UNIQUENESS OF SOLUTIONS FOR SOME ELLIPTIC EQUATIONS WITH A QUADRATIC GRADIENT TERM

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Abstract.

We study a comparison principle and uniqueness of positive solutions for the homogeneous Dirichlet boundary value problem associated to quasi–linear elliptic equations with lower order terms. A model example is given by

$$-\Delta u + \lambda \frac{|\nabla u|^2}{u^r} = f(x)\,, \qquad \lambda, r > 0\,.$$

The main feature of these equations consists in having a quadratic gradient term in which singularities are allowed. The arguments employed here also work to deal with equations having lack of ellipticity or some dependence on u in the right hand side. Furthermore, they could be applied to obtain uniqueness results for nonlinear equations having the p–Laplacian operator as the principal part.

Our results improve those already known, even if the gradient term is not singular.

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1. Introduction

In this paper, we consider the following Dirichlet problem

$$\text{(1)} \qquad \begin{cases} -\mathrm{div}(\alpha(u)\nabla u) + \beta(u)|\nabla u|^2 = f(x,u) & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Here the real continuous functions α and β are defined in an open interval $I =]0, b[$ (the value $b = +\infty$ is not excluded), while the function $f : \Omega \times I \to \mathbb{R}$ satisfies the Carathéodory condition. We also assume that $\beta(s) \geq 0$ and $\alpha(s) > 0$ for every $s \in I$. We point out that α and β may be singular in the extremes of the interval.

Our aim is to prove a comparison principle for solutions of this quasilinear problem and, as a consequence, uniqueness of solutions such that $u(x) \in I$ for almost every $x \in \Omega$. (For the precise meaning of our concept of solution, see Definition 2.1 below).

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As a model problem with singular terms we consider for $\lambda, r > 0$ the boundary value problem

$$\left\{\begin{array}{rcll} -\Delta u + \lambda \dfrac{|\nabla u|^2}{u^r} & = & f(x)\,, & \text{in } x \in \Omega\,, \\ u & = & 0\,, & \text{on } x \in \partial\Omega\,. \end{array}\right. \tag{2}$$

The study of this kind of singular problems is very recent. Indeed, the existence of solution is proved in [3, 5, 6] for the case that $r \le 1$ and $f(x) \in L^q(\Omega)$, with $q > N/2$, and

$$\inf_{\omega} f > 0, \quad \forall \omega \subset\subset \Omega. \tag{3}$$

Recently, in [15] this result is improved by replacing condition (3) by the weaker condition that $0 \le f \in L^{(2^*/r)'}(\Omega)$ with $f \not\equiv 0$ and $\lambda < 1/2$ in the case $r = 1$. On the other hand, a similar equation is studied in [24]. The case $r > 1$ is studied by the first time in [4] where, among other more general results, existence of solutions is proved for every $f \in L^{(2^*)'}(\Omega)$ satisfying (3) provided that $r < 2$, while for $r \ge 2$, it is shown that in general there is nonexistence of solutions.

However, at least for our knowledge, the study of the uniqueness of solutions for (2) is completely new. Even in the case of nonsingular terms, i.e. if α and β are continuous in $[0, +\infty[$, there are only few results concerning the uniqueness. Indeed, in [8] (see also [7] for some extended results allowing L^N–dependence, instead of L^∞–one, in x), it is proved a comparison principle for solutions of (1) whose simplest version requires $\alpha = 1$ and β a C^1–function satisfying

$$\beta(0) = 0, \quad \beta'(s) > 0, \quad \forall s > 0. \tag{4}$$

A simple comparison principle can also be found in the proof of Theorem 3.1 of [28] (see also [27]) for the case $\alpha \equiv 1$ and β nondecreasing.

Other uniqueness results for equations with a gradient term can be found in [31], where the assumption $\beta(s) \le 0$ is considered, [26] for uniqueness of the zero solution for a sign-changing nonsingular function β and $f(x, s)s \le 0$, [9] for the case of subquadratic terms in ∇u, [11, 12, 13] for equations with different dependence on ∇u and [14, 22, 29] for quasilinear equations with no quadratic term in ∇u.

Here we will proved uniqueness of solutions $u \in H_0^1(\Omega)$ for (2) provided either that $r < 1$ or that $r = 1$ and $\lambda < 1$. We prove this result (see Corollary 2.12) as a consequence of a general uniqueness theorem, Theorem 2.9, which is deduced from a comparison principle (see Theorem 2.7) for the general problem (1). We point out that Theorem 2.9 handles also the case of nonsingular terms and, in this case, we improve the result of [8] since we do not require assumption (4). See Corollary 2.10 and Remark 2.8 for more details.

Furthermore, adapting ideas of [21], we study the uniqueness of solutions in the class of bounded solutions, i.e. in $H_0^1(\Omega) \cap L^\infty(\Omega)$. In this case, we improve the results of [7, 8] (see Theorem 3.1 and Remark 3.2-2).

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2. Comparison principle and uniqueness result

We begin by introducing some notation which will be used throughout this paper. For each $k > 0$, we define the truncation at levels $\pm k$ by

$$T_k(s) = (\operatorname{sign} s)(k \wedge |s|), \qquad s \in \mathbb{R}.$$

To prove our results, we fix a point $a \in I$, and define an auxiliary function $\gamma : I \longrightarrow \mathbb{R}$ as

$$\gamma(s) = \int_a^s \frac{\beta(t)}{\alpha(t)}\, dt\,, \qquad s \in I\,. \tag{5}$$

We state what we understand by sub- and supersolution.

Definition 2.1. *By a subsolution (respectively, supersolution) of problem* (1) *we mean a measurable function* $u : \Omega \to \mathbb{R}$ *such that* $u(x) \in I$ *for almost every* $x \in \Omega$, $T_k(u) \in H_0^1(\Omega)$ *for* $0 < k < b$ *and the following conditions hold.*

$$\alpha(u)|\nabla u| \in L^2(\Omega), \quad \beta(u)|\nabla u|^2 \in L^1(\Omega), \quad f(\cdot, u) \in L^1(\Omega), \tag{6}$$

$$\int_\Omega \alpha(u)\nabla u \cdot \nabla v + \int_\Omega \beta(u)|\nabla u|^2 v \overset{(\geq)}{\leq} \int_\Omega f(x,u)v, \tag{7}$$

for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *with* $v \geq 0$. *A solution is a function which is both a subsolution and a supersolution for* (1).

Remarks 2.2.
(1) We point out that the condition $T_k(u) \in H_0^1(\Omega)$ for every $k \in I$ implies that a gradient ∇u can be defined (see [10]): It is the only measurable function $v : \Omega \to \mathbb{R}^N$ satisfying

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}\,, \qquad \forall k \in I$$

This hypothesis, that the truncations of the solution are in the "energy space" $H_0^1(\Omega)$, is quite natural when dealing with elliptic problems having a non regular datum (see [10] and [23]) or either a noncoercive principal term (see [16], [2], [20] and [30]). In our setting truncations are also used in a essential way in order to prove our main tool (see Proposition 2.3 below).

(2) Observe that every term in (7) is well–defined because of condition (6). It is worthwhile to point out that it is natural to require some additional integrability to f in order to assure the existence of solution u such that $\alpha(u)|\nabla u| \in L^2(\Omega)$. For instance, , in the case that $\alpha \equiv 1$, $\beta \equiv 0$ and $f(x,u) = f(x)$, we have to impose that $f \in L^m(\Omega)$ with $m \geq 2N/(N+2)$ to guarantee the existence of solutions u belonging to the space $H_0^1(\Omega)$ (i.e., such that $\alpha(u)|\nabla u| \in L^2(\Omega)$) of the homogeneous Dirichlet boundary value problem associated to the equation $-\Delta u = f$. Remind that, in general this is not so for the case $f \in L^m(\Omega)$ with $1 < m < 2N/(N+2)$.

The following result will be used several times in the sequel. It states an inequality where the quadratic term on the gradient is cancelled. (Similar cancellation results can be found in [20] and [31].) Formally, the idea of its proof is to take, for $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $v = e^{-\gamma(u)}w$ as test function in (7). However, observe that, since the function $t \mapsto e^{-\gamma(t)}$ may be unbounded, this choice of test function could be impossible. Thus, we are forced to take a suitable truncature of it.

Proposition 2.3. *Let u be a subsolution (respectively a supersolution) of (1) for which there exists $\delta \in]0,b[$ such that*

$$\begin{gathered}\alpha(T_\delta(u))e^{-\gamma(T_\delta(u))}|\nabla T_\delta(u)| \in L^2(\Omega)\\ \beta(T_\delta(u))|\nabla T_\delta(u)|^2 e^{-\gamma(T_\delta(u))},\ f(x,T_\delta(u))e^{-\gamma(T_\delta(u))} \in L^1(\Omega)\,.\end{gathered} \tag{8}$$

Then the inequality

$$\int_\Omega \alpha(u)e^{-\gamma(u)}\nabla u\cdot\nabla w \overset{(\geq)}{\leq} \int_\Omega f(x,u)e^{-\gamma(u)}w \tag{9}$$

holds for every $w \in H_0^1(\Omega)\cap L^\infty(\Omega)$, with $w \geq 0$.

Remarks 2.4.
(1) We point out that assumption (8) in the above proposition is straightforwardly satisfied if the function β/α is integrable in a right neighborhood of 0. Indeed, under this assumption (8), the function γ is bounded in $]0,a[$. Consequently, taking into account that $\gamma \geq 0$ in $[a,b[$, we deduce that $e^{-\gamma(u)}$ is bounded and by using the integrability of $\alpha(u)^2|\nabla u|^2$, $\beta(u)|\nabla u|$ and $f(x,u)$ (remind that u is a subsolution of (1)), we see that (8) is trivially satisfied.
(2) Even more, if $\beta/\alpha \in L^1(I)\cap L^\infty(I)$, then $e^{-\gamma(u)} \in H^1(\Omega)\cap L^\infty(\Omega)$, and so the proof below can be simplified by taking directly $v = e^{-\gamma(u)}w$ as test function in (7).

Proof. We prove only the assertion of the theorem for a subsolution u and leave the corresponding one for a supersolution to the reader. Let $w \in H_0^1(\Omega)\cap L^\infty(\Omega)$ be fixed. For every $\epsilon \in]0,\min\{b-a,\delta\}[$, we denote

$$b_\epsilon = \begin{cases} b-\epsilon, & \text{if } b<+\infty,\\ a+\frac{1}{\epsilon}, & \text{if } b=+\infty,\end{cases}$$

and

$$\tau_\epsilon(s) = (s\vee\epsilon)\wedge b_\epsilon\,, \qquad \forall s>0\,. \tag{10}$$

Note that $a < b_\epsilon$ and thus $\gamma(b_\epsilon) \geq 0$. Since $\tau_\epsilon(u) \geq \epsilon$ and γ is increasing, we have $e^{-\gamma(\tau_\epsilon(u))} \leq e^{-\gamma(\epsilon)}$. On the other hand, by the positivity of α, we also have $\dfrac{\beta(\tau_\epsilon(u))}{\alpha(\tau_\epsilon(u))} \leq \dfrac{\max_{t\in[\epsilon,b_\epsilon]}\beta(t)}{\min_{t\in[\epsilon,b_\epsilon]}\alpha(t)}$, and then $\dfrac{\beta(\tau_\epsilon(u))}{\alpha(\tau_\epsilon(u))}e^{-\gamma(\tau_\epsilon(u))}|\nabla\tau_\epsilon(u)|$ belongs to $L^2(\Omega)$. It follows that

$$e^{-\gamma(\tau_\epsilon(u))}w \in H_0^1(\Omega)\cap L^\infty(\Omega)\,.$$

Thus, we may choose $v = e^{-\gamma(\tau_\epsilon(u))}w$ as test function in (7) to get

$$\begin{aligned}&\int_\Omega \alpha(u)e^{-\gamma(\tau_\epsilon(u))}\nabla u\cdot\nabla w\\ &\quad-\int_\Omega \alpha(u)\frac{\beta(\tau_\epsilon(u))}{\alpha(\tau_\epsilon(u))}\left[e^{-\gamma(\tau_\epsilon(u))}w\right]|\nabla\tau_\epsilon(u)|^2 + \int_\Omega \beta(u)|\nabla u|^2 e^{-\gamma(\tau_\epsilon(u))}w\\ &\qquad\qquad\leq \int_\Omega f(x,u)e^{-\gamma(\tau_\epsilon(u))}w,\end{aligned} \tag{11}$$

for every $\epsilon \in]0, \min\{b-a,\delta\}[$. In order to take the limit as ϵ goes to 0, we point out that $e^{-\gamma(b_\epsilon)} \le 1$ and

$$
\begin{aligned}
-\int_\Omega \alpha(u)\frac{\beta(\tau_\epsilon(u))}{\alpha(\tau_\epsilon(u))}\left[e^{-\gamma(\tau_\epsilon(u))}w\right]|\nabla\tau_\epsilon(u)|^2 + \int_\Omega \beta(u)|\nabla u|^2 e^{-\gamma(\tau_\epsilon(u))}w \\
= \int_{\{u<\epsilon\}} \beta(u)|\nabla u|^2 e^{-\gamma(\epsilon)}w + \int_{\{u>b_\epsilon\}} \beta(u)|\nabla u|^2 e^{-\gamma(b_\epsilon)}w \\
= \int_{\{u<\epsilon\}} \beta(T_\delta(u))|\nabla T_\delta(u)|^2 e^{-\gamma(\epsilon)}w + \int_{\{u>b_\epsilon\}} \beta(u)|\nabla u|^2 e^{-\gamma(b_\epsilon)}w \\
\le \|w\|_\infty \int_{\{u<\epsilon\}} \beta(T_\delta(u))|\nabla T_\delta(u)|^2 e^{-\gamma(T_\delta(u))} + \|w\|_\infty \int_{\{u>b_\epsilon\}} \beta(u)|\nabla u|^2 .
\end{aligned}
$$

Using (8), we have $\beta(T_\delta(u))|\nabla T_\delta(u)|^2 e^{-\gamma(T_\delta(u))} \in L^1(\Omega)$ and thus

$$\lim_{\epsilon\to 0} \int_{\{u<\epsilon\}} \beta(T_\delta(u))|\nabla T_\delta(u)|^2 e^{-\gamma(T_\delta(u))} = 0.$$

On the other hand, by the definition of subsolution of (1), $\beta(u)|\nabla u|^2 \in L^1(\Omega)$ and then

$$\lim_{\epsilon\to 0} \int_{\{u>b_\epsilon\}} \beta(u)|\nabla u|^2 = 0.$$

Therefore, by (11), we deduce that

$$\int_\Omega \alpha(u)e^{-\gamma(\tau_\epsilon(u))}\nabla u\cdot\nabla w \le \int_\Omega f(x,u)e^{-\gamma(\tau_\epsilon(u))}w + \omega(\epsilon)\,, \tag{12}$$

where $\lim_{\epsilon\to 0}\omega(\epsilon) = 0$. Using now that $e^{-\gamma(s)} \le e^{-\gamma(\delta)}$ for every $s \in]\delta, b[$, we obtain

$$
\begin{aligned}
\alpha(u)e^{-\gamma(u)}|\nabla u| &= \alpha(u)e^{-\gamma(u)}|\nabla u|\chi_{\{u<\delta\}} + \alpha(u)e^{-\gamma(u)}|\nabla u|\chi_{\{u\ge\delta\}} \\
&\le \alpha(T_\delta(u))e^{-\gamma(T_\delta(u))}|\nabla T_\delta(u)| + \alpha(u)e^{-\gamma(\delta)}|\nabla u| \qquad (13)
\end{aligned}
$$

and we deduce from (8) and the definition of subsolution that $\alpha(u)e^{-\gamma(u)}|\nabla u| \in L^2(\Omega)$. Similarly, $f(x,u)e^{-\gamma(u)} \in L^1(\Omega)$. As a consequence, we may let ϵ tend to 0 in (12) by applying Lebesgue's Theorem and we conclude that

$$\int_\Omega \alpha(u)e^{-\gamma(u)}\nabla u\cdot\nabla w \le \int_\Omega f(x,u)e^{-\gamma(u)}w$$

as desired. □

In the following results, we make two additional assumptions.

(H1) The function $t \mapsto \alpha(t)e^{-\gamma(t)}$ is integrable in a right neighborhood of zero.

(H2) The function $s \mapsto f(x,s)\,e^{-\gamma(s)}$ is decreasing for a.e. $x \in \Omega$.

Remark 2.5. Assumption (H1) on the function $\alpha(s)e^{-\gamma(s)}$ prevents large singularities at 0. To illustrate this fact, suppose that $\alpha(s) = 1$ and $\beta(s) = \lambda/s^r$, with $\lambda, r > 0$. Then, if $r \neq 1$, we have $\alpha(s)e^{-\gamma(s)} = e^{-\frac{\lambda}{1-r}\left(s^{1-r}-a^{1-r}\right)}$, while, for the case $r = 1$, $\alpha(s)e^{-\gamma(s)} = a^r/s^\lambda$. Consequently, the function becomes integrable on every interval containing 0 if and only if either $r < 1$ or $\lambda < 1$ and $r = 1$.

Remark 2.6. If we assume that the function $f(x, s)$ is continuously differentiable with respect to the second variable, condition (H2) becomes

$$\frac{\partial f}{\partial s}(x,s) \le \frac{\beta(s)}{\alpha(s)} f(x,s) \tag{14}$$

for every $s \in I$ and a.e. $x \in \Omega$.

In particular, if f is nonnegative and does not depends on $s \in I$, i.e. if $f(x, s) = f(x) \ge 0$, then (H2) is always satisfied.

On the other hand, assuming that $f(x, s) = \lambda\, h(x)g(s)$, with h positive and $\lambda > 0$, a sufficient condition for (14) is $g'(s) \le \dfrac{\beta(s)}{\alpha(s)} g(s)$ for all $s \in I$. Some particular cases of functions g explain this. First, we consider $g(s) = (1+s)^\theta$, with $\theta > 0$. The sufficient condition becomes $\frac{\theta}{1+s} \le \frac{\beta(s)}{\alpha(s)}$. On the other hand, if $g(s) = \log(1 + s)$, then it turns into $\frac{1}{(1+s)\log(1+s)} \le \frac{\beta(s)}{\alpha(s)}$, while, if $g(s) = \log(1 + \log(1 + s))$, then it becomes $\frac{1}{(1+s)(1+\log(1+s))\,\log(1+\log(1+s))} \le \frac{\beta(s)}{\alpha(s)}$. In this way, we see that there is some connection between how singular can be $\frac{\beta}{\alpha}$ and the function g.

Now, we state our comparison principle.

Theorem 2.7. *Assume that (H1) and (H2) hold. Let $u, \widetilde{u} \in H^1(\Omega)$ be respectively a subsolution and a supersolution of problem* (1) *satisfying* (8). *If $u \le \widetilde{u}$ on $\partial\Omega$ (in the sense that $(u - \widetilde{u})^+ \in H^1_0(\Omega)$), then $u \le \widetilde{u}$.*

Remark 2.8. Observe that if the functions α and β are continuous in $[0, +\infty[$ (i.e., $b = +\infty$, and moreover $\alpha(0)$ and $\beta(0)$ are defined) and $\alpha(0) > 0$, then β/α is integrable in a neighborhood of $s = 0$ and, by Remark 2.4-(1), we deduce that (8) holds for both a subsolution u and a supersolution $\widetilde{u}$ of (1). Consequently, in this case, the above comparison principle improves the one given in [8, Theorem 2.6] where the authors impose, in addition, that ($\alpha = 1$ and that) there exist a positive constant n and a continuous function z such that $\exp\left[-n^{-1}\int_0^t z\right] \in L^\infty(0, +\infty) \cap L^1(0, +\infty)$ with

$$\beta'(s) - \frac{1}{2n}\left[2\beta(s) - z(s)\right]^2 > 0, \quad \forall s > 0.$$

Proof. We use an auxiliary function defined thanks to (H1) by

$$\Psi(s) = \int_0^s \alpha(t)e^{-\gamma(t)}\, dt\,, \qquad \forall s \in I\,. \tag{15}$$

Observe that if u is a subsolution of (1) satisfying (8), then, due to the L^2–integrability of the function $\alpha(u)e^{-\gamma(u)}|\nabla u|$ (remind (13)), we derive that $\Psi(u) \in H^1(\Omega)$. An analogous argument also gives $\Psi(\widetilde{u}) \in H^1(\Omega)$ for a supersolution of (1) satisfying (8). Since $u \le \widetilde{u}$ on $\partial\Omega$ and using that Ψ is increasing, we have $[\Psi(u) - \Psi(\widetilde{u})]^+ \in H^1_0(\Omega)$ and we deduce that

$$T_k\left[\Psi(u) - \Psi(\widetilde{u})\right]^+ \in H^1_0(\Omega) \cap L^\infty(\Omega)\,, \quad \forall k > 0.$$

Applying Proposition 2.3, it yields
$$\int_\Omega \alpha(u)e^{-\gamma(u)}\nabla u\cdot\nabla T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+ \le \int_\Omega f(x,u)e^{-\gamma(u)}T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+ .$$
Having in mind $\nabla\Psi(u)=\Psi'(u)\nabla u=\alpha(u)e^{-\gamma(u)}\nabla u$ we obtain
$$\int_\Omega \nabla\Psi(u)\cdot\nabla T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+ \le \int_\Omega f(x,u)\,e^{-\gamma(u)}T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+ .$$
Similarly, we get
$$\int_\Omega \nabla\Psi(\widetilde{u})\cdot\nabla T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+ \ge \int_\Omega f(x,\widetilde{u})e^{-\gamma(\widetilde{u})}T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+ .$$
Substracting the previous inequalities, we have
$$\int_\Omega |\nabla T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+|^2 \le \int_\Omega \big(f(x,u)\,e^{-\gamma(u)}-f(x,\widetilde{u})\,e^{-\gamma(\widetilde{u})}\big)T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+ .$$

On account that Ψ is (strictly) increasing and that $s\mapsto f(x,s)\,e^{-\gamma(s)}$ is decreasing, we deduce that the integrand in the right hand side of the above equality is nonpositive. Consequently, we obtain $\int_\Omega |\nabla T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+|^2=0$ and so, by Poincaré's inequality, $T_k\left[\Psi(u)-\Psi(\widetilde{u})\right]^+=0$ for all $k>0$. Hence, $\Psi(u)\le\Psi(\widetilde{u})$ a.e. in Ω and, using again that the function Ψ is strictly increasing, one deduces that $u\le\widetilde{u}$ a.e. in Ω. □

As a straightforward consequence of the above result, we obtain the desired uniqueness result.

Theorem 2.9. *If we assume that (H1) and (H2) hold, then problem* (1) *has at most a solution u satisfying* (8). □

By Remark 2.8 we deduce the following improvement of [8].

Corollary 2.10. *Assume that $f(x,s)=f(x)\ge 0$, that α and β are continuous in the interval $[0,+\infty[$ and that $\alpha(0)>0$. Then problem* (1) *has at most a solution u.* □

Remark 2.11. Existence results of bounded and unbounded solutions to elliptic equations with a quadratic gradient term without singularities have extensively been obtained since the eighties of the last century when several articles were published by Boccardo, Murat and Puel (see, for instance, [17] and [18]). On the other hand, (nonsingular) elliptic equations having a principal term with degenerate coercivity has been studied in the nineties (see [16], [2] and references therein). In some sense, these two features are combined in [19], where the authors look for minima of noncoercive functionals whose corresponding Euler–Lagrange equations have both a noncoercive principal term and a quadratic gradient term. The more general (nonvariational) equation in (1) are handled in [20] under the assumption $\beta/\alpha\in L^1(]-\infty,0[\cup]0,+\infty[)$. In that paper the following existence results are proved: If $m>\frac{N}{2}$, then there exists a bounded weak solution (Theorem 2.4 in [20]) and if $\frac{N}{2}>m\ge\frac{2N}{N+2}$, then there exists a weak solution u satisfying $\alpha(u)|\nabla u|\in L^2(\Omega)$ (Theorem 3.2 and Proposition 3.7 in [20]). The general case

$f \in L^m(\Omega)$, with $m > 1$ has been studied in [30]. Actually, the authors of this paper consider more general operators, of p–Laplacian type, and more general assumptions on β/α.

By Remark 2.5, we get also uniqueness of solutions for the singular model problem.

Corollary 2.12. *If $f(x) \geq 0$ and $\lambda, r > 0$, then problem* (2) *has at most one solution satisfying* (8) *provided either that $r < 1$ or that $r = 1$ and $\lambda < 1$.*

3. Another uniqueness result

We study in this section the uniqueness of bounded solutions of (1), i.e. of solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We take advantage of the restriction of the class of functions where we look for uniqueness to improve the condition (H2) and we assume that Ω is of class C^1. The result follows the arguments by H. Brézis and L. Oswald in [21]. In the sequel, the function Ψ is that defined by (15).

Theorem 3.1. *Let $I =]0, +\infty[$ and assume (H1), that Ω is of class C^1 and that the following conditions hold:*

- **(H3)** *The function $s \mapsto \frac{f(x,s)\, e^{-\gamma(s)}}{\Psi(s)}$ is decreasing for almost all $x \in \Omega$.*
- **(H4)** *For every $s \in I$, the function $f(x,s)$ is bounded from below and there exists a continuous increasing function g such that $f(x,s) \leq g(s)$, for a.e. $x \in \Omega$ and every $s \in I$.*

Then there exists at most a bounded solution u of problem (1) *satisfying* (8)*.*

Remarks 3.2.

1. Since the statement of Theorem 3.1 refers to uniqueness of bounded solutions, there is no need of using truncations at level δ in (8). Indeed, if u is a bounded function, then condition (8) is equivalent to impose that $\alpha(u)e^{-\gamma(u)}|\nabla u| \in L^2(\Omega)$ and $\beta(u)|\nabla u|^2 e^{-\gamma(u)}$, $f(x,u)e^{-\gamma(u)} \in L^1(\Omega)$.
2. As in Remark 2.8, if the functions α and β are continuous on $[0, +\infty[$ and $\alpha(0) > 0$, then (8) holds for every solution of (1). Thus, Theorem 3.1 improves Theorem 2.2 in [7] (an extension of Theorem 2.3 in [8]) where the authors study the case $\alpha \equiv 1$ and a general quadratic term $H(x, u, \nabla u)$ instead of $\beta(u)|\nabla u|^2$. In particular, for the case $H(x, u, \nabla u) = \beta(u)|\nabla u|^2$, they additionally impose either that β is bounded or that there exist $k \in \mathbb{R}$ and $m > 0$ such that
$$k[\beta(s) - k] + \beta'(s) \geq m[\beta(s) - k]^2, \quad \forall s > 0.$$

Proof. Let us consider two bounded solutions u and v of problem (1) satisfying the assumptions of the theorem. We point out that $\Psi(u), \Psi(v) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (see the proof of Theorem 2.7) and that $\Psi(u), \Psi(v) > 0$ in Ω. Thus, using (9) for the solution v with $w = \Psi(v)$, we obtain

$$\int_\Omega \alpha(v)e^{-\gamma(v)}\nabla v \cdot \nabla \Psi(v) = \int_\Omega e^{-\gamma(v)} f(x,v)\, \Psi(v)\,.$$

Performing easy manipulations, this equality becomes

$$\int_\Omega |\nabla \Psi(v)|^2 = \int_\Omega \frac{e^{-\gamma(v)} f(x,v)}{\Psi(v)} \Psi(v)^2\,. \tag{16}$$

On the other hand, applying Proposition 2.3 with $w \in C_0^\infty(\Omega)$ and $w \geq 0$, we deduce that
$$\int_\Omega \alpha(u)e^{-\gamma(u)}\nabla u \cdot \nabla w = \int_\Omega f(x,u)e^{-\gamma(u)}w\,,$$
i.e.,
$$-\Delta\Psi(u) = f(x,u)e^{-\gamma(u)}, \quad x \in \Omega.$$
Observe that by the monotonicity of $\frac{f(x,s)\,e^{-\gamma(s)}}{\Psi(s)}$ and $g(s)$, we have
$$g(\|u\|_\infty)e^{-\gamma(u)} \geq f(x,u)e^{-\gamma(u)} \geq \frac{f(x,\|u\|_\infty)e^{-\gamma(\|u\|_\infty)}}{\Psi(\|u\|_\infty)}\Psi(u)\,, \quad \text{a.e. } x \in \Omega$$
and, since $f(x,\|u\|_\infty)$ is bounded from below and (H1) holds, we derive that
$$M \geq f(x,u)e^{-\gamma(u)} \geq -M\Psi(u)\,, \quad \text{a.e. } x \in \Omega$$
for some positive constant M depending on $\|u\|_\infty$. As a consequence, the right hand term of the equations satisfied by $\Psi(u)$ is bounded and so $\Psi(u)$ belongs to $W^{2,p}(\Omega)$ for every $p < \infty$ (see [25]). In particular, since Ω has the strong local Lipschitz property [1, Paragraph 4.5], by Morrey's inequality (cf. [1, Theorem 5.4 Part II]), $\Psi(u) \in C^1(\overline{\Omega})$. Taking into account that Ω satisfies the interior sphere condition in every point of $\partial\Omega$ and using that $-\Delta\Psi(u) + M\Psi(u) \geq 0$ and the Hopf lemma [25, Lemma 3.4], we deduce that $\frac{\partial\Psi(u)}{\partial\nu} < 0$ on $\partial\Omega$ and therefore $\frac{\Psi(v)^2}{\Psi(u)} \in H_0^1(\Omega)\cap L^\infty(\Omega)$.

Using now identity (9) for the solution u with $w = \frac{\Psi(v)^2}{\Psi(u)}$ we get
$$\begin{aligned}
-\int_\Omega \alpha(u)e^{-\gamma(u)}\left(\frac{\Psi(v)}{\Psi(u)}\right)^2\Psi'(u)|\nabla u|^2 \\
+ 2\int_\Omega \alpha(u)e^{-\gamma(u)}\frac{\Psi(v)}{\Psi(u)}\Psi'(v)\nabla u\cdot\nabla v \\
= \int_\Omega \frac{e^{-\gamma(u)}f(x,u)}{\Psi(u)}\Psi(v)^2\,,
\end{aligned}$$
which can be written as
$$-\int_\Omega \left(\frac{\Psi(v)}{\Psi(u)}\right)^2|\nabla\Psi(u)|^2 + 2\int_\Omega \frac{\Psi(v)}{\Psi(u)}\nabla\Psi(u)\cdot\nabla\Psi(v) = \int_\Omega \frac{e^{-\gamma(u)}f(x,u)}{\Psi(u)}\Psi(v)^2\,.$$
Subtracting (16), we deduce that
$$\int_\Omega \left(\frac{\Psi(v)}{\Psi(u)}\nabla\Psi(u) - \nabla\Psi(v)\right)^2 = -\int_\Omega \left(\frac{e^{-\gamma(u)}f(x,u)}{\Psi(u)} - \frac{e^{-\gamma(v)}f(x,v)}{\Psi(v)}\right)\Psi(v)^2\,,$$
and so
$$\int_\Omega \left(\frac{e^{-\gamma(u)}f(x,u)}{\Psi(u)} - \frac{e^{-\gamma(v)}f(x,v)}{\Psi(v)}\right)\Psi(v)^2 \leq 0\,. \tag{17}$$
Exchanging the rôles of u and v we get
$$\int_\Omega \left(\frac{e^{-\gamma(u)}f(x,u)}{\Psi(u)} - \frac{e^{-\gamma(v)}f(x,v)}{\Psi(v)}\right)\Psi(u)^2 \geq 0\,, \tag{18}$$
and, therefore, subtracting (17) from (18) we obtain
$$\int_\Omega \Big(\frac{e^{-\gamma(u)}f(x,u)}{\Psi(u)} - \frac{e^{-\gamma(v)}f(x,v)}{\Psi(v)}\Big)(\Psi(u)^2 - \Psi(v)^2) \geq 0\,.$$

Since the function Ψ is increasing and the function $s \mapsto \frac{f(x,s)\,e^{-\gamma(s)}}{\Psi(s)}$ is decreasing, it follows that $\Psi(u) = \Psi(v)$ a.e. in Ω. Using again that the function Ψ is increasing, we conclude the proof.

□

Remark 3.3. Condition (H3) appearing in the above result turns out to be weaker than assumption (H2) in Theorem 2.9. Indeed, if the function $s \mapsto f(x,s)\,e^{-\gamma(s)}$ is decreasing, then the function $s \mapsto \frac{f(x,s)\,e^{-\gamma(s)}}{\Psi(s)}$ will also be decreasing. However, while the above proof may only be applied to equations whose principal term is controlled by (almost) linear operators, the argument of Theorem 2.8 is essentially nonlinear and can also be applied to equations of p-Laplacian type.

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