Existence results for some quasi-linear parabolic problems with a quadratic gradient term and source

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Abstract.-

In this article we deal with a Cauchy-Dirichlet quasilinear parabolic problem containing a gradient term with quadratic growth and source; namely,

$$\begin{cases} u_t - \Delta u + |u|^{2\gamma-2}u|\nabla u|^2 = |u|^{p-2}u & \text{in } Q := \Omega\times]0,+\infty[; \\ u(x,t) = 0 & \text{on } S := \partial\Omega\times]0,+\infty[; \\ u(x,0) = u_0(x) & \text{in } \Omega; \end{cases}$$

with Ω a bounded open set of $I\!R^N$. We prove that if $p \geq 1$, $\gamma \geq 1/2$ and $p < 2\gamma + 2$, then there exists a global weak solution for all $u_0 \in L^1(\Omega)$. We also see that there exists a nonnegative solution if $u_0 \geq 0$.

1 Introduction and assumptions.

Consider the following quasilinear parabolic problem

$$\begin{cases} u_t - \Delta u + u^{2\gamma-1}|\nabla u|^q = u^{p-1} & \text{in } Q := \Omega\times]0,+\infty[; \\ u = 0 & \text{on } S := \partial\Omega\times]0,+\infty[; \\ u(x,0) = u_0(x) \geq 0 & \text{in } \Omega; \end{cases} \tag{1}$$

where Ω is a bounded open set in $I\!R^N$, whose boundary is denoted by $\partial\Omega$, $p, q \geq 1$ and $\gamma \geq 1/2$.

For the concrete case $\gamma = 1/2$, problem (1) was introduced by M. Chipot and F.B. Weissler in [4] in order to investigate the effect of a damping term on existence or nonexistence of classical solutions. Several authors have studied the existence of non global positive classical solutions, giving conditions for blow-up under certains assumptions on p, q, N and Ω; see [1] and the references therein. Global existence for nonnegative initial data has been proved in the case $q \geq p > 1$.

On the other hand, it is remarked in [12] that problem (1) does not admit global classical solutions in the case $p > 2$, $\gamma \geq 1/2$ and $2\gamma + 2 < p$.

A related problem has been studied in [1], where a degenerate case is considered. More concretely, if the term Δu is replaced by $\Delta(u^m)$ in problem (1), the existence of global weak solutions for nonnegative initial data in $L^{m+1}(\Omega)$ is proved in [1] under the following assumptions: Ω a smooth bounded domain, $m \geq 1$, $(2\gamma + q - 1)/q > m/2$, $1 \leq q < 2$ and $2 \leq p < 2\gamma + q$.

We point out that in [11] and [1] a model in population dynamics is described by this type of equations. The model is as follows: Consider a population of a biological species living on a territory $\Omega \subset I\!R^N$ and denote by $u(.,t)$ the space density of the population at time $t \geq 0$. The evolution of this density is the result of three types of mechanisms: displacements, births and deaths. Displacements are measured by $-\mathrm{div}\,\phi$, where ϕ is the flow of individuals; we will take $\phi = -c_1 \nabla u$. On the other hand, the contributions by accidental deaths should be of the form $-c_2 D(|\nabla u^s|)$, where D is an increasing function; we will suppose that $D(z) = z^2$. Finally, the contribution of births is assumed to be proportional to the number of cuples (or, more generally, of r-tuples), so the population supply due to births will be given by $c_3 u^r$. Therefore, summing up the different contributions one obtains the equation

$$u_t = c_1 \Delta u - c_2 |\nabla u^s|^2 + c_3 u^r.$$

Consequently, under the hipotheses that every death is accidental and that there is a non-viable environment in the boundary zone (we have homogeneous Dirichlet's boundary condition), the solution of our equation (1) describes the evolution of the population density.

The aim of this paper is to prove the existence of global weak solutions of problem (1) for nonnegative initial data $u_0 \in L^1(\Omega)$ in the limit case $q = 2$. To be more precise, we deal with the following problem

$$\begin{cases} u_t - \Delta u + u|u|^{2\gamma-2}|\nabla u|^2 = u|u|^{p-2} & \text{in } Q := \Omega \times]0, +\infty[; \\ u(x,t) = 0 & \text{on } S := \partial\Omega \times]0, +\infty[; \\ u(x,0) = u_0(x) & \text{in } \Omega; \end{cases} \tag{2}$$

where Ω is a bounded open subset of $I\!R^N$ and under the hypotheses

$$p \geq 1, \;\; \gamma \geq 1/2 \;\; \text{and} \;\; p < 2\gamma + 2. \tag{H}$$

We remark that no regularity assumption is required on the boundary of the open set Ω.

The methods used here to prove the existence result are different from those of [1], which do not work for the limit case $q = 2$. Our starting point is to solve problem (2) for bounded initial data. To obtain the existence of weak solutions for integrable initial data, we apply the time-regularizing convolution operator introduced in [6] (and in [9] for nonzero initial data).

This article is organized as follows. In Section 2 we define the concept of weak solution we use and prove the existence of global weak solutions for $u_0 \in L^\infty(\Omega)$. Section 3 is devoted to an initial datum $u_0 \in L^1(\Omega)$: we define weak solution in this context and prove the existence of a global weak solution.

2 Bounded data

In this section we are going to see that if the initial datum u_0 is bounded, there exists a global weak solution of problem (2) in the following sense.

Definition 2.1 Let $u_0 \in L^\infty(\Omega)$. By a weak solution of problem (2) in the set $Q_T = \Omega\times]0,T[$ we mean a function $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$, such that $u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$, $|u|^{p-1} \in L^1(Q_T)$, $|u|^{2\gamma-1}|\nabla u|^2 \in L^1(Q_T)$ and

$$\int_\Omega u(T)\phi(T) - \int_0^T \langle u, \phi_t\rangle + \int_{Q_T} \nabla u\cdot\nabla\phi + \int_{Q_T} |u|^{2\gamma-2}u|\nabla u|^2\phi = \int_{Q_T} |u|^{p-2}u\phi + \int_\Omega u_0\phi(0)$$

for all $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$ such that $\phi_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$.

By a global weak solution of (2), we mean a solution in Q_T for all $T > 0$.

Remark 2.1 (1) If v belongs to $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$ and its distributional derivative in time is such that $v_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$, it is well known that $v \in C([0,T];L^2(\Omega))$. As a consequence, the functions $\phi(0)$ and $\phi(T)$ in the above definition have sense and the meaning of the initial condition $u(0) = u_0$ is clear.

(2) Since $\phi_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$, $\phi_t = \beta_1 + \beta_2$ where $\beta_1 \in L^2(0,T;H^{-1}(\Omega))$ and $\beta_2 \in L^1(Q_T)$. We use the notation

$$\int_0^T \langle u, \phi_t\rangle = \int_0^T \langle u, \beta_1\rangle_{H_0^1,H^{-1}} + \int_{Q_T} u\beta_2$$

in the above definition.

As mentioned above, in this section we prove that there exists a weak solution of problem (2) in each Q_T for u_0 bounded. To this end, we will use the main result in [3] and then an L^∞-estimate procedure introduced by Aronson and Serrin (see [2]). We remark that, since these results hold under more general hypotheses, our results also apply not just to the Laplacian but also to operators satisfying the hypotheses in [3] (see also [8]) and [2].

Lemma 2.1 *Let* $T > 0$ *and let* $b \in C(I\!R) \cap L^\infty(I\!R)$. *For every* $u_0 \in L^\infty(\Omega)$, *there exists* $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$, *weak solution of the problem*

$$\begin{cases} u_t - \Delta u + u|u|^{2\gamma-2}|\nabla u|^2 = b(u) & \textit{in } Q_T := \Omega\times]0,T[; \\ u = 0 & \textit{on } S_T := \partial\Omega\times]0,T[; \\ u(x,0) = u_0(x) & \textit{in } \Omega; \end{cases} \tag{3}$$

such that $u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$.

Moreover, if $b(0) \geq 0$ *and* $u_0 \geq 0$, *then the weak solution can also be taken nonnegative.*

Proof: Let $M > 0$ be such that $|b(s)| \leq M$ for all $s \in I\!R$. For each $k \in I\!N$, we consider the following approximating problem:

$$\begin{cases} u_t - \Delta u + T_k(u)|T_k(u)|^{2\gamma-2}|\nabla u|^2 = b(u) & \text{in } Q_T := \Omega\times]0,T[; \\ u = 0 & \text{on } S_T := \partial\Omega\times]0,T[; \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4}$$

where T_k is the function of a real variable defined by

$$T_k(s) = \max(-k, \min(k,s)).$$

Define two real functions by $\psi(t) = Mt + \|u_0\|_\infty$ and $\varphi = -\psi$. It is easy to check that φ is a subsolution and ψ is a supersolution of problem (4). By Theorem (1.1) in [3], there is a weak solution of (4) which satisfies $\varphi \leq u \leq \psi$ in Q_T. Taking $k > \|\psi\|_\infty$, it follows that $T_k(u) = u$ and consequently u is a weak solution of (3).

When $b(0) \geq 0$ and $u_0 \geq 0$, we only have to notice that $\varphi(t) = 0$ defines a subsolution of problem (4); so that we may take a nonnegative weak solution.

Theorem 2.1 *Let* $T > 0$. *For every* $u_0 \in L^\infty(\Omega)$, *there exists* u *belonging to* $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$, *such that* $u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$, *which is a weak solution of problem (2).*

Furthermore, this weak solution may be chosen nonnegative when $u_0 \geq 0$.

Proof: Consider the following approximating problems:

$$\begin{cases} u_t - \Delta u + u|u|^{2\gamma-2}|\nabla u|^2 = T_n(|u|^{p-2}u) & \text{in } Q_T := \Omega\times]0,T[; \\ u = 0 & \text{on } S_T := \partial\Omega\times]0,T[; \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{5}$$

By Lemma 2.1, there exists $u_n \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$, such that $(u_n)_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$, which is a weak solution of problem (5).

Taking u_n as test function in the weak formulation of (5), we get

$$\int_\Omega u_n(T)^2 - \int_0^T \langle u_n, (u_n)_t\rangle + \int_{Q_T} |\nabla u_n|^2 + \int_{Q_T} |u_n|^{2\gamma}|\nabla u_n|^2 =$$

$$= \int_{Q_T} T_n(|u_n|^{p-2}u_n)u_n + \int_\Omega u_0^2 \le \int_{Q_T} |u_n|^p + \int_\Omega u_0^2.$$

Since $\int_0^T \langle u_n, (u_n)_t\rangle = \frac{1}{2}[\int_\Omega u_n(T)^2 - \int_\Omega u_0^2]$, it follows that

$$\frac{1}{2}\int_\Omega u_n(T)^2 + \int_{Q_T} |\nabla u_n|^2 + \int_{Q_T} |u_n|^{2\gamma}|\nabla u_n|^2 \le \int_{Q_T} |u_n|^p + \frac{1}{2}\int_\Omega u_0^2. \tag{6}$$

Applying Poincaré's inequality and dropping nonnegative terms, we obtain

$$C_1 \int_{Q_T} |u_n|^{2\gamma+2} \le \int_{Q_T} |u_n|^{2\gamma}|\nabla u_n|^2 \le \int_{Q_T} |u_n|^p + \frac{1}{2}\int_\Omega u_0^2.$$

Since $p < 2\gamma + 2$, it follows from Young's inequality that

$$\int_{Q_T} |u_n|^{2\gamma+2} \le C_2,$$

where C_2 only depends on $\|u_0\|_\infty$ and $|\Omega|$.

Now, we prove that the sequence $(u_n)_n$ is bounded in any $L^\lambda(Q_T)$ with $\lambda < \infty$. To this end, we state the following claim:

If for some $s > 0$ the sequence $(u_n)_n$ is bounded in $L^{p+s}(Q_T)$, by a constant only depending on the parameters of our problem, then it is also bounded in $L^{2(\gamma+1)+s}(Q_T)$ by a similar constant.

Take $|u_n|^s u_n$ as test function in the weak formulation of (5), then

$$\frac{1}{s+2}\int_\Omega |u_n(T)|^{s+2} + \int_{Q_T} \nabla u_n \cdot \nabla(|u_n|^s u_n) + \int_{Q_T} |u_n|^{2\gamma+s}|\nabla u_n|^2 \le$$

$$\le \int_{Q_T} |u_n|^{p+s} + \frac{1}{s+2}\int_\Omega |u_0|^{s+2}.$$

Thus, by Poincaré's inequality, it follows that

$$\int_{Q_T} |u_n|^{2(\gamma+1)+s} \le C_3 \int_{Q_T} |u_n|^{2\gamma+s}|\nabla u_n|^2 \le C_3 \int_{Q_T} |u_n|^{p+s} + \frac{C_3}{s+2}\int_\Omega |u_0|^{s+2}.$$

Thus, our claim is proved.

As a consequence, an iterative procedure gives us that $(u_n)_n$ is bounded in $L^\lambda(Q_T)$ for all $\lambda < \infty$. Indeed, if we consider s_1 such that $p + s_1 = 2\gamma + 2$, taking into account that $(u_n)_n$ is bounded in $L^{2\gamma+2}(Q_T)$, then it is bounded in $L^{4(\gamma+1)-p}(Q_T)$. Now, consider s_2 such that $p + s_2 = 4(\gamma+1) - p$ and deduce that $(u_n)_n$ is bounded in $L^{6(\gamma+1)-2p}(Q_T)$. Hence, it is straightforward that the sequence $(u_n)_n$ is bounded in $L^{2(\gamma+1)+s_k}(Q_T)$ for all $k \in I\!N$, where $s_k = k(2\gamma + 2 - p)$. Since $s_k \to \infty$, it follows that $(u_n)_n$ is bounded in any $L^\lambda(Q_T)$, as desired.

Next, take $G_k(u_n)$ as test function in the weak formulation of (5), where the function G_k is defined by $G_k(r) = r - T_k(r)$. Then, denoting by I_k the primitive of G_k such that $I_k(0) = 0$, we get

$$\int_\Omega I_k(u_n)(T) + \int_{Q_T} |\nabla G_k(u_n)|^2 + \int_{Q_T} u_n|u_n|^{2\gamma-1}G_k(u_n) =$$

$$= \int_{Q_T} u_n|u_n|^{p-2} G_k(u_n) + \int_\Omega I_k(u_0)$$

and so

$$\int_\Omega I_k(u_n)(T) + \int_{Q_T} |\nabla G_k(u_n)|^2 \leq \int_{Q_T} u_n|u_n|^{p-2} G_k(u_n) + \int_\Omega I_k(u_0). \tag{7}$$

Observe also that $(u_n|u_n|^{p-2})_{n=1}^\infty$ is bounded in any $L^\lambda(Q_T)$ for $\lambda > \frac{N}{2}+1$. From this fact and (7), using the L^∞-estimate procedure introduced by Aronson and Serrin in [2], we deduce that $(u_n)_n$ is bounded in $L^\infty(Q_T)$. Taking n large enough, we get $T_n(|u_n|^{p-2}u_n) = |u_n|^{p-2}u_n$, so that we conclude that u_n is a weak solution of problem (2).

Finally, if the initial datum u_0 is nonnegative, then (by Lemma 2.1) each u_n can be taken nonnegative and so is the obtained weak solution.

3 L^1 data

In this section we use the following definition of weak solution:

Definition 3.1 Given $u_0 \in L^1(\Omega)$, by a weak solution of problem (2) in Q_T we mean a function $u \in C([0,T]; L^1(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ satisfying $|u|^{p-1} \in L^1(Q_T)$, $|u|^{2\gamma-1}|\nabla u|^2 \in L^1(Q_T)$ and

$$-\int_{Q_T} u\phi_t + \int_{Q_T} \nabla u\cdot\nabla\phi + \int_{Q_T} |u|^{2\gamma-2}u|\nabla u|^2\phi = \int_{Q_T} |u|^{p-2}u\phi + \int_\Omega u_0\phi(0)$$

for all $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$ such that $\phi(T) = 0$ in Ω.

As above, a global weak solution of (2) is a solution in Q_T for all $T > 0$.

Theorem 3.1 *For every $u_0 \in L^1(\Omega)$, there exists a weak solution of problem (2). This weak solution can be nonnegative if u_0 is so.*

Proof: Let $T > 0$ be fixed and let $(u_{0n})_{n=1}^\infty$ be a sequence in $H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$u_{0n} \to u_0 \qquad \text{in} \qquad L^1(\Omega) \tag{8}$$

and $\|u_{0n}\|_1 \leq \|u_0\|_1$ for all $n \in I\!N$. Consider the following approximating problems in Q_T:

$$\begin{cases} (u_n)_t - \Delta u_n + u_n|u_n|^{2\gamma-2}|\nabla u_n|^2 = |u_n|^{p-2}u_n & \text{in } Q_T := \Omega\times]0,T[; \\ u_n = 0 & \text{on } S_T := \partial\Omega\times]0,T[; \\ u_n(x,0) = u_{0n}(x) & \text{in } \Omega. \end{cases} \tag{9}$$

By Theorem 2.1, there exists u_n which is a weak solution of problem (9); observe that if $u_0 \geq 0$, then we may pick $u_n \geq 0$. Taking $T_1(u_n)\chi_{(0,t)}$ as test function in the weak formulation of (9), it follows that

$$\begin{aligned} &\int_\Omega J_1(u_n(t)) + \int_{Q_t} |\nabla T_1(u_n)|^2 + \int_{Q_t} T_1(u_n)u_n|u_n|^{2\gamma-2}|\nabla u_n|^2 = \\ &= \int_{Q_t} T_1(u_n)u_n|u_n|^{p-2} + \int_\Omega J_1(u_{0n}), \end{aligned} \tag{10}$$

where we denote $J_1(r)=\int_0^r T_1(s)ds$.

1.- A priori estimates

Since

$$\int_{Q_T} T_1(u_n)u_n|u_n|^{2\gamma-1}|\nabla u_n|^2 \geq \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{2\gamma-1}|\nabla u_n|^2,$$

having in mind (10) for $t=T$, it follows that

$$\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{2\gamma-1}|\nabla u_n|^2 \leq \int_{Q_T} T_1(u_n)u_n|u_n|^{p-2} + \int_\Omega J_1(u_{0n}) \leq$$

$$\leq |Q_T| + \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{p-1} + \int_\Omega |u_{0n}| \leq C + C\int_{\{|u_n|>1\}\cap Q_T} (|u_n|-1)^{p-1}.$$

Consequently, denoting $G_1(r)=r-T_1(r)$, we get the inequality

$$\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{2\gamma-1}|\nabla u_n|^2 \leq C + C\int_{\{|u_n|>1\}\cap Q_T} |G_1(u_n)|^{p-1},$$

which yields

$$\Big(\gamma+(1/2)\Big)^{-2}\int_{Q_T}\Big|\nabla|G_1(u_n)|^{\gamma+(1/2)}\Big|^2 = \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{2\gamma-1}|\nabla u_n|^2 \leq$$

$$\leq C + C\int_{Q_T} |G_1(u_n)|^{p-1}.$$

Now Poincaré's inequality implies

$$\int_{Q_T} |G_1(u_n)|^{2\gamma+1} \leq C + C\int_{Q_T} |G_1(u_n)|^{p-1},$$

and from here, using Young's inequality, we deduce

$$\int_{Q_T} |G_1(u_n)|^{2\gamma+1} \leq C$$

for all $n\in I\!N$; consequently

$$\int_{Q_T} |u_n|^{2\gamma+1} \leq C \qquad \text{for all} \qquad n\in I\!N, \tag{11}$$

where C only depends on the parameters γ, p, $|Q_T|$ and $\|u_0\|_1$. Obviously, since $p<2\gamma+2$, we also have

$$\int_{Q_T} u_n^{p-1} \leq C \qquad \text{for all} \qquad n\in I\!N. \tag{12}$$

Moreover, since the sequence $(u_{0n})_{n=1}^\infty$ is bounded in $L^1(\Omega)$ by a constant only depending on $\|u_0\|_1$, we obtain that the right-hand side in the equality (10) is bounded. Hence, the following estimates hold:

$$\sup_{t\in[0,T]}\int_\Omega |u_n(t)| \leq C \qquad \text{for all} \qquad n\in I\!N, \tag{13}$$

$$\int_{Q_T} |\nabla T_1(u_n)|^2 \le C \qquad \text{for all} \qquad n \in I\!N, \tag{14}$$

and

$$\int_{Q_T} T_1(u_n) u_n |u_n|^{2\gamma-2} |\nabla u_n|^2 \le C \qquad \text{for all} \qquad n \in I\!N. \tag{15}$$

This last estimate implies

$$\int_{Q_T} |\nabla G_1(u_n)|^2 \le \int_{Q_T} T_1(u_n) u_n |u_n|^{2\gamma-2} |\nabla(u_n)|^2 \le C \qquad \text{for all} \qquad n \in I\!N.$$

So that, it follows from this fact and (14) that

$$\int_{Q_T} |\nabla u_n|^2 \le C \qquad \text{for all} \qquad n \in I\!N. \tag{16}$$

Furthermore, (15) and (16) imply

$$\int_{Q_T} |u_n|^{2\gamma-1} |\nabla u_n|^2 \le C \qquad \text{for all} \qquad n \in I\!N. \tag{17}$$

for all $n \in I\!N$.

Going back again to the equation (9), we get the boundedness of the sequence $((u_n)_t)_{n=1}^\infty$ in the space $L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$. Using this fact and (16), we obtain from [10], Corollary 4, that $(u_n)_{n=1}^\infty$ is relatively compact in $L^2(Q_T)$. Summing up, there exists a function $u \in L^2(0,T;H_0^1(\Omega))$ and a subsequence, still denoted by $(u_n)_{n=1}^\infty$, such that

$$u_n \rightharpoonup u \qquad \text{weakly in} \qquad L^2(0,T;H_0^1(\Omega)) \tag{18}$$

and

$$u_n \to u \qquad \text{in} \qquad L^2(Q_T) \qquad \text{and a.e. in} \qquad Q_T. \tag{19}$$

Thus, in particular, u can be taken nonnegative if $u_0 \ge 0$.

We also deduce that

$$|u_n|^{p-1} \to |u|^{p-1} \qquad \text{in} \qquad L^1(Q_T) \tag{20}$$

Indeed, because of (12), we just have to show that the sequence $(|u_n|^{p-1})_{n=1}^\infty$ is equi-integrable, but it is straighforward taking (11) and Hölder's inequality into account.

2.- Convergence of gradients

Our aim is to prove that

$$\nabla T_k(u_n) \to \nabla T_k(u) \qquad \text{in} \qquad L^2(Q_T) \qquad \text{for all} \quad k \in I\!N, \tag{21}$$

where $0 < \epsilon < T$. From this fact we also deduce that

$$\nabla u_n \to \nabla u \qquad \text{a.e. in} \qquad Q_T. \tag{22}$$

To prove (21), we have to regularize our approximating sequence. We begin by decomposing $(u_n)_t = \beta_{1n} + \beta_{2n}$ where $\beta_{1n} \in L^2(0,T;H^{-1}(\Omega))$ and $\beta_{2n} \in L^1(Q_T)$. Now applying [3] Lemma 2.2 to each $u_n - u_{0n}$ and then adding u_{0n} to the obtained sequence, we can consider a sequence $(z_{n\sigma})_{\sigma=1}^{\infty}$ in $L^2([0,T];H_0^1(\Omega))$ such that $z_{n\sigma}(0)=u_{on}$, and $(z_{n\sigma})_t=\beta_{1n\sigma}+\beta_{2n\sigma}$, where $\beta_{1n\sigma}\in L^2(0,T;H^{-1}(\Omega))$, $\beta_{2n\sigma}\in L^1(Q_T)$, and satisfying the following convergences as σ goes to infinity:

$$\begin{cases} z_{n\sigma}\to u_n \quad \text{in} \quad L^2(0,T;H_0^1(\Omega)) \\ \beta_{1n\sigma}\to \beta_{1n} \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)) \\ \beta_{2n\sigma}\to \beta_{2n} \quad \text{in} \quad L^1(Q_T). \end{cases} \tag{23}$$

On the other hand, we regularize the initial datum by taking $v_{0\nu}\in H_0^1(\Omega)$ such that

$$\begin{cases} 0\le v_{0\nu}\le k \quad \text{for all} \quad \nu\in I\!N \\ v_{0\nu}\to T_k u_0 \quad \text{in} \quad L^1(\Omega) \\ \dfrac{1}{\nu}\displaystyle\int_\Omega |\nabla v_{0\nu}|^2\to 0 \quad \text{as } \nu\to\infty; \end{cases} \tag{24}$$

for which is enough to consider the solution of the following problem

$$\begin{cases} -\frac{1}{\nu}\Delta v_{0\nu}+v_{0\nu}=T_k(u_0) \quad \text{in} \quad \Omega \\ v_{0\nu}=0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Furthermore, we consider the time-regularization function introduced in [6] (see also [9]): for a fixed $\nu\in I\!N$ and a given function $w\in L^2(0,T;H_0^1(\Omega))$, we set

$$w_\nu(t)=\nu\int_0^t w(x,s)e^{\nu(s-t)}ds+e^{-\nu t}v_{0\nu}$$

for $t\in[0,T]$. Applying this regularization to the truncatures $T_k(u_m)$ and having in mind (24), we have that

$$\begin{cases} |(T_k(u_m))_\nu|\le k \\ \Big((T_k(u_m))_\nu\Big)_t=-\nu(T_k(u_m))_\nu+\nu T_k(u_m) \\ (T_k(u_m))_\nu\to T_k(u_m) \quad \text{in} \quad L^2(0,T;H_0^1(\Omega)) \text{ as } \nu\to\infty. \end{cases} \tag{25}$$

Given a number ϵ, $0<\epsilon<T$, we consider the following two real functions ψ and φ. On the one hand, ψ is a nonnegative and decreasing function such that $\psi\in C^1([0,T])$, $\psi(T)=0$ and $\psi(t)=1$ for all $t\in[0,T-\epsilon]$. On the other hand, φ is a Lipschitz continuous function on $I\!R$ satisfying $\varphi(0)=0$ and $\varphi(s)s\ge 0$ for all $s\in I\!R$. Moreover, from now on, we denote by $o(m,n,\nu)$ any quantity I satisfying

$$\lim_{\nu\to\infty}\Big(\lim_{n\to\infty}\big(\lim_{m\to\infty} I\big)\Big)=0;$$

likewise $o(n,\nu)$ denotes a quantity such that $\lim_{\nu\to\infty}(\lim_{n\to\infty} I)=0$.

Our next step is to prove that

$$\langle (u_n)_t,\psi\;\varphi(T_k(u_n)-(T_k(u_m))_\nu)\rangle\ge o(m,n,\nu). \tag{26}$$

For the sake of brevity, we set $w = \psi\,\varphi(T_k(u_n) - (T_k(u_m))_\nu)$ and $w_\sigma = \psi\,\varphi(T_k(z_{n\sigma}) - (T_k(u_m))_\nu)$. Then

$$\langle (u_n)_t, w\rangle = \lim_{\sigma\to\infty}\langle (z_{n\sigma})_t, w_\sigma\rangle = \lim_{\sigma\to\infty}(I_1 + I_2 + I_3),$$

where

$$I_1 = \int_{Q_T}(T_k(z_{n\sigma}) - (T_k(u_m))_\nu)_t w_\sigma$$

$$I_2 = \int_{Q_T}((T_k(u_m))_\nu)_t w_\sigma$$

$$I_3 = \int_{Q_T}(G_k(z_{n\sigma}))_t w_\sigma.$$

We are going to handle the above integrals separately.

Let ϕ be the primitive of φ such that $\phi(0) = 0$; since φ is nondecreasing, ϕ a nonnegative function. Moreover, taking into account that $\psi' \le 0$, we have that

$$I_1 = \int_{Q_T}\psi\,\frac{d}{dt}\Big(\phi(T_k(z_{n\sigma}) - (T_k(u_m))_\nu)\Big) =$$

$$= -\int_{Q_T}\psi'\,\phi(T_k(z_{n\sigma}) - (T_k(u_m))_\nu) - \int_\Omega \phi(T_k(u_{0n}) - T_k(v_{0\nu}))$$

$$\ge -\int_\Omega \phi(T_k(u_{0n}) - T_k(v_{0\nu})).$$

From here, by (25), it follows that

$$I_1 \ge o(m, n, \nu). \tag{27}$$

On the other hand, it is not difficult to see that

$$\lim_{n\to\infty}\Big(\lim_{m\to\infty}\big(\lim_{\sigma\to\infty} I_2\big)\Big) \ge 0. \tag{28}$$

We next decompose I_3 integrating by parts as follows:

$$I_3 = -\int_{Q_T} G_k(z_{n\sigma})\psi'\,\varphi(T_k(z_{n\sigma}) - (T_k(u_m))_\nu) -$$

$$-\int_{Q_T} G_k(z_{n\sigma})\psi\,\varphi'(T_k(z_{n\sigma}) - (T_k(u_m))_\nu)\,(T_k(z_{n\sigma}) - (T_k(u_m))_\nu)_t -$$

$$-\int_\Omega G_k(u_{on})\varphi(T_k(u_{0n}) - T_k(v_{0\nu})) = J_1 + J_2 + J_3.$$

Then,

$$\lim_{n\to\infty}\Big(\lim_{m\to\infty}\big(\lim_{\sigma\to\infty} J_1\big)\Big) = -\int_{Q_T} G_k(u)\psi'\,\varphi(T_k(u) - (T_k(u))_\nu) =$$

$$= -\int_{\{|u|>k\}}(u - k\ \mathrm{sign}u)\psi'\,\varphi(T_k(u) - (T_k(u))_\nu).$$

Now, since $|(T_k(u))_\nu| \le k$, the last integral is nonnegative and so it yields

$$\lim_{n\to\infty}\Big(\lim_{m\to\infty}\big(\lim_{\sigma\to\infty} J_1\big)\Big) \ge 0. \tag{29}$$

Proceeding in a similar way, it is easy to see

$$\begin{cases} \lim_{n\to\infty}\Big(\lim_{m\to\infty}\left(\lim_{\sigma\to\infty} J_2\right)\Big) \geq 0 \\ \\ \lim_{\nu\to\infty}\left(\lim_{n\to\infty} J_3\right) = 0. \end{cases} \tag{30}$$

From (27), (28), (29) and (30), it follows that (26) is proved.

Taking w as test function in the weak formulation of (9); by (26), we have

$$\begin{aligned} o(n,m,\nu) + \int_{Q_T} \nabla u_n \cdot \nabla w + \int_{Q_T} |u_n|^{2\gamma-2} u_n |\nabla u_n|^2 w \leq \\ \leq \int_{Q_T} |u_n|^{p-2} u_n w. \end{aligned} \tag{31}$$

We prove (21) by taking limits in the above inequality and for doing so we now study each integral in (31) separately.

First of all, we deal with the right hand side of (31). Since $(T_k(u_n))_\nu \to (T_k(u))_\nu$ a.e. in Q_T, and $|u_n|^{p-1}|w| \leq |u_n|^{p-1}\varphi(2k) \in L^1(Q_T)$, by Lebesgue's dominated convergence theorem,

$$\lim_{\nu\to\infty}\Big(\lim_{n\to\infty}\big(\lim_{m\to\infty}\int_{Q_T}|u_n|^{p-2}u_n w\ \big)\Big) = 0. \tag{32}$$

We next turn to consider the last term in the left hand side of (31). Note that

$$\lim_{m\to\infty}\int_{Q_T}|u_n|^{2\gamma-2}u_n|\nabla u_n|^2 w =$$

$$\begin{aligned} = \int_{\{|u_n|<k\}} |T_k(u_n)|^{2\gamma-2} T_k(u_n)\psi\ \varphi(T_k(u_n) - (T_k(u))_\nu)|\nabla T_k(u_n)|^2 + \\ + \int_{\{|u_n|\geq k\}} |u_n|^{2\gamma-2} u_n \psi\ \varphi(T_k(u_n) - (T_k(u))_\nu)|\nabla u_n|^2, \end{aligned}$$

where the last term in the above formula is nonnegative; moreover,

$$\begin{aligned} \Big|\int_{\{|u_n|<k\}} |T_k(u_n)|^{2\gamma-2} T_k(u_n)\psi\ \varphi(T_k(u_n) - (T_k(u))_\nu)|\nabla T_k(u_n)|^2\Big| \leq \\ \leq \int_{Q_T} k^{2\gamma-1}\psi|\varphi(T_k(u_n) - (T_k(u))_\nu)|\ |\nabla T_k(u_n)|^2 \leq \\ \leq 2\int_{Q_T} k^{2\gamma-1}\psi|\varphi(T_k(u_n) - (T_k(u))_\nu)|\ |\nabla T_k(u_n) - \nabla T_k(u)|^2 + \\ +2\int_{Q_T} k^{2\gamma-1}\psi|\varphi(T_k(u_n) - (T_k(u))_\nu)|\ |\nabla T_k(u)|^2. \end{aligned}$$

Since

$$\lim_{\nu\to\infty}\Big(\lim_{n\to\infty} 2\int_{Q_T} k^{2\gamma-1}\psi|\varphi(T_k(u_n) - (T_k(u))_\nu)|\ |\nabla T_k(u)|^2\Big) = 0,$$

it follows that

$$\begin{aligned} \lim_{m\to\infty}\int_{Q_T}|u_n|^{2\gamma-2}u_n\psi\ \varphi(T_k(u_n) - (T_k(u_m))_\nu)|\nabla u_n|^2 \geq \\ \geq -2\int_{Q_T} k^{2\gamma-1}\psi|\varphi(T_k(u_n) - (T_k(u))_\nu)|\ |\nabla T_k(u_n) - \nabla T_k(u)|^2 + o(n,\nu). \end{aligned} \tag{33}$$

We finally take limits in the term $\int_{Q_T} \nabla u_n \cdot \nabla w$ of (31). Having in mind that

$$\nabla (T_k(u_m))_\nu \rightharpoonup \nabla (T_k(u))_\nu \qquad \text{weakly in} \qquad L^2(Q_T) \quad \text{as} \quad m\to\infty,$$

we get that

$$\lim_{m\to\infty}\int_{Q_T}\nabla u_n\cdot\nabla w=\int_{Q_T}\psi\nabla u_n\cdot\nabla(T_k(u_n)-(T_k(u))_\nu)\ \varphi'(T_k(u_n)-(T_k(u))_\nu).$$

By denoting

$$H_1=\int_{Q_T}\psi\nabla T_k(u_n)\cdot\nabla(T_k(u_n)-(T_k(u))_\nu)\ \varphi'(T_k(u_n)-(T_k(u))_\nu)$$

and

$$H_2=\int_{Q_T}\psi\nabla G_k(u_n)\cdot\nabla(T_k(u_n)-(T_k(u))_\nu)\ \varphi'(T_k(u_n)-(T_k(u))_\nu),$$

it yields $\lim_{m\to\infty}\int_{Q_T}\nabla u_n\cdot\nabla w=H_1+H_2$.

Obviously,

$$\lim_{n\to\infty}H_2=\int_{Q_T}\psi\nabla G_k(u)\cdot\nabla(T_k(u)-(T_k(u))_\nu)\ \varphi'(T_k(u)-(T_k(u))_\nu)=$$

$$=-\int_{Q_T}\psi\nabla G_k(u)\cdot\nabla((T_k(u))_\nu)\ \varphi'(T_k(u)-(T_k(u))_\nu).$$

Thus,

$$\lim_{\nu\to\infty}(\lim_{n\to\infty}H_2)=-\int_{Q_T}\psi\nabla G_k(u)\cdot\nabla T_k(u)\ \varphi'(0)=0.$$

With respect to H_1, we obtain that

$$\begin{aligned}H_1=&\int_{Q_T}\psi\nabla T_k(u_n)\cdot\nabla(T_k(u_n)-(T_k(u)))\ \varphi'(T_k(u_n)-(T_k(u))_\nu)+\\&+\int_{Q_T}\psi\nabla T_k(u_n)\cdot\nabla(T_k(u)-(T_k(u))_\nu)\ \varphi'(T_k(u_n)-(T_k(u))_\nu).\end{aligned}\tag{34}$$

On the one hand, we have

$$\int_{Q_T}\psi\nabla T_k(u_n)\cdot\nabla(T_k(u_n)-(T_k(u)))\ \varphi'(T_k(u_n)-(T_k(u))_\nu)=$$

$$=\int_{Q_T}\psi|\nabla(T_k(u_n)-(T_k(u)))|^2\ \varphi'(T_k(u_n)-(T_k(u))_\nu)+$$

$$+\int_{Q_T}\psi\nabla T_k(u)\cdot\nabla(T_k(u_n)-(T_k(u)))\ \varphi'(T_k(u_n)-(T_k(u))_\nu)$$

and it is straightforward that

$$\lim_{\nu\to\infty}\Big(\lim_{n\to\infty}\int_{Q_T}\psi\nabla T_k(u)\cdot\nabla(T_k(u_n)-(T_k(u)))\ \varphi'(T_k(u_n)-(T_k(u))_\nu)\Big)=0.$$

On the other hand,

$$\lim_{\nu\to\infty}\Big(\lim_{n\to\infty}\int_{Q_T}\psi\nabla T_k(u_n)\cdot\nabla(T_k(u)-(T_k(u))_\nu)\ \varphi'(T_k(u_n)-(T_k(u))_\nu)\Big)=0.$$

Hence, it follows from (34) that

$$H_1 = \int_{Q_T} \psi |\nabla (T_k(u_n) - (T_k(u)))|^2 \; \varphi'(T_k(u_n) - (T_k(u))_\nu) + o(n,\nu).$$

As a consequence, we get

$$\begin{aligned} &\lim_{m\to\infty} \int_{Q_T} \nabla u_n \cdot \nabla w = H_1 + H_2 = \\ &= \int_{Q_T} \psi |\nabla (T_k(u_n) - (T_k(u)))|^2 \; \varphi'(T_k(u_n) - (T_k(u))_\nu) + o(n,\nu). \end{aligned} \tag{35}$$

Taking into account (32), (33) and (35), we may take limits in (31) obtaining

$$\int_{Q_T} \psi |\nabla (T_k(u_n) - (T_k(u)))|^2 \Phi \le o(m,n,\nu), \tag{36}$$

where $\Phi = \varphi'(T_k(u_n) - (T_k(u))_\nu) - 2k^{2\gamma-1}|\varphi(T_k(u_n) - (T_k(u))_\nu)|$. Choosing $\varphi(s) = se^{\lambda s^2}$, with λ satisfying $\varphi'(s) - 2k^{2\gamma-1}|\varphi(s)| \ge 1/2$, it follows from (36) that

$$\int_{Q_{T-\epsilon}} |\nabla (T_k(u_n) - (T_k(u)))|^2 \le \int_{Q_T} \psi |\nabla (T_k(u_n) - (T_k(u)))|^2 \le o(m,n,\nu)$$

and so

$$\nabla T_k(u_n) \to \nabla T_k(u) \qquad \text{in} \qquad L^2(Q_{T-\epsilon}).$$

Observe that we may always extend our problem considering $Q_{T+\epsilon}$ instead of Q_T (as in [7] or in [5]); therefore, working as before we have that (21) holds true.

3.- u is a weak solution

In order to prove that u is a weak solution of problem (2) in Q_T several facts are needed:

1.- $|u|^{2\gamma-1}|\nabla u|^2 \in L^1(Q_T)$,

2.- $u \in C([0,T]; L^1(\Omega))$ and

3.- the weak formulation holds.

The first condition is a consequence of proving

$$|u_n|^{2\gamma-1}|\nabla u_n|^2 \to |u|^{2\gamma-1}|\nabla u|^2 \qquad \text{in} \qquad L^1(Q_T). \tag{37}$$

By (19) and (22), we already know that this sequence converges a.e. in Q_T so, on account of Vitali's theorem, only the proof of the equi-integrability is necessary. Let E be a measurable subset of Q_T, then

$$\begin{aligned} &\int_E |u_n|^{2\gamma-1}|\nabla u_n|^2 = \int_{E\cap\{|u_n|<k\}} |u_n|^{2\gamma-1}|\nabla u_n|^2 + \int_{E\cap\{|u_n|\ge k\}} |u_n|^{2\gamma-1}|\nabla u_n|^2 \le \\ &\le k^{2\gamma-1} \int_E |\nabla T_k(u_n)|^2 + \int_{\{|u_n|\ge k\}} |u_n|^{2\gamma-1}|\nabla u_n|^2. \end{aligned} \tag{38}$$

We estimate the last integral by taking $T_1(G_{k-1}(u_n))$ in the weak formulation of problem (9). Indeed, denoting $\Theta(r) = \int_0^r T_1(G_{k-1}(s))\, ds$, we obtain

$$\int_\Omega \Theta(u_n(t)) + \int_{Q_T} \nabla u_n \cdot \nabla T_1(G_{k-1}(u_n)) + \int_{Q_T} |u_n|^{2\gamma-2} u_n T_1(G_{k-1}(u_n)) |\nabla u_n|^2 =$$

$$= \int_{Q_T} |u_n|^{2p-2} u_n T_1(G_{k-1}(u_n)) + \int_\Omega \Theta(u_{0n}). \tag{39}$$

Now observe that

$$\int_{\{|u_n|\geq k\}} |u_n|^{2\gamma-1} |\nabla u_n|^2 = \int_{\{|u_n|\geq k\}} |u_n|^{2\gamma-2} u_n T_1(G_{k-1}(u_n)) |\nabla u_n|^2 \leq$$

$$\leq \int_{\{|u_n|\geq k\}} |u_n|^{2\gamma-2} u_n T_1(G_{k-1}(u_n)) |\nabla u_n|^2.$$

From here, using (39), it yields

$$\int_{\{|u_n|\geq k\}} |u_n|^{2\gamma-1} |\nabla u_n|^2 \leq \int_{Q_T} |u_n|^{p-2} u_n T_1(G_{k-1}(u_n)) + \int_\Omega \Theta(u_{0n}) \leq$$

$$\leq \int_{\{|u_n|\geq k-1\}} |u_n|^{p-1} + \int_{\{|u_n|\geq k-1\}} |u_{0n}|.$$

So, it follows from (8) and (20) that

$$\lim_{k\to\infty} \int_{\{|u_n|\geq k\}} |u_n|^{2\gamma-1} |\nabla u_n|^2 = 0.$$

On the other hand, since $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^2(Q_T)$, we also have that

$$\lim_{|E|\to 0} \int_E |\nabla T_k(u_n)|^2 = 0.$$

Hence, going back to (38), we can conclude that

$$\lim_{|E|\to 0} \int_E |u_n|^{2\gamma-1} |\nabla u_n|^2 = 0$$

and so this sequence is equi-integrable.

Since $u_n \in C([0,T]; L^2(\Omega))$, in order to see that $u \in C([0,T]; L^1(\Omega))$, we only have to prove that

$$u_n \to u \qquad \text{in} \qquad C([0,T]; L^1(\Omega)). \tag{40}$$

To do this fix $t \in [0,T]$, and take $T_k(u_n - u_m)\chi_{(0,t)}$ as test function in the weak formulation of u_n and $-T_k(u_n - u_m)\chi_{(0,t)}$ in that of u_m; adding up both identities we deduce that

$$\int_\Omega J_k(u_n(t) - u_m(t)) + \int_{Q_t} \nabla(u_n - u_m) \cdot \nabla T_k(u_n - u_m) +$$

$$+ \int_{Q_t} \Big[|u_n|^{2\gamma-2} u_n |\nabla u_n|^2 - |u_m|^{2\gamma-2} u_m |\nabla u_m|^2 \Big] T_k(u_n - u_m) =$$

$$= \int_{Q_t} \Big[|u_n|^{2p-2} u_n - |u_m|^{2p-2} u_m \Big] T_k(u_n - u_m) + \int_\Omega J_k(u_{0n} - u_{0m}),$$

where J_k is the primitive of T_k such that $J_k(0) = 0$. From here, we obtain a suitable inequality by taking into account that for every $r \in I\!R$, $J_k(r)/k \uparrow |r|$ as $k \downarrow 0$. Indeed, we first disregard a nonnegative term and perform easy manipulations, getting

$$\int_\Omega J_k(u_n(t) - u_m(t)) \le k \int_{Q_T} \left| |u_n|^{2\gamma-2} u_n |\nabla u_n|^2 - |u_m|^{2\gamma-2} u_m |\nabla u_m|^2 \right| +$$

$$+k \int_{Q_T} \left| |u_n|^{p-2} u_n - |u_m|^{p-2} u_m \right| + k \int_\Omega |u_{0n} - u_{0m}|.$$

Next, we divide this inequality by k and let k go to 0 by applying the monotone convergence theorem, obtaining

$$\int_\Omega |u_n(t) - u_m(t)| \le \int_{Q_T} \left| |u_n|^{2\gamma-2} u_n |\nabla u_n|^2 - |u_m|^{2\gamma-2} u_m |\nabla u_m|^2 \right| +$$

$$+ \int_{Q_T} \left| |u_n|^{p-2} u_n - |u_m|^{p-2} u_m \right| + \int_\Omega |u_{0n} - u_{0m}|.$$

Hence,

$$\sup_{t\in[0,T]} \int_\Omega |u_n(t) - u_m(t)| \le \int_{Q_T} \left| |u_n|^{2\gamma-2} u_n |\nabla u_n|^2 - |u_m|^{2\gamma-2} u_m |\nabla u_m|^2 \right| +$$

$$+ \int_{Q_T} \left| |u_n|^{p-2} u_n - |u_m|^{p-2} u_m \right| + \int_\Omega |u_{0n} - u_{0m}|.$$

Thus, it follows from (8), (20) and (37), that $(u_n)_{n=1}^\infty$ is a Cauchy sequence in $C([0,T]; L^1(\Omega))$ and consequently (40) holds true.

To finish the proof, we consider a test function in the weak formulation of the approximating problem (9) and take limits as n tends to ∞, having in mind (8), (18), (19), (20) and (37). Therefore, we deduce that u is a weak solution of problem (2) and so the proof of theorem (3.1) is concluded.

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References

[1] F. Andreu, J.M. Mazón, F. Simondon and J. Toledo *Global existence for a degenerate nonlinear diffusion problem with nonlinear gradient term and source.* Math. Ann., **314** (1999), 703-728.

[2] D.G. Aronson and J. Serrin *Local behavior of solutions of quasilinear parabolic equations.* Arch. Rational Mech. Anal., **25** (1967), 81-122.

[3] L. Boccardo, F. Murat and J.P. Puel *Existence results for some quasilinear parabolic equations.* Nonlinear Analysis T.M.A., **13** (1989), 373-392.

[4] M. Chipot and F.B. Weissler *Some blow up results for a nonlinear parabolic equation with a gradient term.* SIAM J. Math. Anal. , **20** (1989), 886-907.

[5] A. Dall'Aglio and L. Orsina *Nonlinear parabolic equations with natural growth conditions and L^1 data.* Nonlinear Anal. T.M.A. **27** (1996), 59-73

[6] R. Landes *On the existence of weak solutions for quasilinear parabolic boundary value problems.* Proc. Royal Soc. Edinbourgh Sect. A **89** (1981), 217-237

[7] R. Landes and V. Mustonen *On parabolic initial-boundary value problems with critical growth for the gradient.* Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), 135-158

[8] A. Mokrane *Existence of bounded solutions of some nonlinear parabolic equations.* Proc. Royal Soc. Edinburgh, **107** (1987), 313-326

[9] A. Porretta *Regularity for entropy solutions of a class of parabolic equations with non regular initial datum.* Dynamic Systems and Appl., **7** (1998), 53-71

[10] J. Simon *Compact sets in the space $L^p(0,T;B)$.* Ann. Mat. Pura Appl., **146** (1987), 65-96

[11] Ph. Souplet *Finite Time Blow-up for a Non-linear Parabolic Equation with a Gradient Term and Applications.* Math. Meth. Apll. Sc., **19** (1996), 1317-1333

[12] Ph. Souplet and F.B. Weissler *Self-similar solutions and blow-up for nonlinear parabolic equations.* J. Math. Anal. Appl., **212** (1997), 60-74