac{1-\varepsilon_0}{\varepsilon_1}\right)^{n_{01}^{lk}}$$

Proof: Given any $k \neq l$, all the elements of $\widehat{\sigma}_l^S$ in each block $T_{\alpha\beta}^{lk}$ ($\alpha, \beta = 0, 1$) are constant and equal to α. Then, $h_{\alpha\beta}^{lk}(\widehat{\sigma}_l^S, y)$ is just the number of the $n_{\alpha\beta}^{lk}$ elements with value α in $\widehat{\sigma}_l^S$ (placed at positions $T_{\alpha\beta}^{lk}$) mistransmitted by the channel, where mistransmission takes place with probability ε_α. Alternatively, $n_{\alpha\beta}^{lk} - h_{\alpha\beta}^{lk}(\widehat{\sigma}_l^S, y)$ is the number of elements α properly transmitted (each of them with probability $1-\varepsilon_\alpha$). Since the channel transforms elements independently, the $n_{\alpha\beta}^{lk}$ elements of $\widehat{\sigma}_l^S$ in $T_{\alpha\beta}^{lk}$ become the corresponding $n_{\alpha\beta}^{lk}$ of y with probability

$$(1-\varepsilon_\alpha)^{n_{\alpha\beta}^{lk} - h_{\alpha\beta}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_\alpha^{h_{\alpha\beta}^{lk}(\widehat{\sigma}_l^S,y)}$$

Applying this reasoning to the four blocks $T_{\alpha\beta}^{lk}$, we can write the probability of the noisy channel generating output y when the prototype $\widehat{\sigma}_l^S$ was sent as:

$$\begin{aligned} p(y|\widehat{\sigma}_l^S) &= (1-\varepsilon_0)^{n_{00}^{lk} - h_{00}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_0^{h_{00}^{lk}(\widehat{\sigma}_l^S,y)} \times (1-\varepsilon_0)^{n_{01}^{lk} - h_{01}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_0^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)} \times \\ &\quad (1-\varepsilon_1)^{n_{11}^{lk} - h_{11}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_1^{h_{11}^{lk}(\widehat{\sigma}_l^S,y)} \times (1-\varepsilon_1)^{n_{10}^{lk} - h_{10}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_1^{h_{10}^{lk}(\widehat{\sigma}_l^S,y)} \end{aligned}$$

Similarly:

$$\begin{aligned} p(y|\widehat{\sigma}_k^S) &= (1-\varepsilon_0)^{n_{00}^{lk}-h_{00}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_0^{h_{00}^{lk}(\widehat{\sigma}_k^S,y)} \times (1-\varepsilon_0)^{n_{01}^{lk}-h_{01}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_0^{h_{01}^{lk}(\widehat{\sigma}_k^S,y)} \times \\ & \quad (1-\varepsilon_1)^{n_{11}^{lk}-h_{11}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_1^{h_{11}^{lk}(\widehat{\sigma}_k^S,y)} \times (1-\varepsilon_1)^{n_{10}^{lk}-h_{10}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_1^{h_{10}^{lk}(\widehat{\sigma}_k^S,y)} \end{aligned}$$

and then:

$$\begin{aligned} \frac{p(y|\widehat{\sigma}_l^S)}{p(y|\widehat{\sigma}_k^S)} &= \frac{(1-\varepsilon_0)^{n_{00}^{lk}-h_{00}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_0^{h_{00}^{lk}(\widehat{\sigma}_l^S,y)}}{(1-\varepsilon_0)^{n_{00}^{lk}-h_{00}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_0^{h_{00}^{lk}(\widehat{\sigma}_k^S,y)}} \times \frac{(1-\varepsilon_0)^{n_{01}^{lk}-h_{01}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_0^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)}}{(1-\varepsilon_0)^{n_{01}^{lk}-h_{01}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_0^{h_{01}^{lk}(\widehat{\sigma}_k^S,y)}} \times \\ & \quad \frac{(1-\varepsilon_1)^{n_{11}^{lk}-h_{11}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_1^{h_{11}^{lk}(\widehat{\sigma}_l^S,y)}}{(1-\varepsilon_1)^{n_{11}^{lk}-h_{11}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_1^{h_{11}^{lk}(\widehat{\sigma}_k^S,y)}} \times \frac{(1-\varepsilon_1)^{n_{10}^{lk}-h_{10}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_1^{h_{10}^{lk}(\widehat{\sigma}_l^S,y)}}{(1-\varepsilon_1)^{n_{10}^{lk}-h_{10}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_1^{h_{10}^{lk}(\widehat{\sigma}_k^S,y)}} \end{aligned}$$

By the above Lemma, $h_{00}^{lk}(\widehat{\sigma}_l^S, y) = h_{00}^{lk}(\widehat{\sigma}_k^S, y)$ and $h_{11}^{lk}(\widehat{\sigma}_l^S, y) = h_{11}^{lk}(\widehat{\sigma}_k^S, y)$, then the first and third ratio of the above expression are 1. Hence

$$\frac{p(y|\widehat{\sigma}_l^S)}{p(y|\widehat{\sigma}_k^S)} = \frac{(1-\varepsilon_0)^{n_{01}^{lk}-h_{01}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_0^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)}}{(1-\varepsilon_0)^{n_{01}^{lk}-h_{01}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_0^{h_{01}^{lk}(\widehat{\sigma}_k^S,y)}} \frac{(1-\varepsilon_1)^{n_{10}^{lk}-h_{10}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_1^{h_{10}^{lk}(\widehat{\sigma}_l^S,y)}}{(1-\varepsilon_1)^{n_{10}^{lk}-h_{10}^{lk}(\widehat{\sigma}_k^S,y)}\varepsilon_1^{h_{10}^{lk}(\widehat{\sigma}_k^S,y)}}$$

In addition, the above lemma also expresses the Hamming distances to $\widehat{\sigma}_k^S$ in terms of those to $\widehat{\sigma}_l^S$ and then:

$$\begin{aligned} \frac{p(y|\widehat{\sigma}_l^S)}{p(y|\widehat{\sigma}_k^S)} &= \frac{(1-\varepsilon_0)^{n_{01}^{lk}-h_{01}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_0^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)}}{(1-\varepsilon_0)^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_0^{n_{01}^{lk}-h_{01}^{lk}(\widehat{\sigma}_l^S,y)}} \frac{(1-\varepsilon_1)^{n_{10}^{lk}-h_{10}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_1^{h_{10}^{lk}(\widehat{\sigma}_l^S,y)}}{(1-\varepsilon_1)^{h_{10}^{lk}(\widehat{\sigma}_l^S,y)}\varepsilon_1^{n_{10}^{lk}-h_{10}^{lk}(\widehat{\sigma}_l^S,y)}} \\ &= \left(\frac{\varepsilon_1}{1-\varepsilon_0}\right)^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)+h_{01}^{lk}(\widehat{\sigma}_l^S,y)-n_{01}^{lk}} \left(\frac{\varepsilon_0}{1-\varepsilon_1}\right)^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)+h_{01}^{lk}(\widehat{\sigma}_l^S,y)-n_{10}^{lk}} \\ &= \left(\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}\right)^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)+h_{01}^{lk}(\widehat{\sigma}_l^S,y)} \left(\frac{1-\varepsilon_1}{\varepsilon_0}\right)^{n_{10}^{lk}} \left(\frac{1-\varepsilon_0}{\varepsilon_1}\right)^{n_{01}^{lk}} \end{aligned}$$

and since

$$\frac{p(\widehat{\sigma}_l^S|y)}{p(\widehat{\sigma}_k^S|y)} = \frac{q_l}{q_k}\frac{p(y|\widehat{\sigma}_l^S)}{p(y|\widehat{\sigma}_k^S)}$$

the Proposition holds.

Recall that the Receiver has to take the action in Γ, after hearing an output sequence y, which maximizes his expected payoffs. Equivalently, for each y he chooses the action $\hat{a}_l(y)$ such that

$$\sum\nolimits_{j=1}^{|\Omega|} p(\widehat{\sigma}_j^S|y)u(\hat{a}_l|\omega_j) \geq \sum\nolimits_{j=1}^{|\Omega|} p(\widehat{\sigma}_j^S|y)u(a_k|\omega_j),$$

for any other $k \neq l$ which, given both the linearity of the Receiver's payoff functions in probabilities $\{p(\sigma_l^S|y)\}_l$, $l = 1, \ldots, |\Omega|$, and the matrix payofss is equal to $p(\widehat{\sigma}_j^S|y)M_l \geq p(\widehat{\sigma}_j^S|y)M_k$. The next proposition characterizes the best response of R to any received message y in terms of the Hamming distances among subsequences of y and the standard prototypes of any corpus, the noisy parameters and the payoffs:

Proposition 2. *Given any corpus $\{\widehat{\sigma}_1^S, .., \widehat{\sigma}_{|\Omega|}^S\}$ and any output sequence $y \in \{0,1\}^n$, the action $\widehat{a}_l$ is a best response to y if and only if, the following system of $|\Omega| - 1$ inequalities is satisfied:*

$$\{h_{01}^{lk}(\widehat{\sigma}_l^S, y)+h_{10}^{lk}(\widehat{\sigma}_l^S, y) \leq \frac{Ln\frac{q_k M_k}{q_l M_l}}{Ln\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}}+\frac{n_{10}^{lk}Ln\frac{\varepsilon_0}{1-\varepsilon_1}+n_{01}^{lk}Ln\frac{\varepsilon_1}{1-\varepsilon_0}}{Ln\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}},\ for\ all\ k \neq l\}$$

Proof: Given $y, \widehat{a}_l$ is best response to y if and only if

$$\frac{p(\widehat{\sigma}_l^S|y)}{p(\widehat{\sigma}_k^S|y)} \geq \frac{M_k}{M_l}$$

By Proposition 1, the best response condition can be written as

$$\frac{q_l}{q_k}\left(\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}\right)^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)+h_{10}^{lk}(\widehat{\sigma}_l^S,y)}\left(\frac{1-\varepsilon_1}{\varepsilon_0}\right)^{n_{10}^{lk}}\left(\frac{1-\varepsilon_0}{\varepsilon_1}\right)^{n_{01}^{lk}} \geq \frac{M_k}{M_l}$$

$$\left(\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}\right)^{h_{01}^{lk}(\widehat{\sigma}_l^S,y)+h_{10}^{lk}(\widehat{\sigma}_l^S,y)}\left(\frac{1-\varepsilon_1}{\varepsilon_0}\right)^{n_{10}^{lk}}\left(\frac{1-\varepsilon_0}{\varepsilon_1}\right)^{n_{01}^{lk}} \geq \frac{q_k M_k}{q_l M_l}$$

$$(h_{01}^{lk}(\widehat{\sigma}_l^S, y)+h_{10}^{lk}(\widehat{\sigma}_l^S, y))Ln\left(\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}\right)+n_{10}^{lk}Ln\left(\frac{1-\varepsilon_1}{\varepsilon_0}\right)+n_{01}^{lk}\left(\frac{1-\varepsilon_0}{\varepsilon_1}\right) \geq Ln\frac{q_k M_k}{q_l M_l}$$

$$(h_{01}^{lk}(\widehat{\sigma}_l^S, y)+h_{10}^{lk}(\widehat{\sigma}_l^S, y))Ln\left(\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}\right) \geq Ln\frac{q_k M_k}{q_l M_l}-n_{10}^{lk}Ln\left(\frac{1-\varepsilon_1}{\varepsilon_0}\right)-n_{01}^{lk}\left(\frac{1-\varepsilon_0}{\varepsilon_1}\right)$$

$$(h_{01}^{lk}(\widehat{\sigma}_l^S, y) + h_{10}^{lk}(\widehat{\sigma}_l^S, y)) \leq \frac{Ln\frac{q_k M_k}{q_l M_l} - n_{10}^{lk}Ln\left(\frac{1-\varepsilon_1}{\varepsilon_0}\right) - n_{01}^{lk}\left(\frac{1-\varepsilon_0}{\varepsilon_1}\right)}{Ln\left(\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}\right)}$$

and the proposition holds.

The characterization of the proposition applies to any feasible binary corpus (i. e. encoding rule) used by the sender. In words, $\widehat{a}_l$ is chosen whenever the distances between y and prototype $\widehat{\sigma}_l^S$ in blocks T_{01}^{lk} and T_{10}^{lk}, for all $k \neq l$, $k = \{1, 2, \ldots, \Omega\}\backslash l$, are smaller than some bounds involving both game parameters and encoding parameters. Specifically the first part of the bound for the Hamming distance between y and $\widehat{\sigma}_l^S$:

$$\frac{Ln\frac{q_k M_k}{q_l M_l}}{Ln\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}}$$

refers only to the ratio of expected payoffs and the noise parameters, and is not related to any specific corpus. Meanwhile, the second part

$$\frac{n_{10}^{lk}Ln\frac{\varepsilon_0}{1-\varepsilon_1}+n_{01}^{lk}Ln\frac{\varepsilon_1}{1-\varepsilon_0}}{Ln\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}}$$

depends on the particular encoding rule, specifically on parameters n_{10}^{lk} and n_{01}^{lk}. Notice that if the encoding rule or corpus satisfies that $n_{10}^{lk}=n_{01}^{lk}=m$ for all $l,k=1,\ldots,|\Omega|$, then

$$\frac{n_{10}^{lk}Ln\frac{\varepsilon_0}{1-\varepsilon_1}+n_{01}^{lk}Ln\frac{\varepsilon_1}{1-\varepsilon_0}}{Ln\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}}=\frac{mLn\frac{\varepsilon_0}{1-\varepsilon_1}+mLn\frac{\varepsilon_1}{1-\varepsilon_0}}{Ln\frac{\varepsilon_0}{1-\varepsilon_0}\frac{\varepsilon_1}{1-\varepsilon_1}}=m$$

It is clear that the block coding rule satisfies this property.