GLOBAL EXISTENCE FOR NONLINEAR PARABOLIC PROBLEMS WITH MEASURE DATA. APPLICATIONS TO NON-UNIQUENESS FOR PARABOLIC PROBLEMS WITH CRITICAL GRADIENT TERMS

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ABSTRACT. In the present article we study global existence for a nonlinear parabolic equation having a reaction term and a Radon measure datum:

$$\begin{cases} \big(\varphi(v)\big)_t - \Delta_p v = f(x,t)(1+\varphi(v)) + \mu & \text{in } \Omega\times(0,+\infty)\,,\\ v(x,t) = 0 & \text{on } \partial\Omega\times(0,+\infty)\,,\\ v(x,0) = v_0(x) & \text{in } \Omega\,, \end{cases}$$

where $1 < p < N$, Ω is a bounded open set of $\mathbb{R}^N$ $(N \ge 2)$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the so called p-Laplacian operator, $\varphi(s) = \big(1+\frac{s}{p-1}\big)^{p-1}$, $\varphi(v_0) \in L^1(\Omega)$ and μ is a finite Radon measure and $f \in L^\infty(\Omega\times(0,T))$ for every $T > 0$. Then we apply this existence result to show wild nonuniqueness for a connected nonlinear parabolic problem having a gradient term with natural growth.

1. Introduction and Statement of Main Results

In this paper we will consider two related problems. The first one is a doubly nonlinear parabolic equation having a reaction term and a measure datum:

$$(1)\qquad \begin{cases} \big(\varphi(v)\big)_t - \Delta_p v = f(x,t)\,(1+\varphi(v)) + \mu & \text{in } \Omega\times(0,+\infty)\,,\\ v(x,t) = 0 & \text{on } \partial\Omega\times(0,+\infty)\,,\\ v(x,0) = v_0(x) & \text{in } \Omega\,, \end{cases}$$

where $f \in L^\infty(\Omega\times(0,T))$ for every $T > 0$, $\Delta_p v = \operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big)$, with $1 < p < N$, μ is a Radon measure whose total variation is finite in $\Omega\times(0,T)$ for every $T > 0$, and $\varphi(v_0) \in L^1(\Omega)$; here and in what follows

$$(2)\qquad \varphi(s) = \left[\Big(1+\frac{|s|}{p-1}\Big)^{p-1} - 1\right] \operatorname{sign} s\,.$$

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The related equation,

$$\Delta_p w = \left(w^{p-1}\right)_t$$

has been studied in [23]. The behavior of this equation is absolutely different, for instance the homogeneity implies a classical Harnack inequality that is not true for (1).

The second one is a nonlinear parabolic problem having a gradient term with natural growth:

$$\text{(3)}\qquad \begin{cases} u_t - \Delta_p u = |\nabla u|^p + f(x,t) & \text{in } \Omega \times (0,+\infty)\,, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty)\,, \\ u(x,0) = u_0(x) & \text{in } \Omega\,. \end{cases}$$

Notice that for $p = 2$, this equation is used to model some phenomenon in the physical theory of growth and roughening of surfaces, where it is known as the Kardar–Parisi–Zhang equation (see [21]). A modification of the above problem is studied by Berestycki, Kamin, and Sivashinsky as a model in flame propagation, see [7]. We refer also to [5], [16], [14] and the references therein for existence results to parabolic problems with gradient term.

Assume, for simplicity, that nonnegative data are considered, which imply that all solutions are positive. If $\mu = 0$, then both problems are formally connected by means of the Cole–Hopf change of unknown: $v = (p-1)\big(e^{\frac{u}{p-1}} - 1\big)$ transform a solution u to (3) in a solution v to (1) and vice versa.

The main goal of this work is to analyze questions of regularity, uniqueness and non uniqueness of solutions to problems (1) and (3). This was done, when $p = 2$, in [2] (see [1] and [22] for the elliptic case). In that paper, it was proved that, if $f \geq 0$ and $u_0 \geq 0$, then there exists a one–to–one correspondence between solutions to problem

$$\text{(4)}\qquad \begin{cases} u_t - \Delta u = |\nabla u|^2 + f(x,t) & \text{in } \Omega \times (0,+\infty)\,, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty)\,, \\ u(x,0) = u_0(x) & \text{in } \Omega\,. \end{cases}$$

and weak solutions to problem

$$\text{(5)}\qquad \begin{cases} v_t - \Delta v = f(x,t)\,(1+v) + \mu & \text{in } \Omega \times (0,+\infty)\,, \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty)\,, \\ v(x,0) = e^{u_0(x)} - 1 & \text{in } \Omega\,, \end{cases}$$

via the change of variable $v = e^u - 1$. Formally, if one takes a solution u of (4) and sets $v = e^u - 1$, then v satisfies problem (5) with $\mu = 0$. This what actually happens if one considers locally *bounded* weak solutions of (4). However, if one considers an *unbounded* solution u of (4), such that $u \in L^2((0,T); W_0^{1,2}(\Omega))$, for all $T > 0$, then something subtler occurs, and a *singular*, nonnegative measure μ appears in (5). By "singular", we mean that it is concentrated on a set of zero parabolic capacity (see next Section) contained in the cylinder $Q = \Omega \times (0,+\infty)$. Viceversa, if one takes a distributional solution v of (5), with μ singular and nonnegative, then, by setting $u = \log(1+v)$, u becomes a weak solution of (4): the singular measure μ vanishes in the change of variable. This means that problem (4) has infinitely many weak solutions, which may have prescribed singularities (since "$u = +\infty$" on the arbitrary set where μ is concentrated).

Our aim here is to analyze if there is a similar wild non–uniqueness for problem (3) by checking an analogous correspondence in the general case $1 < p < N$. We begin by considering problem (1) in Section 3. For any bounded Radon measure we define the notion of reachable solution or solution obtained as limit of regular solutions to some approximating problems and we prove the existence of a reachable solution, and in the case where $\mu \in L^1(Q_T)$ an uniqueness result is obtained, in this case through the entropy formulation.

Section 4 deals with the main multiplicity result, namely Theorem 1.2. In the case $p = 2$ these result has been obtained in [2] where a complete classification of the positive solutions is obtained. In the case $p \neq 2$ it is necessary to precise the sense of the solutions that we consider. The main idea is to consider a reachable solution v to problem (1) with a singular measure μ and then to use the Cole–Hopf change of unknown. This needs to be justified using the regularity of the reachable solution v. To this end, we will apply some additional properties of gradients of reachable solutions, proved in Subsection 4.1.

In Subsection 4.3 we prove another multiplicity result by considering singular measures as initial data, then the Cole–Hopf change of function allow us to reach a solution to problem (3).

In Section 5 we treat the inverse question, namely, given a solution u to problem (3), by using the inverse of the Cole–Hopf change, we get a reachable solution v to problem (1) obtaining a precise measure μ which depends on u. We analyze some properties of this measure, it seems to be an open problem to show that μ is a singular measure with respect to the parabolic capacity.

In Section 6 we deal with some fine properties of solutions to problem (3), more precisely under suitable hypotheses on p and u_0, we prove the existence of a finite time extinction phenomenon.

Let now summarize the main results of the article.

Theorem 1.1. *If $\varphi(v_0) \in L^1(\Omega)$, $f \in L^\infty_{\text{loc}}([0,\infty); L^\infty(\Omega))$ and μ is a Radon measure such that $\mu\big|_{Q_T}$ has bounded total variation for every $T > 0$, then there exists a function v which is a reachable solution to problem* (1) *(see Definition 3.3 below).*

When $\mu \in L^1_{\text{loc}}((0,+\infty); L^1(\Omega))$, there is uniqueness of entropy solutions (see Definition 3.12 below)

Theorem 1.2. *Let μ be a positive, singular (with respect to the parabolic p–capacity, see Definition 2.2 below) Radon measure in Q_T such that μ has bounded total variation for every $T > 0$. Let $f(x,t)$ and $u_0(x)$ denote nonnegative functions such that $f \in L^\infty_{\text{loc}}([0,\infty); L^\infty(\Omega))$ and $e^{u_0} \in L^1(\Omega)$.*

Consider v, a reachable solution to problem (1) *with initial datum $v_0 = (p-1)\big(e^{\frac{u_0}{p-1}} - 1\big)$ (see Definition 3.3 below), and set $u = (p-1)\log(\frac{v}{p-1} + 1)$, then $u \in L^p_{\text{loc}}([0,\infty); W^{1,p}_0(\Omega)) \cap \mathcal{C}([0,\infty); L^1(\Omega))$ and is a weak solution of* (3) *with initial datum u_0.*

Theorem 1.3. *Let $u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^p_{\text{loc}}([0,\infty); W^{1,p}_0(\Omega))$ be a weak solution to problem* (3)*, where $f \in L^\infty_{\text{loc}}((0,\infty); L^\infty(\Omega))$ is a nonnegative function. Assume that u satisfies*

$$e^u \in L^\infty_{\text{loc}}((0,+\infty); L^1(\Omega)), \tag{6}$$

then

$$e^{\frac{\beta u}{p-1}} \in L^p_{\text{loc}}((0,+\infty); W^{1,p}_0(\Omega)) \qquad \textit{for all } 0<\beta<\frac{p-1}{p}\,. \tag{7}$$

If we set $v=(p-1)\big(e^{\frac{u}{p-1}}-1\big)$*, it follows that* $v\in L^1_{\text{loc}}((0,\infty);L^1(\Omega))$ *and there exists a bounded positive Radon measure* μ *such that* v *solves*

$$\big(\varphi(v)\big)_t-\Delta_p v=f\,(1+\varphi(v))+\mu \qquad \textit{in } \mathcal{D}'(Q)\,.$$

Moreover μ *can be characterized as a weak limit in the space of bounded Radon measures, as follows:*

$$\mu=\lim_{\epsilon\to0}|\nabla u|^p e^{u/(1+\epsilon u)}\Big(1-\frac{1}{(1+\epsilon u)^2}\Big) \qquad \textit{in } \Omega\times(0,T) \quad \textit{for all } T>0\,. \tag{8}$$

Remarks 1.4.

(1) Expressions $v=(p-1)\big(e^{\frac{u}{p-1}}-1\big)$ define a correspondence between solutions to problem (3) and solutions to problem (1) with a measure μ which depends on u. Moreover if v is a *reachable* solution to problem (1) with μ a singular measure with respect to parabolic capacity, then $u=(p-1)\log(\frac{v}{p-1}+1)$ gives a solution to problem (3). Notice that if $p\neq2$ we don't know if there is a bijection between measures in Theorem 1.2 and solutions.

(2) We point out that uniqueness for problem (1) implies that there is indeed a bijection between measures in Theorem 1.2 and solutions. Nevertheless, we are only able to see uniqueness for problem (1) when the measure is absolutely continuous respect to $\text{cap}_{1,p}$–capacity using the techniques in [19] (see also [27]).

(3) There exists a unique entropy solution to problem (1) when the initial datum belongs to $L^1(\Omega)$ (see Theorem 3.15 below). So that if $\mu=0$ in Theorem 1.2, then the reachable solution we found to problem (1) it is actually the unique entropy solution. As a consequence, the associated solution to problem (3) is unique. Since this solution is the most regular solution, it follows that there exists a unique most regular solution to problem (3).

(4) Given u a solution to problem (3), consider $v=(p-1)\big(e^{\frac{u}{p-1}}-1\big)$ and then define

$$\mu=\big(\varphi(v)\big)_t-\Delta_p\, v-f(x,t)\,(1+\varphi(v)) \qquad \text{in } \mathcal{D}'(Q)\,.$$

Thus, the measure μ is uniquely determined by u. Therefore, although we only show that (8) is satisfied up to subsequences, we may state that (8), indeed, holds true.

(5) In the case $p\neq2$, our results are not as precise as those proved in [2]. Indeed, we do not know whether every solution to problem (3) has the regularity required in (6) of Theorem 1.3. Moreover, even when this is true, we cannot prove that the measure μ defined by (8) is singular with respect to the parabolic capacity (see Definition 2.2 below).

2. Notations, definitions and useful results

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 1$. We denote by Q the cylinder $\Omega \times (0, \infty)$; moreover, for $0 < t_1 < t_2$, we will denote by Q_{t_1}, Q_{t_1,t_2} the cylinders $\Omega \times (0, t_1)$, $\Omega \times (t_1, t_2)$, respectively.

The symbols $L^q(\Omega)$ denote the usual Lebesgue spaces. We define Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ as follows. For a measurable function f we set $\phi_f(k) = |\{x \in \Omega : |f(x)| > k\}|$ where $|E|$ denotes the classical Lebesgue measure of $E \subset \Omega$. We say that f is in the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ if there exists $C > 0$ such that $\phi_f(k) \leq Ck^{-q}$ for every $k > 0$. Notice that in the case where Ω is bounded, then $L^q(\Omega) \subset \mathcal{M}^q(\Omega) \subset L^{q-\varepsilon}(\Omega)$ for all $\varepsilon > 0$.

We will denote by $W_0^{1,q}(\Omega)$ the usual Sobolev space, of measurable functions having weak derivative in $L^q(\Omega)$ and zero trace on $\partial\Omega$. Moreover, we will denote by $W^{-1,q'}(\Omega)$ the dual space of $W_0^{1,q}(\Omega)$. Here q' is Hölder's conjugate exponent of $q > 1$, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$. Finally, if $1 \leq q < N$, we will denote by $q^* = Nq/(N-q)$ its Sobolev conjugate exponent.

If $T > 0$, the spaces $L^r(0,T; L^q(\Omega))$ are defined as follows:

$$L^r(0,T;L^q(\Omega)) = \Big\{u \text{ such that } \int_0^T \Big(\int_\Omega |u(x,t)|^q dx\Big)^{\frac{r}{q}} dt < \infty\Big\}.$$

It is clear that for $q, r \geq 1$, $L^r(0,T;L^q(\Omega))$ is a Banach space equipped with the norm

$$||u||_{L^r(0,T;L^q(\Omega))} = \Big(\int_0^T \Big(\int_\Omega |u(x,t)|^q dx\Big)^{\frac{r}{q}} dt\Big)^{\frac{1}{r}}.$$

In the same way we define the space $L^r(0,T;W_0^{1,q}(\Omega))$. We refer to [20] for more details.

For the sake of brevity, instead of writing "$u(x,t) \in L^r(0,\tau;W_0^{1,q}(\Omega))$ for every $\tau > 0$", we shall write $u(x,t) \in L^r_{\text{loc}}([0,\infty);W_0^{1,q}(\Omega))$.

Throughout this paper, we will use two auxiliary real functions: given $k > 0$, we define

$$T_k(s) = \begin{cases} s\,, & \text{if } |s| \leq k\,; \\ k\frac{s}{|s|}\,, & \text{if } |s| > k\,; \end{cases}$$

$$G_k(s) = s - T_k(s)\,.$$

Next we will introduce the notion of parabolic capacity. In the case $p = 2$, the parabolic capacity was defined by M. Pierre in [28]; Droniou, Porretta and Prignet in [19] generalized the definition of M. Pierre for general $p > 1$. This notion will clarify the meaning of "singular" measure.

Consider $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with the natural norm

$$||\phi||_V = ||\phi||_{W_0^{1,p}(\Omega)} + ||\phi||_{L^2},$$

then for $T > 0$, we define the Banach space $\mathbf{W}_T$ by setting

$$\mathbf{W}_T = \{u \in L^p(0,T;V),\ u_t \in L^{p'}(0,T;V')\},$$

equipped with the norm defined by

$$||u||_{\mathbf{W}_T} = ||u||_{L^p(0,T;V)} + ||u_t||_{L^{p'}(0,T;V')}.$$

We remark that $\mathbf{W}_T \subset \mathcal{C}([0,T];L^2(\Omega))$ with continuous embedding.

Definition 2.1. *If $U \subset Q_T$ is an open set, we define*

$$\mathrm{cap}_{1,p}(U) = \inf\{\|u\|_{\mathbf{W}_T} : u \in \mathbf{W}_T,\ u \geq \chi_U \text{ almost everywhere in } \quad Q_T\}$$

(we will use the convention that $\inf\emptyset = +\infty$*), then for any Borelian subset $B \subset Q_T$ the definition is extended by setting:*

$$\mathrm{cap}_{1,p}(B) = \inf\left\{\mathrm{cap}_{1,p}(U),\ U \textit{ open subset of } Q_T,\ B \subset U\right\}.$$

We refer to [19] for the main properties of this capacity. We observe that, if $B \subset Q_T \subset Q_{\tilde{T}}$, then the capacity of B is the same in Q_T and in $Q_{\tilde{T}}$, therefore we will not specify the value of T when speaking of a Borel set compactly contained in Q_T for some $T > 0$.
We recall that, given a Radon measure μ on Q and a Borel set $E \subset Q$, then μ is said to be concentrated on E if $\mu(B) = \mu(B \cap E)$ for every Borel set B.

Definition 2.2. *Let μ be a positive Radon measure in Q, we say that μ is singular if it is concentrated on a subset $E \subset Q$ such that*

$$\mathrm{cap}_{1,p}(E \cap Q_\tau) = 0, \textit{ for every } \tau > 0.$$

In the case where $E \equiv]t_1, t_2[\times B$, then $\mathrm{cap}_{1,p}(E) = 0$ if and only if $\mathrm{cap}^e_{1,p}(B) = 0$ where $\mathrm{cap}^e_{1,p}$ is the elliptic capacity defined in $W^{1,p}_0(\Omega)$. So any measure μ concentrated on the set $B \times (0,+\infty)$, where $B \subset \Omega$ is a borelian set with $\mathrm{cap}^e_{1,p}(B) = 0$, is singular with respect to the parabolic capacity $\mathrm{cap}_{1,p}$.

3. Global existence for the parabolic problem with a Radon measure

We will consider the general problem

$$(9) \qquad \begin{cases} \big(\varphi(v)\big)_t - \Delta_p v &= f(x,t)\,(1+\varphi(v)) + \mu & \text{in } \Omega \times (0,+\infty)\,, \\ v(x,t) &= 0 & \text{on } \partial\Omega \times (0,+\infty)\,, \\ v(x,0) &= v_0(x) & \text{in } \Omega\,, \end{cases}$$

where $p > 1$,

$$\varphi(s) = \left[\left(1 + \frac{|s|}{p-1}\right)^{p-1} - 1\right] \operatorname{sign} s\,,$$

$\varphi(v_0) \in L^1(\Omega)$, and for every $T > 0$, we have $f \in L^\infty(Q_T)$ and μ is a Radon measure whose total variation is finite in Q_T.

This whole Section will be devoted to giving suitable definitions of solutions and to proving Theorem 1.1.

3.1. **Weak solutions.**

Definition 3.1. *Assume that* $\mu \in L^\infty_{\mathrm{loc}}([0,+\infty) : L^\infty(\Omega))$ *and* $\varphi(v_0) \in L^\infty(\Omega)$.

We say that $v \in L^p_{\mathrm{loc}}([0,+\infty); W_0^{1,p}(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,+\infty); L^\infty(\Omega))$ *is a weak solution to* (9) *if*

(1) *The function* $\varphi(v) \in \mathcal{C}([0,+\infty); L^q(\Omega))$ *for all* $q < \infty$ *(so that the initial datum has sense) and satisfies* $(\varphi(v))_t \in L^{p'}_{\mathrm{loc}}([0,+\infty); W^{-1,p'}(\Omega))$.

(2) *For every* $\phi \in L^p_{\mathrm{loc}}([0,+\infty); W_0^{1,p}(\Omega))$ *and every* $T > 0$, *the following equality holds:*

$$\int_0^T \langle \varphi(v)_t, \phi\rangle + \int_0^T \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla \phi = \int_0^T \int_\Omega f(1+\varphi(v))\phi + \int_0^T \int_\Omega \mu\, \phi\,. \tag{10}$$

Theorem 3.2. *Assuming* $\mu \in L^\infty_{\mathrm{loc}}([0,+\infty) : L^\infty(\Omega))$ *and* $\varphi(v_0) \in L^\infty(\Omega)$, *there exists a unique weak solution to problem* (9) *in the sense of Definition 3.1.*

Proof. In [3], the authors find a function $v \in L^p_{\mathrm{loc}}([0,+\infty); W_0^{1,p}(\Omega))$ such that $\varphi(v) \in L^\infty_{\mathrm{loc}}([0,+\infty); L^{p'}(\Omega))$, $(\varphi(v))_t \in L^{p'}_{\mathrm{loc}}([0,+\infty); W^{-1,p'}(\Omega))$ and (10) holds, which satisfies the initial datum in the following sense:
For every $T > 0$ and every $\phi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap W^{1,1}(0,T; L^\infty(\Omega))$ with $\phi(T) = 0$,

$$\int_0^T \langle \varphi(v)_t, \phi\rangle + \int_0^T \int_\Omega \big(\varphi(v(t)) - \varphi(v_0)\big)\phi_t = 0\,.$$

We will check

(1) $v \in L^\infty_{\mathrm{loc}}([0,+\infty); L^\infty(\Omega))$
(2) $\varphi(v) \in \mathcal{C}([0,+\infty); L^q(\Omega))$, for all $q < \infty$
(3) The solution is unique.

(1) In order to find an L^∞–estimate, we may follow the arguments in [17]. To this end, we fix $T > 0$ and $k > \max\{1, \|v_0\|_\infty\}$, and denote $A_k = \{(x,t) \in Q_T : |v(x,t)| > k\}$. Given $t \in [0,T]$, by a standard approximate procedure, we can take $G_k(v(x,\tau))\chi_{[0,t]}(\tau)$ as test function obtaining

$$\int_\Omega H_k(v(t))\, dx - \int_\Omega H_k(v_0)\, dx + \int_0^t \int_\Omega |\nabla G_k(v)|^p = \int_0^t \int_\Omega f(1+\varphi(v))G_k(v) + \int_0^t \int_\Omega G_k(v)\mu\,,$$

where $H_k(s) = \displaystyle\int_0^s \varphi'(\sigma)G_k(\sigma)\, d\sigma$. On account of $H_k(v_0) = 0$, it yields

$$\max_{t\in[0,T]} \int_\Omega H_k(v(t))\, dx + \int_0^T \int_\Omega |\nabla G_k(v)|^p \le \int_0^T \int_\Omega \Big(|f|\big(1+|\varphi(v)|\big) + |\mu|\Big)|G_k(v)|\,. \tag{11}$$

Considering separately the cases $p \ge 2$ and $1 < p < 2$, we may perform some manipulations in (11) and prove, as in [17], that for every $\epsilon > 0$ there exists a positive constant $C(\epsilon)$ such that

$$\max_{t\in[0,T]} \int_\Omega w_k(x,t)^p\, dx + \int_0^T \int_\Omega |\nabla w_k|^p \le C(\epsilon) \int_{A_k} \Big(|f|\, |w_k|^{p+\epsilon} + k^p\Big)\,,$$

where w_k is defined by

$$w_k = \begin{cases} |G_k(v)|\,, & \text{if } p \geq 2\,; \\ \varphi'(v)^{1/p} G_k(v)^{2/p}\,, & \text{if } 1 < p < 2\,. \end{cases}$$

Now one follows the argument in [17] and deduces the required L^∞–estimate.

(2) The solution obtained in [3] is a limit of functions which are piecewise constant in the variable t. Once this solution is obtained, consider the bounded function $g = f(1 + \varphi(v)) + \mu$ and, for a given $h > 0$, approximate this g by a function g_h which is constant in t in each interval $((k-1)h, kh)$ and here take the value g_h^k. Then we define v_h as a step function in time which in the time interval $((k-1)h, kh)$ takes the value v_h^k obtained as the solution of the following elliptic equation:

$$\frac{\varphi(v_h^k) - \varphi(v_h^{k-1})}{h} - \Delta_p v_h(t) = g_h^{k-1}\,,$$

where $v_h^0 = v_0$, the initial datum.

In [6], Section 7, the following estimate is proved

$$\|\varphi(v_1) - \varphi(v_2)\|_1 \leq \|f_1 - f_2\|_1\,,$$

where v_i is a solution of $\varphi(v_i) - \lambda\,\mathrm{div}\,(\nabla v_i|^{p-2}\nabla v_i) = f_i$, with $\lambda > 0$. Now following the proof of Crandall–Liggett's Theorem (see [12]), we have that those steps functions v_h uniformly converge to v and the limit satisfies $\varphi(v) \in \mathcal{C}([0,\infty); L^1(\Omega))$. Since $\varphi(v) \in L^\infty_{\mathrm{loc}}((0,\infty); L^\infty(\Omega)) \cap \mathcal{C}([0,\infty); L^1(\Omega))$, it easily follows that $\varphi(v) \in \mathcal{C}([0,+\infty); L^q(\Omega))$ for all finite q.

(3) Assume that v is a solution with initial datum v_0 and w is a solution with initial datum w_0. Fix $t > 0$ and $k > 0$. Take $\frac{1}{k}T_k(\varphi(v) - \varphi(w))$ as test function in the weak formulation of v, it yields

$$\frac{1}{k}\int_0^t \int_\Omega \langle \varphi(v)_t, T_k(\varphi(v) - \varphi(w))\rangle + \frac{1}{k}\int_0^t \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla T_k(\varphi(v) - \varphi(w))$$
$$= \frac{1}{k}\int_0^t \int_\Omega (f(1 + \varphi(v)) + \mu)\, T_k(\varphi(v) - \varphi(w))\,.$$

Analogously, we obtain

$$\frac{1}{k}\int_0^t \int_\Omega \langle \varphi(w)_t, T_k(\varphi(w) - \varphi(v))\rangle + \frac{1}{k}\int_0^t \int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla T_k(\varphi(w) - \varphi(v))$$
$$= \frac{1}{k}\int_0^t \int_\Omega (f(1 + \varphi(v)) + \mu)\, T_k(\varphi(w) - \varphi(v))\,.$$

We add both expressions to get

$$\text{(12)}\quad \frac{1}{k}\int_0^t\int_\Omega \langle \big(\varphi(v)-\varphi(w)\big)_t, T_k(\varphi(v)-\varphi(w))\rangle + \frac{1}{k}\int_0^t\int_\Omega \big(|\nabla v|^{p-2}\nabla v - |\nabla w|^{p-2}\nabla w\big)\cdot \nabla T_k(\varphi(v)-\varphi(w)) = \frac{1}{k}\int_0^t\int_\Omega f(\varphi(v)) - \varphi(w))\, T_k(\varphi(v)-\varphi(w))\,.$$

In order to analyze the second term, we write it depending on $\varphi(v)$ and $\varphi(w)$, obtaining

$$\big(|\nabla v|^{p-2}\nabla v - |\nabla w|^{p-2}\nabla w\big)\cdot \nabla T_k(\varphi(v)-\varphi(w)) = \Big(\frac{|\nabla\varphi(v)|^{p-2}}{(1+|\varphi(v)|)^{p-2}}\nabla\varphi(v) - \frac{|\nabla\varphi(w)|^{p-2}}{(1+|\varphi(w)|)^{p-2}}\nabla\varphi(w)\Big)\cdot\nabla T_k(\varphi(v)-\varphi(w))\cdot\nabla T_k(\varphi(v)-\varphi(w)) \geq -\Big|\frac{1}{(1+|\varphi(v)|)^{p-2}} - \frac{1}{(1+|\varphi(w)|)^{p-2}}\Big|\,|\nabla\varphi(w)|^{p-1}\,|\nabla T_k(\varphi(v)-\varphi(w))|\,.$$

Observe that the function $s\mapsto \dfrac{1}{(1+|s|)^{p-2}}$ is Lipschitz–continuous, since its derivative is bounded by $|p-2|$, so that we deduce that

$$\Big|\frac{1}{(1+|\varphi(v)|)^{p-2}} - \frac{1}{(1+|\varphi(w)|)^{p-2}}\Big|\,|\nabla\varphi(w)|^{p-1}\,|\nabla T_k(\varphi(v)-\varphi(w))| \leq |p-2|\,\big|\varphi(v)-\varphi(w)\big|\big|\nabla\varphi(w)|^{p-1}\big|\nabla T_k(\varphi(v)-\varphi(w))\big| \leq k\,|p-2|\,|\nabla\varphi(w)|^{p-1}\big|\nabla T_k(\varphi(v)-\varphi(w))\big|\,,$$

holds in the set $\{|\varphi(v)-\varphi(w)|<k\}$. Hence the elliptic term in (12) may be estimated as follows

$$-\frac{1}{k}\int_0^t\int_\Omega \big(|\nabla v|^{p-2}\nabla v - |\nabla w|^{p-2}\nabla w\big)\cdot\nabla T_k(\varphi(v)-\varphi(w)) \leq |p-2|\int_{\{|\varphi(v)-\varphi(w)|<k\}} |\nabla\varphi(w)|^{p-1}\big|\nabla(\varphi(v)-\varphi(w))\big| \leq |p-2|\Big(\int_{\{|\varphi(v)-\varphi(w)|<k\}}|\nabla\varphi(w)|^p\Big)^{1/p'}\Big(\int_{\{|\varphi(v)-\varphi(w)|<k\}}\big|\nabla(\varphi(v)-\varphi(w))|^p\Big)^{1/p}.$$

Observe that the right hand side in the above equation tends to 0 when k goes to 0: we will denote it by writing $\omega(k)$. Therefore, equation (12) becomes

$$\frac{1}{k}\int_0^t\int_\Omega\langle\varphi(v)_t-\varphi(w)_t, T_k(\varphi(v)-\varphi(w))\rangle \leq \frac{1}{k}\int_0^t\int_\Omega |f|(\varphi(v))-\varphi(w))T_k(\varphi(w)-\varphi(v))+\omega(k) \leq \int_0^t\int_\Omega |f|\,|\varphi(v))-\varphi(w)|+\omega(k)\,.$$

Hence, denoting $J_k(s)=\int_0^s T_k(\sigma)\,d\sigma$, it follows that

$$\frac{1}{k}\int_\Omega J_k(\varphi(v(t))-\varphi(w(t))) - \frac{1}{k}\int_\Omega J_k(\varphi(v_0)-\varphi(w_0)) \leq \int_0^t\int_\Omega |f|\,|\varphi(w))-\varphi(v)|+\omega(k)\,,$$

Letting k goes to 0, the following inequality holds:

$$\int_\Omega |\varphi(v(t)) - \varphi(w(t))| \le \int_\Omega |\varphi(v_0) - \varphi(w_0)| + \|f\|_\infty \int_0^t \int_\Omega |\varphi(w) - \varphi(v)| \,.$$

Finally, applying Gronwall's lemma, we obtain

$$\int_\Omega |\varphi(v(t)) - \varphi(w(t))| \le e^{t\|f\|_\infty} \int_\Omega |\varphi(v_0) - \varphi(w_0)| \;.$$

Therefore, $v_0 = w_0$ implies $\varphi(v) = \varphi(w)$ and, since φ is increasing, $v = w$. ■

3.2. **Reachable solutions.** We now introduce the notion of reachable solution for parabolic equations with measure data. For elliptic equations this notion was introduce in [18] to deal with solutions which can be reached by approximation (see also [13] for a similar concept).

Definition 3.3. *Assume that μ is a Radon measure whose total variation is finite in each Q_T and $\varphi(v_0) \in L^1(\Omega)$.*

We say that v is a reachable solution to (9) if

(1) $T_k(v) \in L^p_{\mathrm{loc}}([0,+\infty); W_0^{1,p}(\Omega))$ for all $k > 0$.
(2) $f(1 + \varphi(v)) \in L^1_{\mathrm{loc}}([0,+\infty); L^1(\Omega))$.
(3) For all $t > 0$ there exist both one-side limits $\lim_{\tau\to t^\pm} \varphi(v(\cdot,\tau))$ weakly- in the sense of measures.*
(4) $\varphi(v(\cdot,t)) \to \varphi(v_0(\cdot))$ weakly- in the sense of measures as $t \to 0$.*
(5) There exist three sequences $\{v_n\}_n$ in $L^p_{\mathrm{loc}}([0,+\infty); W_0^{1,p}(\Omega))$, $\{h_n\}_n$ in $L^\infty_{\mathrm{loc}}([0,+\infty); L^\infty(\Omega))$ and $\{g_n\}_n$ in $L^\infty(\Omega)$ such that each v_n is a weak solution to problem

$$\left\{\begin{array}{rcll} \big(\varphi(v_n)\big)_t - \Delta_p v_n &=& f(x,t)\,(1 + \varphi(v_n)) + h_n & \text{in } \Omega \times (0,+\infty)\,, \\ v_n(x,t) &=& 0 & \text{on } \partial\Omega \times (0,+\infty)\,, \\ v_n(x,0) &=& \varphi^{-1}\big(g_n(x)\big) & \text{in } \Omega\,, \end{array}\right. \tag{13}$$

and satisfying

(a) $g_n \to \varphi(v_0)$ in $L^1(\Omega)$.
(b) $h_n \overset{*}{\rightharpoonup} \mu$ as measures.
(c) $|\nabla v_n|^{p-2}\nabla v_n \to |\nabla v|^{p-2}\nabla v$ strongly in $L^\sigma_{\mathrm{loc}}((0,+\infty); L^\sigma(\Omega))$ for $1 \le \sigma < \dfrac{N+p}{N+p-1}$.
(d) The sequence $\{\varphi(v_n)\}$ is bounded in $L^\infty_{\mathrm{loc}}([0,+\infty); L^1(\Omega))$ and $\varphi(v_n) \to \varphi(v)$ strongly in $L^q_{\mathrm{loc}}([0,+\infty); W_0^{1,q}(\Omega))$ for all $1 \le q < \dfrac{N+p}{N+1}$.

Remarks 3.4.

(1) As a consequence of (5) (d), we obtain $\varphi(v) \in L^\infty_{\mathrm{loc}}([0,+\infty); L^1(\Omega))$.
(2) The estimates stated in Proposition 3.6 bellow imply that every reachable solution v satisfies
  (a) $|v|^\beta \in L^p(0,T; W_0^{1,p}(\Omega))$ for all $0 < \beta < (p-1)/p$
  (b) $\varphi(v) \in L^\sigma_{\mathrm{loc}}((0,\infty); L^\sigma(\Omega))$ for all $1 \le \sigma < \frac{N+p}{N}$

(c) $\int_{Q_T} \frac{|\nabla v|^p}{(1+|v|)^{\alpha+1}} < \infty$ for all $\alpha > 0$

(3) Since each v_n is a solution to (13) in the sense of distributions, letting n go to ∞, it follows that a reachable solution to (9) is always a distributional solution.

(4) Furthermore, (3) and (4) in the above definition allow us to give sense to good representatives of the solutions (that is, solutions can be defined for fixed $t > 0$). As a consequence, different weak formulations can be stated (see Corollary 3.11 below).

We will now prove the existence of a reachable solution to problem (1).

3.3. **A priori estimates.** Let us consider the following approximating problems:

$$\left\{ \begin{array}{rcll} \varphi(v_n)_t - \Delta_p v_n &=& f\,(1+\varphi(v_n)) + h_n, & (x,t) \in Q, \\ v_n(x,t) &=& 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ v_n(x,0) &=& \varphi^{-1}(g_n(x)), & x \in \Omega, \end{array} \right. \tag{14}$$

where $g_n \to \varphi(v_0)$ strongly in $L^1(\Omega)$ and $h_n \to \mu$ in the weak-$*$ sense in Q. The existence of weak solutions to these problems follows from Theorem 3.2.

The following Lemma can easily be proved by approximation.

Lemma 3.5. *Let $u \in L^p(0,T;W_0^{1,p}(\Omega))$ satisfy $\varphi(u)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, where φ is defined by (2). Assume that $\phi(s) : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz–continuous function such that $\phi(0) = 0$. Then, if we define*

$$\Phi(s) = \int_0^s \varphi'(\sigma)\,\phi(\sigma)\,d\sigma\,, \tag{15}$$

the following integration by parts formula holds:

$$\int_{t_1}^{t_2} \langle \varphi(u)_t, \phi(u) \rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,dt = \int_\Omega \Phi(u(x,t_2))\,dx - \int_\Omega \Phi(u(x,t_1))\,dx\,, \tag{16}$$

for every $0 \le t_1 < t_2$.

Proposition 3.6. *Let $\{v_n\}$ be a sequence of solutions of the approximate problems* (14) *and let $T > 0$. Then, for each $0 < \beta < (p-1)/p$, the sequence $\{|v_n|^\beta\}_n$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, and the sequence $\{\varphi(v_n)\}_n$ is bounded in $L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;W^{1,q}(\Omega)) \cap L^\sigma(Q_T)$ for all $1 \le q < \frac{N+p}{N+1}$ and for all $1 \le \sigma < \frac{N+p}{N}$.*

Moreover,

$$\int_{Q_T} \frac{|\nabla v_n|^p}{(1+|v_n|)^{\alpha+1}} \le C \qquad \text{for all } \alpha > 0\,. \tag{17}$$

Furthermore, the sequence $\{|\nabla v_n|\}_n$ is bounded in $M^q(Q_T)$ for all $q = \frac{(N+p)(p-1)}{N+p-1}$ and $\{v_n\}_n$ is bounded in $M^\sigma(Q_T)$ for all $\sigma = \frac{(N+p)(p-1)}{N}$.

Proof. We will begin by proving several basic estimates. Consider $\phi_k(v_n) = T_k(v_n)\chi_{(0,t)}$, with $k > 0$ and $0 < t \le T$, as test function in the weak formulation of (14). Then, using Lemma

3.5, we have

$$\int_\Omega \Phi_k(v_n(x,t))\,dx - \int_\Omega \Phi_k(\varphi^{-1}(g_n(x)))\,dx + \int_{Q_t} |\nabla T_k(v_n)|^p \leq k \int_{Q_t} |f|(1+|\varphi(v_n)|) + k|\mu|(Q_t)\,, \tag{18}$$

where $\Phi_k(s) = \int_0^s T_k(\sigma)\,\varphi'(\sigma)\,d\sigma$ (note that this defines an even, nonnegative function). Since $\Phi_k(s) \leq k|\varphi(s)|$, it follows from (18) that

$$\int_\Omega |\Phi_k(v_n(x,t))|\,dx + \int_{Q_t} |\nabla T_k(v_n)|^p \leq ck \int_{Q_t} (1+|\varphi(v_n)|) + ck\,. \tag{19}$$

Now, dropping a nonnegative term, dividing by k and letting k goes to 0, it yields

$$\int_\Omega |\varphi(v_n(x,t))|\,dx \leq c \int_{Q_t} |\varphi(v_n)| + c\,.$$

Thus, Gronwall's Lemma implies that

$$\sup_{t\in[0,T]} \int_\Omega |\varphi(v_n(x,t))|\,dx \leq C\,. \tag{20}$$

Moreover, going back to (19) we get

$$\int_{Q_T} |\nabla T_k(v_n)|^p \leq Ck\,. \tag{21}$$

In order to prove the next estimate, we define $\phi(s) = \big(1-(1+|s|)^{-\alpha}\big)$ sign s, with $0<\alpha$, and we take $\phi(v_n)\chi_{(0,T)}$ as test function in the approximating problems. Then we reach the inequality

$$\int_\Omega \Phi(v_n(T))dx - \int_\Omega \Phi(\varphi^{-1}(g_n))dx + \alpha \int_{Q_T} \frac{|\nabla v_n|^p}{(1+|v_n|)^{\alpha+1}} \leq \int_{Q_T} |f|(1+|\varphi(v_n)|) + ||\mu||\,, \tag{22}$$

where Φ is defined by (15). Hence, disregarding a nonnegative term and having (20) in mind, (17) follows.

Observe that if $0<\alpha<p-1$ and we define auxiliary functions by $1+w_n = (1+\frac{|v_n|}{p-1})^{\frac{p-1-\alpha}{p}}$, then (17) becomes

$$\int_{Q_T} |\nabla w_n|^p \leq C \quad \text{for all } n\in\mathbb{N}\,,$$

and so the sequence $\{|v_n|^\beta\}_n$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for all $0<\beta<\frac{p-1}{p}$.

Next, we are going to estimate the sequence of approximate solutions in Marcinkiewicz spaces following the arguments in [4], but we remark that the procedure introduced in [11] could also be applied when $p > 2-\frac{1}{N+1}$. Observe that one obtains from (20) and the definition of φ that there exists a positive constant C such that

$$|\{x\in\Omega \,:\, |v_n(x,t)|>k\}| \leq \frac{C}{k^{p-1}} \quad \text{for almost all } t\in[0,T]\,,\ \text{all } k>0 \text{ and all } n\in\mathbb{N}\,. \tag{23}$$

On the other hand, by Sobolev's inequality and (21),

$$(24) \quad \int_0^T \big(|\{x\in\Omega \,:\, |v_n(x,t)|\ge k\}|\big)^{p/p^*}\,dt \le \int_0^T \left(\frac{\|T_k(v_n(x,t))\|_{p^*}^{p^*}}{k^{p^*}}\right)^{p/p^*} dt$$

$$\le C\int_0^T \frac{\|\nabla T_k(v_n(x,t))\|_p^p}{k^p}\,dt \le \frac{C}{k^{p-1}} \qquad \text{for all } k>0 \text{ and all } n\in\mathbb{N}\,.$$

Therefore, (23) and (24) imply the following estimate

$$(25) \quad |\{(x,t)\in Q_T \,:\, |v_n(x,t)|\ge k\}|$$

$$= \int_0^T \big(|\{x\in\Omega \,:\, |v_n(x,t)|\ge k\}|\big)^{1-(p/p^*)}\big(|\{x\in\Omega \,:\, |v_n(x,t)|\ge k\}|\big)^{p/p^*}\,dt$$

$$\le \left(\frac{C}{k^{p-1}}\right)^{p/N}\left(\frac{C}{k^{p-1}}\right) = \frac{C}{k^{\frac{(p-1)(N+p)}{N}}} \qquad \text{for all } k>0 \text{ and all } n\in\mathbb{N}\,.$$

A similar estimate for the gradients is now easy to obtain. Indeed, for every $h,k>0$, we have

$$|\{(x,t)\in Q_T \,:\, |\nabla v_n(x,t)|\ge h\}|$$

$$\le |\{(x,t)\in Q_T \,:\, |v_n(x,t)|\ge k\}| + |\{(x,t)\in Q_T \,:\, |\nabla T_k(v_n(x,t))|\ge h\}|$$

$$\le \frac{C}{k^{\frac{(p-1)(N+p)}{N}}} + \int_{Q_T}\frac{|\nabla T_k(v_n)|^p}{h^p} \le \frac{C}{k^{\frac{(p-1)(N+p)}{N}}} + \frac{Ck}{h^p}\,,$$

and taking $k=h^{N/(N+p-1)}$ it yields

$$(26) \quad |\{(x,t)\in Q_T \,:\, |\nabla v_n(x,t)|\ge h\}| \le \frac{C}{h^{(p-1)(N+p)/(N+p-1)}} \qquad \text{for all } h>0 \text{ and all } n\in\mathbb{N}\,.$$

It follows from (25) and (26) that the sequences of approximate solutions and their gradients are bounded in some Lebesgue spaces:

$$(27) \qquad \int_{Q_T}|v_n|^\rho \le C \quad \text{for all } 0<\rho<\frac{(N+p)(p-1)}{N}$$

$$(28) \qquad \int_{Q_T}|\nabla v_n|^r \le C \quad \text{for all } 0<r<\frac{(N+p)(p-1)}{N+p-1}$$

When $p \geq 2$, it leads to an estimate of the sequence $\{\varphi(v_n)\}_n$ in $L^q(0,T;W^{1,q}(\Omega))$ for all $1 \leq q < \frac{N+p}{N+1}$; indeed, by Hölder's inequality,

$$\int_{Q_T} |\nabla\varphi(v_n)|^q = \int_{Q_T} \Big(1 + \frac{|v_n|}{p-1}\Big)^{q(p-2)} |\nabla v_n|^q$$
$$\leq \Big(\int_{Q_T} |\nabla v_n|^r\Big)^{q/r} \Big(\int_{Q_T} \big(1 + \frac{|v_n|}{p-1}\big)^{rq(p-2)/(r-q)}\Big)^{(r-q)/r},$$

and it follows from (27) and (28) that we have an estimate whenever $\dfrac{rq(p-2)}{r-q} < \dfrac{(N+p)(p-1)}{N}$ and $q < r < \dfrac{(N+p)(p-1)}{N+p-1}$; that is, when $q < \dfrac{N+p}{N+1}$. If $1 < p < 2$, then this argument does not work and we need a different approach. Let λ be a fixed positive number to be determined. Then

$$\int_{Q_T} |\nabla\varphi(v_n)|^q = \int_{Q_T} \frac{|\nabla v_n|^q}{\Big(1 + \frac{|v_n|}{p-1}\Big)^{q(2-p)}}$$
$$= \int_{Q_T} \frac{|\nabla v_n|^q}{\Big(1 + \frac{|v_n|}{p-1}\Big)^{q(2-p)+\lambda}} \Big(1 + \frac{|v_n|}{p-1}\Big)^{\lambda}$$
$$\leq \Big(\int_{Q_T} \frac{|\nabla v_n|^p}{\Big(1 + \frac{|v_n|}{p-1}\Big)^{p(2-p)+\frac{p\lambda}{q}}}\Big)^{q/p} \Big(\int_{Q_T} \Big(1 + \frac{|v_n|}{p-1}\Big)^{\frac{\lambda p}{p-q}}\Big)^{(p-q)/p}.$$

By (17) and (27), the right-hand side is bounded if we check

$$\begin{cases} p(2-p) + \frac{p\lambda}{q} > 1 \\ \frac{p\lambda}{p-q} < \frac{(N+p)(p-1)}{N} \end{cases}$$

Taking $\lambda = \dfrac{(p-1)^2(N+p)}{p(N+1)}$, the above inequalities hold if and only if $q < \dfrac{N+p}{N+1}$. So that in any case the sequence $\{\varphi(v_n)\}_n$ is bounded in $L^q(0,T;W^{1,q}(\Omega))$ for all $1 \leq q < \frac{N+p}{N+1}$. ■

We remark that it also follows from (28) that the sequence $\{|\nabla v_n|^{p-1}\}_n$ is bounded in $L^\sigma(Q_T)$ for all $1 \leq \sigma < \frac{N+p}{N+p-1}$. The equations (14) then give us an estimate of $\{\big(\varphi(v_n)\big)_t\}$ in $L^1(Q_T) + L^\sigma(0,T;W^{-1,\sigma}(\Omega))$ for some $\sigma > 1$. Since $\{\varphi(v_n)\}$ is bounded in $L^q(0,T;W^{1,q}(\Omega))$ for some $q > 1$, applying a result by Simon (see [32]), we obtain that the sequence $\{\varphi(v_n)\}_n$ is compact in $L^1(Q_T)$ and so we may extract a subsequence (also labelled by n) such that $\varphi(v_n)$ converges pointwise to a measurable function. Therefore, there exists a measurable function v_T such that $v_n \to v_T$ a.e. in Q_T. Applying a diagonal argument we obtain a measurable function v such that

$$v_n \to v \qquad \text{a.e. in } Q\,. \tag{29}$$

Moreover, the pointwise convergence and estimate (21) implies that the same subsequence satisfies

$$T_k(v_n) \rightharpoonup T_k(v) \qquad \text{weakly in } L^p_{\text{loc}}(0,\infty;W_0^{1,p}(\Omega))\,, \tag{30}$$

for all $k>0$.

Finally, we also point out that (27) implies an estimate of the sequence $\{\varphi(v_n)\}_n$ in $L^\sigma_{\text{loc}}((0,+\infty);L^\sigma(\Omega))$ for all $\sigma<\frac{N+p}{N}$. So that, by (29), it satisfies

$$\varphi(v_n)\to\varphi(v) \qquad \text{strongly in } L^\sigma_{\text{loc}}((0,+\infty);L^\sigma(\Omega)) \tag{31}$$

for all $1\le\sigma<\frac{N+p}{N}$. In particular, we deduce that

$$f(1+\varphi(v_n))\to f(1+\varphi(v)) \qquad \text{strongly in } L^1_{\text{loc}}([0,+\infty);L^1(\Omega))\,. \tag{32}$$

3.4. **Pointwise convergence of the gradients.** In this subsection we are going to see that the sequence of gradients $\{\nabla v_n\}_n$ converges to ∇v pointwise in Q. As a consequence we will obtain that the sequence $\{v_n\}_n$ satisfies condition (5) (c) in Definition 3.3.

Proposition 3.7. *Let $\{v_n\}_n$ be a sequence of solutions of the approximate problems* (14). *Assume that this sequence converges pointwise to a measurable function v and that* (30) *holds. Let $T>0$ and $k'>0$. Then, up to subsequences,*

$$\nabla T_{k'}(v_n)\to\nabla T_{k'}(v) \qquad \textit{almost everywhere in } Q_T\,. \tag{33}$$

As a consequence of diagonal arguments, we also obtain

$$\nabla T_{k'}(v_n)\to\nabla T_{k'}(v) \qquad \textit{almost everywhere in } Q \;\; \forall k>0\,, \tag{34}$$

$$\nabla v_n\to\nabla v \qquad \textit{almost everywhere in } Q\,. \tag{35}$$

Proof. To begin with, fix $k'>0$ and $T>0$, and denote $k=\varphi(k')$, $w_n=\varphi(v_n)$ and $w=\varphi(v)$. By (29), we already know that

$$w_n\to w \qquad \text{a.e. in } Q_T\,.$$

We point out that $\left(1+\frac{|v_n|}{p-1}\right)^{2-p}\nabla T_k w_n=\nabla T_{k'}v_n$ and $\left(1+\frac{|v_n|}{p-1}\right)^{2-p}\neq 0$, so that the convergence almost everywhere of $\nabla T_k w_n$ in Q_T implies the same convergence of $\nabla T_{k'}v_n$. Thus, our aim will be to prove that the sequence $\{\nabla T_k w_n\}_n$ converges to $\nabla T_k w$ pointwise in Q_T. To this end, some preliminaries are necessary. We have to introduce the time–regularization of functions due to Landes (see [24], [25]). For every $\nu\in\mathbb{N}$, we define $(T_k w)_\nu$ as the solution of the Cauchy problem

$$\begin{cases} \dfrac{1}{\nu}[(T_k w)_\nu]_t+(T_k w)_\nu=T_k w;\\[2ex] (T_k w)_\nu(0)=T_k(g_\nu). \end{cases}$$

Then, using the assumptions on the approximations of the initial datum, one has:

$$(T_k w)_\nu\in L^p(0,T;W_0^{1,p}(\Omega)) \qquad ((T_k w)_\nu)_t\in L^p(0,T;W_0^{1,p}(\Omega))\,,$$
$$\|(T_k w)_\nu\|_{L^\infty(Q_T)}\le\|T_k w\|_{L^\infty(Q_T)}\le k,$$

and as ν goes to infinity

$$(T_k w)_\nu\to T_k w \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)).$$

From now on, we will denote by $\omega(n,\nu,\varepsilon)$ any quantity such that

$$\lim_{\varepsilon\to0^+}\limsup_{\nu\to\infty}\limsup_{n\to\infty}|\omega(n,\nu,\varepsilon)|=0\,.$$

If the quantities we are dealing with do not depend on some parameters, we will omit them. So $\omega(n,\nu)$ will denote a quantity which goes to zero as first n and then ν go to infinity, and $\omega(\nu)$ will denote a quantity which goes to zero as ν goes to infinity. On the other hand, $\omega^{\nu,\varepsilon}(n)$ will denote a quantity which goes to zero as n goes to infinity, for every fixed ε and ν.

Moreover, observe that

$$\varphi^{-1}(s)=(p-1)\left[\Big(1+|s|\Big)^{\frac{1}{p-1}}-1\right]\ \text{sign}\ s\,,$$

and so

$$\big(\varphi^{-1}\big)'(s)=\big(1+|s|\big)^{\frac{2-p}{p-1}}\,.$$

As a consequence, the problem satisfied by w_n is the following

$$\text{(36)}\qquad\begin{cases}(w_n)_t-\text{div}\left(\big(1+|w_n|\big)^{2-p}|\nabla w_n|^{p-2}\nabla w_n\right)=f\,(1+w_n)+h_n,\ (x,t)\in Q_T,\\ w_n(x,t)=0,\ (x,t)\in\partial\Omega\times(0,T),\\ w_n(x,0)=g_n(x),\ x\in\Omega,\end{cases}$$

with $g_n\to\varphi(v_0)$ strongly in $L^1(\Omega)$ and $h_n\to\mu$ in the weak-$*$ sense in Q_T.

We are now able to begin with the proof, the first step consists in proving the estimate (40) below. Taking $T_\varepsilon\big(w_n-(T_kw)_\nu\big)$ as test function in (36), we obtain

$$\boxed{\text{A}}\qquad\int_0^T\langle(w_n)_t\,,\,T_\varepsilon\big(w_n-(T_kw)_\nu\big)\rangle\,dt$$

$$\boxed{\text{B}}\qquad+\iint_{Q_T}\big(1+|w_n|\big)^{2-p}\,|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla T_\varepsilon\big(w_n-(T_kw)_\nu\big)$$

$$=\iint_{Q_T}f\,(1+w_n)\,T_\varepsilon\big(w_n-(T_kw)_\nu\big)+\iint_{Q_T}h_n\,T_\varepsilon\big(w_n-(T_kw)_\nu\big)\,.\qquad\boxed{\text{C}}$$

We will analyze the integrals which appear in the previous equality.

Integral $\boxed{\text{A}}$: We will see that

$$\text{(37)}\qquad\boxed{\text{A}}\geq\omega(n,\nu,\varepsilon)\,.$$

Indeed,

$$
\begin{aligned}
&\int_0^T \langle (w_n)_t\,,\, T_\varepsilon(w_n-(T_kw)_\nu)\rangle\,dt\\
&=\int_0^T \langle (w_n-(T_kw)_\nu)_t\,,\, T_\varepsilon(w_n-(T_kw)_\nu)\rangle\,dt+\iint_{Q_T}((T_kw)_\nu)_t\,T_\varepsilon(w_n-(T_kw)_\nu)\\
&=\int_\Omega J_\varepsilon(w_n(T)-(T_kw)_\nu(T))\,dx-\int_\Omega J_\varepsilon(g_n-T_kg_\nu)\,dx\\
&\qquad\qquad\qquad\qquad\qquad+\nu\iint_{Q_T}(T_kw-(T_kw)_\nu)\,T_\varepsilon(w_n-(T_kw)_\nu)\\
&=\boxed{\text{A1}}+\boxed{\text{A2}}+\boxed{\text{A3}},
\end{aligned}
$$

where

$$J_\varepsilon(s)=\int_0^s T_\varepsilon(\sigma)\,d\sigma\,.$$

Note that

$$0\le J_\varepsilon(s)\le\varepsilon s\,.$$

Therefore the first integral $\boxed{\text{A1}}$ is nonnegative. On the other hand, as far as the integral $\boxed{\text{A2}}$ is concerned, we can write

$$
\begin{aligned}
0&\le\int_\Omega J_\varepsilon(g_n-T_kg_\nu)\,dx\\
&=\int_\Omega J_\varepsilon(\varphi(v_0)-T_kg_\nu)\,dx+\omega^{\nu,\varepsilon}(n)\\
&=\int_\Omega J_\varepsilon(\varphi(v_0)-T_k\varphi(v_0))\,dx+\omega^{\varepsilon}(n,\nu)\\
&\le\varepsilon\int_\Omega \varphi(v_0)\,dx+\omega^{\varepsilon}(n,\nu)\\
&=\omega(n,\nu,\varepsilon)\,.
\end{aligned}
$$

Finally

$$\boxed{\text{A3}}=\nu\iint_{Q_T}(T_kw-(T_kw)_\nu)\,T_\varepsilon(w-(T_kw)_\nu)+\omega^{\nu,\varepsilon}(n)\ge\omega^{\nu,\varepsilon}(n)\,.$$

Indeed, the function in the last integral is always nonnegative (this is obvious on the set where $|w|\le k$, while on the remaining set this follows from the inequality $|(T_kw)_\nu|\le k$). Therefore (37) is proved.

Integral $\boxed{\text{B}}$: In this step, we will prove that

$$\text{(38)}\quad \boxed{\text{B}}\ge\iint_{\{|w_n|\le k\}}\big(1+|w_n|\big)^{2-p}|\nabla T_k(w_n)|^{p-2}\nabla T_k(w_n)\cdot\nabla T_\varepsilon\big(T_k(w_n)-(T_kw)_\nu\big)+\omega^\varepsilon(n,\nu)\,.$$

Indeed,

$$
\begin{aligned}
\boxed{\mathrm{B}} &= \iint_{\{|w_n|\le k\}} \big(1+|w_n|\big)^{2-p}|\nabla T_k(w_n)|^{p-2}\nabla T_k(w_n)\cdot\nabla T_\varepsilon\big(T_k(w_n)-(T_kw)_\nu\big)\\
&\quad+\iint_{\{|w_n|>k\}} \big(1+|w_n|\big)^{2-p}|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla T_\varepsilon\big(w_n-(T_kw)_\nu\big)\\
&\ge \iint_{\{|w_n|\le k\}} \big(1+|w_n|\big)^{2-p}|\nabla T_k(w_n)|^{p-2}\nabla T_k(w_n)\cdot\nabla T_\varepsilon\big(T_k(w_n)-(T_kw)_\nu\big)\\
&\quad-\iint_{\{|w_n|>k\,,\ |w_n-(T_kw)_\nu|\le\varepsilon\}} \big(1+|w_n|\big)^{2-p}|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla (T_kw)_\nu\,.
\end{aligned}
$$

Since $\|(T_kw)_\nu\|_\infty\le k$, the last integrand is different from zero only on the set where $|w_n|\le k+\varepsilon$, therefore the last integrand is bounded by

$$
\begin{aligned}
&c_1(k,p)\left[\iint_Q|\nabla T_{k+\varepsilon}w_n|^p\right]^{\frac{p-1}{p}}\left[\iint_Q|\nabla(T_kw)_\nu|^p\chi_{\{|w_n|>k\}}\right]^{\frac1p}\\
&\quad\le c_2(k,p,\varepsilon)\left[\iint_Q|\nabla(T_kw)_\nu|^p\chi_{\{|w_n|>k\}}\right]^{\frac1p}\\
&\quad= c_2(k,p,\varepsilon)\left[\iint_Q|\nabla(T_kw)_\nu|^p\chi_{\{|w|>k\}}\right]^{\frac1p}+\omega^{\nu,\varepsilon}(n)\\
&\quad= c_2(k,p,\varepsilon)\left[\iint_Q|\nabla T_kw|^p\chi_{\{|w|>k\}}\right]^{\frac1p}+\omega^{\varepsilon}(n,\nu)=\omega^{\varepsilon}(n,\nu)\,.
\end{aligned}
$$

Here we have used the a.e. convergence of $\chi_{\{|w_n|>k\}}$ to $\chi_{\{|w|>k\}}$ as $n\to+\infty$, which holds for all k except a countable set (see for instance [10]).
Integrals $\boxed{\mathrm{C}}$: These terms are easy to estimate, since $f\in L^\infty(Q_T)$, and both sequences $\{1+w_n\}_n$ and $\{h_n\}_n$ are bounded in $L^1(Q_T)$. Hence,

$$
\boxed{\mathrm{C}}\le c\,\varepsilon\,, \tag{39}
$$

for some positive constant c only depending on the parameters of our problem.

Therefore, our analysis of the terms $\boxed{\mathrm{A}}$, $\boxed{\mathrm{B}}$ and $\boxed{\mathrm{C}}$ yields the following estimate

$$
\begin{aligned}
&\iint_{Q_T}\big(1+|T_k(w_n)|\big)^{2-p}|\nabla T_k(w_n)|^{p-2}\nabla T_k(w_n)\cdot\nabla T_\varepsilon\big(T_kw_n-(T_kw)_\nu\big)\\
&\quad=\iint_{\{|w_n|\le k\}}\big(1+|w_n|\big)^{2-p}|\nabla T_k(w_n)|^{p-2}\nabla T_k(w_n)\cdot\nabla T_\varepsilon\big(w_n-(T_kw)_\nu\big)\le\omega(n,\nu,\varepsilon)\,.
\end{aligned} \tag{40}
$$

We now define the function

$$
a_k(s,\xi)=\begin{cases}(1+|s|)^{2-p}|\xi|^{p-2}\xi\,, & \text{if } |s|\le k\,;\\[2mm] (1+k)^{2-p}|\xi|^{p-2}\xi\,, & \text{if } |s|>k\,.\end{cases} \tag{41}
$$

Observe that $-\operatorname{div} a_k(u,\nabla u)$ supply us a Leray–Lions type operator defined in $L^p(0,T;W_0^{1,p}(\Omega))$; to see it, we just have to consider separately the cases $1<p<2$ and $p>2$.

With this notation, our next step is to see that estimate (44) below holds true. We remark that, on account of (41), the inequality (40) becomes

$$\iint_{Q_T} a_k(T_k(w_n),\nabla T_k(w_n))\cdot\nabla\big(T_kw_n-(T_kw)_\nu\big)\chi_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}\le\omega(n,\nu,\varepsilon)\,. \tag{42}$$

This implies that

$$\begin{aligned}\iint_{Q_T} & a_k(T_k(w_n),\nabla T_k(w_n))\cdot\nabla\big(T_kw_n-T_kw\big)\chi_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}\\
&=\iint_{Q_T} a_k(T_k(w_n),\nabla T_k(w_n))\cdot\nabla\big(T_kw_n-(T_kw)_\nu\big)\chi_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}\\
&\quad+\iint_{Q_T} a_k(T_k(w_n),\nabla T_k(w_n))\cdot\nabla\big((T_kw)_\nu-T_kw\big)\chi_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}\\
&\le\omega(n,\nu,\varepsilon)+\iint_{Q_T} a_k(T_k(w_n),\nabla T_k(w_n))\cdot\nabla\big((T_kw)_\nu-T_kw\big)\chi_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}\,.\end{aligned}$$

Thus, since $\{a_k(T_k(w_n),\nabla T_k(w_n))\}_n$ is bounded in $L^{p'}(Q_T)$ and $\nabla(T_kw)_\nu\to\nabla T_kw$ strongly in $L^p(Q_T)$, it follows that

$$\iint_{Q_T} a_k(T_k(w_n),\nabla T_k(w_n))\cdot\nabla\big((T_kw)_\nu-T_kw\big)\chi_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}=\omega^\varepsilon(n,\nu)\,.$$

Therefore, we have obtained

$$\iint_{Q_T} a_k(T_k(w_n),\nabla T_k(w_n))\cdot\nabla\big(T_kw_n-T_kw\big)\chi_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}\le\omega(n,\nu,\varepsilon)\,. \tag{43}$$

On the other hand, we have that

$$\iint_{Q_T} a_k(T_k(w_n),\nabla T_kw)\cdot\nabla\big(T_kw_n-T_kw\big)\chi_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}=\omega^{\nu,\varepsilon}(n)\,,$$

since $a_k(T_k(w_n),\nabla T_kw)\to a_k(T_k(w),\nabla T_kw)$ strongly in $L^{p'}(Q_T)$ and $\nabla T_kw_n\to\nabla T_kw$ weakly in $L^p(Q_T)$. So that we deduce from (43) that

$$\begin{aligned}\iint_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}\Big(a_k(T_k(w_n),\nabla T_k(w_n))-a_k(T_k(w_n),\nabla T_kw)\Big)\cdot\nabla\big(T_kw_n-T_kw\big)&\\
\le\omega(n,\nu,\varepsilon)\,.&\end{aligned} \tag{44}$$

Denoting $\Psi_{n,k}=\Big(a_k(T_k(w_n),\nabla T_k(w_n))-a_k(T_k(w_n),\nabla T_kw)\Big)\cdot\nabla\big(T_kw_n-T_kw\big)$, we have seen that

$$\iint_{\{|T_k(w_n)-(T_kw)_\nu|\le\varepsilon\}}\Psi_{n,k}\le\omega(n,\nu,\varepsilon)\,. \tag{45}$$

Our aim is to prove that, for fixed $0 < \theta < 1$,

$$\lim_{n\to\infty} \iint_{Q_T} \Psi_{n,k}^{\theta} = 0\,. \tag{46}$$

To see it, we consider that $\varepsilon > 0$ is such that

$$\chi_{\{|T_k(w_n)-(T_kw)_\nu|>\varepsilon\}} \to \chi_{\{|T_k(w)-(T_kw)_\nu|>\varepsilon\}} \qquad \text{strongly in } L^\rho(Q_T)\,, \forall \rho \geq 1\,. \tag{47}$$

Since almost every $\varepsilon > 0$ satisfies that condition (see [10]), we will assume that ε tends to 0 satisfying (47). Now we split the integral in (46) into two parts.

$$\iint_{Q_T} \Psi_{n,k}^{\theta} = \iint_{\{|T_k(w_n)-(T_kw)_\nu|\leq\varepsilon\}} \Psi_{n,k}^{\theta} + \iint_{\{|T_k(w_n)-(T_kw)_\nu|>\varepsilon\}} \Psi_{n,k}^{\theta}\,. \tag{48}$$

Applying Hölder's inequality the first integral may estimate taking (45) into account

$$\iint_{\{|T_k(w_n)-(T_kw)_\nu|\leq\varepsilon\}} \Psi_{n,k}^{\theta} \leq |Q_T|^{1-\theta}\Big(\iint_{\{|T_k(w_n)-(T_kw)_\nu|\leq\varepsilon\}} \Psi_{n,k}\Big)^{\theta} = \omega(n,\nu,\varepsilon)\,,$$

As far as the second integral in (48) is concerned, we apply that $\{\Psi_{n,k}^{\theta}\}_n$ is bounded in $L^{1/\theta}(Q_T)$ and $\chi_{\{|T_k(w_n)-(T_kw)_\nu|>\varepsilon\}} \to \chi_{\{|T_k(w)-(T_kw)_\nu|>\varepsilon\}}$ strongly in $L^{1/(1-\theta)}(Q_T)$. So that

$$\iint_{\{|T_k(w_n)-(T_kw)_\nu|>\varepsilon\}} \Psi_{n,k}^{\theta} = \omega(n,\nu)\,.$$

Therefore, we deduce that

$$\iint_{Q_T} \Psi_{n,k}^{\theta} \leq \omega(n,\nu,\varepsilon)\,.$$

Having in mind that $\Psi_{n,k} \geq 0$, we get $\displaystyle\iint_{Q_T} \Psi_{n,k}^{\theta} = \omega(n,\nu,\varepsilon)$ and so (46) holds true.

In summary, we have proved that

$$\lim_{n\to\infty} \iint_{Q_T} \Big[\Big(a_k(T_k(w_n),\nabla T_k(w_n)) - a_k(T_k(w_n),\nabla T_k w)\Big)\cdot\nabla\big(T_kw_n - T_kw\big)\Big]^{\theta} = 0\,,$$

where $0 < \theta < 1$ and the function a_k defines a Leray–Lions type operator. It is well known (see [10]) that it implies that $\nabla T_k w_n \to \nabla T_k w$ almost everywhere in Q_T, as desired. ■

Corollary 3.8. *Let $\{v_n\}_n$ be a sequence of solutions of the approximate problems* (14). *Assume that this sequence converges pointwise to a measurable function v and that* (30) *holds. Then*

$$|\nabla v_n|^{p-2}\nabla v_n \to |\nabla v|^{p-2}\nabla v \textit{ strongly in } L^\sigma_{\rm loc}((0,+\infty);L^\sigma(\Omega)) \textit{ for } 1 \leq \sigma < \frac{N+p}{N+p-1}.$$

Proof. This result is a straightforward consequence of (28) and the pointwise convergence of the gradients. ■

Corollary 3.9. *The following equality holds*

$$-\int_\Omega \varphi(v_0(x))\Phi(x,0)\,dx - \int_0^T\int_\Omega \varphi(v)\Phi_t + \int_0^T\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\Phi = \int_0^T\int_\Omega f\big(1+\varphi(v)\big)\Phi + \int_0^T\int_\Omega \Phi\,d\mu\,, \tag{49}$$

for every $T>0$ and every $\Phi\in C^\infty(\overline{Q}_T)$, with $\Phi(\cdot,t)\in C_0(\Omega)$ for all $t\in(0,T)$ and $\Phi(x,T)=0$ for all $x\in\Omega$.

Proof. Just take such a Φ as test function in (14) and apply $g_n\to\varphi(v_0)$ strongly in $L^1(\Omega)$, $h_n\to\mu$ in the weak-$*$ sense in Q, (31), (32) and Corollary 3.8. ■

3.5. **Continuity in the time variable.** This subsection is devoted to prove (3) and (4) in Definition 3.3. Denoting by $C_0(\Omega)$ the space of all continuous functions $u:\overline{\Omega}\to\mathbb{R}$ satisfying $u\big|_{\partial\Omega}=0$, we recall that the dual space of $C_0(\Omega)$ is the space of all Radon measures on Ω with finite total variation. So that, we will see that

$$\text{For every } T>0\,,\ \text{there exist } \lim_{t\to T^\pm}\int_\Omega \varphi(v(x,t))\phi(x)\,dx\,. \tag{50}$$

$$\lim_{t\to 0^+}\int_\Omega \varphi(v(x,t))\phi(x)\,dx = \int_\Omega \varphi(v_0(x))\phi(x)\,dx\,, \tag{51}$$

for every $\phi\in C_0(\Omega)$. We obtain them as a consequence of the following claim.

Lemma 3.10. *Let $T>0$ and $\phi\in C_0^\infty(\Omega)$ be fixed. Then the function defined by*

$$\xi(t)=\int_\Omega \varphi(v(x,t))\phi(x)\,dx$$

is of bounded variation.

Proof. For every $\psi\in C_0^\infty(0,T)$ such that $\|\psi\|_\infty\le 1$, we take $\Phi(x,t)=\phi(x)\psi(t)$ as test function in (49) getting

$$-\int_0^T\int_\Omega \varphi(v(x,t))\phi(x)\,\psi'(t)\,dx\,dt + \int_0^T\int_\Omega |\nabla v(x,t)|^{p-2}\nabla v(x,t)\cdot\nabla\phi(x)\,\psi(t)\,dx\,dt = \int_0^T\int_\Omega f(x,t)\big(1+\varphi(v(x,t))\big)\phi(x)\psi(t)\,dx\,dt + \int_0^T\int_\Omega \phi(x)\psi(t)\,d\mu\,.$$

Thus,

$$\left|\int_0^T \xi(t)\psi'(t)\,dt\right| \le \int_0^T\int_\Omega |\nabla v(x,t)|^{p-1}|\nabla\phi(x)|\,dx\,dt + \int_0^T\int_\Omega |f(x,t)|\,\big(1+|\varphi(v(x,t))|\big)\,|\phi(x)|\,dx\,dt + \int_0^T\int_\Omega |\phi(x)|\,d\mu = M\,,$$

for every $\psi\in C_0^\infty(0,T)$ such that $\|\psi\|_\infty\le 1$. Hence, ξ' is a bounded measure on $(0,T)$ and ξ is a function of bounded variation. ■

As a straightforward consequence of Lemma 3.10, we obtain:

(52) $$\text{Fixed } T > 0\,, \text{ there exists } \lim_{t\to T^\pm} \int_\Omega \varphi(v(x,t))\phi(x)\,dx\,, \quad \text{for all } \phi \in C_0^\infty(\Omega)\,.$$

(53) $$\text{There exists } \lim_{t\to 0^+} \int_\Omega \varphi(v(x,t))\phi(x)\,dx\,, \quad \text{for all } \phi \in C_0^\infty(\Omega)\,.$$

Now let $\phi \in C_0(\Omega)$ be fixed. To see (50) and (51), consider a sequence $\phi_n \in C_0^\infty(\Omega)$ such that $\phi_n \to \phi$ uniformly on $\overline{\Omega}$.

Proof of (50). Recall that (52) implies that there exists

(54) $$L_n^\pm = \lim_{t\to T^\pm} \int_\Omega \varphi(v(x,t))\phi_n(x)\,dx\,.$$

Limit (50) will be obtained by applying the uniform convergence of ϕ_n to ϕ. We begin by observing that the sequence $\{L_n^\pm\}_n$ is a Cauchy sequence since

$$|L_n^\pm - L_m^\pm| \le \|\phi_n - \phi_m\|_\infty \|\varphi(v)\|_{L^\infty(0,T+1;L^1(\Omega))}$$

and $\{\phi_n\}_n$ is a Cauchy sequence. So that there exists $L^\pm = \lim_{n\to\infty} L_n^\pm$.

Given $\epsilon > 0$, fix $n \in \mathbb{N}$ big enough to have

$$\|\phi_n - \phi_m\|_\infty \|\varphi(v)\|_{L^\infty(0,T+1;L^1(\Omega))} \le \frac{\epsilon}{2}\,,$$

for all $m \ge n$. Hence, $|L_n^\pm - L^\pm| \le \epsilon/2$ and

$$\Big| \int_\Omega \varphi(v(x,t))\phi(x)\,dx - \int_\Omega \varphi(v(x,t))\phi_n(x)\,dx \Big| \le \frac{\epsilon}{2}\,.$$

Therefore,

$$\Big| \int_\Omega \varphi(v(x,t))\phi(x)\,dx - L^\pm \Big| \le \Big| \int_\Omega \varphi(v(x,t))\phi_n(x)\,dx - L_n^\pm \Big| + \epsilon\,,$$

so that (50) follows from (54).■

Proof of (51). As in the previous proof, it follows from (53) that there exists

(55) $$L_0 = \lim_{t\to 0^+} \int_\Omega \varphi(v(x,t))\phi(x)\,dx\,.$$

So, to prove (51), all we have to see is

(56) $$L_0 = \int_\Omega \varphi(v_0(x))\phi(x)\,dx\,.$$

For each $n \in \mathbb{N}$, we consider the function defined by $\xi_n(t) = \displaystyle\int_\Omega \varphi(v(x,t))\phi_n(x)\,dx$ and the set

$$E_n = \{t \in (0,+\infty)\ :\ t \text{ is not a Lebesgue point of } \xi_n\}\,.$$

Denoting $E = \big(\cup_{n=1}^{\infty} E_n\big) \cup \{t \in (0,+\infty) \, : \, |\mu|(\{t\}\times\Omega) > 0\}$, we have that E is a null set. Let $t \notin E$, fix $\epsilon > 0$ and consider the real function defined by

$$\eta_\epsilon(s) = \begin{cases} 1\,, & \text{if } 0 \le s \le t\,; \\ \frac{1}{\epsilon}(t-s)+1\,, & \text{if } t \le s \le t+\epsilon\,; \\ 0\,, & \text{if } s \ge t+\epsilon\,. \end{cases} \tag{57}$$

Making a standard approximation procedure, we may take $\Phi(x,s) = \phi_n(x)\eta_\epsilon(s)$ as test function in (49), it yields

$$\begin{aligned} -\int_\Omega \varphi(v_0(x))\phi_n(x)\,dx + \frac{1}{\epsilon}\int_t^{t+\epsilon}\int_\Omega \varphi(v(x,s))\phi_n(x)\,dx\,ds \\ + \int_0^{t+\epsilon}\int_\Omega \eta_\epsilon(s)|\nabla v(x,s)|^{p-2}\nabla v(x,s)\cdot\nabla\phi_n(x)\,dx\,ds \\ = \int_0^{t+\epsilon}\int_\Omega \eta_\epsilon(s) f(x,s)\big(1+\varphi(v(x,s))\big)\phi_n(x)\,dx\,ds + \int_0^{t+\epsilon}\int_\Omega \eta_\epsilon(s)\phi_n(x)\,d\mu\,. \end{aligned} \tag{58}$$

Since t is a Lebesgue point of ξ_n, we have

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_t^{t+\epsilon}\int_\Omega \varphi(v(x,s))\phi_n(x)\,dx\,ds = \int_\Omega \varphi(v(x,t))\phi_n(x)\,dx\,.$$

On the other hand,

$$\limsup_{\epsilon\to 0^+}\Big|\int_0^{t+\epsilon}\int_\Omega \eta_\epsilon(s)\phi_n(x)\,d\mu - \int_0^t\int_\Omega \phi_n(x)\,d\mu\Big| \le \limsup_{\epsilon\to 0^+}\|\phi_n\|_\infty|\mu|\big((t,t+\epsilon)\times\Omega\big) = 0$$

since $\bigcap_{\epsilon>0}\big((t,t+\epsilon)\times\Omega\big) = \emptyset$. Thus, it follows that

$$\lim_{\epsilon\to 0^+}\int_0^{t+\epsilon}\int_\Omega \eta_\epsilon(s)\phi_n(x)\,d\mu = \int_0^t\int_\Omega \phi_n(x)\,d\mu\,.$$

Letting ϵ go to 0 in (58), we obtain

$$\begin{aligned} \int_\Omega \varphi(v(x,t))\phi_n(x)\,dx - \int_\Omega \varphi(v_0(x))\phi_n(x)\,dx \\ + \int_0^t\int_\Omega |\nabla v(x,s)|^{p-2}\nabla v(x,s)\cdot\nabla\phi_n(x)\,dx\,ds \\ = \int_0^t\int_\Omega f(x,s)\big(1+\varphi(v(x,s))\big)\phi_n(x)\,dx\,ds + \int_0^t\int_\Omega \phi_n(x)\,d\mu\,. \end{aligned} \tag{59}$$

The next step is let $t \notin E$ go to 0. Observe that it follows from $\bigcap_{t>0}\big((0,t)\times\Omega\big) = \emptyset$ that

$$\lim_{t\to 0^+}|\mu|\big((0,t)\times\Omega\big) = 0\,,$$

and so

$$\lim_{t\to 0^+}\int_0^t\int_\Omega \phi_n(x)\,d\mu = 0\,.$$

Therefore, we may let $t \notin E$ go to 0 in (59) and get

$$\lim_{\substack{t\to 0^+\\ t\notin E}}\int_\Omega \varphi(v(x,t))\phi_n(x)\,dx = \int_\Omega \varphi(v_0(x))\phi_n(x)\,dx\,.$$

The uniform convergence of ϕ_n to ϕ implies that

$$\lim_{\substack{t\to 0^+\\ t\notin E}}\int_\Omega \varphi(v(x,t))\phi(x)\,dx = \int_\Omega \varphi(v_0(x))\phi(x)\,dx\,.$$

Finally, it follows from (55) that (56) holds, so that (51) is completely proved.■

Proposition 3.11. *Denoting by $\varphi(v(x,T^+))$ and $\varphi(v(x,T^-))$ the one–side limits $\lim_{t\to T^\pm}\varphi(v(\cdot,t))$, the equalities*

$$\begin{aligned}(60)\quad &\int_\Omega \varphi(v(x,T^\pm))\Phi(x,T)\,dx - \int_\Omega \varphi(v_0(x))\Phi(x,0)\,dx - \int_0^T\int_\Omega \varphi(v)\Phi_t\\ &\quad + \int_0^T\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\Phi = \int_0^T\int_\Omega f\big(1+\varphi(v)\big)\Phi + \int_0^{T^\pm}\int_\Omega \Phi\,d\mu\,,\end{aligned}$$

hold for every $T>0$ and every $\Phi\in C^1(\Omega\times(0,T))$ such that $\Phi(\cdot,t)\in C_0(\Omega)$ for all $t\in(0,T]$.

Proof. We prove that equation (60) holds for the right–side limit. Consider $\epsilon>0$ and the real function defined by

$$(61)\qquad \eta_\epsilon(s) = \begin{cases} 1\,, & \text{if } 0\le s\le T\,;\\ \frac{1}{\epsilon}(T-s)+1\,, & \text{if } T\le s\le T+\epsilon\,;\\ 0\,, & \text{if } s\ge T+\epsilon\,.\end{cases}$$

For every $\Phi\in C^1(Q_T)$, we extend it to $Q_{T+\epsilon}$ by defining $\Phi(x,s)=\Phi(x,T)$ for $s\in[T,T+\epsilon]$. Making a standard approximation procedure, we take $\eta_\epsilon(s)\Phi(x,s)$ as test function in (49) replacing T with $T+\epsilon$. Then we obtain

$$\begin{aligned}&-\int_\Omega \varphi(v_0(x))\Phi(x,0)\,dx + \frac{1}{\epsilon}\int_T^{T+\epsilon}\int_\Omega \varphi(v(x,s))\Phi(x,T)\,dx\,ds - \int_0^T\int_\Omega \varphi(v)\Phi_t\\ &\quad + \int_0^{T+\epsilon}\int_\Omega \eta_\epsilon(s)|\nabla v(x,s)|^{p-2}\nabla v(x,s)\cdot\nabla\Phi(x,s)\,dx\,ds\\ &= \int_0^{T+\epsilon}\int_\Omega \eta_\epsilon(s) f(x,s)\big(1+\varphi(v(x,s))\big)\Phi(x,s)\,dx\,ds + \int_0^{T+\epsilon}\int_\Omega \eta_\epsilon(s)\Phi(x,s)\,d\mu\,.\end{aligned}$$

Next we may argue as in the proof of 51 to let ϵ go to 0, it yields (60) for the right–side limit.

To prove that equation (60) holds for the left–side limit, the same argument works. We just have to consider $\epsilon \in (0,T)$ and, instead of (61), the real function defined by

$$\eta_\epsilon(s) = \begin{cases} 1\,, & \text{if } 0 \le s \le T-\epsilon\,;\\ \frac{1}{\epsilon}(T-\epsilon-s)+1\,, & \text{if } T-\epsilon \le s \le T\,;\\ 0\,, & \text{if } s \ge T\,. \end{cases}$$

■

3.6. L^1 **case: entropy solutions.** In this subsection, we define entropy solutions to parabolic equations to get uniqueness of solutions with L^1–data. This formulation was introduced in [6] for elliptic equations, and then extended to parabolic equations in [4] and [30]; a different formulation for parabolic equations, called renormalized solution, was introduced in [8] (see also [19] and [26]).

Definition 3.12. *Assume that $\mu \in L^1_{\rm loc}([0,+\infty);L^1(\Omega))$ and $\varphi(v_0) \in L^1(\Omega)$.*

We say that v is an entropy solution to (9) *if*

(1) $\varphi(v) \in \mathcal{C}([0,+\infty);L^1(\Omega))$.
(2) $T_k(v) \in L^p_{\rm loc}([0,+\infty);W^{1,p}_0(\Omega))$ for all $k>0$.
(3) $f(1+\varphi(v)) \in L^1_{\rm loc}([0,+\infty);L^1(\Omega))$.
(4) For every $\phi \in L^p_{\rm loc}([0,+\infty);W^{1,p}_0(\Omega)) \cap L^\infty_{\rm loc}([0,+\infty);L^\infty(\Omega))$ satisfying that its distributional derivative ϕ_t belongs to $L^{p'}_{\rm loc}([0,+\infty);W^{-1,p'}(\Omega)) + L^1_{\rm loc}([0,+\infty);L^1(\Omega))$, and every $k,T>0$, the following equality holds:

$$\int_\Omega J_k(\varphi(v(T))-\phi(T)) + \int_0^T\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla T_k(\varphi(v)-\phi)$$
$$= -\int_0^T \langle \phi_t, T_k(\varphi(v)-\phi)\rangle + \int_0^T\int_\Omega \Big(f(1+\varphi(v))+\mu\Big)T_k(\varphi(v)-\phi) + \int_\Omega J_k(\varphi(v_0)-\phi(0))\,,$$

where $J_k(s) = \int_0^s T_k(\sigma)\,d\sigma$.

Remark 3.13. We point out that a result by A. Porretta (see [29]) implies that test functions ϕ in (4) belong to $\mathcal{C}([0,+\infty);L^1(\Omega))$, so that $\phi(T)$ and $\phi(0)$ have sense.

Theorem 3.14. *Assume that $\mu \in L^1_{\rm loc}([0,+\infty);L^1(\Omega))$, and consider v a reachable solution to* (9) *(obtained by approximation of μ as a measure). Let $\{v_n\}_n$, $\{g_n\}_n$ and $\{h_n\}_n$ be as in the definition 3.3*

Assume that $h_n \rightharpoonup \mu$ weakly in $L^1(Q_T)$ for all T. Then

(1) $\varphi(v_n) \to \varphi(v)$ in $L^\infty_{\rm loc}([0,+\infty;L^1(\Omega))$, and so $\varphi(v) \in \mathcal{C}([0,+\infty);L^1(\Omega))$.
(2) v is an entropy solution to (9).

Proof. To prove (1), first observe that condition (5) of 3.3 implies $\varphi(v_n) \to \varphi(v)$ strongly in $L^1(Q_T)$ for all $T>0$. For fixed $T>0$ we follow the arguments in [15] Proposition 6.4.

(2) Fix $T>0$ and $k>0$ and let $\phi \in L^p_{\rm loc}([0,+\infty);W^{1,p}_0(\Omega)) \cap L^\infty_{\rm loc}([0,+\infty);L^\infty(\Omega))$ satisfy that its distributional derivative ϕ_t belongs to $L^{p'}_{\rm loc}([0,+\infty);W^{-1,p'}(\Omega)) + L^1_{\rm loc}([0,+\infty);L^1(\Omega))$.

Consider $T_k(\varphi(v_n)-\phi)$ as test function in the weak formulation of v_n and integrate by parts to get

$$\int_\Omega J_k(\varphi(v_n(T))-\phi(T))-\int_\Omega J_k(g_n-\phi(0))+\int_{Q_T}|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla T_k(\varphi(v_n)-\phi)$$
$$=-\int_{Q_T}\phi_t\,T_k(\varphi(v_n)-\phi)+\int_{Q_T}\Big(f\big(1+\varphi(v_n)\big)+h_n\Big)T_k(\varphi(v_n)-\phi)\,.$$

Letting n goes to ∞, we prove the desired formulation. ■

Theorem 3.15. *If $\mu\in L^1_{\rm loc}([0,+\infty);L^1(\Omega))$, then there exists a unique entropy solution to* (9).

Proof. The existence of an entropy solution is a consequence of the existence of a reachable solution, and Theorem 3.14. It remains to prove the uniqueness.

Let $\hat v$ be an entropy solution with initial datum $\hat v_0$ and let v a reachable solution with initial datum v_0. Fix $T>0$. We may assume that the sequence $\{h_n\}_n$ in $L^\infty_{\rm loc}([0,+\infty);L^\infty(\Omega))$ converges to μ in $L^1(Q_T)$ and a.e. in Q_T and, by Theorem 3.14, the sequence $\{v_n\}_n$ in $L^p_{\rm loc}([0,+\infty);W^{1,p}_0(\Omega))$ is such that $\varphi(v_n)\to\varphi(v)$ in $\mathcal{C}([0,T];L^1(\Omega))$.

Take $\varphi(v_n)$ as test function in the entropy formulation of $\hat v$ and take $T_k(\varphi(v_n)-T_h(\varphi(\hat v)))$ in the weak formulation of v_n. Adding both identities, it yields

$$\begin{aligned}(62)\quad &\int_\Omega J_k(\varphi(\hat v(T))-\varphi(v_n(T)))-\int_\Omega J_k(\varphi(\hat v_0)-g_n)\\ &+\int_{Q_T}|\nabla\hat v|^{p-2}\nabla\hat v\cdot\nabla T_k(\varphi(\hat v)-\varphi(v_n))+\int_{Q_T}|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla T_k(\varphi(v_n)-T_h(\varphi(\hat v)))\\ &=-\int_{Q_T}\Big\langle\big(\varphi(v_n)\big)_t\,,\big(T_k(\varphi(\hat v)-\varphi(v_n))+T_k(\varphi(v_n)-T_h(\varphi(\hat v)))\big)\Big\rangle\\ &+\int_{Q_T}\Big(f\big(1+\varphi(\hat v)\big)+\mu\Big)T_k(\varphi(\hat v)-\varphi(v_n))+\int_{Q_T}\Big(f\big(1+\varphi(v_n)\big)+h_n\Big)T_k(\varphi(v_n)-T_h(\varphi(\hat v)))\,.\end{aligned}$$

We will analyze each line of the above equality, when h goes to infinity; so that we may assume that $h>k+\|\varphi(v_n)\|_\infty$. We begin by considering the second line. Observe that then

$$\int_{\{|\varphi(\hat v)|>h\}}|\nabla\hat v|^{p-2}\nabla\hat v\cdot\nabla T_k(\varphi(\hat v)-\varphi(v_n))+\int_{\{|\varphi(\hat v)|>h\}}|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla T_k(\varphi(v_n)-T_h(\varphi(\hat v)))=0\,.$$

Hence, we only have to study the term

$$\int_{\{|\varphi(\hat v)|\le h\}}\Big(|\nabla\hat v|^{p-2}\nabla\hat v-|\nabla v_n|^{p-2}\nabla v_n\Big)\cdot\nabla T_k(\varphi(\hat v)-\varphi(v_n))\,,$$

and this can be analyzed as in the proof of Theorem 3.2. In the end, we obtain

$$\begin{aligned}(63)\quad &-\frac1k\int_{\{|\varphi(\hat v)|\le h\}}\Big(|\nabla\hat v|^{p-2}\nabla\hat v-|\nabla v_n|^{p-2}\nabla v_n\Big)\cdot\nabla T_k(\varphi(\hat v)-\varphi(v_n))\\ &\le|p-2|\Big(\int_{\{|\varphi(\hat v)-\varphi(v_n)|<k\}}|\nabla\varphi(v_n)|^p\Big)^{1/p'}\Big(\int_{\{|\varphi(\hat v)-\varphi(v_n)|<k\}}|\nabla T_k(\varphi(v_n)-\varphi(\hat v))|^p\Big)^{1/p}.\end{aligned}$$

We next turn our attention to the third line in (62). Since

$$\lim_{h\to\infty} T_k(\varphi(v_n) - T_h(\varphi(\hat v))) = T_k(\varphi(v_n) - \varphi(\hat v)) \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega))$$

and $\big(\varphi(v_n)\big)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, one deduces

$$\lim_{h\to\infty} \int_{Q_T} \Big\langle \big(\varphi(v_n)\big)_t \,, \big(T_k(\varphi(\hat v) - \varphi(v_n)) + T_k(\varphi(v_n) - T_h(\varphi(\hat v)))\big)\Big\rangle = 0\,. \tag{64}$$

The fourth line is easy to deal with, since we have pointwise convergence. We get

$$\begin{aligned}\lim_{h\to\infty} \Big[\int_{Q_T} \Big(f\big(1+\varphi(\hat v)\big) + \mu\Big) T_k(\varphi(\hat v) - \varphi(v_n)) \\ + \int_{Q_T} \Big(f\big(1+\varphi(v_n)\big) + h_n\Big) T_k(\varphi(v_n) - T_h(\varphi(\hat v)))\Big] \\ = \int_{Q_T} \Big(f\big(\varphi(\hat v) - \varphi(v_n)\big) + \big(\mu - h_n\big)\Big) T_k(\varphi(\hat v) - \varphi(v_n)) \\ \le k\|f\|_\infty \int_{Q_T} \big|\varphi(\hat v) - \varphi(v_n)\big| + k\int_{Q_T} \big|\mu - h_n\big|\,.\end{aligned} \tag{65}$$

Therefore, having in mind (63), (64) and (65), and dividing by k, identity (62) becomes

$$\begin{aligned}&\frac{1}{k}\int_\Omega J_k(\varphi(\hat v(T)) - \varphi(v_n(T))) - \frac{1}{k}\int_\Omega J_k(\varphi(\hat v_0) - g_n) \\ &\le |p-2| \Big(\int_{\{|\varphi(\hat v) - \varphi(v_n)|<k\}} |\nabla\varphi(v_n)|^p\Big)^{1/p'} \Big(\int_{\{|\varphi(\hat v) - \varphi(v_n)|<k\}} |\nabla T_k(\varphi(v_n) - \varphi(\hat v))|^p\Big)^{1/p} \\ &\qquad + \|f\|_\infty \int_{Q_T} \big|\varphi(\hat v) - \varphi(v_n)\big| + \int_{Q_T} \big|\mu - h_n\big|\,.\end{aligned}$$

Now we let k tends to 0, and so we arrive at the following inequality

$$\begin{aligned}&\int_\Omega |\varphi(\hat v(T)) - \varphi(v_n(T))| \\ &\qquad \le \int_\Omega |\varphi(\hat v_0) - g_n| + \|f\|_\infty \int_0^T \int_\Omega |\varphi(\hat v) - \varphi(v_n)| + \int_0^T\int_\Omega |\mu - h_n|\,.\end{aligned}$$

By Gronwall's Lemma, we obtain

$$\int_\Omega |\varphi(\hat v(T)) - \varphi(v_n(T))| \le e^{T\|f\|_\infty} \int_\Omega |\varphi(\hat v_0) - g_n| + \int_0^T \Big(e^{(T-t)\|f\|_\infty} \int_\Omega |\mu - h_n|\Big)\, dt$$

Finally, letting n goes to ∞, it follows that

$$\int_\Omega |\varphi(\hat v(T)) - \varphi(v(T))| \le e^{T\|f\|_\infty} \int_\Omega |\varphi(\hat v_0) - \varphi(v_0)|\,.$$

Therefore, $\varphi(\hat v_0) = \varphi(v_0)$ implies $\varphi(\hat v(T)) = \varphi(v(T))$ and, since φ is increasing, $\hat v(T) = v(T)$ for all $T > 0$. ■

4. Nonuniqueness for the problem having a gradient term

This Section is devoted to prove the multiplicity result. Throughout it we will assume that all data in problem (9) (f, v_0 and μ) are nonnegative and, as a consequence, so is the solution.

4.1. **Strong convergence of the truncations.** This subsection is devoted to proving the following result.

Proposition 4.1. *Assume that μ is a positive measure which is singular with respect to the p-capacity. Let v be a reachable solution of problem (1), and let $\{v_n\}_n$, $\{h_n\}_n$ and $\{g_n\}_n$ be as in Definition 3.3.*

Then, for every $T > 0$ and $k > 0$,

$$T_k(v_n) \to T_k(v) \qquad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega))\,. \tag{66}$$

The technique that we will use is adapted from the paper [26] by Petitta. We will need the following lemma, which is Lemma 5 in [26]. Here it is essential that μ is concentrated on a set E of zero p-capacity.

Lemma 4.2. *For every $\delta > 0$, there exists a compact set $K_\delta \subset E$ such that*

$$\mu(E \setminus K_\delta) \le \delta\,, \tag{67}$$

and there exists $\psi_\delta \in C_0^1(Q)$ such that

$$0 \le \psi_\delta \le 1\,, \qquad \psi_\delta \equiv 1 \text{ on } K_\delta\,. \tag{68}$$

$$\|\psi_\delta\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le \delta\,, \qquad \|(\psi_\delta)_t\|_{L^{p'}(0,T;W_0^{-1,p'}(\Omega))+L^1(Q_T)} \le \delta\,. \tag{69}$$

Therefore, $\psi_\delta \to 0$ $*$-weakly in $L^\infty(Q_T)$ and, up to subsequences, almost everywhere as $\delta \to 0^+$. Moreover,

$$\iint_{Q_T} (1-\psi_\delta)\, h_n\, dx = \omega(n,\delta)\,, \qquad \iint_{Q_T} (1-\psi_\delta)\, d\mu \le \delta\,. \tag{70}$$

Proof of Proposition 4.1. Proving (66) is the same as proving that, for every $k > 0$,

$$T_k w_n \to T_k w \qquad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega))\,, \tag{71}$$

where $w_n = \varphi(v_n)$ and $w = \varphi(v)$. To prove this, it will be enough to show that

$$\limsup_{n\to+\infty} \iint_{Q_T} |\nabla T_k w_n|^p \le \iint_{Q_T} |\nabla T_k w|^p\,. \tag{72}$$

For $m > 2k$, we set

$$H_m(s) = \begin{cases} 1 & \text{if } |s| \le m \\ \dfrac{2m-s}{m} & \text{if } m < |s| < 2m \\ 0 & \text{if } |s| \ge 2m\,. \end{cases}$$

Then

$$\iint_{Q_T} |\nabla T_k w_n|^p = \iint_{Q_T} |\nabla T_k w_n|^p H_m(w_n)$$
$$= \iint_{Q_T} |\nabla T_{2m} w_n|^{p-2} \nabla T_{2m} w_n \cdot \nabla (T_k w)_\nu \, H_m(w_n)$$
$$+ \iint_{Q_T} |\nabla w_n|^{p-2} \nabla w_n \cdot (\nabla T_k w_n - \nabla (T_k w)_\nu) \, H_m(w_n)$$
$$= \iint_{Q_T} |\nabla T_k w|^p + \iint_{Q_T} |\nabla w_n|^{p-2} \nabla w_n \cdot (\nabla T_k w_n - \nabla (T_k w)_\nu) \, H_m(w_n) + \omega(n,\nu)\,,$$

where we have used the convergence $\nabla T_{2m} w_n \to \nabla T_{2m} w$ a.e., and the a priori estimate $\|T_{2m} w_n\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le c(m)$. Therefore, in order to prove (72), it is enough to show that

$$\iint_{Q_T} |\nabla w_n|^{p-2} \nabla w_n \cdot (\nabla T_k w_n - \nabla (T_k w)_\nu) \, H_m(w_n) \le \omega(n,\nu)\,. \tag{73}$$

If ψ_δ is the function defined in Lemma 4.2, we can write

$$\iint_{Q_T} |\nabla w_n|^{p-2} \nabla w_n \cdot (\nabla T_k w_n - \nabla (T_k w)_\nu) \, H_m(w_n)$$
$$= \iint_{Q_T} |\nabla T_k w_n|^p \, \psi_\delta \qquad \boxed{\mathrm{A}}$$
$$- \iint_{Q_T} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla (T_k w)_\nu \, H_m(w_n) \, \psi_\delta \qquad \boxed{\mathrm{B}}$$
$$+ \iint_{Q_T} |\nabla w_n|^{p-2} \nabla w_n \cdot (\nabla T_k w_n - \nabla (T_k w)_\nu) \, H_m(w_n)(1-\psi_\delta)\,. \qquad \boxed{\mathrm{C}}$$

The integral $\boxed{\mathrm{B}}$ is the easiest to estimate:

$$\boxed{\mathrm{B}} = -\iint_{Q_T} |\nabla T_{2m} w_n|^{p-2} \nabla T_{2m} w_n \cdot \nabla (T_k w)_\nu \, H_m(w_n) \, \psi_\delta$$
$$= -\iint_{Q_T} |\nabla T_{2m} w|^{p-2} \nabla T_{2m} w \cdot \nabla (T_k w)_\nu \, H_m(w) \, \psi_\delta + \omega^\nu(n)$$
$$= -\iint_{Q_T} |\nabla T_k w|^p \, \psi_\delta + \omega(n,\nu)$$
$$= \omega(n,\nu,\delta)\,,$$

by the properties of ψ_δ.

We wish to show that

$$\boxed{\mathrm{A}} \le \omega(n,\delta)\,. \tag{74}$$

To this aim, we use $(k - T_k w_n)\,\psi_\delta$ as test function in (36), and we obtain:

$$
\begin{aligned}
-\iint_{Q_T} \Gamma_k(w_n)\,(\psi_\delta)_t + \iint_{Q_T} (1+|w_n|)^{2-p}(k - T_k w_n)\,|\nabla w_n|^{p-2}\nabla w_n \cdot \nabla \psi_\delta \\
- \iint_{Q_T} (1+|w_n|)^{2-p}\,|\nabla T_k w_n|^p\,\psi_\delta \\
= \iint_{Q_T} f\,(1+w_n)(k - T_k w_n)\,\psi_\delta + \iint_{Q_T} (k - T_k w_n)\,h_n\,\psi_\delta \geq 0\,,
\end{aligned}
$$

where

$$
\Gamma_k(s) = \int_0^s (k - T_k\sigma)\,d\sigma\,.
$$

Therefore

$$
\begin{aligned}
c(k)\,\boxed{\text{A}} \quad \leq \quad & -\iint_{Q_T} \Gamma_k(w_n)\,(\psi_\delta)_t & \boxed{\text{A1}} \\
& + \iint_{Q_T} (k - T_k w_n)\,|\nabla w_n|^{p-2}\nabla w_n \cdot \nabla \psi_\delta & \boxed{\text{A2}}
\end{aligned}
$$

One has

$$
\boxed{\text{A1}} = -\iint_{Q_T} \Gamma_k(w)\,(\psi_\delta)_t + \omega^\delta(n) = \omega(n,\delta)\,,
$$

by the properties of ψ_δ (note that $\Gamma_k(w)$ is a bounded function). On the other hand

$$
\begin{aligned}
\boxed{\text{A2}} &= \iint_{Q_T} (k - T_k w_n)|\nabla T_k w_n|^{p-2}\nabla T_k w_n \cdot \nabla \psi_\delta \\
&= \iint_{Q_T} (k - T_k w)\,|\nabla T_k w|^{p-2}\nabla T_k w \cdot \nabla \psi_\delta + \omega^\delta(n) = \omega(n,\delta)\,.
\end{aligned}
$$

Therefore (74) is proven.

In order to study the integral $\boxed{\text{C}}$, we write

$$
\begin{aligned}
(75) \quad \boxed{\text{C}} = & \iint_{\{|w_n|\leq k\}} |\nabla T_k w_n|^{p-2}\nabla T_k w_n \cdot (\nabla T_k w_n - \nabla (T_k w)_\nu)\,(1-\psi_\delta) & \boxed{\text{C1}} \\
& - \iint_{\{|w_n|>k\}} |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla (T_k w)_\nu\,H_m(w_n)\,(1-\psi_\delta) & \boxed{\text{C2}}
\end{aligned}
$$

The term $\boxed{\text{C2}}$ is easy to estimate:

$$\boxed{\text{C2}} = -\iint_{\{|w_n|>k\}} |\nabla T_{2m} w_n|^{p-2} \nabla T_{2m} w_n \cdot \nabla (T_k w)_\nu \, H_m(w_n)\,(1-\psi_\delta)$$
$$= -\iint_{\{|w|>k\}} |\nabla T_{2m} w|^{p-2} \nabla T_{2m} w \cdot \nabla (T_k w)_\nu \, H_m(w)\,(1-\psi_\delta) + \omega^{\nu,\delta}(n)$$
$$= -\iint_{\{|w|>k\}} |\nabla w|^{p-2} \nabla w \cdot \nabla T_k w\,(1-\psi_\delta) + \omega^{\delta}(n,\nu) = \omega^{\delta}(n,\nu)\,,$$

since the last integral is zero.

To estimate $\boxed{\text{C1}}$, we use $U_{n,\nu,h}\,(1-\psi_\delta)$ as test function in (36), where

$$U_{n,\nu,h} = T_{2k}(w_n - T_h w_n + T_k w_n - (T_k w)_\nu)\,,$$

and $h > 2k$ (note that $\nabla U_{n,\nu,h} = 0$ on the set where $w_n > h + 4k$) and we obtain

$$(76) \quad \int_0^T \langle (w_n)_t\,,\, U_{n,\nu,h}\,(1 - \psi_\delta)\rangle\,dt \qquad \boxed{\text{I}}$$
$$+ \iint_{Q_T} (1+w_n)^{2-p} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla U_{n,\nu,h}\,(1-\psi_\delta) \qquad \boxed{\text{II}}$$
$$- \iint_{Q_T} (1+w_n)^{2-p} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \psi_\delta\, U_{n,\nu,h} \qquad \boxed{\text{III}}$$
$$= \iint_{Q_T} f\,(1+w_n)\, U_{n,\nu,h}\,(1-\psi_\delta) \qquad \boxed{\text{IV}}$$
$$+ \iint_{Q_T} U_{n,\nu,h}\,(1-\psi_\delta)\, h_n \qquad \boxed{\text{V}}$$

To begin with, we observe that

$$\Big|\boxed{\text{V}}\Big| \le 2k \iint_{Q_T} (1-\psi_\delta)\, h_n = \omega(n,\delta)\,.$$

On the other hand,

$$\boxed{\text{IV}} = \iint_{Q_T} f\,(1+w)\, T_{2k}(w - T_h w + T_k w - (T_k w)_\nu)\,(1-\psi_\delta) + \omega^{\nu,h,\delta}(n)$$
$$= \iint_{Q_T} f\,(1+w)\, T_{2k}(w - T_h w)\,(1-\psi_\delta) + \omega^{h,\delta}(n,\nu)$$
$$= \omega^{\delta}(n,\nu,h)\,.$$

As far as the term $\boxed{\text{III}}$ is concerned, we observe that, by Corollary 3.8, $(1+w_n)^{2-p}|\nabla w_n|^{p-2}\nabla w_n = |\nabla v_n|^{p-2}\nabla v_n$ converges strongly in $L^1(Q_T)$ to $(1+w)^{2-p}|\nabla w|^{p-2}\nabla w$, therefore

$$\boxed{\text{III}} = -\iint_{Q_T} (1+w)^{2-p} |\nabla w|^{p-2} \nabla w \cdot \nabla \psi_\delta\, T_{2k}(w - T_h w + T_k w - (T_k w)_\nu) + \omega^{\delta}(n) = \omega(n,\delta)\,.$$

Now we turn to estimate the term $\boxed{\text{II}}$; one has

$$\boxed{\text{II}} \geq \quad c(k)\boxed{\text{C1}} \quad + \iint\limits_{\{w_n>k\,,\,|w_n-T_hw_n+T_kw_n-(T_kw)_\nu|<2k\}} (1+w_n)^{2-p}|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla(w_n-T_hw_n)\,(1-\psi_\delta)$$
$$- \iint\limits_{\{w_n>k\,,\,|w_n-T_hw_n+T_kw_n-(T_kw)_\nu|<2k\}} (1+w_n)^{2-p}|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla(T_kw)_\nu\,(1-\psi_\delta)$$
$$\geq \quad c(k)\boxed{\text{C1}} \quad - \iint\limits_{\{w_n>k\,,\,|w_n-T_hw_n+T_kw_n-(T_kw)_\nu|<2k\}} (1+w_n)^{2-p}|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla(T_kw)_\nu\,(1-\psi_\delta)$$

The last integral can be estimated using $|\nabla T_{h+4k}w_n|^{p-1} \rightharpoonup |\nabla T_{h+4k}w|^{p-1}$ weakly in $L^{p'}(Q_T)$, due to the pointwise convergence of the gradients, as follows:

$$\iint\limits_{\{w_n>k\,,\,|w_n-T_hw_n+T_kw_n-(T_kw)_\nu|<2k\}} (1+w_n)^{2-p}|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla(T_kw)_\nu\,(1-\psi_\delta)$$
$$\leq \iint\limits_{\{w_n>k\}} (1+T_{h+4k}w_n)^{2-p}|\nabla T_{h+4k}w_n|^{p-1}\,|\nabla(T_kw)_\nu|$$
$$= \iint\limits_{\{w>k\}} (1+T_{h+4k}w)^{2-p}|\nabla T_{h+4k}w|^{p-1}\,|\nabla T_kw| + \omega(n,\nu) = \omega(n,\nu)\,.$$

Therefore, we have shown that

$$\boxed{\text{II}} \geq \quad c(k)\boxed{\text{C1}} \quad + \omega(n,\nu)\,.$$

Finally, we will show that

$$\boxed{\text{I}} \geq \omega(n,\nu,h)\,. \tag{77}$$

Once this is done, by putting together all the estimates for integrals $\boxed{\text{I}}$–$\boxed{\text{V}}$, we obtain that the term $\boxed{\text{C1}}$ in (75) satisfies

$$\boxed{\text{C1}} \leq \omega(n,\nu,\delta)\,.$$

This in turn will imply

$$\boxed{\text{C}} \leq \omega(n,\nu,\delta)\,,$$

and this will complete the proof of (73), and therefore of Proposition 4.1. In order to prove (77), we observe that

$$U_{n,\nu,h} = T_{h+k}(w_n-(T_kw)_\nu) - T_{h-k}(w_n-T_kw_n)\,.$$

Therefore

$$
\begin{aligned}
\boxed{\mathrm{I}} = &\int_0^T \langle (w_n - (T_k w)_\nu)_t \, , \, T_{h+k}(w_n - (T_k w)_\nu) \, (1-\psi_\delta) \rangle \, dt \\
&+ \iint_{Q_T} ((T_k w)_\nu)_t \, T_{h+k}(w_n - (T_k w)_\nu) \, (1-\psi_\delta) \\
&- \int_0^T \langle (w_n)_t \, , \, T_{h-k}(w_n - T_k w_n) \, (1-\psi_\delta) \rangle \, dt \\
&= \int_0^T \langle \big(J_{h+k}(w_n - (T_k w)_\nu)\big)_t \, , (1-\psi_\delta) \rangle \, dt && \boxed{\text{I-a}} \\
&\quad + \iint_{Q_T} ((T_k w)_\nu)_t \, T_{h+k}(w_n - (T_k w)_\nu) \, (1-\psi_\delta) && \boxed{\text{I-b}} \\
&\quad - \int_0^T \langle \big(H_{h,k}(w_n)\big)_t \, , (1-\psi_\delta) \rangle \, dt \, , && \boxed{\text{I-c}}
\end{aligned}
$$

where

$$
J_{h+k}(s) = \int_0^s T_{h+k}(\sigma) \, d\sigma \, , \qquad H_{h,k}(s) = \int_0^s T_{h-k}(\sigma - T_k \sigma) \, d\sigma \, .
$$

Let us remark that, for fixed h and k, the functions J_{h+k} and $H_{h,k}$ have linear growth. Observe also that in the integrals of $\boxed{\mathrm{I}}$, we can not pass to the limit when considering the value of a function at the point T. As far as the integral $\boxed{\text{I-a}}$ is concerned, we can write:

$$
\begin{aligned}
\boxed{\text{I-a}} = \int_\Omega J_{h+k}(w_n(T) - (T_k w)_\nu(T)) \, dx - \int_\Omega J_{h+k}(w_0 - T_k w_0) \, dx \\
+ \iint_{Q_T} J_{h+k}(w - T_k w) \, (\psi_\delta)_t + \omega^{h,\delta}(n,\nu) \, ,
\end{aligned}
$$

where $w_0 = \varphi(v_0)$. Similarly,

$$
\boxed{\text{I-c}} = -\int_\Omega H_{h,k}(w_n(T)) \, dx + \int_\Omega H_{h,k}(w_0) \, dx - \iint_{Q_T} H_{h,k}(w) \, (\psi_\delta)_t + \omega^{h,\delta}(n) \, .
$$

Moreover, by the definition of $(T_k w)_\nu$,

$$
\begin{aligned}
\boxed{\text{I-b}} &= \nu \iint_{Q_T} (T_k w - (T_k w)_\nu) \, T_{h+k}(w_n - (T_k w)_\nu) \, (1-\psi_\delta) \\
&= \nu \iint_{Q_T} (T_k w - (T_k w)_\nu) \, T_{h+k}(w - (T_k w)_\nu) \, (1-\psi_\delta) + \omega^{\nu,h,\delta}(n) \geq \omega^{\nu,h,\delta}(n) \, ,
\end{aligned}
$$

since the last integral is nonnegative. Therefore we have shown that

$$\boxed{\mathrm{I}} \geq \omega^{h,\delta}(n,\nu) + \int_\Omega \big[J_{h+k}(w_n(T) - (T_k w)_\nu(T)) - H_{h,k}(w_n(T))\big]\,dx \qquad \boxed{\alpha}$$

$$+ \iint_{Q_T} \big[J_{h+k}(w - T_k w) - H_{h,k}(w)\big]\,(\psi_\delta)_t \qquad \boxed{\beta}$$

$$+ \int_\Omega \big[H_{h,k}(w_0) - J_{h+k}(w_0 - T_k w_0)\big]\,dx\,, \qquad \boxed{\gamma}$$

Let us show that the term $\boxed{\alpha}$ is nonnegative. For fixed z such that $0 \leq z \leq k$, let us define

$$R(s) := J_{h+k}(s - z) - H_{h,k}(s)\,.$$

We want to show that $R(s) \geq 0$ for every $s \geq 0$. Indeed, this is easy to see for $0 \leq s \leq k$, while, for $s > k$,

$$R'(s) = T_{h+k}(s - z) - T_{h-k}(s - k) = T_{2k}(s - T_h(s) + k - z) \geq 0\,.$$

Let's now look at term $\boxed{\gamma}$. If we define

$$P_h(s) := H_{h,k}(s) - J_{h+k}(s - T_k s)\,,$$

then

$$P_h'(s) = T_{h-k}(s - T_k s) - T_{h+k}(s - k) = -T_{2k}(s - T_h s + T_k s - k)\,,$$

therefore $|P_h'(s)| \leq 2k$, which implies $|P_h(w_0)| \leq 2k\,|w_0|$. Since $P_h(s)$ goes to zero pointwise for $h \to +\infty$, one has

$$\boxed{\gamma} = \omega^\delta(h)$$

by Lebesgue's dominated convergence theorem. For similar reasons, the same holds for the integral $\boxed{\beta}$. This concludes the proof of (77) and of Proposition 4.1. □

4.2. **Proof of Theorem** (1.2)**.** Since v is a reachable solution, we may find three sequences: $\{v_n\}_n$, $\{h_n\}_n$ and $\{g_n\}_n$ as in Definition 3.3.

Notice that then, for every $T > 0$, $T_k(v_n) \to T_k(v)$ strongly in $L^p(0,T; W_0^{1,p}(\Omega))$, by Proposition 4.1.

We set $u_n = (p-1)\log(\frac{v_n}{p-1} + 1)$, then by a direct computation one can check that

$$(u_n)_t - \Delta_p u_n = |\nabla u_n|^p + f + \frac{h_n}{1 + \varphi(v_n)} \text{ in } \mathcal{D}'(Q)\,. \tag{78}$$

Fixed $T > 0$, observe that, by using the definition of v_n and having in mind

$$\Big(\frac{v_n}{p-1} + 1\Big)^{p-1} \to \Big(\frac{v}{p-1} + 1\Big)^{p-1} \quad \text{strongly in } L^1(Q_T)\,,$$

we may conclude easily that

$$u_n \to u \text{ in } L^q(Q_T)\,, \quad \text{for all } q < \infty\,. \tag{79}$$

We claim that

$$\frac{h_n}{1 + \varphi(v_n)} \to 0 \text{ in } \mathcal{D}'(Q). \tag{80}$$

To prove the claim let $\phi(x,t)$ be a function in $\mathcal{C}_0^\infty(Q)$; we want to prove that

$$\lim_{n\to\infty}\iint_Q \phi \frac{h_n}{\varphi(v_n)+1}\,dx = 0\,.$$

We assume that supp $\phi \subset Q_T$, and we use the assumption on μ: let $A\subset Q_T$ be such that $\mathrm{cap}_p(A)=0$ and $\mu_s \llcorner Q_T$ is concentrated on A. Then for all $\varepsilon>0$ there exists an open set $U_\varepsilon \subset Q_T$ such that $A\subset U_\varepsilon$ and $\mathrm{cap}_p(U_\varepsilon)\le \varepsilon/2$. Thus, we can find a function $\psi_\varepsilon \in \mathbf{W}_T$ such that $\psi_\varepsilon \ge \chi_{U_\varepsilon}$ and $||\psi_\varepsilon||_{\mathbf{W}_T}\le\varepsilon$.

Let us take $\dfrac{\psi_\varepsilon}{1+\varphi(v_n)}$ as test function in (13) to get

$$\iint_{Q_T}\frac{\varphi(v_n)_t\psi_\varepsilon}{1+\varphi(v_n)} + \iint_{Q_T}\frac{|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla\psi_\varepsilon}{1+\varphi(v_n)} - \iint_{Q_T}\psi_\varepsilon\frac{\varphi'(v_n)|\nabla v_n|^p}{(1+\varphi(v_n))^2} = \iint_{Q_T} f\psi_\varepsilon + \iint_{Q_T}\frac{h_n\psi_\varepsilon}{1+\varphi(v_n)}\,.$$

Having in mind that $f,\psi_\varepsilon\ge 0$, two terms can be dropped and so we obtain

$$\iint_{U_\varepsilon}\frac{h_n}{1+\varphi(v_n)} \le \iint_{Q_T}\frac{h_n\psi_\varepsilon}{1+\varphi(v_n)} \le \iint_{Q_T}\frac{\varphi(v_n)_t\psi_\varepsilon}{1+\varphi(v_n)} + \iint_{Q_T}\frac{|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla\psi_\varepsilon}{1+\varphi(v_n)}\,. \tag{81}$$

To estimate the first integral in the right hand side of (81),observe that

$$\begin{aligned}\iint_{Q_T}\frac{\varphi(v_n)_t\psi_\varepsilon}{1+\varphi(v_n)} &= \int_\Omega \log\big(1+\varphi(v_n(x,T))\big)\psi_\varepsilon(x,T)\,dx - \int_\Omega \log\big(1+g_n(x)\big)\psi_\varepsilon(x,0)\,dx \\ &\qquad - \iint_{Q_T}\log\big(1+\varphi(v_n)\big)(\psi_\varepsilon)_t \\ &\le \int_\Omega \log\big(1+\varphi(v_n(x,T))\big)\psi_\varepsilon(x,T)\,dx - \iint_{Q_T}\log\big(1+\varphi(v_n)\big)(\psi_\varepsilon)_t = \boxed{\mathrm{I1}} - \boxed{\mathrm{I2}}\end{aligned}$$

Using Hölder's inequality we obtain that

$$\boxed{\mathrm{I1}} \le \Big(\int_\Omega \log^2(1+\varphi(v_n(x,T)))\,dx\Big)^{\frac12}\Big(\int_\Omega|\psi_\varepsilon(x,T)|^2\,dx\Big)^{\frac12} \le C\Big(\int_\Omega|\psi_\varepsilon(x,T)|^2\,dx\Big)^{\frac12}$$

where in the last estimate we have used the inequality $\log^2(s+1)\le s+c$ and the bound

$$\max_{t\in[0,T]}\int_\Omega \varphi(v_n(x,t))\,dx \le C(T)\,.$$

It follows that

$$\boxed{\mathrm{I1}} \le C\max_{t\in[0,T]}\Big(\int_\Omega|\psi_\varepsilon(x,t)|^2\,dx\Big)^{\frac12} \le C\,||\psi_\varepsilon||_{\mathbf{W}_T}\le C\,\varepsilon,$$

by the fact that $\mathbf{W}_T \subset \mathcal{C}([0,T];L^2(\Omega))$ with a continuous inclusion. We now estimate $\boxed{\text{I2}}$.

$$|\boxed{\text{I2}}| = \Big| \iint_{Q_T} \log(1+\varphi(v_n))\,(\psi_\varepsilon)_t\,dx\,dt\Big| \\ \le \|\log(1+\varphi(v_n))\|_{L^p(0,T;V)} \|(\psi_\varepsilon)_t\|_{L^{p'}(0,T;V')} \\ \le \varepsilon\,\|\log(1+\varphi(v_n))\|_{L^p(0,T;V)}\,.$$

Therefore, we only have to show that the last norm is bounded. Since the function belongs to $L^q(Q_T)$ for every $q<\infty$, we only have to prove that

$$\iint_{Q_T} \big|\nabla\big(\log(1+\varphi(v_n))\big)\big|^p \le C\,.$$

This is due to the estimate (17) since

$$\iint_{Q_T} \big|\nabla\big(\log(1+\varphi(v_n))\big)\big|^p = \iint_{Q_T} \frac{(\varphi'(v_n))^p|\nabla v_n|^p}{(1+\varphi(v_n))^p} = \iint_{Q_T} \frac{|\nabla v_n|^p}{(1+\frac{v_n}{p-1})^p}.$$

Therefore, it yields

$$\iint_{Q_T} \frac{\varphi(v_n)_t\psi_\varepsilon}{1+\varphi(v_n)} \le C\varepsilon\,. \tag{82}$$

As far as the second integral in the right hand side of (81) is concerned, it is enough to apply Hölder's inequality to get

$$\iint_{Q_T} \frac{|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla\psi_\varepsilon}{1+\varphi(v_n)} \le \Big(\iint_{Q_T} \frac{|\nabla v_n|^p}{(1+\varphi(v_n))^{p'}}\Big)^{1/p'}\Big(\iint_{Q_T} |\nabla\psi_\varepsilon|^p\Big)^{1/p} \le C\varepsilon\,, \tag{83}$$

by (17). It follows from (81), (82) and (83) that

$$\iint_{U_\varepsilon} \frac{h_n}{1+\varphi(v_n)}\,dx\,dt \le C\varepsilon\,.$$

Now, by the property of μ we deduce that

$$\Big|\iint_{Q_T} \phi\frac{h_n}{1+\varphi(v_n)}\,dx\,dt\Big| \\ \le ||\phi||_\infty \iint_{U_\varepsilon} \frac{h_n}{1+\varphi(v_n)}\,dx\,dt + \iint_{Q_T\setminus U_\varepsilon} |\phi|\,h_n\,dx\,dt \le C||\phi||_\infty\,\varepsilon + \iint_{Q_T\setminus U_\varepsilon} |\phi|\,h_n\,dx\,dt\,.$$

Since $h_n \rightharpoonup \mu$ weakly in $\mathcal{M}(Q_T)$ and μ is concentrated on $A\subset U_\varepsilon$, we conclude that

$$\iint_{Q_T\setminus U_\varepsilon} |\phi|\,h_n\,dx\,dt \to 0 \qquad \text{as } n\to\infty.$$

By the arbitrariness of ε we get the desired result, hence the claim (80) follows.
To complete the proof we have just to show that

$$|\nabla u_n|^p \to |\nabla u|^p \text{ in } L^1(Q_T)$$

that means that

$$\frac{|\nabla v_n|^p}{(\frac{v_n}{p-1}+1)^p} \to \frac{|\nabla v|^p}{(\frac{v}{p-1}+1)^p} \text{ in } L^1(Q_T).$$

Since the sequence $\dfrac{|\nabla v_n|^p}{(\frac{v_n}{p-1}+1)^p}$ converges a.e. in Q_T to $\dfrac{|\nabla v|^p}{(\frac{v}{p-1}+1)^p}$, then by Vitali's theorem we only have to prove that it is equi-integrable. Let $E \subset Q_T$ be a measurable set. Then, for every $\delta \in (0, p-1)$ and $k > 0$,

$$\begin{aligned}\iint_E \frac{|\nabla v_n|^p}{(\frac{v_n}{p-1}+1)^p}\,dx\,dt &= \iint_{E\cap\{v_n\le k\}} \frac{|\nabla v_n|^p}{(\frac{v_n}{p-1}+1)^p}\,dx\,dt + \iint_{E\cap\{v_n> k\}} \frac{|\nabla v_n|^p}{(\frac{v_n}{p-1}+1)^p}\,dx\,dt\\ &\le \iint_E |\nabla T_k(v_n)|^p\,dx\,dt + \frac{1}{(\frac{k}{p-1}+1)^\delta}\iint_{Q_T} \frac{|\nabla v_n|^p}{(\frac{v_n}{p-1}+1)^{p-\delta}}\,dx\,dt\,.\end{aligned}$$

Since $p-\delta > 1$, then using (17), it follows that the last integral is uniformly bounded with respect to n, therefore choosing k large, the corresponding term can be made very small. Moreover, since $T_k(v_n) \to T_k(v)$ strongly on $L^p(0,T; W_0^{1,p}(\Omega))$ for any $k > 0$, it follows that the integral $\int_E |\nabla T_k(v_n)|^p\,dx\,dt$ is uniformly small if $|E|$ is small enough. The equi–integrability of $|\nabla u_n|^p$ follows immediately. Hence we may let n go to ∞ in (78) and so check that u solves

$$u_t - \Delta_p u = |\nabla u|^p + f \text{ in } \mathcal{D}'(Q).$$

■

4.3. **Nonuniqueness induced by singular perturbations of the initial data.** In this subsection we will prove another multiplicity result for problem (3) by considering a suitable class of singular measures. Without loss of generality we will assume that $f \equiv 0$. Let us begin with the following definition.

Definition 4.3. *Let μ be a spatial Radon measure on Ω whose total variation is finite. We say that v is a reachable solution to problem*

$$\text{(84)}\qquad \begin{cases} (\varphi(v))_t - \Delta_p v = 0 \text{ in } \mathcal{D}'(Q),\\ v(x,t) = 0 \text{ on } \partial\Omega\times(0,\infty),\\ \varphi(v(x,0)) = \mu,\end{cases}$$

if

1. *$T_k(v) \in L^p_{\rm loc}((0,+\infty); W_0^{1,p}(\Omega))$ for all $k > 0$.*
2. *For all $t > 0$ there exist both one–side limits $\lim_{\tau\to t^\pm}\varphi(v(\cdot,\tau))$ weakly-* in the sense of measures.*
3. *$\varphi(v(\cdot,t)) \rightharpoonup \mu$ weakly-* in the sense of measures as $t \to 0$.*
4. *There exist two sequences $\{v_n\}_n$ in $L^p_{\rm loc}([0,+\infty); W_0^{1,p}(\Omega))$ and $\{h_n\}_n$ in $L^\infty(\Omega)$ such that v_n is a weak solution to problem*

$$\text{(85)}\qquad \begin{cases} \big(\varphi(v_n)\big)_t - \Delta_p v_n = 0 & \text{in } \Omega\times(0,+\infty)\,,\\ v_n(x,t) = 0 & \text{on } \partial\Omega\times(0,+\infty)\,,\\ \varphi(v_n(x,0)) = h_n(x) & \text{in } \Omega\,,\end{cases}$$

and satisfying

(a) $h_n \rightharpoonup \mu$ weakly- in the sense of measures as $t \to 0$.*

(b) $|\nabla v_n|^{p-2}\nabla v_n \to |\nabla v|^{p-2}\nabla v$ strongly in $L^\sigma_{\rm loc}((0,+\infty); L^\sigma(\Omega))$ for $1 \le \sigma < \dfrac{N+p}{N+p-1}$.

(c) The sequence $\{\varphi(v_n)\}$ *is bounded in* $L^\infty_{\rm loc}([0,+\infty);L^1(\Omega))$ *and* $\varphi(v_n)\to\varphi(v)$ *strongly in* $L^q_{\rm loc}([0,+\infty);W^{1,q}_0(\Omega))$ *for all* $1\le q<\dfrac{N+p}{N+1}$.

Combining the arguments introduced in Subsection 3.2, the result obtained in [9] and [31] we get the existence of a reachable solution v enjoying the above properties. Nevertheless, some changes are needed to prove the strong convergence of the truncations. First of all we now take

$$(T_kw)_\nu(x,t)=\nu\int_0^t e^{\nu(s-t)}T_kw(x,s)\,ds\,.$$

Moreover, following [9], we have to replace the sequence $\{\psi_\delta\}_\delta$ with another sequence $\{\rho_\delta\}_\delta$, which will next be defined. Assume that μ is concentrated on a subset $E\subset\Omega$ such that $|E|=0$. Then, for every $\delta>0$, there exists a compact set $K_\delta\subset E$ such that

$$|E\setminus K_\delta|\le\delta\,,$$

and there exists $\phi_\delta\in C^\infty_0(\Omega)$ such that

$$0\le\phi_\delta\le1\,,\qquad \phi_\delta\equiv1 \text{ on } K_\delta\,.$$

Now consider the solution $\rho_\delta(x,t)$ to problem

$$\begin{cases}(\rho_\delta)_t-\Delta_p\,\rho_\delta &= 0 \quad \text{in } \Omega\times(0,+\infty)\,,\\ \rho_\delta(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,+\infty)\,,\\ \rho_\delta(x,0) &= \phi_\delta \quad \text{in } \Omega\,,\end{cases}\tag{86}$$

It is straightforward that $\rho_\delta\to0$ strongly in $L^p(0,T;W^{1,p}_0(\Omega))$ and $\rho_t\to0$ strongly in $L^{p}(0,T;W^{-1,p}(\Omega))$. Then the same technics of subsection 4.1 allow us to deduce that $T_k(v_n)\to T_k(v)$ strongly in $L^p(0,T;W^{1,p}_0(\Omega))$ for all $T>0$ and $k>0$.

Once we have reachable solutions in this case, we are able to state and prove the next multiplicity result.

Theorem 4.4. *Let* μ *be a bounded positive measure in* Ω*, concentrated on a subset* $E\subset\subset\Omega$ *such that* $|E|=0$ *. Let* v *be a reachable solution to problem* (84)*. Define* $u=(p-1)\log(\frac{v}{p-1}+1)$*, then* $u\in L^p_{\rm loc}([0,\infty);W^{1,p}_0(\Omega))$ *and verifies*

$$\begin{cases}u_t-\Delta_pu &= |\nabla u|^p \quad \textit{in } \mathcal{D}'(Q)\,,\\ u(x,t) &= 0 \quad \textit{on } \partial\Omega\times(0,\infty)\,,\\ u(x,0) &= 0 \quad \textit{in } \Omega\,.\end{cases}\tag{87}$$

Proof. Let $\{h_n\}_n\subset L^\infty(\Omega)$ be such that $h_n\ge0$ and $h_n\rightharpoonup\mu$ weakly-* in the sense of measures as $n\to\infty$. Consider v_n the unique solution to the approximated problem (85). We set $u_n=(p-1)\log(\frac{v_n}{p-1}+1)$, it is clear that $u_n\in L^p_{\rm loc}([0,\infty);W^{1,p}_0(\Omega))$, and u_n solves

$$\begin{cases}(u_n)_t-\Delta_pu_n &= |\nabla u_n|^p \quad \text{in } \mathcal{D}'(Q)\,,\\ u_n(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,\infty)\,,\\ u_n(x,0) &= \log(h_n+1) \quad \text{in } \Omega.\end{cases}\tag{88}$$

Following the same argument used in the proof of Theorem 1.2 and using the same techniques developed in Section 3 we obtain that $|\nabla u_n|^p \to |\nabla u|^p$ strongly in $L^1(Q_T)$ for all $T > 0$.

We claim that $\log(h_n+1) \to 0$ strongly in $L^1(\Omega)$. It is clear that $\{\log(h_n+1)\}_n$ is bounded in $L^q(\Omega)$ for all $1 \le q < \infty$. Since μ is concentrated on a set $E \subset\subset \Omega$ with $|E| = 0$, it follows that, for $\varepsilon \in (0,1)$, there exists an open set U_ε such that $E \subset U_\varepsilon \subset \Omega$ and $|U_\varepsilon| \le \varepsilon$. Recalling that $\log(1+s) \le Cs^{1/2}$ and $\log(1+s) \le s$, for $s \ge 0$, one has

$$\int_\Omega \log(h_n+1)dx = C\int_{U_\varepsilon} h_n^{1/2}dx + \int_{\Omega\setminus U_\varepsilon} h_n dx$$
$$\le C\Big(\int_{U_\varepsilon} h_n dx\Big)^{1/2}|U_\varepsilon|^{1/2} + \int_{\Omega\setminus U_\varepsilon} h_n dx \le C\varepsilon^{1/2} + \int_{\Omega\setminus U_\varepsilon} h_n dx\,.$$

Since μ is concentrated on E, the last integral goes to 0 as n goes to ∞ and then the claim follows.

Hence, u solves problem (87). ■

5. Generation of a solution with measure data from a solution of the problem with gradient term

Proof of Theorem (1.3). Let $T > 0$ be fixed. We begin by proving that, for each $\delta \in (0,1]$, there is a positive constant such that

$$\int_{Q_T} |\nabla u|^p e^{\frac{\delta u}{1+\epsilon u}}\Big(1 - \frac{\delta}{(1+\epsilon u)^2}\Big) \le C \qquad \forall \epsilon > 0\,. \tag{89}$$

Since $|\nabla u|^p \in L^1(Q_T)$, then we reach that u is the entropy solution to problem (3), thus we can use $\big(e^{\frac{\delta u}{1+\epsilon u}} - 1\big)\chi_{(0,T)}$ as test function in (3) to get

$$\int_\Omega \Psi_{\epsilon\delta}(u(T))\,dx - \int_\Omega \Psi_{\epsilon\delta}(u_0)\,dx + \int_{Q_T} \frac{\delta}{(1+\epsilon u)^2} e^{\frac{\delta u}{1+\epsilon u}}|\nabla u|^p$$
$$= \int_{Q_T} \Big(e^{\frac{\delta u}{1+\epsilon u}} - 1\Big)|\nabla u|^p + \int_{Q_T} f\Big(e^{\frac{\delta u}{1+\epsilon u}} - 1\Big)\,, \tag{90}$$

where $\Psi_{\epsilon\delta}(s) = \displaystyle\int_0^s \Big(e^{\frac{\delta\sigma}{1+\epsilon\sigma}} - 1\Big)\,d\sigma$. We remark that $0 \le \Psi_{\epsilon\delta}(s) \le \frac{1}{\delta}e^{\delta s} \le \frac{1}{\delta}e^s$ for all $s \ge 0$. So it follows that

$$\int_\Omega \Psi_{\epsilon\delta}(u(T))\,dx - \int_\Omega \Psi_{\epsilon\delta}(u_0)\,dx \le \frac{1}{\delta}\sup_{t\in[0,T]}\int_\Omega e^u\,dx \le C\,.$$

Since $u \in L^p([0,T]; W_0^{1,p}(\Omega))$, (90) becomes

$$\int_{Q_T} f\Big(e^{\frac{\delta u}{1+\epsilon u}} - 1\Big) + \int_{Q_T} |\nabla u|^p e^{\frac{\delta u}{1+\epsilon u}}\Big(1 - \frac{\delta}{(1+\epsilon u)^2}\Big) \le \int_{Q_T} |\nabla u|^p + C \le C\,, \tag{91}$$

and so $f\Big(e^{\frac{\delta u}{1+\epsilon u}} - 1\Big) \ge 0$ implies that (89) is proved.Fixed $\delta < 1$, then we can pass to the limit in ε to reach that $\int_{Q_T} |\nabla u|^p e^{\delta u} \le C$ for all $\delta < 1$. Thus the regularity estimate (7) follows. Let us take $\delta = 1$, then

$$\int_{Q_T} |\nabla u|^p e^{\frac{u}{1+\epsilon u}} \Big(1 - \frac{1}{(1+\epsilon u)^2}\Big) \le C \qquad \forall \epsilon > 0\,.$$

Hence, up to subsequences, there exists a positive Radon measure μ such that

$$\mu = \lim_{\epsilon\to 0} |\nabla u|^p e^{\frac{u}{1+\epsilon u}} \Big(1 - \frac{1}{(1+\epsilon u)^2}\Big)\,, \tag{92}$$

the limit being taken weakly-* in the sense of measures in each Q_T.

Now consider the following auxiliary functions

$$v_\epsilon(x,t) = \int_0^{u(x,t)} e^{\frac{s}{(p-1)(1+\epsilon s)}}\, ds \qquad \text{and} \qquad w_\epsilon(x,t) = \int_0^{u(x,t)} e^{\frac{s}{1+\epsilon s}}\, ds\,, \tag{93}$$

and observe that $v_\epsilon(x,t) \uparrow v(x,t)$ and $w_\epsilon(x,t) \uparrow \varphi(v(x,t))$. Then performing easy computations, it yields

$$(w_\epsilon)_t + \Delta_p v_\epsilon = f\, e^{\frac{u}{1+\epsilon u}} + |\nabla u|^p e^{\frac{u}{1+\epsilon u}} \Big(1 - \frac{1}{(1+\epsilon u)^2}\Big)\,.$$

Thus, for every $\phi \in C_0^\infty(\Omega \times (0,+\infty))$, we have

$$-\int_{Q_T} w_\epsilon \phi_t + \int_{Q_T} |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \cdot \nabla\phi = \int_{Q_T} f\, e^{\frac{u}{1+\epsilon u}} \phi + \int_{Q_T} |\nabla u|^p e^{\frac{u}{1+\epsilon u}} \Big(1 - \frac{1}{(1+\epsilon u)^2}\Big)\phi\,, \tag{94}$$

where $T > 0$ is such that $\operatorname{supp}\phi \subset \Omega \times (0,T)$.

In order to let ϵ go to 0 in (94), we will analyze each term in (94). Since

$$\varphi(v) = e^u - 1 \in L^\infty((0,T); L^1(\Omega))\,,$$

Levi's monotone convergence Theorem implies $w_\epsilon \to \varphi(v)$ in $L^1(Q_T)$, and it follows that

$$\lim_{\epsilon\to 0} \int_{Q_T} w_\epsilon \phi_t = \int_{Q_T} \varphi(v)\phi_t\,. \tag{95}$$

To handle the second term of (94), we have to see that $e^u |\nabla u|^{p-1} \in L^1(Q_T)$. To this end, we apply the Gagliardo–Nirenberg inequality to the function

$$e^{\beta u/(p-1)} \in L^p((0,T); W^{1,p}(\Omega)) \cap L^\infty((0,T); L^{(p-1)/\beta}(\Omega)) \qquad \forall \beta < \frac{p-1}{p}\,,$$

getting

$$e^u \in L^\sigma(Q_T) \quad \text{for all } \sigma < 1 + \frac{p}{N}\,. \tag{96}$$

So if $\delta > 1 - \frac{p}{N(p-1)}$, then $\delta + (1-\delta)p < 1 + \frac{p}{N}$ holds; thus (96) implies $e^{\delta u}\, e^{(1-\delta)pu} \in L^1(Q_T)$. Since we also have $e^{\delta u}\, |\nabla u|^p \in L^1(Q_T)$ for all $\delta < 1$, by (7), it follows from Hölder's inequality (using $e^{\delta u}$ as a weight function) that

$$e^u |\nabla u|^{p-1} = e^{\delta u}\, |\nabla u|^{p-1}\, e^{(1-\delta)u} \in L^1(Q_T)\,,$$

and therefore

$$(97)\quad \lim_{\epsilon\to 0}\int_{Q_T} |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon\cdot\nabla\phi = \lim_{\epsilon\to 0}\int_{Q_T} e^{\frac{u}{1+\epsilon u}}\,|\nabla u|^{p-2}\nabla u\cdot\nabla\phi$$
$$= \int_{Q_T} e^{u}\,|\nabla u|^{p-2}\nabla u\cdot\nabla\phi = \int_{Q_T} |\nabla v|^{p-2}\nabla v\cdot\nabla\phi\,.$$

The first term in the right hand side of (94) can be easily handled: by (91) there is a positive constant satisfying

$$\int_{Q_T} f\,e^{\frac{u}{1+\epsilon u}} \le C \qquad \forall\epsilon > 0\,,$$

thus applying Levi's monotone convergence Theorem, it yields

$$(98)\qquad \lim_{\epsilon\to 0}\int_{Q_T} f\,e^{\frac{u}{1+\epsilon u}}\phi = \int_{Q_T} f\,e^{u}\phi = \int_{Q_T} f\,(1+\varphi(v))\phi\,.$$

Finally, we may let ϵ goes to 0 in the last term, by the definition of μ:

$$(99)\qquad \lim_{\epsilon\to 0}\int_{Q_T} |\nabla u|^p e^{\frac{u}{1+\epsilon u}}\Big(1-\frac{1}{(1+\epsilon u)^2}\Big)\phi = \int_{Q_T}\phi\,d\mu\,.$$

Hence, having in mind (95), (97), (98) and (99), one deduces from (94) that

$$\varphi(v)_t - \Delta_p v = f\,(1+\varphi(v)) + \mu$$

in the sense of distributions.
Notice that for all $k>0$, we have

$$|\nabla u|^p e^{\frac{u}{1+\varepsilon u}}\Big(1-\frac{1}{(1+\varepsilon u)^2}\Big)\chi_{\{u<k\}} \to 0$$

strongly in $L^1(Q_T)$ as $\varepsilon\to 0$ and then in this *formal sense* we could say that the measure μ is concentred in the set $\{u=\infty\}$. To show that μ is a singular measure seems to be an open problem.

Corollary 5.1. *There is, at most, a weak solution u of problem 3 satisfying*

$$e^{u/p}-1\in L^p_{\mathrm{loc}}((0,\infty);W_0^{1,p}(\Omega))\cap L^\infty_{\mathrm{loc}}((0,\infty);L^p(\Omega))\,.$$

Proof. If u is a weak solution to problem 3 satisfying the regularity stated above, then

$$e^u|\nabla u|^p\in L^1_{\mathrm{loc}}((0,\infty);L^1(\Omega))\,,$$

and it follows from Lebesgue's Theorem that

$$|\nabla u|^p e^{\frac{u}{1+\varepsilon u}}\Big(1-\frac{1}{(1+\varepsilon u)^2}\Big)\to 0 \quad \text{strongly in } L^1_{\mathrm{loc}}((0,\infty);L^1(\Omega))\,.$$

By Theorem 1.3, $v=(p-1)\big(e^{\frac{u}{p-1}}-1\big)$ solves problem

$$\begin{cases} \big(\varphi(v)\big)_t - \Delta_p v &= f(x,t)\,(1+\varphi(v)) & \text{in } \Omega\times(0,+\infty)\,,\\ v(x,t) &= 0 & \text{on } \partial\Omega\times(0,+\infty)\,,\\ v(x,0) &= (p-1)\big(e^{\frac{u_0(x)}{p-1}}-1\big) & \text{in } \Omega\,,\end{cases}$$

so that Theorem 3.14 implies that v is an entropy solution. Therefore, uniqueness for problem (3) follows from Theorem 3.15. ■

6. Finite time extinction

In this section we will show that in the case where $p < 2$ and $f(x,t) \equiv 0$, the "regular" solutions of (3) become zero in finite time, provided the initial datum u_0 is summable enough.

Let us begin by defining the meaning of "regular" solutions to (3).

Definition 6.1. *We say that u is a regular solution to problem (3) in Q_T if*

$$e^{u/(p-1)} - 1 \in L^p((0,T); W_0^{1,p}(\Omega)) \cap \mathcal{C}([0,T]; L^p(\Omega)),\ u_t \in L^{p'}((0,T); W^{-1,p'}(\Omega))$$

and for all $\phi \in L^p((0,T); W_0^{1,p}(\Omega))$ we have

$$\int_0^T \langle u_t, \phi\rangle + \int_0^T \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi = \int_0^T \int_\Omega |\nabla u|^p \phi. \tag{100}$$

We are able now to state the next result.

Theorem 6.2. *Assume that $\dfrac{2N}{N+2} \le p < 2$. If u is the regular solution of problem (3) in the sense of Definition 6.1, then there exists a positive, finite time t_0, depending on N, p and u_0 such that $u(x,t) \equiv 0$ for $t > t_0$.*

Proof. Since u is a regular solution to (3), we have

$$\int_\Omega e^{\frac{p}{p-1}u_0(x)}\,dx < \infty\,. \tag{101}$$

Multiplying (3) by $e^u(e^{u/(p-1)} - 1)$ and integrating in x, one obtains:

$$\frac{d}{dt}\int_\Omega \Psi(u(x,t))\,dx + c(p)\int_\Omega \left|\nabla(e^{\frac{u(x,t)}{p-1}} - 1)\right|^p dx = 0\,, \tag{102}$$

where

$$\Psi(s) = \int_0^s e^\sigma(e^{\frac{\sigma}{p-1}} - 1)\,d\sigma\,.$$

It is easy to check that

$$\Psi(s) \le c(p)(e^{\frac{s}{p-1}} - 1)^2 \qquad \text{for every } s \ge 0.$$

Therefore, by Sobolev's and Hölder's inequalities,

$$\begin{aligned}\int_\Omega \left|\nabla(e^{\frac{u(x,t)}{p-1}} - 1)\right|^p dx &\ge c_1(N,p)\left[\int_\Omega \left(e^{\frac{u(x,t)}{p-1}} - 1\right)^{p^*} dx\right]^{p/p^*} \\ &\ge c_2(N,p,|\Omega|)\left[\int_\Omega \left(e^{\frac{u(x,t)}{p-1}} - 1\right)^2 dx\right]^{p/2} \ge c_3(N,p,|\Omega|)\left[\int_\Omega \Psi(u(x,t))\,dx\right]^{p/2},\end{aligned}$$

where we have used the inequality $p^* \ge 2$, which is true under our assumptions on p. Then, if we set

$$\xi(t) = \int_\Omega \Psi(u(x,t))\,dx\,,$$

it follows from (102) that

$$\frac{\xi'(t)}{\xi(t)^{p/2}} \le -c_4 < 0\,.$$

Note that the assumption on u_0 corresponds to $\xi(0) < \infty$. Integrating in t, one obtains

$$\frac{2}{2-p}\big(\xi(t)^{\frac{2-p}{2}} - \xi(0)^{\frac{2-p}{2}}\big) \le -c_4 t\,.$$

Thus, as long as $\xi(t) > 0$, one has

$$\xi(t)^{\frac{2-p}{2}} \le \xi(0)^{\frac{2-p}{2}} - c_4\frac{2-p}{2}\,t\,.$$

Therefore, $\xi(t) \equiv 0$ for t large enough. ■

Theorem 6.3. *Assume now that* $1 < p < \dfrac{2N}{N+2}$ *and*

$$\int_\Omega e^{\frac{(2-p)(N-p)}{p(p-1)}u_0(x)}\,dx < \infty\,. \tag{103}$$

If u is the regular solution of problem (3)*, then there exists a positive, finite time t_0, depending on N, p and the integral* (103) *such that $u(x,t) \equiv 0$ for $t > t_0$.*

Proof. By an approximation argument and using the hypothesis on u_0 we reach that

$$\int_\Omega e^{\frac{(2-p)(N-p)}{p(p-1)}u(x,t)}\,dx + \int_0^T\int_\Omega |\nabla u(x,t)|^p e^{\frac{(2-p)(N-p)}{p(p-1)}u(x,t)}\,dx < \infty.$$

Multiplying (100) by $e^u(e^{u/(p-1)}-1)^\alpha$, with $\alpha \ge 1$, one obtains:

$$\frac{d}{dt}\int_\Omega \Psi(u(x,t))\,dx + \frac{\alpha}{p-1}\int_\Omega |\nabla u(x,t)|^p e^{\frac{p}{p-1}u(x,t)}\big(e^{\frac{u(x,t)}{(p-1)}}-1\big)^{\alpha-1}\,dx = 0\,, \tag{104}$$

where

$$\Psi(s) = \int_0^s e^\sigma(e^{\frac{\sigma}{p-1}}-1)^\alpha\,d\sigma\,.$$

Equality (104) can be read as

$$\frac{d}{dt}\int_\Omega \Psi(u(x,t))\,dx + c(\alpha,p)\int_\Omega \big|\nabla\big(e^{\frac{u(x,t)}{(p-1)}}-1\big)^{\frac{\alpha+p-1}{p}}\big|^p\,dx = 0\,, \tag{105}$$

On the other hand, it is easy to see that

$$\Psi(s) \le c(\alpha,p)(e^{\frac{s}{p-1}}-1)^{\alpha+1} \qquad \text{for every } s \ge 0.$$

Indeed, one has

$$\Psi(s) \sim \frac{p-1}{\alpha+1}(e^{\frac{s}{p-1}}-1)^{\alpha+1} \qquad \text{as } s \to 0^+\,,$$

$$\Psi(s) \sim e^{\frac{\alpha+p-1}{p-1}s} \qquad \text{as } s \to +\infty\,.$$

Therefore, if we choose

$$\alpha = \frac{2N}{p} - (N+1) \ge 1\,,$$

then, by Sobolev's and Hölder's inequalities,

$$\int_\Omega \left|\nabla(e^{\frac{u(x,t)}{p-1}}-1)^{\frac{\alpha+p-1}{p}}\right|^p dx \geq c_1(\alpha,N,p)\left[\int_\Omega \left(e^{\frac{u(x,t)}{p-1}}-1\right)^{\frac{(\alpha+p-1)p^*}{p}} dx\right]^{p/p^*}$$

$$= c_1(\alpha,N,p)\left[\int_\Omega \left(e^{\frac{u(x,t)}{p-1}}-1\right)^{\alpha+1} dx\right]^{p/p^*} \geq c_2(\alpha,N,p)\left[\int_\Omega \Psi(u(x,t))\,dx\right]^{p/p^*}.$$

Therefore we have shown that

$$\frac{d}{dt}\int_\Omega \Psi(u(x,t))\,dx + c_3(\alpha,N,p)\left[\int_\Omega \Psi(u(x,t))\,dx\right]^{p/p^*} \leq 0\,,$$

and from here one can conclude as in the previous Theorem, provided

$$\int_\Omega \Psi(u(x,0))\,dx < +\infty\,,$$

which is equivalent to (103). ■

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