BOUNDED SOLUTIONS TO THE 1–LAPLACIAN EQUATION WITH A CRITICAL GRADIENT TERM

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This work was started by the three authors. When Fuensanta Andreu passed away, the research was completed by the other two authors, who are responsible for the final form of the paper. They dedicate this work to Fuensanta.

She is and will be much missed.

* * *

Abstract. In this paper, we study the Dirichlet problem for an elliptic equation, in which the 1–Laplacian operator and a first order term appear. We introduce a suitable definition of solution and prove the existence of, at least, one bounded solution in $BV(\Omega)$ having no jump part. Moreover, a uniqueness result for small positive data is proved, and explicit examples of solutions are shown.

1. Introduction

This paper is concerned with the Dirichlet problem for an elliptic equation which involves the 1–Laplacian operator and lower order terms; namely:

$$
(1) \qquad \begin{cases} u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = |Du| + f\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,; \end{cases}
$$

where $f \in L^m(\Omega)$, with $m > N$, and $\Omega \subset \mathbb{R}^N$ is a bounded open set having Lipschitz–continuous boundary $\partial\Omega$. The natural space to study problems where the 1–Laplacian appears is $BV(\Omega)$, the space of functions $u \in L^1(\Omega)$ whose distributional derivatives are Radon measures with finite total variation. In this paper, we will provide a suitable notion of solution in $BV(\Omega)$, we will prove the existence of one solution,

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analyze some of its properties (in particular, we will show that this solution has no jumps) and obtain a partial uniqueness result. As far as we know, this paper is the first work that considers the 1–Laplacian and a gradient term not satisfying the "good" sign condition as in equation (3) below.

Recently, the Dirichlet problem for several elliptic and parabolic equations involving the 1–Laplacian operator has been analyzed (see [18] and references therein). The asymptotic behaviour of the corresponding p–Laplacian problems as p goes to 1, have been considered in [23, 15, 29, 30] (see also [25, 22]). In turn, several authors have focused their research on finding solutions to the 1–Laplacian problem; they include [3, 5, 8, 9, 16, 17], and references therein. Other related works are [26, 21]. The interest in this framework comes, on the one hand, from an optimal design problem in the theory of torsion and related geometrical problems (see [24]) and, on the other, from the variational approach to image restoration (see [6]).

The simplest of the elliptic problems involving the 1–Laplacian is obviously the Dirichlet problem for the equation

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right) = f\,. \tag{2}$$

It is worth noting some features of solutions to problem (2) (see [29, 30]).

(i) Existence of BV–solutions is only guaranteed for data small enough.
(ii) Solutions can be infinite in a set of positive measure
(iii) The boundary condition only holds in a weak sense (see (10) in Definition 1 below); in general, $u|_{\partial\Omega} \neq 0$.
(iv) There is no uniqueness at all: given a solution u, we also have that $g(u)$ is a solution, for every increasing function g.

Other equations have been studied which, besides the 1–Laplacian, include lower order terms. We point out that these lower order terms have a regularizing effect and some of the above features do not hold. We also remark that an anisotropic case has been considered and it also shows a regularizing effect (see [28]).

The presence of a zero order term u in the left hand side, namely, the equation

$$u - \mathrm{div}\left(\frac{Du}{|Du|}\right) = f\,,$$

has been analyzed as a way of approaching the study of the parabolic problem of the total variation flow (see [6]). For this equation, the boundary condition is still verified in a weak sense, but the above features (i), (ii) and (iv) do not hold since solutions satisfy the following properties

- There is always a solution, regardless of the size of the datum.
- Solutions belong to $BV(\Omega)$ (and, in general, have a non empty jump part).
- There is uniqueness of solution.

A gradient term $|Du|$ satisfying the "good" sign condition has been studied in [27]:

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right)+|Du|=f\geq 0\,. \tag{3}$$

This equation, with $f\equiv 0$, occurs in the level set formulation of the inverse mean curvature flow (see [21]). Related developments can be found in [31]; the framework of these works, however, is different since Ω is unbounded and the datum vanishes. It is shown in [27] that the solution behaves very differently from (2). Indeed,

- There is always a solution, even in the case where the datum is large.
- Solutions belong to $BV(\Omega)$, but they have no jump part.
- The boundary condition holds in the trace sense, that is, the value is attained pointwise on the boundary.
- There is uniqueness of solution.

Note that in this case, the gradient term acts as an absorption term and a regularizing effect can be expected.

Our aim is to study problem (1), in which the gradient term acts as a source term, at least in the case where f is positive. We will analyze whether the features already seen in the above equations also hold for the solutions to (1). On the positive side, we will prove that

- There is always a solution, regardless of the size of the datum.
- Solutions belong to $BV(\Omega)$ and they have no jump part.

The last property is due to presence of the gradient term. This might be surprising, since that term has not always the "good" sign. On the other hand:

- The boundary condition holds in a weak sense since the value is not always attained on the boundary.
- We are not able to prove a full uniqueness result, only a partial one.

As always when studying equations where the 1–Laplacian is involved, we have to give a sense to the quotient $\frac{Du}{|Du|}$, where in general

Du is not a function but a Radon measure. Following [3], the meaning of that quotient is given through a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ which satisfies $\|\mathbf{z}\|_\infty \le 1$ and $(\mathbf{z}, Du) = |Du|$. Moreover, the definition of the pairing $(\mathbf{z}, Du)$ relies on the theory of bounded divergence–measure vector fields by Anzellotti [7] and Chen–Frid [14]. This theory also defines a notion of a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ (that we will consider to give sense to the boundary condition in a weak sense), and provides a Green's formula.

When dealing with an elliptic problem involving the p–Laplacian and a gradient term with natural growth, a change of unknown can be performed: $v = (p-1)\big(e^{u/(p-1)} - 1\big)$, which generalizes that by Cole–Hopf. This change allows us to obtain a simplified equation without gradient terms since $-\Delta_p u = |\nabla u|^p + f$ becomes $-\Delta_p v = f\big(1 + \frac{v}{p-1}\big)^{p-1}$ (see [20], and also [1], [12] and [32]). Unluckily, this change of unknown does not work for the 1–Laplacian. Thus, we will consider approximating problems involving the p–Laplacian and a gradient term with natural growth, and perform the Cole–Hopf change of variable for these approximating problems and look for a priori estimates which must not depend on p.

This paper is organized as follows. The next section is devoted to preliminaries: we introduce our notation, and auxiliary results on BV–functions and L^∞–divergence–measure vector fields. In Section 3, we define our notion of solution and prove the main existence result. Section 4 is devoted to uniqueness and Section 5 includes some explicit examples of radial solutions to problem (1).

2. Preliminaries

In this section we introduce some notation and some preliminary results that we will use in what follows. Throughout this paper, we will consider $N \ge 2$, and $\mathcal{H}^{N-1}(E)$ will denote the $(N-1)$–dimensional Hausdorff measure of a set E and $|E| = \mathcal{L}^N(E)$ its Lebesgue measure.

2.1. **Notation.** In this paper, Ω will always denote an open subset of $\mathbb{R}^N$ with Lipschitz boundary. Thus, an outward normal unit vector $\nu(x)$ is defined for $\mathcal{H}^{N-1}$–almost every $x \in \partial\Omega$. We will make use of the usual Lebesgue and Sobolev spaces, denoted by $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively (see, for instance, [13]).

We recall that for a Radon measure μ in Ω and a Borel set $A \subseteq \Omega$ the measure $\mu \llcorner A$ is defined by $(\mu \llcorner A)(B) = \mu(A \cap B)$ for any Borel set $B \subseteq \Omega$. If a measure μ is such that $\mu = \mu \llcorner A$ for a certain Borel set A, the measure μ is said to be concentrated on A.

Two auxiliary real functions will often be used:

$$T_k(s) = (|s| \wedge k)\,\mathrm{sign}\,(s)\,, \quad G_k(s) = s - T_k(s)\,, \quad s \in \mathbb{R}\,,\ k > 0\,.$$

2.2. **The space BV and lower semicontinuous functionals.** The space $BV(\Omega)$ of *functions of bounded variation* is defined as the space of functions $u \in L^1(\Omega)$ whose distributional gradient Du is a vector valued Radon measure on Ω with finite total variation.

We recall that the functional defined by

$$u \mapsto \int_\Omega |Du| \tag{4}$$

is lower semicontinuous with respect to the convergence in $L^1(\Omega)$. Similarly, if we fix $\varphi \in C_0^1(\Omega)$, with $\varphi \geq 0$, the functional defined by

$$u \mapsto \int_\Omega \varphi |Du|\,,$$

is lower semicontinuous in $L^1(\Omega)$. Furthermore, every function $u \in BV(\Omega)$ has a trace defined on $\partial\Omega$, and the functional defined by

$$u \mapsto \int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1}\,, \tag{5}$$

is also lower semicontinuous in $L^1(\Omega)$.

For every $u \in BV(\Omega)$, the Radon measure Du can be decomposed into its absolutely continuous and singular parts with respect to the Lebesgue measure: $Du = D^a u + D^s u$. So, for each measurable set E, we have $D^a u(E) = \int_E \nabla u(x)\, d\mathcal{L}^N(x)$, where ∇u is the Radon–Nikodým derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$.

We denote by S_u the set of all $x \in \Omega$ such that x is not a Lebesgue point of u. We say that $x \in S_u$ is an *approximate jump point of* u if there exist $u^+(x) > u^-(x) \in \mathbb{R}$ and $\nu_u(x) \in S^{N-1}$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{|B_\rho^+(x, \nu_u(x))|} \int_{B_\rho^+(x,\nu_u(x))} |u(y) - u^+(x)|\, dy = 0$$

$$\lim_{\rho \downarrow 0} \frac{1}{|B_\rho^-(x, \nu_u(x))|} \int_{B_\rho^-(x,\nu_u(x))} |u(y) - u^-(x)|\, dy = 0,$$

where

$$B_\rho^+(x, \nu_u(x)) = \{y \in B_\rho(x)\ :\ \langle y - x, \nu_u(x)\rangle > 0\}$$

and

$$B_\rho^-(x, \nu_u(x)) = \{y \in B_\rho(x)\ :\ \langle y - x, \nu_u(x)\rangle < 0\}.$$

We denote by J_u the set of approximate jump points of u. By the Federer-Vol'pert Theorem [2, Theorem 3.78], we know that S_u is countably $\mathcal{H}^{N-1}$–rectifiable and $\mathcal{H}^{N-1}(S_u \backslash J_u) = 0$. Moreover, $Du \llcorner J_u =$

$(u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u$. Using S_u and J_u, we may split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$ defined by

$$D^j u = D^s u \llcorner J_u \quad \text{and} \quad D^c u = D^s u \llcorner (\Omega \backslash S_u)\,,$$

respectively. Thereby

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

Moreover, if $x \in J_u$, then $\nu_u(x) = \frac{Du}{|Du|}(x)$, $\frac{Du}{|Du|}$ being the Radon–Nikodým derivative of Du with respect to its total variation $|Du|$.

If x is a Lebesgue point of u, then $u^+(x) = u^-(x)$ for any choice of the normal vector and we say that x is an approximate continuity point of u. We define the approximate limit of u by $\tilde{u}(x) = u^+(x) = u^-(x)$. The *precise representative* $u^* : \Omega\backslash(S_u\backslash J_u) \to \mathbb{R}$ of u is defined as equal to $\tilde{u}$ on $\Omega\backslash S_u$ and equal to $\frac{u^+ + u^-}{2}$ on J_u. It is well known (see for instance [2, Corollary 3.80]) that if ρ is a symmetric mollifier, then the mollified functions $u \star \rho_\epsilon$ converges pointwise to u^* in its domain.

We also need a chain rule for functions in $BV(\Omega)$. Since we will only apply it for functions having empty jump set, we will state it only in this simple case. If $u \in BV(\Omega) \cap L^\infty(\Omega)$ is such that $D^j u = 0$, and f is a real Lipschitz–continuous function, then $v = f \circ u$ belongs to $BV(\Omega)$ and $Dv = f'(u^*)Du$.

For further information concerning functions of bounded variation we refer to [2] or [19].

2.3. **Green's formula.** In order to give sense to our notion of solution, we have to define certain pairings between vector fields and derivatives of BV–functions, and to state a Green's formula. This theory was introduced in [7] and, from a different point of view, it is also studied in [14]. We will denote by $\mathcal{DM}_\infty(\Omega)$ the space of all vector fields $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ such that its distributional divergence $\operatorname{div} \mathbf{z}$ is a Radon measure whose total variation is finite. Throughout this subsection, we will consider a vector field $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. In [14, Proposition 3.1] is proved that if $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$, then the total variation of the measure $|\operatorname{div} \mathbf{z}|$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$. Since the precise representative u^* is $\mathcal{H}^{N-1}$–a.e. equal to a bounded Borel function, it yields that u^* is summable with respect to the measure $\operatorname{div} \mathbf{z}$.

In the spirit of [7], in [30] the following distribution on Ω is defined. For every $\varphi \in C_0^\infty(\Omega)$, we write

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u^* \varphi \operatorname{div} \mathbf{z} - \int_\Omega u\mathbf{z} \cdot \nabla\varphi\,. \tag{6}$$

Note that the previous remark implies that every term in the above definition has sense.

Then it was proved in [30] that

$$|\langle(\mathbf{z}, Du), \varphi\rangle| \le \|\varphi\|_\infty \|\mathbf{z}\|_{L^\infty(U)} \int_U |Du|$$

holds for every open set $U \subset \Omega$ and every $\varphi \in C_0^\infty(U)$. Therefore, $(\mathbf{z}, Du)$ is a Radon measure, and $|(\mathbf{z}, Du)| \le \|\mathbf{z}\|_\infty |Du|$ as measures.

On the other hand, for every $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$, a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ is defined in [7] and denoted by $[\mathbf{z}, \nu]$. With the above definitions, the following Green formula holds

$$\int_\Omega u^* \, d\mu + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} [\mathbf{z}, \nu] u \, d\mathcal{H}^{N-1}\,, \tag{7}$$

where $\mu = \operatorname{div} \mathbf{z}$.

We will also need the following technical results. The proof of the first one can be found in [27, Proposition 2.3].

Proposition 1. *Let $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and assume that $u, w \in BV(\Omega) \cap L^\infty(\Omega)$ satisfy $D^j u = D^j w = 0$, then*

$$(w\mathbf{z}, Du) = w^*(\mathbf{z}, Du) \quad \textit{as Radon measures.}$$

Proposition 2. *For every $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$, $[u\mathbf{z}, \nu] = u[\mathbf{z}, \nu]$ holds $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$.*

Proof. Given $\phi \in C(\partial\Omega)$, we may find a function $\varphi \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ satisfying $\varphi|_{\partial\Omega} = \phi$. For instance, this function φ can be taken as the solution to the Dirichlet problem for Laplace's equation (recall that bounded Lipschitz domains satisfy, uniformly, an exterior cone condition and so every continuous function on the boundary is the trace of a harmonic function on Ω).

It is straightforward that the next equalities hold in the sense of distributions:

$$\operatorname{div}(u\mathbf{z}) = u^* \operatorname{div} \mathbf{z} + (\mathbf{z}, Du)$$

$$(\mathbf{z}, D(u\varphi)) = \varphi(\mathbf{z}, Du) + u\mathbf{z} \cdot \nabla\varphi$$

Then, applying Green's formula twice,

$$\begin{aligned}\int_{\partial\Omega}\phi[u\mathbf{z},\nu]\,d\mathcal{H}^{N-1} &= \int_\Omega \varphi\mathrm{div}\,(u\mathbf{z}) + \int_\Omega u\mathbf{z}\cdot\nabla\varphi \\ &= \int_\Omega \varphi u^*\mathrm{div}\,\mathbf{z} + \int_\Omega \varphi(\mathbf{z},Du) + \int_\Omega u\mathbf{z}\cdot\nabla\varphi \\ &= \int_\Omega \varphi u^*\mathrm{div}\,\mathbf{z} + \int_\Omega (\mathbf{z},D(u\varphi)) \\ &= \int_{\partial\Omega}\phi u[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\,.\end{aligned}$$

Since this equality holds for all $\phi\in C(\partial\Omega)$, the proof is completed. ■

3. Main Result

We introduce the following concept of solution to problem (1).

Definition 1. Given $f\in L^m(\Omega)$, with $m>N$, we say that u is a *weak solution* of problem (1), if $u\in BV(\Omega)\cap L^\infty(\Omega)$ is such that $D^ju=0$ and there exists a vector field $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$, with $\|\mathbf{z}\|_\infty\le 1$, satisfying

$$u-\mathrm{div}\,\mathbf{z}=|Du|+f\quad\text{in }\ \mathcal{D}'(\Omega)\,, \tag{8}$$

$$(\mathbf{z},Du)=|Du|\quad\text{as measures in }\ \Omega\,, \tag{9}$$

and

$$[\mathbf{z},\nu]\in\ \mathrm{sign}\,(-u)\ \ \mathcal{H}^{N-1}\text{–a.e. on }\ \partial\Omega \tag{10}$$

Remark 1. It follows from the definition that

$$-\mathrm{div}\,(e^u\mathbf{z})=(f-u)e^u \tag{11}$$

holds in the sense of distributions. To see this, first consider $\varphi\in C_0^\infty(\Omega)$ and obtain

$$\langle(\mathbf{z},De^u),\varphi\rangle=-\int_\Omega\varphi(e^u)^*\ \mathrm{div}\,\mathbf{z}-\int_\Omega e^u\,\mathbf{z}\cdot\nabla\varphi$$

and so

$$-\mathrm{div}\,(e^u\mathbf{z})=-(e^u)^*\ \mathrm{div}\,\mathbf{z}-(\mathbf{z},De^u) \tag{12}$$

holds in the sense of distributions. Now apply Proposition 2.2 in [27] to get that the Radon–Nikodým derivative of $(\mathbf{z},Du)$ with respect to $|Du|$ is equal to the Radon–Nikodým derivative of $(\mathbf{z},De^u)$ with respect to $|De^u|$. Thus, (9) implies $(\mathbf{z},De^u)=|De^u|$. We point out that (8) states an equality between two measures and the function $(e^u)^*$ is summable with respect to each of them. Hence, the previous equality, the chain rule and (8) yield

$$-\operatorname{div}(e^u\mathbf{z}) = -(e^u)^* \ \operatorname{div}\mathbf{z} - |De^u| \\ = (e^u)^*\big(-\operatorname{div}\mathbf{z} - |Du|\big) = e^u(f-u)$$

as measures.

Remark 2. As a consequence of Green's formula, we may deduce a variational formulation of solution. Indeed, for every $v \in BV(\Omega)\cap L^\infty(\Omega)$, multiplying by $u-v$ in (8) and taking (9) and (10) into account, we obtain

$$\int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} - \int_\Omega (\mathbf{z}, Dv) + \int_{\partial\Omega} v[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} \\ = \int_\Omega (u-v)^*|Du| + \int_\Omega (f-u)(u-v)\,. \tag{13}$$

Remark 3. We explicitly observe that, if f is nonnegative, then u is nonnegative. Indeed, take $u\chi_{\{u<0\}}$ as test function in (11) to obtain

$$\int_{\{u<0\}} u^2 e^u \\ = -\int_{\{u<0\}} (e^u)^*(\mathbf{z}, Du) - \int_{\partial\Omega\cap\{u<0\}} e^u|u|\, d\mathcal{H}^{N-1} + \int_{\{u<0\}} f\, u\, e^u \le 0\,,$$

since each term in the right hand side is nonpositive. Hence, $u \ge 0$.

Theorem 1. *If $f \in L^m(\Omega)$, for some $m > N$, then there exists a weak solution to problem* (1).

Proof. The proof will be divided in several steps.

Step 1. *Approximating problems.*

To prove the existence of solution to problem (1) we consider approximating problems related with the p-Laplacian:

$$\begin{cases} u_p - \Delta_p(u_p) = |\nabla u_p|^p + f & \text{in } \Omega \\ u_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where, as usual, $\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. It is well known (see [10] and [11]), that for every $p > 1$ there exists a bounded, weak solution u_p of (14).

Step 2. *L^∞–estimate.*

This subsection is devoted to prove the following result:

Proposition 3. *There exists a constant C_1, depending only on N, $\|f\|_m$ and $|\Omega|$ such that*

$$\|u_p\|_\infty \le C_1\,, \tag{15}$$

for all $p \in (1, \frac{3}{2})$.

To this aim we will need some auxiliary results. The first lemma shows that the constant in Sobolev's inequality does not depend on p for p close to 1.

Lemma 1. *There exists a positive constant $C_2 = C_2(N) > 0$ such that*

$$\int_\Omega |\nabla v|^p \ge C_2 \Big(\int_\Omega |v|^{p^*} \Big)^{p/p^*}$$

for every $p \in (1, \frac{3}{2})$, for every $v \in W_0^{1,p}(\Omega)$.

Proof. As it is well–known (see, for instance, [13]), one has

$$\int_\Omega |\nabla v|^p \ge \Big(\frac{N-p}{p(N-1)} \Big)^p \Big(\int_\Omega |v|^{p^*} \Big)^{p/p^*}$$

and it is straightforward to check that the last constant is greater than $\left(\frac{2N-3}{3N-2}\right)^2$, for $1 < p < \frac{3}{2}$. □

Lemma 2. *There exists a positive constant $C_3 > 0$ such that*

$$e^{2s} - 1 \le C_3 \big(e^{\frac{2s}{p}} - 1 \big)^p$$

for every $p \in (1, 2)$, for every $s \ge 1$. For instance, one may take $C_3 = e + 1$.

Lemma 3. *(see [33]) Let ϕ be a nonnegative, nonincreasing function defined on the half line $[k_0, \infty)$. Suppose that there exist positive constants A, γ, β, with $\beta > 1$, such that*

$$\phi(h) \le \frac{A}{(h-k)^\gamma} \phi(k)^\beta$$

for every $h > k \ge k_0$. Then $\phi(k) = 0$ for every $k \ge k_1$, where

$$k_1 = k_0 + A^{1/\gamma} 2^{\beta/(\beta-1)} \phi(k_0)^{(\beta-1)/\gamma}\,.$$

Proof of Proposition 3. For $k \ge k_0$ (with k_0 to be chosen hereafter) we use $(e^{2|G_k u_p|}-1)\operatorname{sign} u_p$ as a test function in (14). After cancelling the integral which comes from the first order term, we obtain

$$
\underbrace{\int_{A_k} |u_p|\left(e^{2|G_k u_p|}-1\right)}_{(A)} + \underbrace{\int_{A_k} e^{2|G_k u_p|}|\nabla u_p|^p}_{(B)} \le \underbrace{\int_{A_k} |f|\left(e^{2|G_k u_p|}-1\right)}_{(C)}, \tag{16}
$$

where

$$
A_k = A_{k,p} = \left\{x \in \Omega \ : \ |u_p(x)| > k\right\}.
$$

Let us estimate the integrals in the previous formula. Using Lemma 1, we obtain

$$
\begin{aligned}
(B) &= \left(\frac{p}{2}\right)^p \int_{A_k} \left|\nabla\left(e^{\frac{2|G_k u_p|}{p}}-1\right)\right|^p \\
&\ge \frac{1}{4}\int_{A_k} \left|\nabla\left(e^{\frac{2|G_k u_p|}{p}}-1\right)\right|^p \ge \frac{C_2}{4}\left(\int_{A_k}\left(e^{\frac{2|G_k u_p|}{p}}-1\right)^{p^*}\right)^{p/p^*}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
(C) &\le \int_{A_{k+1}} |f|\left(e^{2|G_k u_p|}-1\right) + (e^2-1)\int_{A_k\setminus A_{k+1}} |f| \\
&\le \underbrace{\int_{A_{k+1}} |f|\left(e^{2|G_k u_p|}-1\right)}_{(C')} + (e^2-1)\,\|f\|_m |A_k|^{1-\frac{1}{m}}
\end{aligned}
$$

Since $1 < m' < \frac{p^*}{p}$, using Hölder's, interpolation and Young's inequalities, and Lemmas 2 and 1, we obtain

$$
\begin{aligned}
(C') &\le \|f\|_m \,\|e^{2|G_k u_p|}-1\|_{L^{m'}(A_{k+1})} \\
&\le \|f\|_m \,\|e^{2|G_k u_p|}-1\|^{\frac{N}{pm}}_{L^{p^*/p}(A_{k+1})}\left(\int_{A_k} e^{2|G_k u_p|}-1\right)^{1-\frac{N}{pm}} \\
&\le \varepsilon\,\|e^{2|G_k u_p|}-1\|_{L^{p^*/p}(A_{k+1})} + \varepsilon^{-\frac{N}{pm-N}}\,\|f\|_m^{\frac{pm}{pm-N}} \int_{A_k}\left(e^{2|G_k u_p|}-1\right) \\
&\le \varepsilon C_3 \left(\int_{A_k}\left(e^{\frac{2|G_k u_p|}{p}}-1\right)^{p^*}\right)^{p/p^*} + \varepsilon^{-\frac{N}{pm-N}}\,\|f\|_m^{\frac{pm}{pm-N}} \int_{A_k}\left(e^{2|G_k u_p|}-1\right),
\end{aligned}
$$

where ε is any positive constant. Therefore, if we choose $\varepsilon \leq \frac{C_2}{8C_3}$, we obtain

$$(C') \leq \frac{(B)}{2} + \varepsilon^{-\frac{N}{pm-N}} \|f\|_m^{\frac{pm}{pm-N}} \int_{A_k} \left(e^{2|G_k u_p|} - 1\right),$$

On the other hand, if we also choose $\varepsilon \leq 1$, then $\varepsilon^{-\frac{N}{pm-N}} \leq \varepsilon^{-\frac{N}{m-N}}$; moreover,

$$\|f\|_m^{\frac{pm}{pm-N}} \leq (1 + \|f\|_m)^{\frac{m}{m-N}} \qquad \text{for all } p \in (1, \frac{3}{2}).$$

Therefore, if we choose

$$k_0 = \varepsilon^{-\frac{N}{m-N}} (1 + \|f\|_m)^{\frac{m}{m-N}},$$

then we can write

$$(C') \leq \frac{(B)}{2} + (A).$$

Hence, going back to (16) and using our estimate of term (B), we have proved that

$$\left(\int_{A_k} \left(e^{\frac{2|G_k u_p|}{p}} - 1\right)^{p^*}\right)^{p/p^*} \leq c\,\|f\|_m |A_k|^{1-\frac{1}{m}}, \tag{17}$$

where c does not depend on p. Since

$$e^{\frac{2s}{p}} - 1 \geq \frac{2s}{p} \geq s, \qquad \text{for all } s \geq 0,$$

from (17) one obtains easily, for every $h > k \geq k_0$,

$$(h-k)^p \, |A_h|^{p/p^*} \leq c\,\|f\|_m |A_k|^{1-\frac{1}{m}},$$

and therefore

$$|A_h| \leq \tilde{c}\, \frac{\|f\|_m^{p^*/p}}{(h-k)^{p^*}} \, |A_k|^{\frac{p^*}{m'p}},$$

where $\tilde{c} = (1+c)^{\frac{2N}{2N-3}}$ is independent on p. It follows from Lemma 3 that

$$\|u_p\|_\infty \leq c_1 (1 + \|f\|_m)^{\frac{m}{m-N}} + \tilde{c}^{1/p^*}\, 2^{\frac{p^*}{p^*-m'p}} |\Omega|^{\frac{p^*-m'p}{p^* m' p}} \, \|f\|_m^{1/p},$$

where the constants are independent on p. Therefore, one obtains

$$\|u_p\|_\infty \leq c_1 (1 + \|f\|_m)^{\frac{m}{m-N}} + c_2 (1 + |\Omega|)^{\frac{1^*-m'}{1^* m'}} \, (1 + \|f\|_m),$$

for all $p \in (1, \frac{3}{2})$. □

Step 3. *BV–estimate and identification of a candidate to solution.*

Taking $e^{u_p}-1$ as test function in (14), we may simplify and drop a nonnegative term, obtaining

$$\int_\Omega |\nabla u_p|^p \le \int_\Omega |f||e^{u_p}-1| \le e^{\|u_p\|_\infty}\int_\Omega |f|\,.$$

Therefore, in view of the L^∞-estimate proved in Step 2, there exists a positive constant M (independent on p) such that

$$\int_\Omega |\nabla u_p|^p \le M\,.$$

Applying Young's inequality, we get

$$\int_\Omega |\nabla u_p| \le M+|\Omega| \qquad \forall\, p\,. \tag{18}$$

Thus, $(u_p)_p$ is bounded in $W^{1,1}(\Omega)$ and we may deduce that there exists a function $u\in BV(\Omega)\cap L^\infty(\Omega)$ satisfying, up to subsequences,

$$u_p \to u \qquad \text{strongly in } L^1(\Omega)\,, \tag{19}$$

$$\nabla u_p \rightharpoonup Du \qquad \text{weakly* as measures}\,, \tag{20}$$

$$u_p(x)\to u(x) \qquad \text{pointwise a.e. in } \Omega\,. \tag{21}$$

Step 4. *Weak convergence in $L^1(\Omega;\mathbb{R}^N)$ of the sequence $\big(|\nabla u_p|^{p-2}\nabla u_p\big)_p$ to some $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$, with $\|\mathbf{z}\|_\infty\le 1$.*

We begin by proving that $\big(|\nabla u_p|^{p-2}\nabla u_p\big)_p$ is weakly relatively compact in $L^1(\Omega;\mathbb{R}^N)$. To this end, using (18) and Hölder's inequality, we see that

$$\int_\Omega |\nabla u_p|^{p-1} \le \left(\int_\Omega |\nabla u_p|^p\right)^{\frac{p-1}{p}} |\Omega|^{\frac1p} \le (M+|\Omega|)^{\frac{p-1}{p}}|\Omega|^{\frac1p}\le C,$$

where C does not depend on p. On the other hand, for p close to 1 and any measurable subset $E\subset\Omega$,

$$\left|\int_E |\nabla u_p|^{p-2}\nabla u_p\right| \le \int_E |\nabla u_p|^{p-1} \le (M+|\Omega|)^{\frac{p-1}{p}}|E|^{\frac1p}.$$

Thus, $\big(|\nabla u_p|^{p-2}\nabla u_p\big)_p$ is bounded and equi–integrable in $L^1(\Omega,\mathbb{R}^N)$, and so is weakly relatively compact in $L^1(\Omega,\mathbb{R}^N)$. We do not relabel and assume that the whole "sequence" converges. Therefore, there exists $\mathbf{z}\in L^1(\Omega,\mathbb{R}^N)$ such that

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z} \quad \text{weakly in } L^1(\Omega,\mathbb{R}^N)\,. \tag{22}$$

Now, on account of (18), we can follow the arguments in [4] and prove that

$$\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N) \quad \text{with} \quad \|\mathbf{z}\|_\infty \le 1. \tag{23}$$

Furthermore, it follows from (14) and the previous steps that the term $-\operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right)$ is bounded in $L^1(\Omega)$. Therefore, up to subsequences, it converges weakly* in the sense of measures to some measure, which must be $-\operatorname{div}\mathbf{z}$. It yields that $-\operatorname{div}\mathbf{z}$ is a Radon measure having finite total variation.

Step 5. *$-div\,(e^u\mathbf{z}) = (f-u)e^u$ holds in $\mathcal{D}'(\Omega)$.*

Taking $e^{u_p}\varphi$, with $\varphi \in C_0^\infty(\Omega)$, as test function in (14) we obtain

$$\int_\Omega u_p e^{u_p}\varphi + \int_\Omega e^{u_p}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi + \int_\Omega \varphi e^{u_p}|\nabla u_p|^p \\ = \int_\Omega \varphi e^{u_p}|\nabla u_p|^p + \int_\Omega f e^{u_p}\varphi\,.$$

Simplifying and letting p goes to 1, we have

$$\int_\Omega u e^u\varphi + \int_\Omega e^u\mathbf{z}\cdot\nabla\varphi = \int_\Omega f e^u\varphi\,.$$

Step 6. *$u - div\,(\mathbf{z}) \ge |Du| + f$ holds in $\mathcal{D}'(\Omega)$.*

Consider $\varphi \in C_0^\infty(\Omega)$, with $\varphi \ge 0$, as test function in (14). Then we have

$$\int_\Omega u_p\varphi + \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi = \int_\Omega \varphi|\nabla u_p|^p + \int_\Omega f\varphi\,.$$

and so Young's inequality implies

$$\int_\Omega \varphi|\nabla u_p| \le \frac{1}{p}\int_\Omega \varphi|\nabla u_p|^p + \frac{p-1}{p}\int_\Omega\varphi \\ = -\frac{1}{p}\int_\Omega f\varphi + \frac{1}{p}\int_\Omega u_p\varphi + \frac{1}{p}\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi + \frac{p-1}{p}\int_\Omega\varphi\,.$$

Using the lower–semicontinuity, we may pass to the limit:

$$\int_\Omega \varphi|Du| \le -\int_\Omega f\varphi + \int_\Omega u\varphi + \int_\Omega \mathbf{z}\cdot\nabla\varphi\,.$$

Thus,

$$u - \operatorname{div}\mathbf{z} \ge |Du| + f \qquad \text{in } \mathcal{D}'(\Omega)\,. \tag{24}$$

Step 7. $D^j u = 0$.

Since S_u is countably $\mathcal{H}^{N-1}$–rectifiable, there exist countably many hypersurfaces Γ_k of class C^1 such that

$$\mathcal{H}^{N-1}\left(S_u \backslash \bigcup_{k=1}^{\infty} \Gamma_k\right) = 0.$$

(see [19, Theorem 12, Sec. 5.9] or [2, Theorem 3.78 and Chapter 2.9]). Therefore, it is enough to show that

$$|D^j u|(\Gamma_k) = 0 \quad \forall\, k \in \mathbb{N}\,.$$

To this end, we fix Γ_k with an orientation ν. First of all, we observe that

$$|Du| \llcorner \Gamma_k = |D^j u| \llcorner \Gamma_k\,, \tag{25}$$

since both the absolutely continuous part $D^a u$ and the Cantor part $D^c u$ of the gradient vanish on sets which are σ-finite with respect to $\mathcal{H}^{N-1}$ (see [2, Proposition 3.92]).

The proof of Step 7 relies on the following claim:

Lemma 4. *For all $x_0 \in \Gamma_k$, there exists U, an open neighbourhood of x_0, satisfying*

$$|D^j u|(U \cap \Gamma_k) = 0\,. \tag{26}$$

Once Lemma 4 is proven, we will deduce that any compact subset of Γ_k is $|D^j u|$–null, and so $|D^j u|(\Gamma_k) = 0$.

Proof of Lemma 4. Given $x_0 \in \Gamma_k$, there exists an open ball U centered in x_0 such that $U \cap \Gamma_k$ is a C^1 surface compactly contained in Ω. Set $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < d(\overline{U \cap \Gamma_k}, \partial\Omega)$ and define

$$U_n = \{x + t\nu(x) \,:\, x \in U \cap \Gamma_k\,,\ |t| < \frac{1}{n}\}\,, \qquad n \geq n_0. \tag{27}$$

Then U_n is an open generalized cylinder (whose "base" is $U \cap \Gamma_k$) with piecewise-C^1 boundary and

$$\bigcap_{n \geq n_0} U_n = U \cap \Gamma_k\,.$$

We will denote by η the unit outward normal to ∂U_n. Set $\lambda > \|u\|_\infty + 2$. Since $u - \operatorname{div} \mathbf{z} \geq |Du| + f$ as measures, we deduce

$$-\int_{U_n} (u-\lambda)^* \operatorname{div} \mathbf{z} \leq \int_{U_n} (u-\lambda)^* |Du| + \int_{U_n} (f-u)(u-\lambda)\,.$$

By applying Green's formula and $(\mathbf{z}, Du) \geq -|Du|$, it yields

$$\int_{U_n} (\lambda-1-u)^*|Du| - \int_{\partial U_n} (u-\lambda)[\mathbf{z},\eta]\, d\mathcal{H}^{N-1} \leq \int_{U_n} (f-u)(u-\lambda)\,. \tag{28}$$

Now, we are going to analyze each term in the previous equation. It is straightforward that

$$\lim_{n\to\infty} \int_{U_n} (\lambda-1-u)^*|Du| = \int_{U\cap\Gamma_k} (\lambda-1-u)^*|Du|. \tag{29}$$

Moreover, since $|\Gamma_k| = 0$, we also have

$$\lim_{n\to\infty} \int_{U_n} (f-u)(u-\lambda) = \int_{U\cap\Gamma_k} (f-u)(u-\lambda) = 0. \tag{30}$$

To study the remainder term, we split the boundary ∂U_n into three parts:

$$\partial U_n = E_n^+ \cup E_n^- \cup E_n^0\,,$$

where

$$E_n^+ = \left\{ x + \frac{1}{n}\nu(x) \ : \ x \in U \cap \Gamma_k \right\}\,,$$
$$E_n^- = \left\{ x - \frac{1}{n}\nu(x) \ : \ x \in U \cap \Gamma_k \right\}$$

and E_n^0 denotes the "lateral surface" of the generalized cylinder U_n, namely that obtained from $\partial U \cap \Gamma_k$. It satisfies $\cap_{n\geq n_0} E_n^0 = \partial U \cap \Gamma_k$.

It is clear that

$$\lim_{n\to\infty} \int_{E_n^0} (u-\lambda)[\mathbf{z},\eta]\, d\mathcal{H}^{N-1} = 0\,,$$

since u and $[\mathbf{z},\eta]$ are bounded, and $\mathcal{H}^{N-1}(\partial U \cap \Gamma_k) = 0$. Now, suppose for a moment that

$$\lim_{n\to\infty} \int_{E_n^+} (u-\lambda)[\mathbf{z},\eta]\, d\mathcal{H}^{N-1} = \int_{U\cap\Gamma_k} (u_{\Gamma_k}^+ - \lambda)[\mathbf{z},\nu]\, d\mathcal{H}^{N-1}\,, \tag{31}$$

$$\lim_{n\to\infty} \int_{E_n^-} (u-\lambda)[\mathbf{z},\eta]\, d\mathcal{H}^{N-1} = \int_{U\cap\Gamma_k} (u_{\Gamma_k}^- - \lambda)[\mathbf{z},-\nu]\, d\mathcal{H}^{N-1}\,. \tag{32}$$

Then we would deduce

$$\lim_{n\to\infty} \int_{\partial U_n} (u-\lambda)[\mathbf{z},\eta]\, d\mathcal{H}^{N-1} = \int_{U\cap\Gamma_k} (u_{\Gamma_k}^+ - u_{\Gamma_k}^-)[\mathbf{z},\nu]\, d\mathcal{H}^{N-1}\,. \tag{33}$$

Taking (29), (30) and (33) into account, if we let n goes to $+\infty$ in (28), then we get

$$\begin{aligned}\int_{U\cap\Gamma_k}(\lambda-1-u)^*|Du| &\le \int_{U\cap\Gamma_k}(u^+_{\Gamma_k}-u^-_{\Gamma_k})[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\\ &\le \int_{U\cap\Gamma_k}|u^+-u^-|\,d\mathcal{H}^{N-1} = \int_{U\cap\Gamma_k}|Du|\,.\end{aligned}$$

and consequently

$$\int_{U\cap\Gamma_k}(\lambda-2-u)^*|Du|\le 0\,. \tag{34}$$

From the arbitrariness of λ and (25), we deduce $\int_{U\cap\Gamma_k}|D^j u| = 0$. Hence, (26) holds.

Therefore, we only have to prove (31) and (32). We will just see (31), the other one is similar. To this end, define

$$U_n^+ = \{x+t\nu(x)\ :\ x\in U\cap\Gamma_k\,,\ 0<t<\frac{1}{n}\}\,,\quad n\ge n_0\,,$$

and denote by η^+ the unit outward normal to ∂U_n^+. Observe that $\eta^+=-\nu$ on $U\cap\Gamma_k$ and $\eta^+=\eta$ on $\partial U_n^+\backslash(U\cap\Gamma_k)$. Thus,

$$\begin{aligned}&\int_{\partial U_n^+}(u-\lambda)[\mathbf{z},\eta^+]\,d\mathcal{H}^{N-1}\\ &\quad= \int_{E_n^+}(u-\lambda)[\mathbf{z},\eta]\,d\mathcal{H}^{N-1} - \int_{U\cap\Gamma_k}(u^+-\lambda)[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\\ &\qquad\qquad + \int_{E_n^{0+}}(u-\lambda)[\mathbf{z},\eta]\,d\mathcal{H}^{N-1}\,,\end{aligned}$$

where E_n^{0+} is the "lateral surface" of U_n^+. Since the last integral goes to 0 as n tends to ∞, it yields

$$\begin{aligned}\lim_{n\to\infty}\Big[\int_{E_n^+}(u-\lambda)[\mathbf{z},\eta]\,d\mathcal{H}^{N-1} - \int_{\partial U_n^+}(u-\lambda)[\mathbf{z},\eta^+]\,d\mathcal{H}^{N-1}\Big]\\ = \int_{U\cap\Gamma_k}(u^+-\lambda)[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\,.\end{aligned}$$

On the other hand, by Green's formula, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_{\partial U_n^+}(u-\lambda)[\mathbf{z},\eta^+]\,d\mathcal{H}^{N-1}\\ &\qquad= \lim_{n\to\infty}\Big[\int_{U_n^+}(u-\lambda)^*\operatorname{div}\mathbf{z} + \int_{U_n^+}(\mathbf{z},Du)\Big] = 0\,,\end{aligned}$$

due to $\bigcap_{n\geq n_0} U_n^+ = \emptyset$. Therefore, (31) is proved and so the proof of Lemma 4 is finished. $\square$

Step 8. *$u - div\,(\mathbf{z}) = |Du| + f$ holds in $\mathcal{D}'(\Omega)$.*

First observe that (12) holds. Therefore, using inequality (24), Step 5, the chain rule and Anzellotti's inequality yield

$$(e^u)^*|Du| \leq -(e^u)^*\mathrm{div}\,\mathbf{z} - (f-u)e^u = \big(\mathbf{z}, D(e^u)\big) \\ \leq \|\mathbf{z}\|_\infty |D(e^u)| \leq |D(e^u)| = (e^u)^*|Du|\,.$$

Since the above inequalities become equalities, one deduces

$$u - \mathrm{div}\,\mathbf{z} = |Du| + f$$

and

$$\big(\mathbf{z}, D(e^u)\big) = |D(e^u)| \tag{35}$$

as measures.

Step 9. *$(\mathbf{z}, Du) = |Du|$ as measures.*

This follows from the equality (35) and Proposition 2.2 of [27].

Step 10. *$[\mathbf{z}, \nu] \in$ sign $(-u)$ holds $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$.*

Given $\phi \in C(\partial\Omega)$ nonnegative, we may find a nonnegative function $\varphi \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ satisfying $\varphi|_{\partial\Omega} = \phi$, as in the proof of Proposition 2.

Let $m \in \mathbb{N}$ be such that $m > \|u_p\|_\infty$ for p close to 1, and take $u_p|u_p|^{m-1}\varphi$ as test function in (14), then

$$m\int_\Omega |u_p|^{m-1}\varphi\,|\nabla u_p|^p + \int_\Omega |u_p|^{m-1}u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi \\ = \int_\Omega |u_p|^{m-1}u_p\,\varphi\,|\nabla u_p|^p + \int_\Omega |u_p|^{m-1}u_p\,\varphi(f-u_p)\,.$$

Rearranging the terms and applying Young's inequality, we get

$$
\begin{aligned}
&\int_\Omega (m-u_p)|u_p|^{m-1}\varphi\,|\nabla u_p| \\
&\qquad \le \frac{1}{p}\int_\Omega (m-u_p)|u_p|^{m-1}\varphi\,|\nabla u_p|^p \\
&\qquad\qquad + \frac{p-1}{p}\int_\Omega (m-u_p)|u_p|^{m-1}\varphi \\
&\qquad = -\frac{1}{p}\int_\Omega |u_p|^{m-1}u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi \\
&\qquad\qquad + \frac{1}{p}\int_\Omega |u_p|^{m-1}u_p\,\varphi(f-u_p) \\
&\qquad\qquad + \frac{p-1}{p}\int_\Omega (m-u_p)|u_p|^{m-1}\varphi\,.
\end{aligned}
\tag{36}
$$

Since the functional defined by

$$I[v]=\int_\Omega \varphi\,|Dv|+\int_{\partial\Omega}|v|\varphi\,d\mathcal{H}^{N-1}\,,$$

is lower–semicontinuous with respect to L^1–convergence, it follows from (36), the chain rule and the equation satisfied by u that

$$
\begin{aligned}
&\int_\Omega \big((m-u)|u|^{m-1}\big)^*\varphi\,|Du|+\int_{\partial\Omega}(1-\frac{u}{m+1})|u|^m\phi\,d\mathcal{H}^{N-1} \\
&\quad\le -\int_\Omega \mathbf{z}\cdot\nabla\varphi\,|u|^{m-1}u+\int_\Omega (f-u)\varphi\,|u|^{m-1}u \\
&\quad= -\int_\Omega \mathbf{z}\cdot\nabla\varphi\,|u|^{m-1}u-\int_\Omega \varphi\,(|u|^{m-1}u)^*\operatorname{div}\mathbf{z}-\int_\Omega \varphi\,(|u|^{m-1}u)^*|Du|\,.
\end{aligned}
$$

It is easy to check that

$$(\mathbf{z},D(v\varphi))=\varphi(\mathbf{z},Dv)+v\mathbf{z}\cdot\nabla\varphi$$

holds in the sense of distributions, where $v=|u|^{m-1}u$. Therefore, by Green's formula and the chain rule, the previous inequality becomes

$$
\begin{aligned}
&m\int_\Omega \big(|u|^{m-1}\big)^*\varphi\,|Du|+\int_{\partial\Omega}(1-\frac{u}{m+1})|u|^m\phi\,d\mathcal{H}^{N-1} \\
&\qquad\le \int_\Omega \varphi\,(\mathbf{z},D(|u|^{m-1}u))-\int_{\partial\Omega}|u|^{m-1}u\phi\,[\mathbf{z},\nu]\,d\mathcal{H}^{N-1} \\
&\qquad\le \int_\Omega \varphi\,|D(|u|^{m-1}u)|-\int_{\partial\Omega}|u|^{m-1}u\phi\,[\mathbf{z},\nu]\,d\mathcal{H}^{N-1} \\
&\qquad= m\int_\Omega \big(|u|^{m-1}\big)^*\varphi\,|Du|-\int_{\partial\Omega}|u|^{m-1}u\phi\,[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\,,
\end{aligned}
$$

so that

$$\int_{\partial\Omega} \phi\, |u|^{m-1}\Big(|u| + u[\mathbf{z},\nu] - \frac{u|u|}{m+1}\Big)\, d\mathcal{H}^{N-1} \le 0$$

for every nonnegative $\phi \in C(\partial\Omega)$. Thus, we deduce either $u = 0$, $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$ or

$$|u| + u[\mathbf{z},\nu] \le \frac{u|u|}{m+1} \le \frac{\|u\|_\infty^2}{m+1}\,, \qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega\,.$$

Letting m goes to ∞, this last inequality implies

$$|u| + u[\mathbf{z},\nu] \le 0\,, \qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega\,,$$

whence Step 10 follows. ■

Remark 4. In the above Theorem, we have proved that the solution could have a trace different of 0 on the boundary. If it is so, then

$$[\mathbf{z},\nu] = \begin{cases} -1\,, & \text{if } u > 0\,; \\ 1\,, & \text{if } u < 0\,. \end{cases} \tag{37}$$

We explicitly point out that the former case really occurs as it is shown by examples in the next section, but the later does not. Indeed, we will see that the solution cannot attain negative values on the boundary.

In this remark, we will set v_- to denote the negative part of a function v.

To begin with, we take $-(u_p)_-$ as test function in (14), to obtain

$$\int_\Omega (1 + (u_p)_-)|\nabla (u_p)_-|^p = -\int_\Omega (u_p)_-(f - u_p)\,.$$

Then Young's inequality implies

$$\int_\Omega (1 + (u_p)_-)|\nabla (u_p)_-| \\ = -\frac{1}{p}\int_\Omega (u_p)_-(f - u_p) + \frac{p-1}{p}\int_{\{u_p \le 0\}} (1 - u_p)\,. \tag{38}$$

We now apply the lower semicontinuity of the functional defined by

$$J[v] = \int_\Omega |Dv| + \int_{\partial\Omega} |v|\, d\mathcal{H}^{N-1}$$

to the sequence $\frac{((u_p)_-)^2}{2} + (u_p)_-$. It follows from (38) that

$$\int_\Omega (1+u_-)^*|Du_-| + \int_{\partial\Omega} \Big(\frac{(u_-)^2}{2} + u_-\Big)\, d\mathcal{H}^{N-1} \leq -\int_\Omega u_-(f-u)\,. \tag{39}$$

The right–hand side may be manipulated applying the result of Step 8 and Green's formula

$$-\int_\Omega u_-(f-u) = \int_\Omega (u_-)^* \operatorname{div} \mathbf{z} + \int_\Omega (u_-)^*|Du|$$
$$= -\int_\Omega (\mathbf{z}, Du_-) + \int_{\partial\Omega} u_-[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_\Omega u_-^*|Du_-|\,.$$

Thus, (39) becomes

$$\int_{\{u\leq 0\}} (1-u)^*|Du| + \int_{\partial\Omega\cap\{u\leq 0\}} \Big(\frac{u^2}{2} + |u|\Big)\, d\mathcal{H}^{N-1}$$
$$\leq \int_{\{u\leq 0\}} |Du| + \int_{\partial\Omega\cap\{u\leq 0\}} |u|[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} - \int_{\{u\leq 0\}} u^*|Du|\,.$$

Simplifying, we obtain

$$\int_{\partial\Omega\cap\{u\leq 0\}} \frac{u^2}{2}\, d\mathcal{H}^{N-1} + \int_{\partial\Omega\cap\{u\leq 0\}} |u|(1-[\mathbf{z},\nu])\, d\mathcal{H}^{N-1} \leq 0\,.$$

Since every integrand is nonnegative, it follows that

$$u = 0 \quad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega\cap\{u\leq 0\}\,.$$

□

4. A result on uniqueness

This Section is devoted to prove that problem (1) has, at most, a solution when the datum is nonnegative and small enough. We do not know whether the result is true in the general case.

Remark 5. Note that uniqueness only holds for the solution u, not for the vector field $\mathbf{z}$. Indeed, we will show in the next Section that a solution u may be associated to different fields $\mathbf{z}_1$ and $\mathbf{z}_2$. The only fact one can prove is that $\operatorname{div}\mathbf{z}_1 = \operatorname{div}\mathbf{z}_2$.

Proposition 4. *Assume that $0 \leq f(x) \leq \alpha < 2$. There exists, at most, a weak solution to problem* (1).

PROOF. Consider two weak solutions u_1 and u_2 to problem (1); so that there exist two vector fields $\mathbf{z}_1$ and $\mathbf{z}_2$ in such a way that u_i and $\mathbf{z}_i$ satisfy Definition 1, $i = 1, 2$.

Taking $e^{u_1} - e^{u_2}$ as test function in the equations (8) satisfied by u_1 and u_2, one obtains

$$\int_\Omega (u_1 - u_2)(e^{u_1} - e^{u_2}) + \int_\Omega (\mathbf{z}_1 - \mathbf{z}_2, D(e^{u_1} - e^{u_2}))$$
$$- \int_{\partial\Omega} (e^{u_1} - e^{u_2})[\mathbf{z}_1 - \mathbf{z}_2, \nu] \, d\mathcal{H}^{N-1} = \int_\Omega (e^{u_1} - e^{u_2})^* (|Du_1| - |Du_2|) \, .$$

It is easy to check that

$$\int_\Omega (\mathbf{z}_1 - \mathbf{z}_2, D(e^{u_1} - e^{u_2})) \geq 0 \quad \text{and} \quad -\int_{\partial\Omega} (e^{u_1} - e^{u_2})[\mathbf{z}_1 - \mathbf{z}_2, \nu] \, d\mathcal{H}^{N-1} \geq 0 \, ,$$

therefore

$$\int_\Omega (u_1 - u_2)(e^{u_1} - e^{u_2}) \leq \int_\Omega (e^{u_1} - e^{u_2})^* (|Du_1| - |Du_2|) \, . \tag{40}$$

On the other hand, taking $u_1 - u_2$ as test function in the equations (11) satisfied by u_1 and u_2, one deduces

$$\int_\Omega (u_1 e^{u_1} - u_2 e^{u_2})(u_1 - u_2) + \int_\Omega (e^{u_1}\mathbf{z}_1 - e^{u_2}\mathbf{z}_2, D(u_1 - u_2)) \tag{41}$$
$$- \int_{\partial\Omega} (u_1 - u_2)[e^{u_1}\mathbf{z}_1 - e^{u_2}\mathbf{z}_2, \nu] \, d\mathcal{H}^{N-1} = \int_\Omega f(e^{u_1} - e^{u_2})(u_1 - u_2) \, .$$

Since $u_i \geq 0$, for $i = 1, 2$ (see Remark 3), it follows from Proposition 2 that

$$- \int_{\partial\Omega} (u_1 - u_2)[e^{u_1}\mathbf{z}_1 - e^{u_2}\mathbf{z}_2, \nu] \, d\mathcal{H}^{N-1} \geq 0 \, .$$

On the other hand, one has

$$(u_1 e^{u_1} - u_2 e^{u_2})(u_1 - u_2) \geq (e^{u_1} - e^{u_2})(u_1 - u_2)$$

and, using (40) and Proposition 1,

$$\int_\Omega (e^{u_1}\mathbf{z}_1 - e^{u_2}\mathbf{z}_2, D(u_1 - u_2)) \geq \int_\Omega (e^{u_1} - e^{u_2})^* (|Du_1| - |Du_2|)$$
$$\geq \int_\Omega (e^{u_1} - e^{u_2})(u_1 - u_2) \, .$$

Thus, equation (41) becomes

$$2 \int_\Omega (e^{u_1} - e^{u_2})(u_1 - u_2) \leq \alpha \int_\Omega (e^{u_1} - e^{u_2})(u_1 - u_2) \, .$$

Since $\alpha < 2$, this implies $u_1 = u_2$. ∎

5. Examples

In this Section we present some simple examples of solutions in the radial case. Assume that $\Omega = B_R$ is the open ball centered at 0 and having radius R, and f is radially symmetric: $f = f(|x|)$. We look for solutions $u(x) = g(|x|)$, with g decreasing.

Note that in the regions where g is strictly decreasing, one always has $\mathbf{z} = -\frac{x}{|x|}$ and $-\mathrm{div}\,(\mathbf{z}) = \frac{N-1}{|x|}$. Therefore, g satisfies the first order equation

$$g(r) + g'(r) = -\frac{N-1}{r} + f(r)\,. \tag{42}$$

Example 1. Assume first that f is constant: $f(r) \equiv \lambda > 0$. Then we have to consider two cases.

(1) If $0 < \lambda \le \dfrac{N}{R}$, the solution u and its associate vector field $\mathbf{z}$ are given by
$$u \equiv 0\,, \quad \mathbf{z}(x) = -\lambda \frac{x}{N}\,.$$
(2) If $\lambda > \dfrac{N}{R}$, one has
$$u \equiv \lambda - \frac{N}{R} > 0\,, \quad \mathbf{z}(x) = -\frac{x}{R}\,.$$

Remark 6. (i) Note that when λ is small, $u \equiv 0$ is a solution of a nonhomogeneous problem. This is a well–known feature of problems involving the 1–Laplacian (see [6]).
(ii) In the case of large λ, u may lose its boundary value since the trace of u does not vanishes. However, the boundary condition always holds in the sense of (10).
(iii) There is no uniqueness of the vector field $\mathbf{z}$. For instance, for λ small enough, instead of $\mathbf{z}(x) = -\frac{x}{R}$ one could also take $\mathbf{z}(x) = \big(-\frac{Nx_1}{R}, 0, \dots, 0\big)$.

Example 2. Assume now that f is the characteristic function of a smaller ball:
$$f(x) = \lambda\,\chi_{B_\rho}\,, \quad \text{with } 0 < \rho < R\,.$$
For the sake of simplicity, we will only consider the case where $R = 1$ and $\rho = \frac{1}{2}$. The general case can easily be deduced.

One has to distinguish three cases, depending on the size of λ. To this aim, we define
$$\lambda_0 = 2N + (N-1)e^{-1/2} \int_{1/2}^{1} \frac{e^s}{s}\, ds\,.$$

The solution u and its associate vector field $\mathbf{z}$ are as follows:

(1) If $0 < \lambda \le 2N$, then

$$u \equiv 0\,, \quad \mathbf{z}(x) = \begin{cases} -\dfrac{\lambda}{N} x & \text{for } |x| < \dfrac{1}{2} \\ -\dfrac{\lambda x}{N\, 2^N\, |x|^N} & \text{for } |x| > \dfrac{1}{2} \end{cases} .$$

(2) If $2N < \lambda \le \lambda_0$, then

$$u(x) = \begin{cases} \lambda - 2N & \text{for } |x| < \dfrac{1}{2} \\ g(|x|) & \text{for } \dfrac{1}{2} < |x| < r_0 \\ 0 & \text{for } r_0 < |x| < 1 \end{cases}$$

$$\mathbf{z}(x) = \begin{cases} -2x & \text{for } |x| < \dfrac{1}{2} \\ -\dfrac{x}{|x|} & \text{for } \dfrac{1}{2} < |x| < r_0 \\ -r_0^{N-1} \dfrac{x}{|x|^N} & \text{for } r_0 < |x| < 1 \end{cases}$$

where

$$g(r) = e^{-r} \left(e^{1/2}(\lambda - 2N) - (N-1) \int_{1/2}^{r} \frac{e^s}{s}\, ds \right)$$

and r_0 is such that $\frac{1}{2} < r_0 < 1$ and $g(r_0) = 0$.

(3) If $\lambda > \lambda_0$, then

$$u(x) = \begin{cases} \lambda - 2N & \text{for } |x| < \dfrac{1}{2} \\ g(|x|) & \text{for } \dfrac{1}{2} < |x| < 1 \end{cases}$$

$$\mathbf{z}(x) = \begin{cases} -2x & \text{for } |x| < \dfrac{1}{2} \\ -\dfrac{x}{|x|} & \text{for } \dfrac{1}{2} < |x| < 1 \end{cases}$$

where $g(r)$ is the same as before. Observe that in this last case the trace of u on the boundary is $e^{-1/2}(\lambda - \lambda_0) > 0$, so that the boundary datum is not attained.

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