# MULTIPLICITY OF SOLUTIONS TO ELLIPTIC PROBLEMS INVOLVING THE 1–LAPLACIAN WITH A CRITICAL GRADIENT TERM

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Abstract. In the present paper we study the Dirichlet problem for an equation involving the 1–Laplacian and a total variation term as reaction, namely:

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right) = |Du| + f(x)\,.$$

We prove a strong multiplicity result: we show that, for any positive Radon measure concentrated in a set away from the boundary and singular with respect to a certain capacity, there exists an unbounded solution, which is infinite on the set where the measure is concentrated. These results can be viewed as the analogue for the 1–Laplacian operator of some known multiplicity results obtained by Abdellaoui, Dall'Aglio, Peral and by Hamid, Bidaut–Veron. We show explicit examples of multiplicity as well.

## 1. Introduction and Statement of the Main Result

The starting point of this paper lies in the paper [6] by Andreu, Dall'Aglio and Segura de León. In that paper, existence and uniqueness results for problem

$$\begin{cases} u - \mathrm{div}\left(\dfrac{Du}{|Du|}\right) = |Du| + f(x) & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,; \end{cases} \tag{1}$$

were obtained, where $\Omega$ is a bounded open subset of $\mathbb{R}^N$. The main result of that article provides a bounded solution for every datum $f(x)$ belonging to $L^m(\Omega)$, with $m > N$ (see [6, Theorem 1]). Knowing that for small enough data the solution is the trivial one (see [22, Theorem 4.2]), it is further shown that this is the unique bounded solution (see [6, Proposition 4]).

On the other hand, in [13], the authors study problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = |Du| + f(x) & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,; \end{cases} \tag{2}$$

showing a criterion on the datum to determine when the unique solution is the trivial one. The question dealt in the present paper is whether there exists some other solution (which must be unbounded). We will focus on problem (2) for the sake of simplicity.

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In the $p$–Laplacian setting ($p > 1$) problem (2) has been studied in [2, 19, 20]. The main result of the present paper can be seen as a translation of those to our setting. Recall that Abdellaoui, Dall'Aglio and Peral in [2] the following multiplicity result for the Laplacian was proved:

*Let $f \in L^m(\Omega)$ be nonnegative and small enough, where $m > N/2$. For any positive Radon measure $\mu$ which is concentrated on a set which has zero capacity, there exists a solution $u_\mu$ to problem*

$$(3) \qquad \begin{cases} -\Delta u_\mu = |\nabla u_\mu|^2 + f(x) & \text{in } \Omega\,; \\ u_\mu = 0 & \text{on } \partial\Omega \end{cases}$$

*which is unbounded near the set where the measure is concentrated.*

It must be noted that the results in those papers rely on a Cole–Hopf change of unknown, which does not work in the case $p = 1$. The absence of any kind of Cole–Hopf change of unknown is indeed one of the biggest difficulties to deal with multiplicity in our framework. Another difficulty comes from the definition of solution to problems involving the 1–Laplacian operator (as introduced in [4] by Andreu, Ballester, Caselles and Mazón, see also [5]). Indeed, the suitable energy space is $BV(\Omega)$, the space of all functions of bounded variation, and this notion relies on a bounded vector field $\mathbf{z}$ which plays the role of $\dfrac{Du}{|Du|}$ in the sense that it satisfies $\|\mathbf{z}\|_\infty \le 1$ and $(\mathbf{z}, Du) = |Du|$ (where $(\mathbf{z}, Du)$ stands for a type of dot product of $\mathbf{z}$ and $Du$). Moreover, the equation holds, so that $-\operatorname{div}\mathbf{z} = |Du| + f$ is just a Radon measure. In our computations we need to manipulate products of the form $(\mathbf{z}, Du)$. According to the Anzellotti theory, $(\mathbf{z}, Du)$ is well–defined as a Radon measure whenever $\operatorname{div}\mathbf{z}$ is a Radon measure and $u \in BV(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$. This last condition can be relaxed to avoid the continuity of $u$, namely: $\operatorname{div}\mathbf{z}$ a Radon measure and $u \in BV(\Omega) \cap L^\infty(\Omega)$ (see [11], see also [23, 10] ). Nevertheless, this is not enough to deal with unbounded solutions. For this reason, we have to extend the Anzellotti theory (see Section 3 below).

Despite all these difficulties, we can prove the multiplicity of solutions for the 1–Laplacian. Being a very singular operator, the result we obtain is not as sharp as the one stated for the Laplacian. Indeed, measures are assumed to be singular with respect to a stronger capacity and must be zero near the boundary of $\Omega$. Nevertheless, our result is sufficient to show wild multiplicity of solutions:

**Theorem 1.1.** *Let $f \in L^m(\Omega)$, with $m > N$, be nonnegative and small enough to satisfy*

$$\|f\|_m < \Big(\frac{N - m'(N-1)}{N}\Big)^{\frac{m-1}{m}} \frac{|\Omega|^{\frac{N-1}{N} - \frac{1}{m'}}}{S_{N,1}}\,,$$

*where $S_{N,1}$ denotes the best constant in the Sobolev embedding $BV(\Omega) \hookrightarrow L^{N/(N-1)}(\Omega)$.*

*Let $\mu$ be a positive Radon measure which is singular with respect to the $q$–capacity for some $1 < q$ (that is, it is concentrated on a set $E$ of zero $W^{1,q}$–capacity). Assume also that the distance from $E$ to $\partial\Omega$ is positive.*

*Consider a renormalized solution $v_p$ of*

$$
\left\{
\begin{array}{ll}
-\Delta_p(v_p) = f(x)\Big(1+\dfrac{v_p}{p-1}\Big)^{p-1} + \mu & \textit{in } \ \Omega\,; \\
\qquad\qquad v_p = 0 & \textit{on } \partial\Omega\,;
\end{array}
\right.
\tag{4}
$$

*and set $u_p = (p-1)\log\big(1+\frac{v_p}{p-1}\big)$. Then the "sequence" $\{u_p\}$ converges, as $p$ goes to 1, to a solution $u_\mu$ of problem*

$$
\left\{
\begin{array}{ll}
-\mathrm{div}\Big(\dfrac{Du_\mu}{|Du_\mu|}\Big) = |Du_\mu| + f(x) & \textit{in } \ \Omega\,; \\
\qquad\qquad u_\mu = 0 & \textit{on } \partial\Omega\,;
\end{array}
\right.
\tag{5}
$$

*in the sense of Definition 4.1 below. This solution satisfies*

$$
e^{\delta u_\mu} \in BV(\Omega)\,, \quad \textit{for all } 0<\delta<1\,. \tag{6}
$$

*and*

$$
-\mathrm{div}\,(e^{u_\mu}\mathbf{z}) = e^{u_\mu} f + \mu\,, \qquad \textit{in } \mathcal{D}'(\Omega)\,, \tag{7}
$$

*where $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ is the vector field appearing in Definition 4.1 below.*

The plan of this paper is the following. The next section is devoted to preliminaries: we introduce our notation, the notions of capacity and renormalized solution as well as auxiliary results on BV–functions. Section 3 is devoted to extend the Anzellotti theory of $L^\infty$–divergence–measure vector fields to our needs. The multiplicity Theorem 1.1 is proved in Section 4, while the last Section deals with radial explicit solutions.

## 2. Preliminaries

2.1. **General notation.** From now on, we fix an integer $N \geq 2$. The symbol $\mathcal{H}^{N-1}(E)$ stands for the $(N-1)$–dimensional Hausdorff measure of a set $E \subset \mathbb{R}^N$ and $|E|$ for its Lebesgue measure. Moreover, $\Omega$ will always denote an open subset of $\mathbb{R}^N$ with Lipschitz boundary. Thus, an outward normal unit vector $\nu(x)$ is defined for $\mathcal{H}^{N-1}$–almost every $x \in \partial\Omega$.

The space of all $C^\infty$–functions having compact support in $\Omega$ is denoted by $C_0^\infty(\Omega)$. The symbol $L^q(\Omega)$, with $1 \leq q \leq \infty$, denotes the usual Lebesgue space with respect to Lebesgue measure and $q'$ is the conjugate of $q$: $q' = \dfrac{q}{q-1}$. We will denote by $W_0^{1,q}(\Omega)$ the usual Sobolev space, of measurable functions having weak gradient in $L^q(\Omega;\mathbb{R}^N)$ and zero trace on $\partial\Omega$. The dual space of $W_0^{1,q}(\Omega)$ will be denoted by $W^{-1,q'}(\Omega)$, we recall that its elements can be written as $\mathrm{div}\,F$ for some $F \in L^{q'}(\Omega;\mathbb{R}^N)$. Finally, if $1 \leq p < N$, we will denote by $p^* = Np/(N-p)$ its Sobolev conjugate exponent and by $S_{N,p}$ the best constant in the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, that is,

$$
\Big(\int_\Omega |u|^{p^*}\Big)^{p/p^*} \leq S_{N,p}\int_\Omega |\nabla u|^p\,.
$$

The truncation function will be use throughout this paper. Given $k>0$, it is defined by

$$
T_k(s) = \min\{|s|,k\}\ \mathrm{sign}\ (s)\,, \tag{8}
$$

for all $s \in \mathbb{R}$. Moreover we will denote by $G_k(s)$ the function defined by

$$G_k(s) = s - T_k(s)\,.$$

2.2. **Capacity.** Let $1 \le p < N$. For every compact set $K \subset \Omega$, we define its $p$–capacity with respect to $\Omega$ as

$$\mathrm{cap}_{1,p}(K,\Omega) = \inf\left\{ \int_\Omega |\nabla u|^p : u \in W_0^{1,p}(\Omega),\ u \ge \chi_K \text{ almost everywhere in } \Omega \right\}$$

(we will use the convention that $\inf \emptyset = +\infty$). For any open set $U \subset \Omega$, its $p$–capacity is then defined by

$$\mathrm{cap}_{1,p}(U,\Omega) = \sup\left\{\mathrm{cap}_{1,p}(K,\Omega) : K \text{ is a compact subset of } \Omega\right\}.$$

Finally, given a Borelian subset $B \subset \Omega$ the definition is extended by setting:

$$\mathrm{cap}_{1,p}(B,\Omega) = \inf\left\{\mathrm{cap}_{1,p}(U,\Omega) :\ U \text{ open subset of } \Omega,\ B \subset U\right\}.$$

We point out that $p$–capacity is not a Radon measure, although it is an outer measure. Using the definition of capacity, it is easy to see that $1 < p < q$ and $\mathrm{cap}_{1,q}(A,\Omega) = 0$ imply $\mathrm{cap}_{1,p}(A,\Omega) = 0$ as well as $\mathcal{H}^{N-1}(A) = 0$.

2.3. **Radon measures.** We recall that a Radon measure is a distribution of order 0 and that every positive distribution $T$, which is a distribution satisfying $\langle T, \varphi\rangle \ge 0$ for all nonnegative $\varphi \in C_0^\infty(\Omega)$, is a nonnegative Radon measure. Given a Radon measure $\mu$, we denote by $|\mu|$ its total variation. The Lebesgue spaces with respect to $\mu$ are denoted by $L^q(\Omega,\mu)$, where $1 \le q \le \infty$.

For a Radon measure $\mu$ in $\Omega$ and a Borel set $A \subseteq \Omega$ the measure $\mu \llcorner A$ is defined by $(\mu \llcorner A)(B) = \mu(A \cap B)$ for any Borel set $B \subseteq \Omega$. If a measure $\mu$ is such that $\mu = \mu \llcorner A$ for a certain Borel set $A$, the measure $\mu$ is said to be concentrated on $A$.

Let $\mu$ be a Radon measure in $\Omega$, we say that $\mu$ is singular with respect to the $p$–capacity if it is concentrated on a subset $E \subset \Omega$ such that

$$\mathrm{cap}_{1,p}(E,\Omega) = 0\,,$$

and we say that it is absolutely continuous with respect to the $p$–capacity if $\mathrm{cap}_{1,p}(E,\Omega) = 0$ implies $\mu(E) = 0$. Although the $p$–capacity is not a measure, every Radon measure $\mu$ can be decomposed as $\mu = \mu_a + \mu_s$, where $\mu_a$ is absolutely continuous and $\mu_s$ is singular, with respect to the $p$–capacity. Moreover, thanks to [9, Theorem 2.1], every Radon measure $\mu$ which is absolutely continuous with respect to the $p$–capacity can be written as $\mu = f - \operatorname{div} F$, where $f \in L^1(\Omega)$ and $F \in L^{p'}(\Omega;\mathbb{R}^N)$.

2.4. **Definition of renormalized solution.** Two definitions of solution must be considered, those to problem (4) and to problem (5). We point out that definition of a solution to problem (5) relies on the theory of $L^\infty$–divergence–measure vector fields, which will be studied in the next section, so that we postpone the definition of solution to problem (5) to Section 4. We now introduce the concept of renormalized solution to problem (4), we refer to [14] for a detailed study of this concept.

Given the measure $\mu$, we decompose it as $\mu = \mu_0 + \mu_s^+ - \mu_s^-$, where $\mu_0$ is absolutely continuous with respect to the $p$–capacity, while $\mu_s^+$ and $\mu_s^-$ are two nonnegative measures which are concentrated on two disjoint subsets of zero $p$–capacity.

Let $f(x)$ be a function in $L^1(\Omega)$, and $h(s)$ be a real continuous function. A measurable function $v : \Omega \to \mathbb{R}$ is a renormalized solution to problem

$$\begin{cases} -\Delta_p(v) = f(x)h(v) + \mu & \text{in } \Omega\,; \\ v = 0 & \text{on } \partial\Omega\,; \end{cases}$$

if the following conditions hold:

(a) The function $v$ is finite almost everywhere and $T_k(v) \in W_0^{1,p}(\Omega)$ for all $k > 0$. (As a consequence, a generalized gradient $\nabla v$ can be defined, see [8, Lemma 2.1].)

(b) The gradient satisfies $|\nabla v|^{p-1} \in L^q(\Omega)$ for every $q < \frac{N}{N-1}$.

(c) $fh(v) \in L^1(\Omega)$.

(d) For every $S \in W^{1,\infty}(\mathbb{R})$ such that $S'$ has compact support in $\mathbb{R}$ (consequently $S$ is constant for $|s|$ large and so the limits $S(+\infty) = \lim_{s\to+\infty} S(s)$ and $S(-\infty) = \lim_{s\to-\infty} S(s)$ exist), we have

$$\int_\Omega S'(v)\varphi|\nabla v|^p + \int_\Omega S(v)|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi$$
$$= \int_\Omega f(x)h(v)S(v)\varphi + \int_\Omega S(v)\varphi\, d\mu_0 + S(+\infty)\int_\Omega \varphi\, d\mu_s^+ - S(-\infty)\int_\Omega \varphi\, d\mu_s^-$$

for all $\varphi \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > N$, such that $S(v)\varphi \in W_0^{1,p}(\Omega)$.

2.5. $BV$–**functions.** The space $BV(\Omega)$ of *functions of bounded variation* is defined as the space of functions $u \in L^1(\Omega)$ whose distributional gradient $Du$ is a vector valued Radon measure on $\Omega$ with finite total variation. This space is a Banach space with the norm defined by

$$\|u\|_{BV} = \int_\Omega |u|\, dx + |Du|(\Omega)\,.$$

We recall that the notion of trace can be extended to any $u \in BV(\Omega)$ and this fact allows us to interpret it as the boundary values of $u$ and to write $u\big|_{\partial\Omega}$. Moreover, it holds that the trace is a linear bounded operator $BV(\Omega) \to L^1(\partial\Omega)$ which is onto. Using the trace, an equivalent norm in $BV(\Omega)$ can be defined by

$$\|u\| = \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + |Du|(\Omega)\,.$$

For every $u \in BV(\Omega)$, the Radon measure $Du$ can be decomposed into its absolutely continuous and singular parts with respect to the Lebesgue measure: $Du = D^a u + D^s u$. So, for each measurable set $E$, we have $D^a u(E) = \int_E \nabla u(x)\, dx$, where $\nabla u$ is the Radon–Nikodým derivative of the measure $D^a u$ with respect to the Lebesgue measure.

We denote by $S_u$ the set of all $x \in \Omega$ such that $x$ is not a Lebesgue point of $u$. We say that $x \in S_u$ is an *approximate jump point of* $u$ if there exist $u_+(x) > u_-(x) \in \mathbb{R}$ and $\nu_u(x) \in S^{N-1}$ such that

$$\lim_{\rho\downarrow 0} \fint_{B_\rho^+(x,\nu_u(x))} |u(y) - u_+(x)|\, dy = \lim_{\rho\downarrow 0} \fint_{B_\rho^-(x,\nu_u(x))} |u(y) - u_-(x)|\, dy = 0,$$

where

$$B_\rho^+(x,\nu_u(x)) = \{y \in B_\rho(x) \ : \ \langle y - x, \nu_u(x)\rangle > 0\}$$

and

$$B_\rho^-(x,\nu_u(x)) = \{y \in B_\rho(x) \ : \ \langle y - x, \nu_u(x)\rangle < 0\}.$$

We denote by $J_u$ the set of approximate jump points of $u$. By the Federer-Vol'pert Theorem [3, Theorem 3.78], we know that $S_u$ is countably $\mathcal{H}^{N-1}$–rectifiable and $\mathcal{H}^{N-1}(S_u \backslash J_u) = 0$. Moreover, $Du \llcorner J_u = (u_+ - u_-)\nu_u \mathcal{H}^{N-1} \llcorner J_u$. Using $S_u$ and $J_u$, we may split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$ defined by

$$D^j u = D^s u \llcorner J_u \quad \text{and} \quad D^c u = D^s u \llcorner (\Omega \backslash S_u)\,,$$

respectively. Thereby

$$D^j u = (u_+ - u_-)\nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

Moreover, if $x \in J_u$, then $\nu_u(x) = \frac{Du}{|Du|}(x)$, $\frac{Du}{|Du|}$ being the Radon–Nikodým derivative of $Du$ with respect to its total variation $|Du|$.

If $x$ is a Lebesgue point of $u$, then $u_+(x) = u_-(x)$ for any choice of the normal vector and we say that $x$ is an approximate continuity point of $u$. We define the approximate limit of $u$ by $\tilde{u}(x) = u_+(x) = u_-(x)$. The *precise representative* $u^* : \Omega \backslash (S_u \backslash J_u) \to \mathbb{R}$ of $u$ is defined as equal to $\tilde{u}$ on $\Omega \backslash S_u$ and equal to $\frac{u_+ + u_-}{2}$ on $J_u$. It is well known (see for instance [3, Corollary 3.80]) that if $\rho$ is a symmetric mollifier, then the mollified functions $u \star \rho_\epsilon$ converges pointwise to $u^*$ in its domain.

A compactness result in $BV(\Omega)$ will be used several times in what follows. It states that every sequence that is bounded in $BV(\Omega)$ has a subsequence which strongly converges in $L^1(\Omega)$ to a certain $u \in BV(\Omega)$.

To pass to the limit we will often apply that some functionals defined on $BV(\Omega)$ are lower semicontinuous with respect to the convergence in $L^1(\Omega)$. We recall that the functional defined by

$$u \mapsto |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \tag{9}$$

is lower semicontinuous with respect to the convergence in $L^1(\Omega)$. Similarly, if we fix $\varphi \in C_0^1(\Omega)$, with $\varphi \geq 0$, the functional defined by

$$u \mapsto \int_\Omega \varphi \, d|Du|\,,$$

is lower semicontinuous in $L^1(\Omega)$.

We also need a chain rule for functions in $BV(\Omega)$. Since we will only apply it for functions having empty jump set, we will state it only in this simple case. If $u \in BV(\Omega) \cap L^\infty(\Omega)$ is such that $D^j u = 0$, and $f$ is a real Lipschitz–continuous function, then $v = f \circ u$ belongs to $BV(\Omega)$ and $Dv = f'(u^*)Du$.

For further information concerning functions of bounded variation we refer to [3] or [26].

## 3. Extending Anzellotti's theory

In this section we will study some properties involving divergence–measure vector fields and functions of bounded variation. Our aim is to extend the Anzellotti theory introduced in [7].

Following [11] we define $\mathcal{DM}^\infty(\Omega)$ as the space of all vector fields $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ whose divergence in the sense of distributions is a Radon measure with finite total variation, i.e., $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ if and only if $\operatorname{div}\mathbf{z}$ is a Radon measure belonging to $W^{-1,\infty}(\Omega)$.

The theory of $L^\infty$–divergence–measure vector fields is due to G. Anzellotti [7] and, independently, to G.–Q. Chen and H. Frid [11]. In spite of their different points of view, both approaches introduce the normal trace of a vector field through the boundary and establish the same generalized Gauss–Green formula. Both also define the pairing $(\mathbf{z}, Du)$, where $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and $u$ is a certain $BV$–function, as a Radon measure. However, they differ in handling this concept. In the present paper we will need that the "dot product" be defined for every $u \in BV(\Omega)$ and every $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ satisfying a certain condition (see Corollary 3.5 below). We begin by recalling a result proved in [11].

**Proposition 3.1.** *For every $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, the measure $\mu = \operatorname{div}\mathbf{z}$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$. As a consequence, $|\mu|$ is also absolutely continuous with respect to $\mathcal{H}^{N-1}$.*

Consider now $\mu = \operatorname{div}\mathbf{z}$ with $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and let $u \in BV(\Omega)$; then the precise representative $u^*$ of $u$ is equal $\mathcal{H}^{N-1}$–a.e. to a Borel function; that is, to $\lim_{\epsilon\to 0} \rho_\epsilon \star u$, where $(\rho_\epsilon)$ is a symmetric mollifier. Then one deduces from the Proposition 3.1 that $u^*$ is equal $\mu$–a.e. to a Borel function. So, given $u \in BV(\Omega)$, $u^*$ is always $\mu$–measurable. Moreover, $u \in BV(\Omega)\cap L^\infty(\Omega)$ implies $u \in L^\infty(\Omega,\mu) \subset L^1(\Omega,\mu)$.

### 3.1. Preservation of the norm.

We point out that every $\operatorname{div}\mathbf{z}$, with $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, defines a functional on $W_0^{1,1}(\Omega)$ by

$$\langle \operatorname{div}\mathbf{z}, u\rangle_{W^{-1,\infty}(\Omega),W_0^{1,1}(\Omega)} = -\int_\Omega \mathbf{z}\cdot\nabla u\,.$$

To express this functional in terms of an integral with respect to the measure $\mu = \operatorname{div}\mathbf{z}$, we need the following Meyers–Serrin type theorem (see [3, Theorem 3.9] for its extension to $BV$–functions).

**Proposition 3.2.** *Let $\mu = \operatorname{div}\mathbf{z}$, with $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$. For every $u \in BV(\Omega)\cap L^\infty(\Omega)$ there exists a sequence $(u_n)_n$ in $W^{1,1}(\Omega)\cap C^\infty(\Omega)\cap L^\infty(\Omega)$ such that*

(1) $u_n \to u^*$ *in* $L^1(\Omega,\mu)$

(2) $\int_\Omega |\nabla u_n| \to |Du|(\Omega)$.

(3) $u_n|_{\partial\Omega} = u|_{\partial\Omega}$ *for all* $n \in \mathbb{N}$.

(4) $|u_n(x)| \le \|u\|_\infty$ $|\mu|$*–a.e. for all* $n \in \mathbb{N}$.

*Moreover, if $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, then one may find $u_n$ satisfying, instead of (2), the condition*

$$\text{(2')}\ u_n \to u \ \text{in}\ W^{1,1}(\Omega)\,.$$

Since

$$-\int_\Omega \mathbf{z} \cdot \nabla\varphi = \int_\Omega \varphi\, d\mu$$

holds for every $\varphi \in C_0^\infty(\Omega)$, it is easy to obtain this equality for every $W_0^{1,1}(\Omega) \cap C^\infty(\Omega)$. Given $u \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$ and applying Proposition 3.2, we may find a sequence $(u_n)_n$ in $W_0^{1,1}(\Omega) \cap C^\infty(\Omega)$ satisfying (1) and (2'). It follows from

$$-\int_\Omega \mathbf{z} \cdot \nabla u_n = \int_\Omega u_n\, d\mu$$

for every $n \in \mathbb{N}$. letting $n$ go to infinity, that

$$-\int_\Omega \mathbf{z} \cdot \nabla u = \int_\Omega u^*\, d\mu$$

and so

$$\langle \operatorname{div} \mathbf{z}, u\rangle_{W^{-1,\infty}(\Omega), W_0^{1,1}(\Omega)} = \int_\Omega u^*\, d\mu$$

holds for every $u \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$. Then the norm of this functional is given by

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup\left\{ \Big| \int_\Omega u^*\, d\mu \Big| \ :\ \int_\Omega |\nabla u| \le 1 \right\}\,.$$

We have seen that $\mu = \operatorname{div} \mathbf{z}$ can be extended from $W_0^{1,1}(\Omega)$ to $BV(\Omega) \cap L^\infty(\Omega)$. Next, we will prove that this extension preserves the norm.

**Theorem 3.3.** *If $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, then $\mu = \operatorname{div} \mathbf{z}$ can be extended to $BV(\Omega) \cap L^\infty(\Omega)$ in such a way that*

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup\left\{ \Big| \int_\Omega u^*\, d\mu \Big| \ :\ |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \le 1 \right\}\,.$$

Proof. Since we already know that $BV(\Omega) \cap L^\infty(\Omega)$ is a subset of $L^1(\Omega, \mu)$, all we have to prove is

$$\Big| \int_\Omega u^*\, d\mu \Big| \le \|\mu\|_{W^{-1,\infty}(\Omega)} \Big( |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \Big)\,. \tag{10}$$

for all $u \in BV(\Omega) \cap L^\infty(\Omega)$. This inequality will be proved in two steps.

1) Assume first that $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. Applying [7, Lemma 5.5], we find a sequence $(w_n)_n$ in $W^{1,1}(\Omega) \cap C(\Omega)$ such that

(1) $w_n|_{\partial\Omega} = u|_{\partial\Omega}$ .

(2) $\displaystyle\int_\Omega |\nabla w_n| \le \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \frac{1}{n}$ .

(3) $\displaystyle\int_\Omega |w_n| \le \frac{1}{n}$ .

(4) $w_n(x) = 0, \qquad \text{if } \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}$ .

(5) $w_n(x) \to 0, \qquad \text{for all } x \in \Omega$ .

Then it yields

$$\Big| \int_\Omega (u^* - w_n^*)\, d\mu \Big| = \big| \langle \mu, (u - w_n) \rangle_{W^{-1,\infty}(\Omega), W_0^{1,1}(\Omega)} \big| \le \|\mu\|_{W^{-1,\infty}(\Omega)} \int_\Omega |\nabla u - \nabla w_n|$$

$$\le \|\mu\|_{W^{-1,\infty}(\Omega)} \Big( \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \frac{1}{n} \Big) .$$

It follows that

$$\Big| \int_\Omega u^*\, d\mu \Big| \le \Big| \int_\Omega (u^* - w_n^*)\, d\mu \Big| + \Big| \int_\Omega w_n^*\, d\mu \Big| \tag{11}$$

$$\le \|\mu\|_{W^{-1,\infty}(\Omega)} \Big( \int_\Omega |\nabla u|\, dx + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \frac{1}{n} \Big) + \Big| \int_\Omega w_n^*\, d\mu \Big| .$$

Since the sequence $(w_n)_n$ tends pointwise to 0 and it is uniformly bounded in $L^\infty(\Omega)$, by Lebesgue's Theorem,

$$\lim_{n\to\infty} \int_\Omega w_n^*\, d\mu = 0 .$$

Now, taking the limit in (11) we obtain (10).

2) In the general case, we apply Proposition 3.2 and find a sequence $u_n$ in $W^{1,1}(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ satisfying (1)–(4). Then, it follows from

$$\Big| \int_\Omega u_n^*\, d\mu \Big| \le \|\mu\|_{W^{-1,\infty}(\Omega)} \Big( \int_\Omega |\nabla u_n| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \Big) \quad \text{for all } n \in \mathbb{N}$$

that (10) holds. ■

**Corollary 3.4.** *Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ satisfy $\operatorname{div} \mathbf{z} = \nu + f$ for a certain Radon measure $\nu$ and a certain $f \in L^N(\Omega)$. If either $\nu \ge 0$ or $\nu \le 0$, then $\mu = \operatorname{div} \mathbf{z}$ can be extended to $BV(\Omega)$ and*

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup \Big\{ \Big| \int_\Omega u^*\, d\mu \Big| \, : \, |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \le 1 \Big\} .$$

*Moreover,* $BV(\Omega) \hookrightarrow L^1(\Omega, \mu)$.

PROOF. Consider $u \in BV(\Omega)$, write $u^+ = \max\{u, 0\}$ and, for every $k > 0$, apply the previous result to $T_k(u^+)$. Then

$$\Big| \int_\Omega \big(T_k(u^+)\big)^* \, d\mu \Big| \le \|\mu\|_{W^{-1,\infty}(\Omega)} \Big( |DT_k(u^+)|(\Omega) + \int_{\partial\Omega} T_k(u^+) \, d\mathcal{H}^{N-1} \Big) \tag{12}$$

$$\le \|\mu\|_{W^{-1,\infty}(\Omega)} \Big( |Du^+|(\Omega) + \int_{\partial\Omega} u^+ \, d\mathcal{H}^{N-1} \Big) .$$

On the other hand, observe that $u^*$ is a $\nu$–measurable function, so that we obtain

$$\int_\Omega \big(T_k(u^+)\big)^* \, d\mu = \int_\Omega T_k(u^+)^* \, d\nu + \int_\Omega T_k(u^+) f$$

for every $k > 0$. We may apply Levi's Theorem and Lebesgue's Theorem to deduce that

$$\lim_{k \to +\infty} \int_\Omega \big(T_k(u^+)\big)^* \, d\nu = \int_\Omega (u^+)^* \, d\nu \, ;$$

$$\lim_{k \to +\infty} \int_\Omega T_k(u^+) f = \int_\Omega u^+ f \, .$$

Thus,

$$\lim_{k \to +\infty} \int_\Omega \big(T_k(u^+)\big)^* \, d\mu = \int_\Omega (u^+)^* \, d\mu \, .$$

Now taking the limit when $k$ goes to $\infty$ in (12), it yields

$$\Big| \int_\Omega (u^+)^* \, d\mu \Big| \le \|\mu\|_{W^{-1,\infty}(\Omega)} \Big( |Du^+|(\Omega) + \int_{\partial\Omega} u^+ \, d\mathcal{H}^{N-1} \Big) . \tag{13}$$

Assume, in order to be concrete, that $\nu \ge 0$. Since $\int_\Omega (u^+)^* \, d\mu^- = \int_\Omega u^+ f^-$, we already have that $(u^+)^*$ is $\mu^-$–integrable. Hence, as a consequence of (13), we deduce that $(u^+)^*$ is also $\mu^+$–integrable and so $\mu$–integrable.

Since we may prove a similar inequality to $u^- = \max\{-u, 0\}$, adding both inequalities we deduce that $u^*$ is $\mu$–integrable and that

$$\Big| \int_\Omega u^* \, d\mu \Big| \le \|\mu\|_{W^{-1,\infty}(\Omega)} \Big( |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \Big)$$

holds true. ∎

3.2. **A Green formula.** Let $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ and let $u \in BV(\Omega)$. Assume that $\operatorname{div} \mathbf{z} = \nu + f$, with $\nu$ a Radon measure satisfying either $\nu \ge 0$ or $\nu \le 0$, and $f \in L^N(\Omega)$. In the spirit of [7], we define the following distribution on $\Omega$. For every $\varphi \in C_0^\infty(\Omega)$, we write

$$\langle (\mathbf{z}, Du), \varphi \rangle = - \int_\Omega u^* \, \varphi \, d\mu - \int_\Omega u \mathbf{z} \cdot \nabla \varphi \, , \tag{14}$$

where $\mu = \operatorname{div} \mathbf{z}$. Note that the previous subsection implies that every term in the above definition has sense.

**Proposition 3.5.** *Let* $\mathbf{z}$ *and* $u$ *be as above. The distribution* $(\mathbf{z}, Du)$ *defined previously satisfies*

$$|\langle(\mathbf{z}, Du), \varphi\rangle| \leq \|\varphi\|_\infty \|\mathbf{z}\|_{L^\infty(U)} \int_U d|Du| \tag{15}$$

*for all open set* $U \subset \Omega$ *and for all* $\varphi \in C_0^\infty(U)$*.*

*As a consequence, the distribution* $(\mathbf{z}, Du)$ *is actually a Radon measure. Both* $(\mathbf{z}, Du)$ *and its total variation* $|(\mathbf{z}, Du)|$ *are absolutely continuous with respect to the measure* $|Du|$ *and*

$$\left| \int_B d(\mathbf{z}, Du) \right| \leq \int_B d|(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_{L^\infty(U)} \int_B d|Du|$$

*holds for all Borel sets* $B$ *and for all open sets* $U$ *such that* $B \subset U \subset \Omega$*.*

Proof. If $U \subset \Omega$ is an open set and $\varphi \in C_0^\infty(U)$, then it was proved in [23] that

$$|\langle(\mathbf{z}, DT_k(u)), \varphi\rangle| \leq \|\varphi\|_\infty \|\mathbf{z}\|_{L^\infty(U)} \int_U d|DT_k(u)| \leq \|\varphi\|_\infty \|\mathbf{z}\|_{L^\infty(U)} \int_U d|Du| \tag{16}$$

holds for every $k > 0$. On the other hand,

$$\langle(\mathbf{z}, DT_k(u)), \varphi\rangle = -\int_\Omega \big(T_k(u)\big)^* \varphi \, d(\operatorname{div} \mathbf{z}) - \int_\Omega T_k(u) \mathbf{z} \cdot \nabla\varphi \, .$$

We may let $k \to \infty$ in each term on the right hand side, due to $u^* \in L^1(\Omega, \mu)$ and $u \in L^1(\Omega)$. Therefore,

$$\lim_{k\to\infty} \langle(\mathbf{z}, DT_k(u)), \varphi\rangle = \langle(\mathbf{z}, Du), \varphi\rangle \, ,$$

and so (16) implies (15). ■

On the other hand, for every $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ is defined in [7] and denoted by $[\mathbf{z}, \nu]$.

**Proposition 3.6.** *Let* $\mathbf{z}$ *and* $u$ *be as above. With the above definitions, the following Green formula holds*

$$\int_\Omega u^* \, d\mu + \int_\Omega d(\mathbf{z}, Du) = \int_{\partial\Omega} [\mathbf{z}, \nu] u \; d\mathcal{H}^{N-1} \, , \tag{17}$$

*where* $\mu = \operatorname{div} \mathbf{z}$*.*

Proof. Applying the Green formula proved in [23], we obtain

$$\int_\Omega \big(T_k(u)\big)^* \, d\mu + \int_\Omega d(\mathbf{z}, DT_k(u)) = \int_{\partial\Omega} [\mathbf{z}, \nu] T_k(u) \; d\mathcal{H}^{N-1} \, , \tag{18}$$

for every $k > 0$. In the proof of the previous Proposition, we have seen that

$$\lim_{k\to\infty} \int_\Omega d(\mathbf{z}, DT_k(u)) = \int_\Omega d(\mathbf{z}, Du) \, .$$

We may take limits in the other terms since $u^* \in L^1(\Omega, \mu)$ and $u \in L^1(\partial\Omega)$. Hence, letting $k$ go to $\infty$ in (18), we get (17). ■

## 4. Multiplicity of solutions

In this Section, we will assume that $f$ is a nonnegative function belonging to $L^m(\Omega)$, with $m > N$. We also assume that

$$\|f\|_m < \Big(\frac{m-N}{N(m-1)}\Big)^{\frac{m-1}{m}} \frac{|\Omega|^{\frac{1}{m}-\frac{1}{N}}}{S_{N,1}}\,. \tag{19}$$

The constant on the right hand side is obtained in the proof of Theorem 1.1 (see Step 1 below). It could also been deduced from an argument by N. Grenon in [17] and [18], checking the dependence on $p > 1$ of every involved constant and letting $p$ go to 1. (It should be mentioned that Grenon assume $p > \frac{2N}{N+1}$, but this hypothesis can be removed.) We point out that this procedure leads to the same constant.

It is worth showing the translation of condition (19) to the $N$–norm. Indeed, Hölder's inequality implies

$$\|f\|_N \le \|f\|_m |\Omega|^{\frac{1}{N}-\frac{1}{m}} < \Big(\frac{m-N}{N(m-1)}\Big)^{\frac{m-1}{m}} S_{N,1}^{-1} \le S_{N,1}^{-1}\,, \tag{20}$$

since $N \ge 1$. So, [13, Theorem 4.1] allows us to deduce that the only bounded solution to problem (5) is the trivial solution. We now turn to define solution to problem (5), following the concept introduced in [4].

**Definition 4.1.** *Given $f \in L^m(\Omega)$, with $m > N$, we say that $u$ is a solution of problem* (5)*, if $u \in BV(\Omega)$ is such that the jump part satisfies $D^j u = 0$ and there exists a vector field $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$, with $\|\mathbf{z}\|_\infty \le 1$, satisfying*

$$-\operatorname{div} \mathbf{z} = |Du| + f \quad \textit{in} \;\; \mathcal{D}'(\Omega)\,, \tag{21}$$

$$(\mathbf{z}, Du) = |Du| \quad \textit{as measures in} \;\; \Omega\,, \tag{22}$$

*and*

$$[\mathbf{z}, \nu] \in \operatorname{sign}(-u) \quad \mathcal{H}^{N-1}\textit{–a.e. on} \;\; \partial\Omega\,. \tag{23}$$

Thanks to Propositions 3.5, identity (22) has sense. Heuristically, identity (22), jointly with $\|\mathbf{z}\|_\infty \le 1$, leads to $\mathbf{z} = \dfrac{Du}{|Du|}$, while (23) is a weak formulation of the Dirichlet boundary condition.

The proof of our main Theorem below uses the following elementary technical results.

**Lemma 4.2.** *If $1 < \alpha < \frac{N}{N-1}$ and $1 < p < 2$, then*

$$e^{\alpha s} - 1 \le 2\Big(e^{\frac{\alpha}{p} s} - 1\Big)^p + 1$$

*for all $s \ge 0$.*

Proof. Just note that every $x \ge 1$ satisfies

$$x^p = \big((x-1)+1\big)^p \le 2^{p-1}\big((x-1)^p + 1\big) \le 2\big((x-1)^p + 1\big)$$

and we may choose $x = e^{\frac{\alpha}{p} s}$. ∎

**Lemma 4.3.** *If* $1 < \alpha < \frac{N}{N-1}$ *and* $1 < p < 2$*, then*

$$\left(e^{\frac{\alpha}{p}s} - 1\right)^p \le e^{\alpha s} - 1$$

*for all* $s \geq 0$.

PROOF. As in Lemma 4.2 consider $x = e^{\frac{\alpha}{p}s}$ and check that the real function defined by

$$\eta(x) = (x-1)^p - x^p + 1\,, \qquad x \geq 1\,,$$

is increasing. ■

PROOF OF THEOREM 1.1. Fix a positive Radon measure $\mu$ satisfying

(1) $\mu$ is concentrated in a set $A$
(2) $dist(\overline{A}, \partial\Omega) > 0$
(3) There exists $q > 1$ such that $\mathrm{cap}_q(A, \Omega) = 0$.

Since our aim is to let $p$ go to 1, we may take $q$ as small as we want. For instance, we may assume that $q \leq 2$ without loss of generality.

The proof of Theorem 1.1 will be developed in several stages.

*Step 1: Problems with measure datum.* For any $1 < p < q$, consider the problem

$$\begin{cases} -\Delta_p(v_p) = f(x)\Big(1 + \dfrac{v_p}{p-1}\Big)^{p-1} + \mu\,, & \text{in } \Omega\,; \\ v_p = 0\,, & \text{on } \Omega\,. \end{cases} \tag{24}$$

This problem has been studied in [17, Theorem 1.1] and [20, Theorem 6.2]. It follows from (20) that

$$\limsup_{p\to 1} \|f\|_{\frac{N}{p}} S_{N,p} \leq \lim_{p\to 1} \|f\|_N |\Omega|^{\frac{p-1}{N}} S_{N,p} = \|f\|_N S_{N,1} < 1\,,$$

so that $\|f\|_{\frac{N}{p}} S_{N,p} < 1$, for $p$ close enough to 1. Observe that the Hölder and Sobolev inequalities imply

$$\int_\Omega f|w|^p \leq \|f\|_{\frac{N}{p}} S_{N,p} \int_\Omega |\nabla w|^p\,,$$

for all $w \in W_0^{1,p}(\Omega)$. As a consequence, it yields

$$1 < \frac{1}{\|f\|_{\frac{N}{p}} S_{N,p}} \leq \inf\left\{ \frac{\int_\Omega |\nabla w|^p}{\int_\Omega f|w|^p} \ : \quad w \in W_0^{1,p}(\Omega)\,, \quad \int_\Omega f|w|^p \neq 0 \right\}\,.$$

Thus, for $p$ close enough to 1, we may apply [20, Theorem 6.2] and find a renormalized solution to (24). Since data $f$ and $\mu$ are nonnegative, it follows that $v_p \geq 0$.

Taking $1 - \dfrac{1}{(1 + \frac{T_k(v_p)}{p-1})^{p-1}}$ as test function in the renormalized formulation of (24), we obtain

$$\begin{aligned} \int_{\{v_p<k\}} \frac{|\nabla v_p|^p}{(1+\frac{v_p}{p-1})^p} &\leq \int_\Omega f\Big(1 + \frac{v_p}{p-1}\Big)^{p-1} + \mu(\Omega) \\ &\leq \int_\Omega f + \int_\Omega f\Big(\frac{v_p}{p-1}\Big)^{p-1} + \mu(\Omega) \leq \|f\|_1 + \|f\|_m \Big\| \Big(\frac{v_p}{p-1}\Big)^{p-1} \Big\|_{m'} + \mu(\Omega)\,. \end{aligned}$$

Passing to the limit as $k$ goes to $+\infty$, we have

$$\int_\Omega \frac{|\nabla v_p|^p}{\left(1+\frac{v_p}{p-1}\right)^p} \le \|f\|_1 + \|f\|_m \Big\| \Big(\frac{v_p}{p-1}\Big)^{p-1}\Big\|_{m'} + \mu(\Omega)\,. \tag{25}$$

Thus, to go on, we need an estimate of $\|v_p^{p-1}\|_{m'}$, not depending on $p$. This is a consequence of the regularity of renormalized solutions. Indeed, observe that we may choose the truncate $T_k(v_p)$ as test function in the renormalized formulation of (24) and deduce that

$$\int_{\{v_p<k\}} |\nabla v_p|^p \le k\int_\Omega f\Big(1+\frac{v_p}{p-1}\Big)^{p-1} + k\mu(\Omega) \le kM_p$$

where $M_p = \|f\|_1 + \|f\|_m \Big\|\Big(\frac{v_p}{p-1}\Big)^{p-1}\Big\|_{m'} + \mu(\Omega)$. An appeal to [8, Lemma 4.1] leads to

$$|\{v_p^{p-1} > k\}| \le \left(\frac{S_{N,p}\, M_p}{k}\right)^{\frac{N}{N-p}}\,, \quad \text{for all } k>0\,.$$

In the setting of Marcinkiewicz spaces (see, for instance, [15, Appendix: Singular Integrals]), this inequality states that $\big[v_p^{p-1}\big]_{\frac{N}{N-p}} \le S_{N,p}\, M_p$, and so

$$\big[v_p^{m'(p-1)}\big]_{\frac{N}{m'(N-p)}} = \big[v_p^{p-1}\big]^{m'}_{\frac{N}{N-p}} \le (S_{N,p}\, M_p)^{m'}\,.$$

On the other hand, having in mind inequality (6.5) in [15, Appendix], we deduce

$$\begin{aligned}
\int_\Omega |v_p^{p-1}|^{m'} &\le \frac{N}{N-m'(N-p)}|\Omega|^{1-\frac{m'(N-p)}{N}}\big[v_p^{m'(p-1)}\big]_{\frac{N}{m'(N-p)}}\\
&\le \frac{N}{N-m'(N-p)}|\Omega|^{1-\frac{m'(N-p)}{N}}(S_{N,p}\, M_p)^{m'}\\
&= \frac{N}{N-m'(N-p)}|\Omega|^{1-\frac{m'(N-p)}{N}}(S_{N,p})^{m'}\Big(\|f\|_1 + \|f\|_m\Big\|\Big(\frac{v_p}{p-1}\Big)^{p-1}\Big\|_{m'} + \mu(\Omega)\Big)^{m'}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|v_p^{p-1}\|_{m'} \le \\
&\quad \left(\frac{N}{N-m'(N-p)}\right)^{\frac{1}{m'}} |\Omega|^{\frac{1}{m'}-\frac{(N-p)}{N}} S_{N,p}\Big(\|f\|_1 + (p-1)^{-(p-1)}\|f\|_m\|v_p^{p-1}\|_{m'} + \mu(\Omega)\Big).
\end{aligned} \tag{26}$$

Note that a bound for the norm $\|v_p^{p-1}\|_{m'}$ can be obtained if

$$\|f\|_m < \Big(\frac{N-m'(N-p)}{N}\Big)^{\frac{m-1}{m}} \frac{(p-1)^{p-1}|\Omega|^{\frac{N-p}{N}-\frac{1}{m'}}}{S_{N,p}}\,.$$

The limit, as $p$ goes to 1, on the right hand side is straightforward (since $\lim_{p\to1} S_{N,p} = S_{N,1}$) and is given by

$$\Big(\frac{N-m'(N-1)}{N}\Big)^{\frac{m-1}{m}} \frac{|\Omega|^{\frac{N-1}{N}-\frac{1}{m'}}}{S_{N,1}}\,.$$

Hence, our hypothesis (19) allows us to rearrange inequality (26) and obtain a bound for the norm $\|v_p^{p-1}\|_{m'}$ for $p$ close enough to 1. So, there is not loss of generality in assuming that $q$ is small enough to perform the above manipulations for all $1 < p < q$. Therefore, we have found a bound (not depending on $p$) of the right hand side of (25), that is, there exists $M > 0$ such that

$$\|f\|_1 + \|f\|_m \Big\| \Big(\frac{v_p}{p-1}\Big)^{p-1} \Big\|_{m'} + \mu(\Omega) \le M\,, \quad \text{for all } 1 < p < q\,, \tag{27}$$

and so

$$\int_\Omega \frac{|\nabla v_p|^p}{\big(1+\frac{v_p}{p-1}\big)^p} \le M\,, \quad \text{for all } 1 < p < q\,. \tag{28}$$

*Step 2: Problems having gradient terms.* In this step, we are considering the problems

$$\begin{cases} -\Delta_p(u_p) = |\nabla u_p|^p + f(x)\,, & \text{in } \Omega\,; \\ \qquad\qquad u_p = 0\,, & \text{on } \Omega\,. \end{cases} \tag{29}$$

According to the results in [19, 20], the function $u_p = (p-1)\log\big(1+\frac{v_p}{p-1}\big)$ belongs to $W_0^{1,p}(\Omega)$ and is a solution to (29). In terms of these new functions, the estimate (28) becomes

$$\int_\Omega |\nabla u_p|^p \le M\,, \quad \text{for all } 1 < p < q\,. \tag{30}$$

Applying Young's inequality, it follows that

$$\int_\Omega |\nabla u_p| \le \frac{1}{p}\int_\Omega |\nabla u_p|^p + \frac{p-1}{p}|\Omega| \le M + |\Omega|\,, \quad \text{for all } 1 < p < q\,. \tag{31}$$

(Recall that $M$ depends on $\Omega$, $m$, $\mu$ and $f$, but not on $p$.) This $BV$–estimate implies that there exists $u \in BV(\Omega)$ satisfying (up to subsequences)

$$u_p \to u \quad \text{pointwise a.e in } \Omega \tag{32}$$

$$u_p \to u \quad \text{strongly in } L^r(\Omega)\,, \quad 1 \le r < \frac{N}{N-1}\,. \tag{33}$$

On the other hand, estimate (30) is the starting point in [4] to get a suitable vector field $\mathbf{z}$. So, following [4] (see also [24, Theorem 3.5]) we get $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ satisfying $\|\mathbf{z}\|_\infty \le 1$ and

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}\,, \quad \text{weakly in } L^s(\Omega;\mathbb{R}^N)\,, \quad 1 \le s < \infty\,. \tag{34}$$

*Step 3: Passing to the limit in* (29). Let $\varphi \in C_0^\infty(\Omega)$ be nonnegative. Taking $\varphi$ as test function in (29), we have

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi = \int_\Omega \varphi|\nabla u_p|^p + \int_\Omega f\varphi\,.$$

Applying Young's inequality, it yields

$$\begin{aligned} \int_\Omega \varphi|\nabla u_p| + \int_\Omega f\varphi &\le \frac{1}{p}\int_\Omega \varphi|\nabla u_p|^p + \frac{p-1}{p}\int_\Omega\varphi + \int_\Omega f\varphi \\ &\le \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi + \frac{p-1}{p}\int_\Omega\varphi \end{aligned}$$

and appealing to the lower–semicontinuity on the left hand side, we deduce

$$\int_\Omega \varphi\, d|Du| + \int_\Omega f\varphi \le \int_\Omega \mathbf{z}\cdot\nabla\varphi\,.$$

Therefore,

$$|Du| + f \le -\operatorname{div}\mathbf{z}\,,\quad \text{in } \mathcal{D}'(\Omega)\,, \tag{35}$$

and so $\operatorname{div}\mathbf{z}$ is a Radon measure. It has finite total variation since is the distributional limit of $\Delta_p(u_p)$ and this sequence is bounded in $L^1(\Omega)$, due to (30) and equation (29).

On the other hand, $-\operatorname{div}\mathbf{z}$ is a Radon measure which is above an $L^N$–function. Thus, we may apply the results of Section 3 and so we have at our disposal a Green's formula involving the Radon measure $(\mathbf{z}, Du)$ (Proposition 3.6).

*Step 4: Passing to the limit in* (24). We need another estimate, this one to get the convergence of $e^{u_p}$. Fix $0<\delta<1$ and $k>0$, and take $e^{\delta T_k(u_p)}-1$ as test function in the weak formulation of (29). Dropping nonnegative terms it yields

$$\begin{aligned}\delta\int_\Omega e^{\delta T_k(u_p)}|\nabla T_k(u_p)|^p &= \int_\Omega \Big(e^{\delta T_k(u_p)}-1\Big)|\nabla u_p|^p + \int_\Omega f\Big(e^{\delta T_k(u_p)}-1\Big)\\ &\ge \int_\Omega \Big(e^{\delta T_k(u_p)}-1\Big)|\nabla u_p|^p - \int_\Omega f\,.\end{aligned}$$

Then rearranging and taking into account (30) we get

$$\int_\Omega e^{\delta T_k(u_p)}|\nabla u_p|^p - \delta\int_\Omega e^{\delta T_k(u_p)}|\nabla T_k(u_p)|^p \le \int_\Omega |\nabla u_p|^p + \int_\Omega f \le M + \int_\Omega f$$

from where it follows

$$(1-\delta)\int_\Omega e^{\delta T_k(u_p)}|\nabla u_p|^p \le M + \int_\Omega f\,,$$

for all $0<\delta<1$ and all $k>0$. Thanks to Levi's Monotone Convergence Theorem, we may let $k$ go to $+\infty$ and get

$$(1-\delta)\int_\Omega \Big|\nabla\Big(e^{\delta u_p}-1\Big)\Big|^p \le M + \int_\Omega f\,,$$

for all $0<\delta<1$. Hence, as a consequence of Young's inequality we have a $W_0^{1,1}$–estimate of $e^{\delta u_p}-1$, and on account of (32) it yields $e^{\delta u}\in BV(\Omega)$ for all $0<\delta<1$, as well as

$$e^{\delta u_p}\to e^{\delta u}\quad \text{pointwise a.e in } \Omega$$

$$e^{\delta u_p}\to e^{\delta u}\quad \text{strongly in } L^r(\Omega)\,,\quad 1\le r<\frac{N}{N-1}\,.$$

A straightforward consequence of the last convergence is

$$e^{u_p}\to e^{u}\quad \text{strongly in } L^r(\Omega)\,,\quad 1\le r<\frac{N}{N-1}\,. \tag{36}$$

(Nevertheless, we do not claim that $e^u\in BV(\Omega)$, see Remark 4.4 below.)

Now recalling that every renormalized solution is a distributional solution as well, take $\varphi \in C_0^\infty(\Omega)$ as test function in (24). Then

$$\int_\Omega |\nabla v_p|^{p-2}\nabla v_p \cdot \nabla\varphi = \int_\Omega f\Big(1+\frac{v_p}{p-1}\Big)^{p-1}\varphi + \int_\Omega \varphi\, d\mu\,,$$

which in terms of $u_p$ becomes

$$\int_\Omega e^{u_p}|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi = \int_\Omega f e^{u_p}\varphi + \int_\Omega \varphi\, d\mu\,. \tag{37}$$

Our next aim is to let $p$ go to 1, to this end, we are analyzing each term in (37). On the left hand side, we apply (36) and (34) to pass to the limit. On the other hand, (36) implies that $e^{u_p} \to e^u$ in $L^{m'}(\Omega)$, so that we may also pass to the limit on the right hand side. We conclude that

$$-\mathrm{div}\,\big(e^u\mathbf{z}\big) = fe^u + \mu\,, \quad \text{in } \mathcal{D}'(\Omega)\,. \tag{38}$$

We note that our assumptions on $\mu$ imply that $\mu \ll \mathrm{cap}_{1,q}$ and so $\mu \notin L^1(\Omega) + W^{-1,q'}(\Omega)$, by [9, Theorem 2.1]. We deduce from (38) that $e^u \notin L^{q'}(\Omega)$; in particular we have $u \notin L^\infty(\Omega)$.

*Step 5: $D^j u = 0$.* This fact is proved in [6] for a bounded solution to problem (5) through [6, Lemma 2]. The only modification of that proof we need in our setting is to choose $\lambda > 3$ and take $\lambda - T_1(u)$ as test function in $U_n$.

*Step 6: The equation $-\mathrm{div}\,\mathbf{z} = |Du| + f$ holds as measures.* First we claim that

$$-e^u\mathrm{div}\,\mathbf{z} \le e^u|Du| + e^u f + \mu \tag{39}$$

holds as measures. (Here we do not mean that $e^u$ is integrable with respect to the positive Radon measures $-\mathrm{div}\,\mathbf{z}$ and $|Du|$.) To see our claim, for any $k > 0$, our starting point is $(\mathbf{z}, De^{T_k(u)}) \le |De^{T_k(u)}|$ jointly with the equality

$$-e^{T_k(u)}\mathrm{div}\,\mathbf{z} = (\mathbf{z}, De^{T_k(u)}) - \mathrm{div}\,\Big(e^{T_k(u)}\mathbf{z}\Big)\,.$$

Then, by the chain rule,

$$-e^{T_k(u)}\mathrm{div}\,\mathbf{z} \le |De^{T_k(u)}| - \mathrm{div}\,\Big(e^{T_k(u)}\mathbf{z}\Big) = e^{T_k(u)}|DT_k(u)| - \mathrm{div}\,\Big(e^{T_k(u)}\mathbf{z}\Big)\,.$$

We now choose a nonnegative $\varphi \in C_0^\infty(\Omega)$ obtaining

$$-\int_\Omega \varphi e^{T_k(u)} d(\mathrm{div}\,\mathbf{z}) \le \int_\Omega \varphi e^{T_k(u)} d|DT_k u| + \int_\Omega e^{T_k(u)}\mathbf{z}\cdot\nabla\varphi\,.$$

Applying Levi's Monotone Convergence Theorem to the measures $-\mathrm{div}\,\mathbf{z}$ and $|Du|$, let $k$ go to infinity to get

$$-\int_\Omega \varphi e^u d(\mathrm{div}\,\mathbf{z}) \le \int_\Omega \varphi e^u d|Du| + \int_\Omega e^u\mathbf{z}\cdot\nabla\varphi = \int_\Omega \varphi e^u d|Du| + \int_\Omega \varphi e^u f + \int_\Omega \varphi\, d\mu\,,$$

due to (38). Therefore, (39) is proved.

Next, we will study these measures concentrated on the sets $\{u < k\}\backslash A$, with $k > 0$. Having in mind that $\mu \llcorner (\Omega\backslash A) = 0$, it follows that

$$-e^u\mathrm{div}\,\mathbf{z}\llcorner(\{u<k\}\backslash A) \le e^u|Du|\llcorner(\{u<k\}\backslash A) + e^u f\chi_{(\{u<k\}\backslash A)}\,.$$

Observing that every term is finite, we deduce

$$-\mathrm{div}\,\mathbf{z}\llcorner(\{u<k\}\backslash A)\le |Du|\llcorner(\{u<k\}\backslash A)+f\chi_{(\{u<k\}\backslash A)}\,.$$

Letting $k$ go to infinity, it yields

$$-\mathrm{div}\,\mathbf{z}\llcorner(\{u<\infty\}\backslash A)\le |Du|\llcorner(\{u<\infty\}\backslash A)+f\chi_{(\{u<\infty\}\backslash A)}\,.$$

We point out that $\{u=+\infty\}\subset S_u$ satisfies $\mathcal{H}^{N-1}(\{u=+\infty\})=0$, and so it is a null set with respect to all the involved measures. Since $\mathrm{cap}_{1,q}(A,\Omega)=0$, and so $\mathcal{H}^{N-1}(A)=0$, a similar consequence is seen for $A$. Thus,

$$-\mathrm{div}\,\mathbf{z}\le |Du|+f\,,$$

and this inequality and (35) leads to the desired equality.

*Step 7:* $(\mathbf{z},Du)=|Du|$ *as measures.* Given $\varphi\in C_0^\infty(\Omega)$, with $\varphi\ge 0$, take $e^{-u_p}\varphi$ as test function in (29) to get

$$2\int_\Omega e^{-u_p}\varphi|\nabla u_p|^p+\int_\Omega fe^{-u_p}\varphi=\int_\Omega e^{-u_p}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,.$$

It is straightforward that then

$$2\int_\Omega \varphi|\nabla\big(e^{-u_p}\big)|^p+\int_\Omega fe^{-u_p}\varphi\le\int_\Omega e^{-u_p}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,,$$

and, by Young's inequality,

$$2\int_\Omega \varphi|\nabla\big(e^{-u_p}\big)|+\int_\Omega fe^{-u_p}\varphi\le\int_\Omega e^{-u_p}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi+2\frac{p-1}{p}\int_\Omega\varphi\,.$$

The lower semicontinuity of the total variation leads to

$$\begin{aligned}2\int_\Omega \varphi\, d|D\big(e^{-u}\big)|+\int_\Omega fe^{-u}\varphi&\le\int_\Omega e^{-u}\mathbf{z}\cdot\nabla\varphi\\&=-\int_\Omega\varphi\, d\big(\mathrm{div}\,(e^{-u}\mathbf{z})\big)=\int_\Omega\varphi e^{-u}d|Du|+\int_\Omega\varphi e^{-u}f-\int_\Omega\varphi\, d(\mathbf{z},D(e^{-u}))\,,\end{aligned}$$

since

$$\mathrm{div}\,(e^{-u}\mathbf{z})=e^{-u}\mathrm{div}\,\mathbf{z}+\big(\mathbf{z}\,,\,D(e^{-u})\big)\qquad\text{in }\mathcal{D}'(\Omega)\,.$$

Simplifying and applying the chain rule we deduce

$$\int_\Omega\varphi\, d|D\big(e^{-u}\big)|\le-\int_\Omega\varphi\, d(\mathbf{z},D(e^{-u}))\le\int_\Omega\varphi\, d|D(e^{-u})|=\int_\Omega\varphi e^{-u}d|Du|\,,$$

from where $-(\mathbf{z},D(e^{-u}))=|D(e^{-u})|$ follows. As a consequence of the definition of the pairing of a vector field and a gradient, it follows that

$$(\mathbf{z},D(1-e^{-u}))=-(\mathbf{z},D(e^{-u}))=|D(1-e^{-u})|\,.$$

Finally, thanks to [21, Proposition 2.2],we are done.

*Step 8:* $L^\infty$*–estimate near the boundary.* Recall that, on account of our hypothesis $dist(\overline{A},\partial\Omega)>0$, we may apply [12, Theorem 4.3] and deduce that each solution $u_p$ is bounded in any closed subset of $\Omega\backslash\overline{A}$. Let $B_R$ denote a ball of radius $R>0$ such that $\overline{B}_R\cap\overline{A}=\emptyset$ and

$|B_R| < 1$. We explicitly point out that $B_R \cap (\mathbb{R}^N \backslash \Omega)$ can be a non null set since we want to prove regularity up to the boundary. In what follows, we will write

$$A_{k,R}^p = \{x \in B_R \cap \Omega : u_p(x) > k\} \quad \text{and} \quad A_{k,R} = \{x \in B_R \cap \Omega : u(x) > k\}\,.$$

Consider $\varphi \in C_0^\infty(B_R)$, with $0 \le \varphi \le 1$, $1 < \alpha < \frac{N}{N-1}$ and $k > 0$. Our aim is to prove

$$\int_{B_R\cap\Omega} |\nabla(e^{\frac{\alpha}{p}G_k(u_p)} - 1)|^p \varphi^p \le C \int_{A_{k,R}^p} |\nabla\varphi|^p (e^{\alpha G_k(u_p)} - 1) + C\|f\|_m |A_{k.R}^p|^{1/m'}\,, \tag{40}$$

where $C$ is a constant which does not depend on $p$.

To this end, take $(e^{(\alpha-1)G_k(u_p)} - e^{-G_k(u_p)})\varphi^p$ as a test function in problem (24), written in terms of $u_p$. Then it yields

$$\begin{aligned}\int_\Omega e^{u_p} |\nabla G_k(u_p)|^p \big[(\alpha-1)e^{(\alpha-1)G_k(u_p)} + e^{-G_k(u_p)}\big]\varphi^p \\ \le p\int_\Omega e^{u_p}|\nabla G_k(u_p)|^{p-1}(e^{(\alpha-1)G_k(u_p)} - e^{-G_k(u_p)})\varphi^{p-1}|\nabla\varphi|^p \\ + \int_\Omega f e^{u_p}(e^{(\alpha-1)G_k(u_p)} - e^{-G_k(u_p)})\varphi^p\,.\end{aligned}$$

Having in mind that we are actually integrating on the set $\{u_p > k\}$, we have $e^{u_p} = e^{k+G_k(u_p)}$. Thus, dividing the last inequality by $e^k$, we obtain

$$\begin{aligned}\int_\Omega |\nabla G_k(u_p)|^p \big[(\alpha-1)e^{\alpha G_k(u_p)} + 1\big]\varphi^p \\ \le p\int_\Omega |\nabla G_k(u_p)|^{p-1}(e^{\alpha G_k(u_p)} - 1)\varphi^{p-1}|\nabla\varphi|^p + \int_\Omega f(e^{\alpha G_k(u_p)} - 1)\varphi^p\,.\end{aligned} \tag{41}$$

We are now analyzing the first term on the right hand side of (41). Observe that Young's inequality implies

$$\begin{aligned}p\int_\Omega |\nabla G_k(u_p)|^{p-1}(e^{\alpha G_k(u_p)} - 1)\varphi^{p-1}|\nabla\varphi|^p \\ \le (p-1)\int_\Omega |\nabla G_k(u_p)|^p(e^{\alpha G_k(u_p)} - 1)\varphi^p + \int_\Omega |\nabla\varphi|^p(e^{\alpha G_k(u_p)} - 1)\,,\end{aligned}$$

so that one term can be absorbed by the left hand side of (41) becoming

$$(\alpha - p)\int_\Omega |\nabla G_k(u_p)|^p(e^{\alpha G_k(u_p)} - 1)\varphi^p \le \int_\Omega |\nabla\varphi|^p(e^{\alpha G_k(u_p)} - 1) + \int_\Omega f(e^{\alpha G_k(u_p)} - 1)\varphi^p\,.$$

Since there is not loss of generality in assuming $1 < p < \frac{\alpha+1}{2}$ and $1 < p < 2$, we may let $\alpha - p > \frac{\alpha-1}{2}$ and $\alpha^p < \alpha^2$. Furthermore, easy manipulations leads to

$$\begin{aligned}\Big(\frac{\alpha-1}{2\alpha^2}\Big)\int_\Omega |\nabla(e^{\frac{\alpha}{p}G_k(u_p)} - 1)|^p\varphi^p \le \Big(\frac{\alpha-p}{2\alpha^p}\Big)\int_\Omega |\nabla(e^{\frac{\alpha}{p}G_k(u_p)} - 1)|^p\varphi^p \\ \le \int_\Omega |\nabla\varphi|^p(e^{\alpha G_k(u_p)} - 1) + \int_\Omega f(e^{\alpha G_k(u_p)} - 1)\varphi^p\,.\end{aligned} \tag{42}$$

We point out that, in every term of (42), we are integrating on the set $A^p_{k,R}$. Applying Lemma 4.2 to the last term of (42), we get

$$\int_\Omega f(e^{\alpha G_k(u_p)}-1)\varphi^p \le 2\int_{A^p_{k,R}} f(e^{\frac{\alpha}{p}G_k(u_p)}-1)^p\varphi^p + 2\int_{A^p_{k,R}} f\varphi^p\,. \tag{43}$$

To estimate the right hand side of (43), we will apply the Hölder and Sobolev inequalities. (We explicitly point out that it follows from $u_p \in W^{1,p}_0(B_R)\cap L^\infty(B_R)$ that $(e^{\frac{\alpha}{p}G_k(u_p)}-1)\varphi$ belongs to $W^{1,p}_0(B_R)\cap L^\infty(B_R)$, even though $\alpha/p>1$.) Performing those manipulations, we obtain

$$\begin{aligned}
\int_{A^p_{k,R}} f(e^{\frac{\alpha}{p}G_k(u_p)}-1)^p\varphi^p &\le \|f\|_m\left[\int_{A^p_{k,R}}(e^{\frac{\alpha}{p}G_k(u_p)}-1)^{p^*}\varphi^{p^*}\right]^{p/p^*}|A^p_{k,R}|^{\frac{1}{m'}-\frac{p}{p^*}}\\
&\le \|f\|_m S_{N,p}\left[\int_{A^p_{k,R}}|\nabla\big((e^{\frac{\alpha}{p}G_k(u_p)}-1)\varphi\big)|^p\right]|B_R|^{\frac{1}{m'}-\frac{p}{p^*}}\\
&\le \|f\|_m S_{N,p}2^{p-1}\left[\int_{A^p_{k,R}}\varphi^p|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|^p+\int_{A^p_{k,R}}(e^{\frac{\alpha}{p}G_k(u_p)}-1)^p|\nabla\varphi|^p\right]|B_R|^{\frac{1}{m'}-\frac{p}{p^*}}\\
&\le 2(S_{N,1}+1)\|f\|_m\left[\int_{A^p_{k,R}}\varphi^p|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|^p+\int_{A^p_{k,R}}(e^{\frac{\alpha}{p}G_k(u_p)}-1)^p|\nabla\varphi|^p\right]|B_R|^{\frac{1}{m'}-\frac{p}{p^*}}\,,
\end{aligned}\tag{44}$$

here we have estimated the constant taking $p$ close enough to 1. Now we set $\delta=\frac12\left(\frac1N-\frac1m\right)>0$ and note that

$$\frac{1}{m'}-\frac{p}{p^*}=\frac{p}{N}-\frac{1}{m}>\delta\,.$$

Recalling that $|B_R|<1$, we deduce that $|B_R|^{\frac{1}{m'}-\frac{p}{p^*}}\le|B_R|^\delta$. Going back to (44), it yields

$$\begin{aligned}
&\int_{A^p_{k,R}} f(e^{\frac{\alpha}{p}G_k(u_p)}-1)^p\varphi^p\\
&\quad\le 2(S_{N,1}+1)\|f\|_m\left[\int_{A^p_{k,R}}\varphi^p|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|^p+\int_{A^p_{k,R}}(e^{\frac{\alpha}{p}G_k(u_p)}-1)^p|\nabla\varphi|^p\right]|B_R|^\delta\\
&\quad\le 2(S_{N,1}+1)\|f\|_m\left[\int_{A^p_{k,R}}\varphi^p|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|^p+\int_{A^p_{k,R}}(e^{\alpha G_k(u_p)}-1)|\nabla\varphi|^p\right]|B_R|^\delta\,,
\end{aligned}$$

this last inequality due to Lemma 4.3. This inequality implies that (42) is transformed in

$$
\begin{aligned}
\Big(\frac{\alpha-1}{2\alpha^2}\Big)\int_{A^p_{k,R}}|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|^p\varphi^p &\le \int_{A^p_{k,R}}|\nabla\varphi|^p(e^{\alpha G_k(u_p)}-1)\\
&+4(S_{N,1}+1)\|f\|_m|B_R|^\delta\left[\int_{A^p_{k,R}}\varphi^p|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|^p+\int_{A^p_{k,R}}(e^{\alpha G_k(u_p)}-1)|\nabla\varphi|^p\right]\\
&+2\int_{A^p_{k,R}}f\varphi^p\,.
\end{aligned}
$$

Now $R>0$ is chosen small enough to have $4\|f\|_m|B_R|^\delta<\frac{\alpha-1}{4\alpha^2}$. Hence we find a constant $C>0$ independent of $p$ satisfying

$$
\frac{\alpha-1}{4\alpha^2}\int_{A^p_{k,R}}|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|^p\varphi^p\le C\int_{A^p_{k,R}}|\nabla\varphi|^p(e^{\alpha G_k(u_p)}-1)+2\int_{A^p_{k,R}}f\varphi^p\,,
$$

To finish the proof of (40) is enough to apply Hölder's inequality.

The next step is to let $p$ go to 1 in (40). Applying Young's inequality, it follows that

$$
\begin{aligned}
\int_{B_R\cap\Omega}|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|\varphi &\le \frac{1}{p}\int_{B_R\cap\Omega}|\nabla(e^{\frac{\alpha}{p}G_k(u_p)}-1)|^p\varphi^p+\frac{p-1}{p}|B_R|^{p/(p-1)}\\
&\le \frac{C}{p}\int_{B_R\cap\Omega}|\nabla\varphi|^p(e^{\alpha G_k(u_p)}-1)+\frac{C}{p}\|f\|_m|A^p_{k.R}|^{1/m'}+\frac{p-1}{p}
\end{aligned}
$$

Thanks to be $\alpha<\frac{N}{N-1}$, we may use that $e^{\alpha G_k(u_p)}-1$ converges to $e^{\alpha G_k(u)}-1$ in $L^1(B_R\cap\Omega)$. This fact and $u_p\to u$ pointwise in $B_R\cap\Omega$ allow us to pass to the limit on the right hand side. On the left hand side, we deduce that $e^{\frac{\alpha}{p}G_k(u_p)}-1$ converges to $e^{\alpha G_k(u)}-1$ in $L^1(B_R\cap\Omega)$ (it is enough to realize that $|e^{\frac{\alpha}{p}G_k(u_p)}-1|\le|e^{\alpha G_k(u_p)}-1|$ and use a variant of the dominated convergence Theorem) and apply the lower semicontinuity of the total variation. Therefore, we conclude that

$$
\int_{B_R\cap\Omega}\varphi\,d|D(e^{\alpha G_k(u)}-1)|\le C\int_{B_R\cap\Omega}|\nabla\varphi|(e^{\alpha G_k(u)}-1)+C\|f\|_m|A_{k.R}|^{1/m'}\,. \tag{45}
$$

To obtain a Caccioppoli type inequality, consider $0<\rho<R$ and a function $\varphi\in C_0^\infty(B_R)$ such that $0\le\varphi\le1$ and $\varphi\equiv1$ in $B_\rho$, the ball concentric with $B_R$ and having radius $\rho$. We may assume that $|\nabla\varphi|\le\frac{2}{R-\rho}$. Then (45) becomes

$$
\int_{B_\rho\cap\Omega}d|D(e^{\alpha G_k(u)}-1)|\le\frac{C}{R-\rho}\int_{B_R\cap\Omega}(e^{\alpha G_k(u)}-1)+C\|f\|_m|A_{k.R}|^{1/m'}\,. \tag{46}
$$

This Caccioppoli inequality will allow us to apply Stampacchia's Theorem. To begin with, consider $B_{(R+\rho)/2}$, the ball concentric with $B_R$ but having radius $\frac{R+\rho}{2}$, and take the function $\eta\in C_0^\infty(B_{(R+\rho)/2})$ satisfying $0\le\eta\le1$, $\eta\equiv1$ in $B_\rho$ and $|\nabla\eta|\le\frac{2}{R-\rho}$. We do not know that $e^{\alpha G_k(u)}-1\in BV(B_R\cap\Omega)$ yet, so that we do not may apply Sobolev's inequality. Instead, we will consider a suitable truncation. Indeed, we have

$$\begin{aligned}\int_{B_\rho\cap\Omega}(e^{\alpha G_k(T_h(u))}-1) &\le \int_{B_{(R+\rho)/2}\cap\Omega}(e^{\alpha G_k(T_h(u))}-1)\eta\\
&\le |A_{k,R}|^{1/N}\left[\int_{B_{(R+\rho)/2}\cap\Omega}(e^{\alpha G_k(T_h(u))}-1)^{\frac{N}{N-1}}\eta^{\frac{N}{N-1}}\right]^{(N-1)/N}\\
&\le |A_{k,R}|^{1/N}S_{N,1}\int_{B_{(R+\rho)/2}\cap\Omega}d\big|D\big((e^{\alpha G_k(T_h(u))}-1)\eta\big)\big|\\
&\le |A_{k,R}|^{1/N}S_{N,1}\int_{B_{(R+\rho)/2}\cap\Omega}d\big|D\big((e^{\alpha G_k(u)}-1)\eta\big)\big|\,,\end{aligned}$$

and the monotone convergence Theorem gives us the desired inequality

$$\int_{B_\rho\cap\Omega}(e^{\alpha G_k(u)}-1)\le |A_{k,R}|^{1/N}S_{N,1}\int_{B_{(R+\rho)/2}\cap\Omega}d\big|D\big((e^{\alpha G_k(u)}-1)\eta\big)\big|\,.$$

Hence, we deduce from inequality (46) that

$$\begin{aligned}&\int_{B_\rho\cap\Omega}(e^{\alpha G_k(u)}-1)\\
&\le |A_{k,R}|^{1/N}S_{N,1}\left[\int_{B_{(R+\rho)/2}\cap\Omega}\eta\, d\big|D(e^{\alpha G_k(u)}-1)\big|+\int_{B_{(R+\rho)/2}\cap\Omega}(e^{\alpha G_k(u)}-1)|\nabla\eta|\right]\\
&\le |A_{k,R}|^{1/N}S_{N,1}\left[\int_{B_{(R+\rho)/2}\cap\Omega}d\big|D(e^{\alpha G_k(u)}-1)\big|+\frac{2}{R-\rho}\int_{B_{(R+\rho)/2}\cap\Omega}(e^{\alpha G_k(u)}-1)\right]\\
&\le \frac{C}{R-\rho}|A_{k,R}|^{1/N}\int_{B_R\cap\Omega}(e^{\alpha G_k(u)}-1)+C|A_{k,R}|^{\frac1N+\frac{1}{m'}}\,.\end{aligned}\tag{47}$$

To apply Stampacchia's procedure take $0<h<k$ and observe that the following facts hold.

$$\int_{B_R\cap\Omega}(e^{\alpha G_k(u)}-1)\le\int_{B_R\cap\Omega}(e^{\alpha G_h(u)}-1)$$

$$|A_{k,R}|\le\frac{1}{k-h}\int_{B_R\cap\Omega}G_h(u)\le\frac{1}{\alpha(k-h)}\int_{B_R\cap\Omega}(e^{\alpha G_h(u)}-1)$$

Therefore, inequality (47) yields

$$\begin{aligned}&\int_{B_\rho\cap\Omega}(e^{\alpha G_k(u)}-1)\\
&\le\frac{C}{\alpha^{1/N}(k-h)^{1/N}(R-\rho)}\left[\int_{B_R\cap\Omega}(e^{\alpha G_h(u)}-1)\right]^{1+\frac1N}\\
&\quad+\frac{C}{\alpha^{(1/N)+(1/m')}(k-h)^{(1/N)+(1/m')}}\left[\int_{B_R\cap\Omega}(e^{\alpha G_h(u)}-1)\right]^{\frac{1}{m'}+\frac1N}\,.\end{aligned}$$

We point out that $k-h$, $R-\rho$ and $\int_{B_R\cap\Omega}(e^{\alpha G_h(u)}-1)$ can be taken as small as we want in Stampacchia's procedure. Thus, we may unify all the exponents obtaining

$$\int_{B_\rho\cap\Omega}(e^{\alpha G_k(u)}-1)\le \frac{C}{(k-h)^{1+(1/N)}(R-\rho)}\left[\int_{B_R\cap\Omega}(e^{\alpha G_h(u)}-1)\right]^{\frac{1}{m'}+\frac{1}{N}}\,.$$

Applying Stampacchia's Theorem (see [25, Lemma 5.1], and observe that the last exponent is larger than 1) to

$$\varphi(h,R)=\int_{B_R\cap\Omega}(e^{\alpha G_h(u)}-1)\,,$$

then we get $k_0$ such that $\varphi(k,R)=0$ for all $k>k_0$, so that $u\in L^\infty(B_R\cap\Omega)$.

Therefore, we have seen that $u\in L^\infty(B_R\cap\Omega)$ for every ball satisfying $\overline{B}_R\cap\overline{A}=\emptyset$ and $|B_R|<1$. The last step uses a compactness argument to conclude that $u$ is bounded in a strip around $\partial\Omega$.

A further remark is in order: we have deduced that the trace $u\big|_{\partial\Omega}\in L^\infty(\partial\Omega)$, so that all integrals on $\partial\Omega$ that occur in the next Step are well–defined.

*Step 9: Boundary condition.* As a consequence of Step 8, we know that there exists a strip around the boundary $\partial\Omega$ where $u$ is bounded. Let $\phi\in C(\partial\Omega)$ be a nonnegative function. Then there exists a nonnegative $\varphi\in C^\infty(\overline{\Omega})$ such that $\varphi\big|_{\partial\Omega}=\phi$ and that vanishes outside that strip. For instance, we may consider $\varphi=\varphi_1\varphi_2$ where $\varphi_1$ is the solution to the Dirichlet problem for Laplace equation with datum $\phi$, and $\varphi_2$ vanishes outside that strip and satisfies $\varphi_2\big|_{\partial\Omega}\equiv 1$. In other words, we search a smooth function satisfying $\varphi\big|_{\partial\Omega}=\phi$ and such that $u$ is bounded in $\operatorname{supp}\varphi$.

Fix $\lambda,k>0$ and take $(e^{\lambda T_k(u_p)}-1)\varphi$ as a test function in problem (24), written in terms of $u_p$. Then

$$\lambda\int_\Omega e^{u_p}e^{\lambda T_k(u_p)}\varphi|\nabla T_k(u_p)|^p \\ =-\int_\Omega e^{u_p}(e^{\lambda T_k(u_p)}-1)|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi+\int_\Omega fe^{u_p}(e^{\lambda T_k(u_p)}-1)\varphi\,.$$

Now, Young's inequality implies

$$\lambda\int_\Omega e^{(\lambda+1)T_k(u_p)}\varphi|\nabla T_k(u_p)| \\ \le\frac{\lambda}{p}\int_\Omega e^{u_p}e^{\lambda T_k(u_p)}\varphi|\nabla T_k(u_p)|^p+\frac{\lambda(p-1)}{p}\int_\Omega e^{(\lambda+1)T_k(u_p)}\varphi \\ =-\frac{1}{p}\int_\Omega e^{u_p}(e^{\lambda T_k(u_p)}-1)|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi+\frac{1}{p}\int_\Omega fe^{u_p}(e^{\lambda T_k(u_p)}-1)\varphi \\ +\frac{\lambda(p-1)}{p}\int_\Omega e^{(\lambda+1)T_k(u_p)}\varphi\,.$$

Formally adding a null term, we obtain

$$\frac{\lambda}{\lambda+1}\int_\Omega \varphi|\nabla\big(e^{(\lambda+1)T_k(u_p)}-1\big)|+\frac{\lambda}{\lambda+1}\int_{\partial\Omega}\varphi\big(e^{(\lambda+1)T_k(u_p)}-1\big)\,d\mathcal{H}^{N-1}$$
$$\le -\frac{1}{p}\int_\Omega e^{u_p}(e^{\lambda T_k(u_p)}-1)|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi+\frac{1}{p}\int_\Omega f e^{u_p}(e^{\lambda T_k(u_p)}-1)\varphi$$
$$+\frac{\lambda(p-1)}{p}\int_\Omega e^{(\lambda+1)T_k(u_p)}\varphi\,.$$

Recall that, by (36), we have $e^{u_p}\to e^u$ in $L^1(\Omega)$ and, by (32) and Lebesgue's Theorem, we also have $e^{(\lambda+1)T_k(u_p)}-1$ converges to $e^{(\lambda+1)T_k(u)}-1$ in $L^1(\Omega)$ as $p$ goes to 1. By the lower semicontinuity of the functional we may let $p$ go to 1 as well as remove the truncation, since $u$ is bounded in $\operatorname{supp}\varphi$. Therefore,

$$\frac{\lambda}{\lambda+1}\int_\Omega\varphi\,d\big|D\big(e^{(\lambda+1)u}-1\big)\big|+\frac{\lambda}{\lambda+1}\int_{\partial\Omega}\varphi\big(e^{(\lambda+1)u}-1\big)\,d\mathcal{H}^{N-1}$$
$$\le-\int_\Omega e^u(e^{\lambda u}-1)\mathbf{z}\cdot\nabla\varphi+\int_\Omega f e^u(e^{\lambda u}-1)\varphi\,. \tag{48}$$

To perform some manipulations on the right hand side, we use (38) to deduce

$$-\varphi\operatorname{div}(e^u\mathbf{z})=\varphi e^u f\,.$$

Having in mind that $e^u\mathbf{z}$ is bounded in $\operatorname{supp}\varphi$, by Green's formula, Step 7 and the chain rule, we can write

$$\int_\Omega f e^u(e^{\lambda u}-1)\varphi=\int_\Omega d\big(e^u\mathbf{z},D((e^{\lambda u}-1)\varphi)\big)-\int_{\partial\Omega}(e^{\lambda u}-1)\varphi[e^u\mathbf{z},\nu]\,d\mathcal{H}^{N-1}$$
$$=\int_\Omega e^u d\big(\mathbf{z},D((e^{\lambda u}-1)\varphi)\big)-\int_{\partial\Omega}(e^{\lambda u}-1)e^u\varphi[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}$$
$$=\int_\Omega e^u\varphi\,d|D(e^{\lambda u}-1)|+\int_\Omega e^u(e^{\lambda u}-1)\mathbf{z}\cdot\nabla\varphi-\int_{\partial\Omega}(e^{\lambda u}-1)e^u\varphi[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}$$
$$=\frac{\lambda}{\lambda+1}\int_\Omega\varphi\,d\big|D\big(e^{(\lambda+1)u}-1\big)\big|+\int_\Omega e^u(e^{\lambda u}-1)\mathbf{z}\cdot\nabla\varphi-\int_{\partial\Omega}(e^{\lambda u}-1)e^u\varphi[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\,.$$

Inserting this equality in (48) and simplifying, it yields

$$\frac{\lambda}{\lambda+1}\int_{\partial\Omega}\phi\big(e^{(\lambda+1)u}-1\big)\,d\mathcal{H}^{N-1}\le-\int_{\partial\Omega}(e^{\lambda u}-1)e^u\phi[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}\,.$$

Since $\phi$ is an arbitrary nonnegative continuous function, it follows that

$$\frac{\lambda}{\lambda+1}\big(e^{(\lambda+1)u}-1\big)\le-(e^{\lambda u}-1)e^u[\mathbf{z},\nu]\,,\quad \mathcal{H}^{N-1}\text{–a.e. on }\partial\Omega\,.$$

Obviously, this inequality holds on the set where $\{u=0\}\cap\partial\Omega$. On the contrary, in the set $\{u>0\}\cap\partial\Omega$ this inequality implies

$$[\mathbf{z},\nu]\le-\frac{\lambda}{\lambda+1}\frac{e^{(\lambda+1)u}-1}{(e^{\lambda u}-1)e^u}\le-\frac{\lambda}{\lambda+1}\,.$$

Hence, the arbitrariness of $\lambda > 0$ and $|[\mathbf{z}, \nu]| \le 1$ lead to $[\mathbf{z}, \nu] = -1$, so that

$$[\mathbf{z}, \nu] = \text{ sign } (-u) , \qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega .$$

We have proved that $u$ is actually a solution to problem (5), so that the proof of Theorem 1.1 is completely finished. ■

**Remark 4.4.** We explicitly point out that $e^u \notin BV(\Omega)$ for every nontrivial measure $\mu$. Indeed, if $e^u \in BV(\Omega)$, then the following manipulations would hold:

$$\begin{aligned} fe^u + \mu = -\text{div}\left(e^u\mathbf{z}\right) &= -e^u\text{div}\,(\mathbf{z}) - (\mathbf{z}, D(e^u)) = -e^u\text{div}\,(\mathbf{z}) - |D(e^u))| \\ &= -e^u\text{div}\,(\mathbf{z}) - (e^u)|Du| = e^u\big[-\text{div}\,(\mathbf{z}) - |Du|\big] = e^u f , \end{aligned}$$

wherewith $\mu = 0$. Compare this argument with [2, Remark 2.11].

**Remark 4.5.** Assume for a moment that the function $e^{-u}$ is $\mu$–measurable. Having in mind Step 7 and [21, Proposition 2.2], it yields $(\mathbf{z}, De^{T_k(u)}) = |De^{T_k(u)}|$ for any $k > 0$. We are able to see, redoing the same calculations, that (39) becomes an equality. This fact and $-\text{div}\,\mathbf{z} = |Du| + f$ imply that the measure $e^{-u}\mu$ vanishes. Even thought this argument does not work, it suggest that $\mu$ is concentrated on the set $\{u = +\infty\}$. In other words: $A \subset \{u = +\infty\}$. Compare this note with [2, Remark 2.16].

**Remark 4.6.** It is worth noting that we can recover the singular measure from the solution to (5) we have found. The argument is very similar to that of [2] for $p = 2$. To check it, take $e^{\frac{u}{1+\epsilon u}} - 1$, with $\epsilon > 0$, as test function in (5). Then, since $u$ is a solution to problem (5), it follows that

$$|Du|(\Omega) + \int_{\partial\Omega} \big|e^{\frac{u}{1+\epsilon u}} - 1\big|\, d\mathcal{H}^{N-1} = \int_\Omega e^{\frac{u}{1+\epsilon u}}\Big(1 - \frac{1}{(1+\epsilon u)^2}\Big)\, d|Du| + \int_\Omega f(x)\big(e^{\frac{u}{1+\epsilon u}} - 1\big) .$$

Taking into account that $u \in BV(\Omega)$ and $u \in L^\infty(\partial\Omega)$, we deduce that the left hand side is bounded. Thus,

$$\int_\Omega e^{\frac{u}{1+\epsilon u}}\Big(1 - \frac{1}{(1+\epsilon u)^2}\Big)\, d|Du| \le M , \qquad \text{for all } \epsilon > 0 ,$$

and so there exists a measure $\mu$ such that, up to subsequences,

$$|Du|e^{u/(1+\epsilon u)}\Big(1 - \frac{1}{(1+\epsilon u)^2}\Big) \rightharpoonup \mu , \qquad \text{weakly in the sense of measures.}$$

Now, it is easy to check that this measure satisfies equation (7). In fact, $e^{\frac{u}{1+\varepsilon u}} \in BV(\Omega)$ and

$$\begin{aligned} &-\text{div}\left(e^{u/(1+\epsilon u)}\mathbf{z}\right) = -e^{u/(1+\epsilon u)}\text{div}\,\mathbf{z} - (\mathbf{z}, De^{u/(1+\epsilon u)}) \\ &= |Du|e^{u/(1+\epsilon u)} + fe^{u/(1+\epsilon u)} - \frac{1}{(1+\epsilon u)^2}e^{u/(1+\epsilon u)}|Du| = |Du|e^{u/(1+\epsilon u)}\Big(1 - \frac{1}{(1+\epsilon u)^2}\Big) + fe^{u/(1+\epsilon u)} . \end{aligned}$$

It follows from the estimate $e^{u/(1+\epsilon u)} \le e^u \in L^1(\Omega)$ and the monotone convergence Theorem that $e^{u/(1+\epsilon u)} \to e^u$ in $L^1(\Omega)$, wherewith we may let $\epsilon \to 0$ proving that

$$-\text{div}\left(e^u\mathbf{z}\right) = fe^u + \mu , \qquad \text{in } \mathcal{D}'(\Omega).$$

**Remark 4.7.** Roughly speaking, Theorem 1.1 states that for each measure $\mu$ concentrated in a set $\mathcal{H}^{N-1}$–null, we find a solution to problem (5). A few words on the map

$$\mu \text{ singular measure } \mapsto u \text{ solution to (5)} \tag{49}$$

is in order. Although we cannot assert that this is a one–to–one map, we can state that not every measure $\mu$ leads to the same unbounded solution $u$. It is enough to choose two singular measures $\mu_1$ and $\mu_2$ satisfying $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \emptyset$. Applying our Theorem to each $\mu_i$, we obtain an unbounded solution $u_i$ to problem (5), $i = 1, 2$. However, we know that $u_1$ is bounded in $\operatorname{supp}\mu_2$ and $u_2$ is bounded in $\operatorname{supp}\mu_1$, so that these solutions are different.

## 5. Multiplicity of radial solutions

In this section we deal with the case of radial solutions in a ball, so that in the following examples we always assume $\Omega = B_R(0)$ (i.e., the ball centered at the origin having radius $R > 0$) and we search solutions depending on $|x|$. In what follows, we denote $\omega_N = |B_1(0)|$, and $\delta_0$ the Dirac measure concentrated in the origin.

Let us begin by dealing with the homogeneous case.

**Example 5.1.** Assume that $f \equiv 0$, then problem (5) has a trivial solution, given by

$$u(x) \equiv 0\,, \qquad \text{with } \mathbf{z}(x) \equiv 0\,.$$

In the paper [13] it is shown that $u(x) \equiv 0$ is the only bounded solution of (5). On the other hand, we will now show that (5) has infinitely many **unbounded** radial solutions.

A first kind of solution is the following:

$$u(x) = -(N-1)\log\left(\frac{|x|}{\alpha R}\right)\,, \tag{50}$$

for any choice of $\alpha \geq 1$. The corresponding vector field $\mathbf{z}$ is given by

$$\mathbf{z}(x) = -\frac{x}{|x|}\,.$$

Note that this solution is zero on $\partial B_R$ only when $\alpha = 1$.

Then another kind of solution is given by

$$u(x) = \begin{cases} -(N-1)\log\left(\dfrac{|x|}{\rho}\right) & \text{if } 0 < |x| < \rho \\ 0 & \text{if } \rho \leq |x| < R\,, \end{cases} \tag{51}$$

for every $\rho$ such that $0 < \rho < R$. In this case the vector field $\mathbf{z}$ is given by

$$\mathbf{z}(x) = \begin{cases} -\dfrac{x}{|x|} & \text{if } 0 < |x| < \rho \\ -\dfrac{\rho^{N-1}x}{|x|^N} & \text{if } \rho \leq |x| < R\,. \end{cases}$$

It is easy to check that both (50) and (51) are solutions according to Definition 4.1.

All these solutions can be achieved using the procedure of Theorem 1.1. Indeed, consider the singular measure $\mu = C\delta_0$, with $C > 0$, and the approximating problems

$$\begin{cases} -\Delta_p v_p = C\delta_0 & \text{in } B_R(0) \\ v_p = 0 & \text{on } \partial B_R(0)\,. \end{cases}$$

It is easy to check that

$$v_p(x) = \left(\frac{C}{N\omega_N}\right)^{1/(p-1)} \frac{p-1}{N-p}\left(\frac{1}{|x|^{\frac{N-p}{p-1}}} - \frac{1}{R^{\frac{N-p}{p-1}}}\right).$$

Now set

$$\begin{aligned} u_p(x) &= (p-1)\log(1+\frac{v_p}{p-1}) \\ &= (p-1)\log\left[1+\left(\frac{C}{N\omega_N}\right)^{1/(p-1)}\frac{1}{N-p}\left(\frac{1}{|x|^{\frac{N-p}{p-1}}}-\frac{1}{R^{\frac{N-p}{p-1}}}\right)\right]. \end{aligned} \tag{52}$$

We may distinguish two cases according to the size of the constant $C$.

*First case:* $C \geq N\omega_N R^{N-1}$. In this case, it is straightforward that the limit, for $p \to 1$, is

$$u(x) = -(N-1)\log(|x|) + \log\left(\frac{C}{N\omega_N}\right),$$

which can be written as (50) for $\alpha = \frac{1}{R}\left(\frac{C}{N\omega_N}\right)^{\frac{1}{N-1}} \geq 1$.

*Second case:* $0 < C < N\omega_N R^{N-1}$. Here the limit of $u_p(x)$ for $p \to 1$ is given by (51) with $\rho = \left(\frac{C}{N\omega_N}\right)^{\frac{1}{N-1}}$.

Therefore, both types of solutions (50) and (51) correspond to different multiples of the Dirac delta centered at the origin.

Let us also note that the solutions $u_p(x)$ correspond to the unbounded solutions to problem

$$-\Delta_p u_p = |\nabla u_p|^p\,, \qquad \text{in } \Omega = \{x \in \mathbb{R}^N \,:\, |x| < R\}\,,$$

exhibited by Ferone and Murat for $p > 1$ in [16, Remark 2.11], i.e.,

$$u_p(x) = (p-1)\log\left(\frac{|x|^{-(N-p)/(p-1)} - m}{R^{-(N-p)/(p-1)} - m}\right)$$

where $m$ is any constant satisfying $mR^{(N-p)/(p-1)} < 1$. These solutions are the same as (52), where $C$ and $m$ are related by

$$C = \frac{N\omega_N R^{N-p}(N-p)^{p-1}}{\left(1 - mR^{\frac{N-p}{p-1}}\right)^{p-1}}.$$

The previous example can be adapted in order to obtain a multiplicity result in a general open set.

**Example 5.2.** Assume that $\Omega$ is a bounded domain with Lipschitz boundary and $f \equiv 0$, then problem (5) has infinitely many nonnegative unbounded solutions. More precisely, for any $x_0 \in \Omega$, we can find a solution $u(x)$ which is unbounded near $x_0$.

Fix $x_0 \in \Omega$ and choose $\rho > 0$ such that $B_\rho(x_0) \subset\subset \Omega$. We set

$$u(x) = \begin{cases} -(N-1)\,\log\left(\dfrac{|x-x_0|}{\rho}\right) & \text{if } 0 < |x-x_0| < \rho \\ 0 & \text{if } |x-x_0| > \rho\,, \end{cases} \tag{53}$$

with the associated vector field

$$\mathbf{z}(x) = \begin{cases} -\dfrac{x-x_0}{|x-x_0|} & \text{if } 0 < |x-x_0| < \rho \\ -\dfrac{\rho^{N-1}(x-x_0)}{|x-x_0|^N} & \text{if } |x-x_0| > \rho\,. \end{cases}$$

**Example 5.3.** In this example, we will show unbounded radial solutions of problem (5) with constant datum $f \equiv \lambda \in \left]0, \dfrac{N-1}{R}\right]$ in $\Omega = B_R(0)$.

In this case, a solution is given by

$$u(x) = -(N-1)\,\log\left(\frac{|x|}{\alpha R}\right) + \lambda\big(|x| - R\big)\,, \tag{54}$$

for any choice of $\alpha \geq 1$. The corresponding vector field $\mathbf{z}$ is given by

$$\mathbf{z}(x) = -\frac{x}{|x|}\,.$$

Another type of solution is given, for any choice of $\rho \in ]0, R[$, by

$$u(x) = \begin{cases} -(N-1)\,\log\left(\dfrac{|x|}{\rho}\right) + \lambda\big(|x| - \rho\big) & \text{if } 0 < |x| < \rho \\ 0 & \text{if } \rho < |x| < R\,, \end{cases} \tag{55}$$

with the associated vector field

$$\mathbf{z}(x) = \begin{cases} -\dfrac{x}{|x|} & \text{if } 0 < |x| < \rho \\ -\dfrac{\lambda}{N}x - \left(1 - \dfrac{\lambda\rho}{N}\right)\dfrac{\rho^{N-1}x}{|x|^N} & \text{if } \rho < |x| < R\,. \end{cases}$$

The details are left to the reader. Note that $|\mathbf{z}(x)| \leq 1$ due to the inequality $\lambda \leq \frac{N-1}{R}$ As in the first examples, these solutions are related to a singular measure of the form $C\,\delta_0$. In particular, solution (54) is obtained for large values of $C$, while solution (55) corresponds to small values of $C$.

**Example 5.4.** In this final example, we will exhibit an unbounded solution to problem (1) with $f \equiv 0$, that is:

$$\begin{cases} u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = |Du| & \text{in } B_R(0)\,; \\ u = 0 & \text{on } \partial B_R(0)\,; \end{cases}$$

A solution of this problem is defined by

$$u(x) = g\big(|x|\big)\,, \text{ where } g(r) = (N-1)\, e^{-r} \int_r^{\alpha R} \frac{e^s}{s}\, ds\,,$$

for every choice of $\alpha \geq 1$, with $\mathbf{z}(x) = -\dfrac{x}{|x|}$. Another possibility is given by

$$u(x) = \begin{cases} g\big(|x|\big) & \text{if } 0 < |x| < \rho \\ 0 & \text{if } \rho < |x| < R\,, \end{cases}$$

where

$$g(r) = (N-1)\, e^{-r} \int_r^{\rho} \frac{e^s}{s}\, ds$$

with the associated vector field

$$\mathbf{z}(x) = \begin{cases} -\dfrac{x}{|x|} & \text{if } 0 < |x| < \rho \\ -\dfrac{\rho^{N-1} x}{|x|^N} & \text{if } \rho < |x| < R\,. \end{cases}$$

It is important to observe that all unbounded solutions exhibited in this Section satisfy

$$e^u \notin BV(B_R(0))\,, \qquad e^{\delta u} \in W^{1,1}(B_R(0)) \text{ for every } \delta < 1\,.$$

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