Quasi-linear diffusion equations with gradient terms and L^1 data

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Abstract.-

In this article we study the following quasilinear parabolic problem

$$\begin{cases} u_t - \Delta u + |u|^{\beta-2}u|\nabla u|^q = |u|^{\alpha-2}u|\nabla u|^p & \text{in } Q := \Omega\times]0,T[; \\ u(x,t) = 0 & \text{on } S := \partial\Omega\times]0,T[; \\ u(x,0) = u_0(x) & \text{in } \Omega; \end{cases}$$

Ω being a bounded open set of $I\!R^N$ and $T > 0$. We prove that if $\alpha, \beta \geq 1$, $0 \leq p < q$, $1 \leq q \leq 2$, and $\alpha + p < \beta + q$, then there exists a generalized solution for all $u_0 \in L^1(\Omega)$. We also see that there exists a nonnegative solution when $u_0 \geq 0$.

1 Introduction.

Given $T > 0$, consider the following quasilinear parabolic problem

$$\begin{cases} u_t - \Delta u + |u|^{\beta-2}u|\nabla u|^q = |u|^{\alpha-2}u|\nabla u|^p & \text{in } Q := \Omega\times]0,T[; \\ u(x,t) = 0 & \text{on } S := \partial\Omega\times]0,T[; \\ u(x,0) = u_0(x) & \text{in } \Omega; \end{cases} \tag{1}$$

where Ω is a bounded open set in $I\!R^N$, whose boundary is denoted by $\partial\Omega$, $1 \leq q \leq 2$, $0 \leq p < q$ and $\alpha, \beta \geq 1$. (We denote $|\nabla u|^0 = 1$.)

For the concrete case $p = 0$ and $\beta = 1$, and for positive initial data, problem (1) was introduced by M. Chipot and F.B. Weissler in [9] in order to investigate the effect of a damping term on existence or nonexistence of classical solutions. Several authors have studied the existence of non global positive classical solutions, giving conditions for blow-up under certains assumptions on α, q, N and Ω; see [3] and the references there in. Global existence for nonnegative initial data has been proved in the case $q+1 \geq \alpha > 2$ (see [11, 18]). On the other hand, it is observed in [21] that problem (1), with $q = 2$ and $p = 0$, does not admit global classical solution in the case $\alpha > 2$, $\beta \geq 1$ and $\beta + 2 < \alpha$.

For positive initial data and $p = 0$, the degenerate case (the term Δu is replaced by Δu^m in problem (1)) has been studied in [3], where the existence of global weak solutions for nonnegative initial data in $L^{m+1}(\Omega)$ is proved under the following assumptions: Ω a smooth bounded domain, $m \geq 1$, $(\beta+q-1)/q > m/2$, $1 \leq q < 2$ and $2 \leq \alpha < \beta + q$. We remark that the methods used in our paper are different of that of [3] which does not work in the limit case $q = 2$; moreover, we obtain an existence result for, not necessarily positive, initial data in $L^1(\Omega)$.

We point out that in [20] and [3] a model in population dynamics is described by this type of equations.

Problem (1), with $p = 0$ and $q = 2$, has been dealt with in [1] to obtain existence for L^1-initial data. We point out that the technique we use here is different from that employed in [1], which, moreover, does not work when $q < 2$.

Related problems are also studied in [4] in the degenerate case with measure initial data. In contrast with the above references, in [4] it is considered an equation with right hand side depending on the gradient.

The aim of this paper is to prove the existence of a generalized solution of problem (1) for initial data $u_0 \in L^1(\Omega)$ under the following hypotheses on the parameters: $\alpha, \beta \geq 1$, $p \geq 0$, $1 \leq q \leq 2$, and $\alpha+p < \beta+q$. Cases $p > 0$ and $p = 0$ correspond to equations with different behaviour; for instance, when $u_0 \in L^\infty$, $\|u_0\|_\infty$ is a bound in the case $p > 0$ but it is not so if $p = 0$. Nevertheless, the existence result lies on a stability theorem with respect to the initial datum (Theorem 3.1), which deal with both cases in a similar way. No regularity assumption is required on the open set Ω.

This article is organized as follows. In section 2 we define the concept of generalized solution we use and prove that these solutions are solutions in the sense of distributions. Section 3 is devoted to prove the existence of generalized solutions of problem (1) for initial datum $u_0 \in L^1(\Omega)$ by proving our stability result. In Section 4 we give an example which shows that the hypothesis $\alpha + p < \beta + q$ in our stability result cannot be avoided.

2 Generalized solutions

In this section we define and analyze our concept of solution of problem (1). Since our solution does not belong to the "right" space $L^2\big(0, T; H_0^1(\Omega)\big)$, we cannot apply the classical framework of [14] with its weak formulation. Instead, we will use another concept which was introduced in [6] for stationary problems, and in [2] and [17] for evolution ones. (In these papers it is called "entropy" solution, we prefer an alternative denomination in order to not be confused with entropy solutions of conservation laws.) Closely related with entropy solutions is the question of uniqueness. In our situation, it is not clear under which conditions it is possible to obtain uniqueness of solutions; this other interesting question will not be treated here.

In order to introduce this kind of solution, some notation is needed: for each $k > 0$, denote $T_k(r) = (r \wedge k) \vee (-k)$ and J_k the primitive of T_k such that $J_k(0) = 0$.

Definition 2.1 Let $u_0 \in L^1(\Omega)$. By a generalized solution of problem (1) in the set $Q_T = \Omega \times]0, T[$ we mean a function $u \in C\big([0, T]; L^1(\Omega)\big)$, such that $T_k(u) \in L^2\big(0, T; H_0^1(\Omega)\big)$ for all $k > 0$, $u|u|^{\alpha-2}|\nabla u|^p \in L^1(Q_T)$, $u|u|^{\beta-2}|\nabla u|^q \in L^1(Q_T)$ and

$$\int_\Omega J_k\big(u(t) - \phi(t)\big) + \int_0^t \int_\Omega \nabla u \cdot \nabla T_k(u - \phi) + \int_0^t \int_\Omega |u|^{\beta-2}u|\nabla u|^q T_k(u - \phi) =$$

$$= -\int_0^t \langle T_k(u - \phi), \phi_s\rangle + \int_0^t \int_\Omega |u|^{\alpha-2}u|\nabla u|^p T_k(u - \phi) + \int_\Omega J_k\big(u_0 - \phi(0)\big)$$

for all $k > 0$, all $t \in [0, T]$ and all test function $\phi \in L^2\big(0, T; H_0^1(\Omega)\big) \cap L^\infty(Q_T)$ such that its derivative in time in the sense of distributions, ϕ_t, belongs to $L^2\big(0, T; H^{-1}(\Omega)\big) + L^1(Q_T)$.

Remark 2.1 (1) If ϕ belongs to $L^2\big(0, T; H_0^1(\Omega)\big) \cap L^\infty(Q_T)$ and its distributional derivative in time is such that $\phi_t \in L^2\big(0, T; H^{-1}(\Omega)\big) + L^1(Q_T)$, it is well known that $\phi \in C\big([0, T]; L^2(\Omega)\big)$. As a consequence, the functions $\phi(0)$ and $\phi(T)$ in the above definition have sense.

(2) Since $T_k(u) \in L^2\big(0, T; H_0^1(\Omega)\big)$ and $\phi \in L^2\big(0, T; H_0^1(\Omega)\big) \cap L^\infty(Q_T)$, it follows that $T_k(u - \phi) \in L^2\big(0, T; H_0^1(\Omega)\big) \cap L^\infty(Q_T)$ (see [6]).

(3) It follows from $\nabla T_k(u - \phi) = 0$ when $|u - \phi| > k$, that $\nabla T_k(u - \phi) = 0$ when $|u| > M := k + \|\phi\|_\infty$. Thus, $\nabla u \cdot \nabla T_k(u - \phi) = \nabla T_M u \cdot \nabla T_k(u - \phi) \in L^1(Q_T)$ and the second term is well defined.

(4) Since $\phi_t \in L^2\big(0, T; H^{-1}(\Omega)\big) + L^1(Q_T)$, we have $\phi_t = \beta_1 + \beta_2$ where $\beta_1 \in L^2\big(0, T; H^{-1}(\Omega)\big)$ and $\beta_2 \in L^1(Q_T)$. We use the notation

$$\int_0^t \langle T_k(u - \phi), \phi_s\rangle = \int_0^t \langle T_k(u - \phi), \beta_1\rangle_{H_0^1, H^{-1}} + \int_{Q_t} T_k(u - \phi)\beta_2$$

in the above definition.

(5) Taking $\phi = 0$ and $k = 1$ in the generalized formulation, it yields

$$\int_{Q_T} |\nabla T_1(u)|^2 \leq \int_{Q_T} |u|^{\alpha-1}||T_1(u)||\nabla u|^p < \infty.$$

Moreover, we also have

$$\int_{Q_T} \big|\nabla\big(u - T_1(u)\big)\big|^q \leq \int_{Q_T} |u|^{\beta-1}||T_1(u)||\nabla u|^q < \infty.$$

Hence, these estimates imply $\int_{Q_T} |\nabla u|^q < \infty$ and so $u \in L^q\big(0, T; W_0^{1q}(\Omega)\big)$.

(6) Actually, the condition $u|u|^{\beta-2}|\nabla u|^q \in L^1(Q_T)$ in the above definition is redundant since the hypotheses $T_k(u) \in L^2\big(0, T; H_0^1(\Omega)\big)$ for all $k > 0$ and $u|u|^{\alpha-2}|\nabla u|^p \in L^1(Q_T)$ imply $u|u|^{\beta-2}|\nabla u|^q \in L^1(Q_T)$. Indeed, on the one hand,

$$\int_{\{|u| \leq k\}} |u|^{\beta-1}|\nabla u|^q \leq \frac{2-q}{2} \int_{\{|u| \leq k\}} k^{\frac{2(\beta-1)}{2-q}} + \frac{q}{2} \int_{\{|u| \leq k\}} |\nabla u|^2 < +\infty;$$

on the other hand, taking $\phi = 0$ and $t = T$ in the generalized formulation and disregarding non negative terms, it yields

$$\int_{\{|u|>k\}} |u|^{\beta-1}|\nabla u|^q \leq \int_{Q_T} |u|^{\alpha-1}|\nabla u|^p + \int_\Omega |u_0| < +\infty.$$

Thus, $|u|^{\beta-1}|\nabla u|^q \in L^1(Q)$.

Next, we are going to see that generalized solutions satisfy our equation in the sense of distributions. We will first prove that every generalized solution is a kind of "weak solution". (We point out that this is possible since $q \geq 1$; in another case this formulation has no sense, althought the generalized formulation still has it. Nevertheless our methods do not work to obtain existence of solutions when $0 < q < 1$.) In order to see it, we have to regularize our initial datum and apply the time-regularization procedure introduced in [12] (see also [13] and, for non-zero initial datum, [15] and [16]): for a fixed $\nu \in I\!N$ and a given function $w \in L^2(0,T;H_0^1(\Omega))$, we set

$$w_\nu(x,t) = \nu \int_0^t w(x,s)e^{\nu(s-t)}ds \tag{2}$$

for $t \in [0,T]$. This regularization function has the following properties

$$\begin{cases} w_\nu \in C\big([0,T];H_0^1(\Omega)\big) \\ (w_\nu)_t = \nu(w - w_\nu) \quad \text{in the sense of distributions} \\ w_\nu \to w \qquad \text{in} \qquad L^2(0,T;H_0^1(\Omega)) \text{ as } \nu \to \infty. \end{cases} \tag{3}$$

Moreover, $\|w_\nu\|_\infty \leq \|w\|_\infty$ if $w \in L^\infty(Q_T)$ and, when $w \in C([0,T];L^1(\Omega))$, $w_\nu(.,t) \to w(.,t)$ in $L^1(\Omega)$ for $0 < t \leq T$.

Proposition 2.1 *Let $T > 0$. If u is a generalized solution of (1) and $\phi \in L^{q'}\big(0,T;W_0^{1,q'}(\Omega)\big) \cap W^{1,\infty}\big(0,T;L^\infty(\Omega)\big)$, then the following equality holds:*

$$\int_\Omega u(T)\phi(T) + \int_{Q_T} \nabla u \cdot \nabla\phi + \int_{Q_T} |u|^{\beta-2}u|\nabla u|^q\phi =$$

$$= \int_{Q_T} u\phi_t + \int_{Q_T} |u|^{\alpha-2}u|\nabla u|^p\phi + \int_\Omega u_0\phi(0)$$

Proof: Fix $k > 0$ such that $k > \|\phi\|_\infty$ and let $h > k$. Consider a sequence $(\psi_j)_{j=1}^\infty$ in $\mathcal{D}(\Omega)$ such that $\psi_j \to u_0$ in $L^1(\Omega)$.

Now define $\eta_{\nu,j}(u) = \big(T_h(u)\big)_\nu + e^{-\nu t}T_h(\psi_j)$. By (3), $\eta_{\nu,j}(u) \in L^2(0,T;H_0^1(\Omega)) \cap C\big([0,T];L^1(\Omega)\big) \cap L^\infty(Q_T)$ and, in a distributional sense, $(\eta_{\nu,j}(u))_t = \nu(T_h(u) - \eta_{\nu,j}(u)) \in L^\infty(Q_T)$. Thus, if $\phi \in L^{q'}\big(0,T;W_0^{1,q'}(\Omega)\big) \cap W^{1,\infty}\big(0,T;L^\infty(\Omega)\big)$, then $\eta_{\nu,j}(u) - \phi$ may be taken as test function in the generalized formulation of problem

(1) which yields

$$\begin{aligned}
&\int_\Omega J_k\big(u(T)-\eta_{\nu,j}(u)(T)+\phi(T)\big)+\int_{Q_T}\nabla u\cdot\nabla T_k\big(u-\eta_{\nu,j}(u)+\phi\big)+\\
&+\int_{Q_T}u|u|^{\beta-2}|\nabla u|^qT_k\big(u-\eta_{\nu,j}(u)+\phi\big)=\\
&=-\int_{Q_T}\big(\eta_{\nu,j}(u)\big)_tT_k\big(u-\eta_{\nu,j}(u)+\phi\big)+\int_{Q_T}\phi_tT_k\big(u-\eta_{\nu,j}(u)+\phi\big)+\\
&+\int_{Q_T}u|u|^{\alpha-2}|\nabla u|^pT_k\big(u-\eta_{\nu,j}(u)+\phi\big)+\int_\Omega J_k\big(u_0-T_h(\psi_j)+\phi(0)\big).
\end{aligned}\tag{4}$$

We now analyze the following term:

$$\int_{Q_T}\big(\eta_{\nu,j}(u)\big)_tT_k\big(u-\eta_{\nu,j}(u)+\phi\big)=\nu\int_{Q_T}\big(T_h(u)-\eta_{\nu,j}(u)\big)T_k\big(u-\eta_{\nu,j}(u)+\phi\big).$$

Observe that the functions $T_h(u)-\eta_{\nu,j}(u)$ and $u-\eta_{\nu,j}(u)$ have the same sign. Indeed, when $|u|\le h$ both functions coincide and when $|u|>h$, taking into account that $|\eta_{\nu,j}(u)|\le h$, we have that

$$\operatorname{sgn}\big(T_h(u)-\eta_{\nu,j}(u)\big)=\operatorname{sgn}(u)=\operatorname{sgn}\big(u-\eta_{\nu,j}(u)\big).$$

On the other hand, since T_k is an increasing function, $\operatorname{sgn}\,a=\operatorname{sgn}\,\overline{a}$ implies $a\big(T_k(\overline{a}+b)-T_k(b)\big)\ge 0$; that is, $aT_k(\overline{a}+b)\ge aT_k(b)$. Hence,

$$\int_{Q_T}\big(T_h(u)-\eta_{\nu,j}(u)\big)T_k\big(u-\eta_{\nu,j}(u)+\phi\big)\ge\int_{Q_T}\big(T_h(u)-\eta_{\nu,j}(u)\big)T_k\big(\phi\big)=$$

$$=\int_{Q_T}\big(T_h(u)-\eta_{\nu,j}(u)\big)\phi=\frac{1}{\nu}\int_{Q_T}\big(\eta_{\nu,j}(u)\big)_t\phi$$

so that

$$\int_{Q_T}\big(\eta_{\nu,j}(u)\big)_tT_k\big(u-\eta_{\nu,j}(u)+\phi\big)\ge$$

$$\ge\int_\Omega\eta_{\nu,j}(u)(T)\phi(T)-\int_\Omega T_h(\psi_j)\phi(0)-\int_{Q_T}\eta_{\nu,j}(u)\phi_t.$$

Thus, (4) becomes

$$\begin{aligned}
&\int_\Omega J_k\big(u(T)-\eta_{\nu,j}(u)(T)+\phi(T)\big)+\int_{Q_T}\nabla u\cdot\nabla T_k\big(u-\eta_{\nu,j}(u)+\phi\big)+\\
&+\int_{Q_T}u|u|^{\beta-2}|\nabla u|^qT_k\big(u-\eta_{\nu,j}(u)+\phi\big)\le-\int_\Omega\eta_{\nu,j}(u)(T)\phi(T)+\\
&+\int_\Omega T_h(\psi_j)\phi(0)+\int_{Q_T}\eta_{\nu,j}(u)\phi_t+\int_{Q_T}\phi_tT_k\big(u-\eta_{\nu,j}(u)+\phi\big)+\\
&+\int_{Q_T}u|u|^{\alpha-2}|\nabla u|^pT_k\big(u-\eta_{\nu,j}(u)+\phi\big)+\int_\Omega J_k\big(u_0-T_h(\psi_j)+\phi(0)\big).
\end{aligned}\tag{5}$$

In order to take limit as ν goes to ∞ we have to study the term

$$\int_{Q_T}\nabla u\cdot\nabla T_k\big(u-\eta_{\nu,j}(u)+\phi\big)=\int_{\{|u-\eta_{\nu,j}(u)+\phi|<k\}}\nabla u\cdot\nabla\big(u-\eta_{\nu,j}(u)+\phi\big),$$

which can be split up as

$$\int_{Q_T} \nabla u \cdot \nabla T_k\big(u - \eta_{\nu,j}(u) + \phi\big) = I_1 + I_2 + I_3,$$

where $I_1 = \int_{\{|u-\eta_{\nu,j}(u)+\phi|<k\}} \nabla u \cdot \nabla\big(u - \eta_{\nu,j}(u) + \phi\big)\chi_{\{|u-T_h(u)+\phi|>k\}}$,
$I_2 = \int_{\{|u-\eta_{\nu,j}(u)+\phi|<k\}} \nabla u \cdot \nabla\big(T_h(u) - \eta_{\nu,j}(u)\big)\chi_{\{|u-T_h(u)+\phi|\le k\}}$ and
$I_3 = \int_{\{|u-\eta_{\nu,j}(u)+\phi|<k\}} \nabla u \cdot \nabla\big(u - T_h(u) + \phi\big)\chi_{\{|u-T_h(u)+\phi|\le k\}}$. Having in mind $\lim_{\nu\to\infty} \eta_{\nu,j}(u) = T_h(u)$, it is easy to see that $\lim_{\nu\to\infty} I_1 = 0 = \lim_{\nu\to\infty} I_2$ and $\lim_{\nu\to\infty} I_3 = \int_{\{|u-T_h(u)+\phi|<k\}} \nabla u \cdot \nabla\big(u - T_h(u) + \phi\big)$. So that

$$\lim_{\nu\to\infty} \int_{Q_T} \nabla u \cdot \nabla T_k\big(u - \eta_{\nu,j}(u) + \phi\big) = \int_{Q_T} \nabla u \cdot \nabla T_k\big(u - T_h(u) + \phi\big).$$

Thus, by this convergence and Lebesgue's Theorem, we may take limit in (5) first when ν tends to ∞ and then when j goes to ∞, and it follows that

$$\begin{aligned}
&\textstyle\int_\Omega J_k\big(u(T) - T_h(u)(T) + \phi(T)\big) + \int_{Q_T} \nabla u \cdot \nabla T_k\big(u - T_h(u) + \phi\big) + \\
&+ \textstyle\int_{Q_T} u|u|^{\beta-2}|\nabla u|^q T_k\big(u - T_h(u) + \phi\big) \le - \int_\Omega T_h(u)(T)\phi(T) + \\
&+ \textstyle\int_\Omega T_h(u_0)\phi(0) + \int_{Q_T} T_h(u)\phi_t + \int_{Q_T} \phi_t T_k\big(u - T_h(u) + \phi\big) + \\
&+ \textstyle\int_{Q_T} u|u|^{\alpha-2}|\nabla u|^p T_k\big(u - T_h(u) + \phi\big) + \int_\Omega J_k\big(u_0 - T_h(u_0) + \phi(0)\big).
\end{aligned} \tag{6}$$

Notice that

$$\begin{aligned}
&\int_{Q_T} \nabla u \cdot \nabla T_k\big(u - T_h(u) + \phi\big) = \\
&= \int_{\{|u|\le h\}} \nabla u \cdot \nabla\phi + \int_{\{h<|u|<k+h+\|\phi\|_\infty\}\cap\{|u-T_h(u)+\phi|<k\}} \nabla u \cdot \nabla(u + \phi) \ge \\
&\ge \int_{Q_T} \nabla u \cdot \nabla\phi\ \big(\chi_{\{|u|\le h\}} + \chi_{\{h<|u|<k+h+\|\phi\|_\infty\}\cap\{|u-T_h(u)+\phi|<k\}}\big)
\end{aligned}$$

and the last term in the above inequality converges to $\int_{Q_T} \nabla u \cdot \nabla\phi$ when h tends to ∞. As a consequence, we obtain from (6) that

$$\begin{aligned}
&\textstyle\int_\Omega u(T)\phi(T) + \int_\Omega J_k\big(\phi(T)\big) + \int_{Q_T} \nabla u \cdot \nabla\phi + \\
&+ \textstyle\int_{Q_T} u|u|^{\beta-2}|\nabla u|^q T_k\big(\phi\big) \le \int_{Q_T} u\phi_t + \int_{Q_T} \phi_t T_k\big(\phi\big) + \\
&+ \textstyle\int_{Q_T} u|u|^{\alpha-2}|\nabla u|^p T_k\big(\phi\big) + \int_\Omega J_k\big(\phi(0)\big) + \int_\Omega u_0\phi(0).
\end{aligned} \tag{7}$$

Taking now into account that

$$\int_{Q_T} \phi_t T_k\big(\phi\big) = \int_\Omega J_k\big(\phi(T)\big) - \int_\Omega J_k\big(\phi(0)\big),$$

we deduce from (7) that

$$\int_\Omega u(T)\phi(T) + \int_{Q_T} \nabla u \cdot \nabla\phi + \int_{Q_T} u|u|^{\beta-2}|\nabla u|^q\phi \le$$

$$\leq \int_{Q_T} u\phi_t + \int_{Q_T} u|u|^{\alpha-2}|\nabla u|^p\phi + \int_\Omega u_0\phi(0).$$

Finally, the desired equality follows by considering $\pm\phi$.

Corollary 2.1 *Every generalized solution of (1) in Q_T satisfies the equation in the sense of distributions.*

Both type of solutions are different, in general; nevertheless they coincides for bounded solutions.

Proposition 2.2 *Let u belong to $L^2\big(0,T;H_0^1(\Omega)\big)\cap L^\infty(Q_T)$ and be such that*

$$u_t \in L^2\big(0,T;H^{-1}(\Omega)\big) + L^1(Q_T)$$

and

$$|u|^{\beta-1}|\nabla u|^q,\ |u|^{\alpha-1}|\nabla u|^p \in L^1(Q_T).$$

Then u satisfies (1) in the sense of distributions if and only if u is a generalized solution.

Proof: On account of Corollary (2.1), we only have to see that distributional solution implies generalized one. Fix $\phi \in L^2\big(0,T;H_0^1(\Omega)\big)\cap L^\infty(Q_T)$ such that $\phi_t \in L^2\big(0,T;H^{-1}(\Omega)\big)+L^1(Q_T)$, and consider a sequence $(\varphi_n)_{n=1}^\infty$ in $\mathcal{D}(Q_T)$ such that $\varphi_n \to u-\phi$ in $L^2\big(0,T;H_0^1(\Omega)\big)$ and a.e.

Now, let $S:\mathbb{R}\to\mathbb{R}$ be a bounded C^∞-function satisfying $S(0)=0$, $0\leq S'\leq 1$, $S'(s)=0$ for all s big enough, $S(-s)=-S(s)$ for all $s\in\mathbb{R}$, and $S''(s)\leq 0$ for all $s\geq 0$. Taking $S(\varphi_n)$ as test function in the distributional formulation and passing to the limit when n goes to infinity, it yields

$$\int_0^t \langle S(u-\phi),u_s\rangle + \int_0^t\int_\Omega \nabla u\cdot\nabla S(u-\phi) + \int_0^t\int_\Omega |u|^{\beta-2}u|\nabla u|^q S(u-\phi) =$$

$$= \int_0^t\int_\Omega |u|^{\alpha-2}u|\nabla u|^p S(u-\phi)$$

for all $t\in[0,T]$.

Next denote $J_S(s)=\int_0^s S(r)\,dr$ and integrate to get

$$\int_\Omega J_S\big(u(t)-\phi(t)\big) + \int_0^t\int_\Omega \nabla u\cdot\nabla S(u-\phi) + \int_0^t\int_\Omega |u|^{\beta-2}u|\nabla u|^q S(u-\phi) =$$

$$= -\int_0^t \langle S(u-\phi),\phi_s\rangle + \int_0^t\int_\Omega |u|^{\alpha-2}u|\nabla u|^p S(u-\phi) + \int_\Omega J_S\big(u(0)-\phi(0)\big)$$

for all $t\in[0,T]$.

Finally, approximate the truncature T_k by an increasing sequence of functions $(S_m)_{m=1}^\infty$ as in [6, Lemma 3.2] and let m tend to infinity; then u satisfies the generalized formulation.

3 Existence of generalized solutions

This section is devoted to prove a stability result from which the existence of generalized solutions follows.

Theorem 3.1 *Assume that* u_n *is a bounded generalized solution of*

$$\begin{cases} (u_n)_t - \Delta u_n + u_n|u_n|^{\beta-2}|\nabla u_n|^q = |u_n|^{\alpha-2}u_n|\nabla u_n|^p & \text{in } Q_T := \Omega\times]0,T[; \\ u_n = 0 & \text{on } S_T := \partial\Omega\times]0,T[; \\ u_n(x,0) = u_{0n}(x) & \text{in } \Omega; \end{cases} \tag{8}$$

where $\alpha, \beta \geq 1$, $1 \leq q \leq 2$, $0 \leq p < q$, $p + \alpha < q + \beta$, *and* $u_{0n} \in L^\infty(\Omega)$ *for all* $n \in I\!N$.

If

$$u_{0n} \to u_0 \qquad \text{in} \qquad L^1(\Omega), \tag{9}$$

then there exists a subsequence (still denoted by u_n *) and a function* $u : Q_T \to I\!R$ *satisfying*

$$u_n \to u \qquad \text{in} \qquad L^q(0,T;W_0^{1,q}(\Omega)), \tag{10}$$

$$T_k(u_n) \to T_k(u) \qquad \text{in} \qquad L^2(0,T;H_0^1(\Omega)) \qquad \text{for all} \qquad k > 0 \tag{11}$$

$$|u_n|^{\beta-1}|\nabla u_n|^q \to |u|^{\beta-1}|\nabla u|^q \qquad \text{in} \qquad L^1(Q_T), \tag{12}$$

$$|u_n|^{\alpha-1}|\nabla u_n|^p \to |u|^{\alpha-1}|\nabla u|^p \qquad \text{in} \qquad L^1(Q_T), \tag{13}$$

$$u_n \to u \qquad \text{in} \qquad C([0,T];L^1(\Omega)). \tag{14}$$

As a consequence, this function u *is a generalized solution of problem (1).*

Proof: In this proof C will denote a positive constant that only depends on Ω, T, a bound of $\|u_{0n}\|_1$ and on the parameters α, β, p and q. The value of C may vary from line to line.

The following equality will be used several times in what follows,

$$\begin{aligned} &\int_\Omega J_k(u_n(t)) + \int_{Q_t} |\nabla T_k(u_n)|^2 + \int_{Q_t} |u_n|^{\beta-2}u_n T_k(u_n)|\nabla u_n|^q = \\ &= \int_{Q_t} |u_n|^{\alpha-2}u_n T_k(u_n)|\nabla u_n|^p + \int_\Omega J_k(u_{0n}). \end{aligned} \tag{15}$$

To obtain it, fix $t \in [0,T]$ and take $\phi = 0$ as test function in the generalized formulation of (8). Moreover, dividing by k, dropping a nonnegative term and letting $k \to 0^+$, it follows that

$$\int_\Omega |u_n(t)| + \int_{Q_t} |u_n|^{\beta-1}|\nabla u_n|^q \leq \int_{Q_t} |u_n|^{\alpha-1}|\nabla u_n|^p + \int_\Omega |u_{0n}|. \tag{16}$$

1.- A priori estimates

We will prove that

$$\int_{Q_T} |u_n|^{\alpha-1}|\nabla u_n|^p \leq C \qquad \text{for all } n \in I\!N. \tag{17}$$

Assume first that $(\alpha-1)q>(\beta-1)p$. Applying Young's inequality it follows that

$$\int_{Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p\leq\frac{p}{q}\int_{Q_T}|u_n|^{\beta-1}|\nabla u_n|^q+\frac{q-p}{q}\int_{Q_T}|u_n|^{\frac{\alpha q-\beta p}{q-p}-1}. \tag{18}$$

Taking $t=T$ in (16), the above inequality implies

$$\int_{Q_T}|u_n|^{\beta-1}|\nabla u_n|^q\leq\int_{Q_T}|u_n|^{\frac{\alpha q-\beta p}{q-p}-1}+\tfrac{q}{q-p}\int_\Omega|u_{0n}|. \tag{19}$$

Taking into account (9) and applying Poincaré's inequality we get

$$\begin{aligned}\int_{Q_T}\left|u_n\right|^{\beta-1+q}&\leq C\int_{Q_T}\left|\nabla u_n^{\frac{\beta-1}{q}+1}\right|^q=C(\tfrac{\beta-1}{q}+1)^q\int_{Q_T}|u_n|^{\beta-1}\left|\nabla u_n\right|^q\leq\\&\leq C\Big(\int_{Q_T}|u_n|^{\frac{\alpha q-\beta p}{q-p}-1}+1\Big).\end{aligned} \tag{20}$$

Since $p+\alpha<q+\beta$ and $p<q$ imply $\frac{\alpha q-\beta p}{q-p}<q+\beta$, it follows from (20) that

$$\int_{Q_T}|u_n|^{\beta-1+q}\leq C\qquad\text{for all}\qquad n\in I\!N, \tag{21}$$

and so

$$\int_{Q_T}|u_n|^{\frac{\alpha q-\beta p}{q-p}-1}\leq C\qquad\text{for all}\qquad n\in I\!N. \tag{22}$$

Going back to (19), we deduce that

$$\int_{Q_T}|u_n|^{\beta-1}|\nabla u_n|^q\leq C\qquad\text{for all}\qquad n\in I\!N. \tag{23}$$

Now, (18), (22) and (23) imply that (17) holds when $(\alpha-1)q>(\beta-1)p$.

The case $(\alpha-1)q=(\beta-1)p$ is proved in a similar way. Consider finally the case $(\alpha-1)q<(\beta-1)p$. Then we deduce from (15), with $t=T$ and $k=1$, that

$$\begin{aligned}&\int_{Q_T}|\nabla T_1u_n|^2+\int_{\{|u_n|<1\}\cap Q_T}|u_n|^\beta|\nabla u_n|^q+\int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q=\\&=\int_{Q_T}|\nabla T_1u_n|^2+\int_{Q_T}|u_n|^{\beta-2}u_nT_1(u_n)|\nabla u_n|^q\leq\\&\leq\int_{Q_T}|u_n|^{\alpha-2}u_nT_1(u_n)|\nabla u_n|^p+\int_\Omega|u_{0n}|=\\&=\int_{\{|u_n|<1\}\cap Q_T}|u_n|^\alpha|\nabla u_n|^p+\int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p+\int_\Omega|u_{0n}|.\end{aligned} \tag{24}$$

Thus,

$$\begin{aligned}&\int_{Q_T}|\nabla T_1u_n|^2+\int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q\leq\\&\leq\int_{Q_T}|\nabla T_1u_n|^p+\int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p+\int_\Omega|u_{0n}|.\end{aligned}$$

Since $p<2$, using Young's inequality, the first member on the right hand side can be cancelled with the first one on the left. Now, applying again Young's inequality, it yields

$$\begin{aligned}&\int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p\leq\\&\leq\tfrac{p}{q}\int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q+\tfrac{q-p}{q}\int_{\{|u_n|>1\}\cap Q_T}|u_n|^{\frac{\alpha q-\beta p}{q-p}-1};\end{aligned} \tag{25}$$

so that,

$$\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\beta-1}|\nabla u_n|^q \le \int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\frac{\alpha q-\beta p}{q-p}-1} + \frac{q-p}{q}\int_\Omega |u_{0n}| + C.$$

Note that we have $\frac{\alpha q-\beta p}{q-p} - 1 < 0$ and so the right hand side in the above inequality is bounded, that is,

$$\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\beta-1}|\nabla u_n|^q \le C.$$

Hence, the above inequality and (25) imply

$$\textstyle\int_{\{|u_n|>1\}\cap Q_T} |u_n|^{\alpha-1}|\nabla u_n|^p \le C. \tag{26}$$

On the other hand, dropping non negative terms in inequality (24) we obtain

$$\int_{Q_T} |\nabla T_1 u_n|^2 \le \int_{Q_T} |\nabla T_1 u_n|^p + C,$$

so that it follows from Young's inequality that

$$\int_{Q_T} |\nabla T_1 u_n|^p \le C.$$

Therefore, from this and (26), we get that (17) holds true in every case. As a consequence, the right-hand side in the equality (16) is bounded. This fact implies two important estimates:

$$\int_{Q_T} |u_n|^{\beta-1}|\nabla u_n|^q \le C \qquad \text{for all} \qquad n \in I\!N. \tag{27}$$

and

$$\sup_{t\in[0,T]}\int_\Omega |u_n(t)| \le C \qquad \text{for all} \qquad n \in I\!N. \tag{28}$$

Furthermore, from the equality (15) the following estimates also hold,

$$\int_{Q_T} |\nabla T_k(u_n)|^2 \le Ck \qquad \text{for all} \qquad n \in I\!N, \tag{29}$$

and, for $k = 1$,

$$\int_{Q_T} T_1(u_n)u_n|u_n|^{\beta-2}|\nabla u_n|^q \le C \qquad \text{for all} \qquad n \in I\!N. \tag{30}$$

Denoting $G_k(r) = r - T_k(r)$, this last estimate implies

$$\int_{Q_T} |\nabla G_1(u_n)|^q \le \int_{Q_T} T_1(u_n)u_n|u_n|^{\beta-2}|\nabla(u_n)|^q \le C \qquad \text{for all} \qquad n \in I\!N.$$

From this fact and (29) we obtain

$$\int_{Q_T} |\nabla u_n|^q \le C \qquad \text{for all} \qquad n \in I\!N. \tag{31}$$

Moreover, for q close to 1 a better estimate can be obtained; indeed, multiplying problem (8) by $T_1\big(u_n - T_h(u_n)\big)$, $h>0$, and integrating, we have:

$$\int_\Omega J_1\big(u_n(T) - T_h(u_n)(T)\big) + \int_{Q_T} \nabla u_n \cdot \nabla T_1\big(u_n - T_h(u_n)\big) +$$

$$+ \int_{Q_T} u_n |u_n|^{\beta-2} |\nabla u_n|^q T_1\big(u_n - T_h(u_n)\big) \le$$

$$\le \int_{Q_T} u_n |u_n|^{\alpha-2} |\nabla u_n|^p T_1\big(u_n - T_h(u_n)\big) + \int_\Omega J_1\big(u_{0n} - T_h(u_{0n})\big).$$

Thus dropping nonegative terms it yields

$$\int_{\{h<|u_n|<h+1\}} |\nabla u_n|^2 \le \int_{Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p + \int_\Omega |u_{0n} - T_h(u_{0n})|$$

and, as a consequence of (9) and (17),

$$\int_{\{h<|u_n|<h+1\}} |\nabla u_n|^2 \le C$$

for all $h>0$. From this inequality we may follow the procedure introduced by Boccardo and Gallouët in [7] and deduce that, for $1 \le r < (N+2)/(N+1)$,

$$\int_{Q_T} |\nabla u_n|^r \le C \tag{32}$$

for all $n \in I\!N$.

Going back again to (8), we get that the sequence $((u_n)_t)_{n=1}^\infty$ is bounded in the spaces $L^q(0,T;W^{-1,q}(\Omega)) + L^1(Q_T)$ and $L^r(0,T;W^{-1,r}(\Omega)) + L^1(Q_T)$ for $1 \le r < (N+2)/(N+1)$. Using this fact, (31) and (32), we obtain from [19, Corollary 4] that $(u_n)_{n=1}^\infty$ is relatively compact in $L^q(Q_T)$.

Summing up, there exists a function $u \in L^q(0,T;W_0^{1,q}(\Omega))$ and a subsequence, still denoted by $(u_n)_{n=1}^\infty$, such that

$$u_n \rightharpoonup u \qquad \text{weakly in} \qquad L^q(0,T;W_0^{1,q}(\Omega)) \tag{33}$$

and

$$u_n \to u \qquad \text{in} \qquad L^q(Q_T) \qquad \text{and a.e. in} \qquad Q_T. \tag{34}$$

Moreover, by (29), we may assume that

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{weakly in} \qquad L^2(0,T;H_0^1(\Omega)). \tag{35}$$

Finally, assuming $(\alpha-1)q > (\beta-1)p$, we also deduce that

$$|u_n|^{\frac{\alpha q - \beta p}{q-p} - 1} \to |u|^{\frac{\alpha q - \beta p}{q-p} - 1} \qquad \text{in} \qquad L^1(Q_T). \tag{36}$$

Indeed, because of (34), we just have to show that the sequence $(|u_n|^{\frac{\alpha q - \beta p}{q-p} - 1})_{n=1}^\infty$ is equi-integrable, but it is straightforward taking (21) and Hölder's inequality into account.

2.- Convergence of truncations in $L^2\big(0,T;H_0^1(\Omega)\big)$

Our aim is to prove that (11) holds; that is,

$$\nabla T_k(u_n) \to \nabla T_k(u) \qquad \text{in} \qquad L^2(Q_T) \qquad \text{for all} \quad k \in I\!N. \tag{37}$$

From this fact, by applying a diagonal procedure, it yields

$$\nabla u_n \to \nabla u \qquad \text{a.e. in} \qquad Q_T. \tag{38}$$

To prove (37), we have to regularize the initial datum u_0 and to use the time-regularization function given in (2). Let $\psi_j \in \mathcal{D}(\Omega)$ be such that

$$\psi_j \to u_0 \quad \text{in } L^1(\Omega),$$

and let

$$\eta_{\nu j}(u^+) = \big(T_k(u^+)\big)_\nu + e^{-\nu t} T_k(\psi_j^+),$$

which has the following properties (see (3)):

$$(\eta_{\nu j}(u^+))_t = \nu(T_k(u^+) - \eta_{\nu j}(u^+)),$$

$$\eta_{\nu j}(u^+)(0) = T_k(\psi_j^+),$$

$$|\eta_{\nu j}(u^+)| \le k,$$

$$\eta_{\nu j}(u^+) \to T_k(u^+) \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ as } \nu \to \infty.$$

By denoting $\omega(n,\nu,j,h)$ any quantity such that

$$\lim_{h\to\infty}\lim_{j\to\infty}\lim_{\nu\to\infty}\lim_{n\to\infty}\omega(n,\nu,j,h) = 0,$$

all we have to prove is that

$$\int_{Q_T} |\nabla(T_k(u_n) - \eta_{\nu j}(u))|^2 \le \omega(n,\nu,j,h), \tag{39}$$

where h is a parameter that we will consider later. The proof of this fact will be splitted into several stages. We begin by showing that

Claim 1:

$$\int_{Q_T} |\nabla\big(T_k(u_n^+) - \eta_{\nu j}(u^+)\big)^+|^2 \le \omega(n,\nu,j,h)$$

Proof: Consider

$$w_n = T_{2k}\big(u_n^+ - T_h(u_n^+) + \big(T_k(u_n) - \eta_{\nu j}(u^+)\big)^+\big)$$

with $h > k$, and observe that $u_n w_n \ge 0$. Multiplying problem (8) by w_n and integrating, we obtain

$$\begin{aligned} &\textstyle\int_0^T \langle w_n, (u_n)_t\rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n + \int_{Q_T} |u_n|^{\beta-1}|\nabla u_n|^q |w_n| = \\ &= \textstyle\int_{Q_T} |u_n|^{\alpha-1}|\nabla u_n|^p |w_n|. \end{aligned} \tag{40}$$

Let us prove that

$$\int_0^T \langle w_n, (u_n)_t \rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n \le \omega(n, \nu, h). \tag{41}$$

We have to consider three cases; assume first $(\alpha-1)q > (\beta-1)p$. Then using Young's inequality in the right hand side of (40) we obtain

$$\int_0^T \langle w_n, (u_n)_t \rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n + \int_{Q_T} |u_n|^{\beta-1} |\nabla u_n|^q |w_n| \le$$

$$\le \tfrac{p}{q} \int_{Q_T} |u_n|^{\beta-1} |\nabla u_n|^q |w_n| + \tfrac{q-p}{q} \int_{Q_T} |u_n|^{\frac{\alpha q - \beta p}{q-p} - 1} |w_n|$$

and consequently, there exists a constant $C > 0$ such that

$$\int_0^T \langle w_n, (u_n)_t \rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n \le C \int_{Q_T} |u_n|^{\frac{\alpha q - \beta p}{q-p} - 1} |w_n|.$$

Having in mind (36), the properties of $\eta_{\nu j}(u^+)$ and Lebesgue's Theorem, it is easy to see that

$$\lim_{h\to\infty} \lim_{\nu\to\infty} \lim_{n\to\infty} \int_{Q_T} |u_n|^{\frac{\alpha q - \beta p}{q-p} - 1} |w_n| = 0;$$

thus, (41) is proved in this case. The case $(\alpha-1)q = (\beta-1)p$ is similar. Consider next $(\alpha-1)q < (\beta-1)p$, then

$$\int_{Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n| =$$

$$= \int_{\{|u_n|<1\}\cap Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n| + \int_{\{|u_n|\ge 1\}\cap Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n|.$$

The first integral can be manipulated as follows

$$\int_{\{|u_n|<1\}\cap Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n| \le \int_{\{|u_n|<1\}\cap Q_T} |\nabla u_n|^p |w_n| \le$$

$$\le \Big(\int_{Q_T} |\nabla T_1 u_n|^2 \Big)^{p/2} \Big(\int_{Q_T} |w_n|^{\frac{2}{2-p}} \Big)^{(2-p)/2}.$$

With respect to the second integral, we use Young's inequality to get

$$\int_{\{|u_n|\ge 1\}\cap Q_T} |u_n|^{\alpha-1} |\nabla u_n|^p |w_n| \le$$

$$\le \tfrac{p}{q} \int_{\{|u_n|\ge 1\}\cap Q_T} |u_n|^{\beta-1} |\nabla u_n|^q |w_n| + \tfrac{q-p}{q} \int_{\{|u_n|\ge 1\}\cap Q_T} |u_n|^{\frac{\alpha q - \beta p}{q-p} - 1} |w_n| \le$$

$$\le \tfrac{p}{q} \int_{Q_T} |u_n|^{\beta-1} |\nabla u_n|^q |w_n| + \tfrac{q-p}{q} \int_{Q_T} |w_n|.$$

On account of (40), we have

$$\int_0^T \langle w_n, (u_n)_t \rangle + \int_{Q_T} \nabla u_n \cdot \nabla w_n \le$$

$$\le \Big(\int_{Q_T} |\nabla T_1 u_n|^2 \Big)^{p/2} \Big(\int_{Q_T} |w_n|^{\frac{2}{2-p}} \Big)^{(2-p)/2} + \tfrac{q-p}{q} \int_{Q_T} |w_n| \le$$

$$\le \omega(n, \nu, h).$$

Therefore, (41) is proved.

Let us now analyze the term

$$\int_0^T \langle w_n, (u_n)_t\rangle$$

in (41). Note that $w_n = w_n\chi_{\{u_n\geq 0\}}$ and, if $u_n \geq 0$, then

$$w_n = T_{h+k}(u_n - \eta_{\nu j}(u^+))^+ - T_{h-k}(u_n^+ - T_k u_n^+)$$

and so

$$\int_0^T \langle w_n, (u_n)_t\rangle = \int_0^T \langle T_{h+k}(u_n - \eta_{\nu j}(u^+))^+, (u_n)_t\rangle - \int_0^T \langle T_{h-k}(u_n^+ - T_k(u_n^+)), (u_n)_t\rangle. \tag{42}$$

On the one hand,

$$\int_0^T \langle T_{h+k}(u_n - \eta_{\nu j}(u^+))^+, (u_n)_t\rangle =$$

$$= \int_{Q_T} (\eta_{\nu j}(u^+))_t T_{h+k}(u_n - \eta_{\nu j}(u^+))^+ + \int_\Omega J_{h+k}(u_n - \eta_{\nu j}(u^+))^+(T) -$$

$$- \int_\Omega J_{h+k}(u_{0n} - T_k(\psi_j^+))^+ =$$

$$= \nu \int_{Q_T} (T_k(u) - \eta_{\nu j}(u^+)) T_{h+k}(u_n - \eta_{\nu j}(u^+))^+ + \int_\Omega J_{h+k}(u_n - \eta_{\nu j}(u^+))^+(T) -$$

$$- \int_\Omega J_{h+k}(u_{0n} - T_k(\psi_j^+))^+ \geq$$

$$\geq \int_\Omega J_{h+k}(u_n - \eta_{\nu j}(u^+))^+(T) - \int_\Omega J_{h+k}(u_{0n} - T_k(\psi_j^+))^+,$$

having in mind $|\eta_{\nu j}(u^+)| \leq k$ and $(T_k(u) - \eta_{\nu j}(u^+))T_{h+k}(u - \eta_{\nu j}(u))^+ \geq 0$.

In order to handle with the last term in (42), we have to approximate the functions u_n. We begin by splitting up $(u_n)_t = \beta_{1n} + \beta_{2n}$ where $\beta_{1n} \in L^2\big(0,T; H^{-1}(\Omega)\big)$ and $\beta_{2n} \in L^1(Q_T)$. Applying [8, Lemma 2.2] to each $u_n - u_{0n}$ and then adding u_{0n} to the obtained sequence, we may consider a sequence $(z_{n\sigma})_{\sigma=1}^\infty$ in $L^2([0,T]; H_0^1(\Omega))$ such that $z_{n\sigma}(0) = u_{0n}$, and $z_{n\sigma} \to u_n$ in $L^2(0,T; H_0^1(\Omega))$ when σ tends to infinity. Moreover, $(z_{n\sigma})_t = \beta_{1n\sigma} + \beta_{2n\sigma}$, where $\beta_{1n\sigma} \in L^2(0,T; H^{-1}(\Omega))$ and $\beta_{2n\sigma} \in L^1(Q_T)$ satisfy the following convergences as σ goes to infinity:

$$\begin{cases} \beta_{1n\sigma} \to \beta_{1n} & \text{in} \quad L^2(0,T; H^{-1}(\Omega)) \\ \beta_{2n\sigma} \to \beta_{2n} & \text{in} \quad L^1(Q_T); \end{cases} \tag{43}$$

in other words, $\lim_{\sigma\to\infty} (z_{n\sigma})_t = (u_n)_t$ in $L^2(0,T; H^{-1}(\Omega)) + L^1(Q_T)$.

$$\int_0^T \langle T_{h-k}(u_n^+ - T_k(u_n^+)), (u_n)_t\rangle = \lim_{\sigma\to\infty} \int_{Q_T} T_{h-k}(z_{n\sigma}^+ - T_k(z_{n\sigma}^+)), (z_{n\sigma})_t =$$

$$= \lim_{\sigma\to\infty} \int_{Q_T} T_{h-k}(G_k(z_{n\sigma}^+)) G_k(z_{n\sigma})_t =$$

$$= \lim_{\sigma\to\infty} \int_\Omega J_{h-k}(G_k(z_{n\sigma}^+(T)) - \int_\Omega J_{h-k}(G_k(z_{n\sigma}^+(0)) =$$

$$= \int_\Omega J_{h-k}(G_k(u_n^+(T)) - \int_\Omega J_{h-k}(G_k(u_{0n}^+).$$

Therefore, (42) becomes

$$\int_0^T \langle w_n, (u_n)_t \rangle \geq \int_\Omega J_{h+k}(u_n - \eta_{\nu j}(u^+))^+(T) - \int_\Omega J_{h+k}(u_{0n} - T_k(\psi_j^+))^+ -$$

$$- \int_\Omega J_{h-k}(G_k(u_n^+(T)) + \int_\Omega J_{h-k}(G_k(u_{0n}^+).$$

As $|\eta_{\nu j}(u^+)| \leq k$ it follows that

$$J_{h+k}(u_n - \eta_{\nu j}(u^+))^+(T) - J_{h-k}(u_n(T) - k)\chi_{\{u_n(T)\geq k\}} \geq 0,$$

so

$$\int_0^T \langle w_n, (u_n)_t \rangle \geq - \int_\Omega J_{h+k}(u_{0n} - T_k(\psi_j^+))^+ + \int_{\{u_{0n}\geq k\}} J_{h-k}(u_{0n} - k) =$$

$$= - \int_\Omega J_{h+k}(u_{0n} - T_k(\psi_j^+))^+ + \int_{\{u_{0n}\geq k\}} J_{h-k}(u_{0n} - k).$$

Taking limits as n goes to infinity, it follows that

$$\int_0^T \langle w_n, (u_n)_t \rangle \geq \omega(n) - \int_\Omega J_{h+k}(u_0 - T_k(\psi_j^+))^+ +$$

$$+ \int_{\{u_0\geq k\}} J_{h-k}(u_0 - k).$$

Taking now limits as ν and j tend to infinity,

$$\int_0^T \langle w_n, (u_n)_t \rangle \geq \omega(n, \nu, j) - \int_\Omega J_{h+k}(u - T_k(u_0))^+ +$$

$$+ \int_{\{u_0\geq k\}} J_{h-k}(u_0 - k) =$$

$$= \omega(n, \nu, j) + \int_{\{u_0\geq k\}} \left(J_{h-k}(u_0 - k) - J_{h+k}(u_0 - k)^+\right).$$

Since

$$-2k \int_{\{u_0\geq h\}} u_0 \leq \int_{\{u_0\geq k\}} \left(J_{h-k}(u_0 - k) - J_{h+k}(u_0 - k)^+\right) \leq 0,$$

it yields

$$\int_0^T \langle w_n, (u_n)_t \rangle \geq \omega(n, \nu, j, h).$$

Then, by (41) and the above inequality,

$$\int_{Q_T} \nabla u_n \cdot \nabla w_n \leq \omega(n, \nu, j, h). \tag{44}$$

Now, since $\nabla w_n = 0$ when $u_n \geq h + 4k$ and $w_n = w_n \chi_{\{u_n\geq 0\}}$, it follows that

$$\begin{aligned} &\int_{Q_T} \nabla u_n \cdot \nabla w_n = \int_{Q_T} \nabla T_{h+4k}(u_n^+) \cdot \nabla w_n = \\ &= \int_{Q_T} \nabla T_k(u_n^+) \cdot \nabla (T_k(u_n^+) - \eta_{\nu j}(u))^+ + \int_{\{u_n\geq k\}} \nabla T_{h+4k}(u_n^+) \cdot \nabla w_n, \end{aligned} \tag{45}$$

where

$\int_{\{u_n\geq k\}} \nabla T_{h+4k}(u_n^+)\cdot\nabla w_n =$

$= \int_{\{u_n\geq k\}\cap\{u_n-T_h(u_n)-\eta_{\nu j}\leq k\}} \nabla T_{h+4k}(u_n^+)\cdot\nabla(u_n - T_h(u_n) + k - \eta_{\nu j}(u^+)) \geq$

$\geq -\int_{\{u_n\geq k\}\cap\{u_n-T_h(u_n)-\eta_{\nu j}\leq k\}} \nabla T_{h+4k}(u_n^+)\cdot\nabla\eta_{\nu j}(u^+) \geq$

$\geq -\int_{\{u_n\geq k\}} |\nabla T_{h+4k}(u_n^+)||\nabla\eta_{\nu j}(u^+)| \geq$

$\geq -\int_{\{u_n\geq k\}} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+)| - \int_{\{u_n\geq k\}} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+) - \nabla\eta_{\nu j}(u^+)| \geq$

$\geq -\int_{\{u_n\geq k\}} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+)| - \int_{Q_T} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+) - \nabla\eta_{\nu j}(u^+)|,$

with

$$\lim_{n\to\infty}\int_{\{u_n\geq k\}} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+)| = 0,$$

and

$$\lim_{\nu\to\infty}\lim_{n\to\infty}\int_{Q_T} |\nabla T_{h+4k}(u_n^+)||\nabla T_k(u^+) - \nabla\eta_{\nu j}(u^+)| = 0,$$

so that (44) and (45) imply

$$\int_{Q_T} \nabla T_k(u_n^+)\cdot\nabla(T_k(u_n^+) - \eta_{\nu j}(u^+))^+ \leq \omega(n,\nu,j,h).$$

On the other hand, it is easy to see that

$$\lim_{\nu\to\infty}\lim_{n\to\infty}\int_{Q_T} \nabla\eta_{\nu j}(u^+)\cdot\nabla(T_k(u_n^+) - \eta_{\nu j}(u^+))^+ = 0.$$

Therefore,

$$\int_{Q_T} |\nabla(T_k(u_n^+) - \eta_{\nu j}(u^+))^+|^2 \leq \omega(n,\nu,j,h).$$

and claim 1 is proved.

Claim 2:

$$\int_{Q_T} |\nabla T_k(u_n^-)|^2\eta_{\nu j}(u^+) \leq \omega(n,\nu,j,h)$$

Proof: We multiply problem (8) by

$$\theta_n = T_{k^2}\big(-u_n^- + T_h(u_n^-) - T_k(u_n^-)\eta_{\nu j}(u^+)\big),$$

integrate and work as in the above claim to deduce that

$$\int_0^T \langle\theta_n,(u_n)_t\rangle + \int_{Q_T} \nabla u_n\cdot\nabla\theta_n \leq \omega(n,\nu,h). \tag{46}$$

Let us next study the term $\int_0^T\langle\theta_n,(u_n)_t\rangle$. Recall the notation used in the above Claim, there exists a sequence $(z_{n\sigma})_{\sigma=1}^\infty$ in $L^2([0,T];H_0^1(\Omega))$ such that

$z_{n\sigma}(0) = u_{0n}$, $\lim_{\sigma\to\infty} z_{n\sigma} \to u_n$ in $L^2(0,T;H_0^1(\Omega))$ and $\lim_{\sigma\to\infty}(z_{n\sigma})_t = (u_n)_t$ in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$. Now, denote

$$\theta_{n\sigma} = T_{k^2}\big(- z_{n\sigma}^- + T_h(z_{n\sigma}^-) - T_k(z_{n\sigma}^-)\eta_{\nu j}(u^+)\big),$$

which belongs to $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$. Then we have

$$\int_0^T \langle \theta_n, (u_n)_t\rangle = \lim_{\sigma\to\infty}\int_{Q_T}(z_{n\sigma})_t\theta_{n\sigma} = \lim_{\sigma\to\infty}(A_{n\sigma}+B_{n\sigma}), \tag{47}$$

where $A_{n\sigma} = \int_{\{-h<z_{n\sigma}<0\}}(z_{n\sigma})_t\theta_{n\sigma}$ and $B_{n\sigma} = \int_{\{z_{n\sigma}<-h\}}(z_{n\sigma})_t\theta_{n\sigma}$.

On the one hand, note that

$$A_{n\sigma} = \int_{Q_T}(-T_h(z_{n\sigma}^-))_t T_k(-T_h(z_{n\sigma}^-))\eta_{\nu j}(u^+) =$$

$$= \int_{Q_T}\Big(J_k(-T_h(z_{n\sigma}^-))\Big)_t\eta_{\nu j}(u^+) = -\int_{Q_T}J_k(-T_h(z_{n\sigma}^-))\Big(\eta_{\nu j}(u^+)\Big)_t +$$

$$+\int_\Omega J_k(-T_h(z_{n\sigma}^-(T)))\eta_{\nu j}(u^+)(T) - \int_\Omega J_k(-T_h(u_{0n}^-))T_k(\psi_j^+),$$

and so we deduce that

$$\lim_{n\to\infty}\lim_{\sigma\to\infty}A_{n\sigma} = -\int_{Q_T}J_k(-T_h(u^-))\Big(\eta_{\nu j}(u^+)\Big)_t +$$

$$+\int_\Omega J_k(-T_h(u^-(T)))\eta_{\nu j}(u^+)(T) - \int_\Omega J_k(-T_h(u_0^-))T_k(\psi_j^+).$$

Since

$$\int_{Q_T}J_k(-T_h(u^-))\Big(\eta_{\nu j}(u^+)\Big)_t = \nu\int_{Q_T}J_k(-T_h(u^-))\big(-\eta_{\nu j}(u^+)\big) \le 0,$$

it follows that

$$\lim_{n\to\infty}\lim_{\sigma\to\infty}A_{n\sigma} \ge \int_\Omega J_k(-T_h(u^-(T)))\eta_{\nu j}(u^+)(T) - \int_\Omega J_k(-T_h(u_0^-))T_k(\psi_j^+). \tag{48}$$

On the other hand,

$$B_{n\sigma} = \int_{Q_T}\big(-G_h(z_{n\sigma}^-)\big)_t T_{k^2}\big(-G_h(z_{n\sigma}^-) - k\eta_{\nu j}(u^+)\big) =$$

$$= \int_\Omega J_{k^2}\big(-G_h(z_{n\sigma}(T)^-) - k\eta_{\nu j}(u^+)(T)\big) - \int_\Omega J_{k^2}\big(-G_h(u_{0n}^-) - kT_k(\psi_j^+)\big)+$$

$$+k\int_{Q_T}\big(\eta_{\nu j}(u^+))\big)_t T_{k^2}\big(-G_h(z_{n\sigma}^-) - k\eta_{\nu j}(u^+)\big).$$

Thus,

$$\lim_{n\to\infty}\lim_{\sigma\to\infty}B_{n\sigma} =$$

$$= \int_\Omega J_{k^2}\big(-G_h(u(T)^-) - k\eta_{\nu j}(u^+)(T)\big) - \int_\Omega J_{k^2}\big(-G_h(u_0^-) - kT_k(\psi_j^+)\big)+ \tag{49}$$

$$+k\int_{Q_T}\big(\eta_{\nu j}(u^+))\big)_t T_{k^2}\big(-G_h(u^-) - k\eta_{\nu j}(u^+)\big).$$

We point out that the last term in the above expression is equal to

$$k\nu\int_{Q_T}\big(-\eta_{\nu j}(u^+))\big)T_{k^2}\big(-G_h(u^-)-k\eta_{\nu j}(u^+)\big)$$

and so $T_{k^2}\big(-G_h(u^-)-k\eta_{\nu j}(u^+)\big)\leq -k\eta_{\nu j}(u^+)$ implies

$$k\int_{Q_T}\big(\eta_{\nu j}(u^+))\big)_t T_{k^2}\big(-G_h(u^-)-k\eta_{\nu j}(u^+)\big)\geq -\frac{k^2}{2}\int_{Q_T}\big(\eta_{\nu j}(u^+)^2\big)_t.$$

Hence, by (49),

$$\begin{aligned}&\lim_{n\to\infty}\lim_{\sigma\to\infty}B_{n\sigma}\geq\int_\Omega J_{k^2}\big(-G_h(u(T)^-)-k\eta_{\nu j}(u^+)(T)\big)-\\&-\int_\Omega J_{k^2}\big(-G_h(u_0^-)-kT_k(\psi_j^+)\big)-\tfrac{k^2}{2}\int_\Omega\eta_{\nu j}(u^+)^2(T)+\tfrac{k^2}{2}\int_\Omega T_k(\psi_j^+)^2.\end{aligned}\tag{50}$$

Having in mind (48) and (50), it follows from (47) that

$$\begin{aligned}&\textstyle\int_0^T\langle\theta_n,(u_n)_t\rangle\geq\omega(n)+\\&+\textstyle\int_\Omega J_k(-T_h(u^-(T)))\eta_{\nu j}(u^+)(T)-\int_\Omega J_k(-T_h(u_0^-))T_k(\psi_j^+)+\\&+\textstyle\int_\Omega J_{k^2}\big(-G_h(u(T)^-)-k\eta_{\nu j}(u^+)(T)\big)-\int_\Omega J_{k^2}\big(-G_h(u_0^-)-kT_k(\psi_j^+)\big)-\\&-\tfrac{k^2}{2}\textstyle\int_\Omega\eta_{\nu j}(u^+)^2(T)+\tfrac{k^2}{2}\int_\Omega T_k(\psi_j^+)^2.\end{aligned}$$

Taking now $\nu\to\infty$ and then $j\to\infty$, we deduce that

$$\begin{aligned}&\textstyle\int_0^T\langle\theta_n,(u_n)_t\rangle\geq\omega(n,\nu,j)+\\&+\textstyle\int_\Omega J_{k^2}(-G_h(u^-(T))-kT_k(u^+)(T))-\int_\Omega J_{k^2}(-G_h(u_0^-)-kT_k(u_0^+))-\\&-\tfrac{k^2}{2}\textstyle\int_\Omega T_k(u^+(T))^2+\tfrac{k^2}{2}\int_\Omega T_k(u_0^+)^2.\end{aligned}$$

Letting h go to infinity, we have that

$$\begin{aligned}&\textstyle\int_0^T\langle\theta_n,(u_n)_t\rangle\geq\omega(n,\nu,j,h)+\\&+\textstyle\int_\Omega J_{k^2}(-kT_k(u^+)(T))-\int_\Omega J_{k^2}(-kT_k(u_0^+))-\\&-\tfrac{k^2}{2}\textstyle\int_\Omega T_k(u^+(T))^2+\tfrac{k^2}{2}\int_\Omega T_k(u_0^+)^2=\omega(n,\nu,j,h),\end{aligned}$$

because $J_{k^2}(-kT_k(u^+)(T))=\frac{k^2}{2}T_k(u^+(T))^2$ and $J_{k^2}(-kT_k(u_0^+))=\frac{k^2}{2}T_k(u_0^+)^2$.

Hence, from (46), we conclude that

$$\int_{Q_T}\nabla u_n\cdot\nabla\theta_n\leq\omega(n,\nu,j,h).\tag{51}$$

We next turn to study this term. It is straightforward that

$$\begin{aligned}&\textstyle\int_{Q_T}\nabla u_n\cdot\nabla\theta_n=\int_{\{-k<u_n<0\}}|\nabla u_n|^2\eta_{\nu j}(u^+)+\\&+\textstyle\int_{\{-k<u_n<0\}}u_n\nabla u_n\cdot\nabla\eta_{\nu j}(u^+)-k\int_{\{-h<u_n<-k\}}\nabla u_n\cdot\nabla\eta_{\nu j}(u^+)-\\&-\textstyle\int_{\{u_n<-h\}}\nabla u_n^-\cdot\nabla T_{k^2}\big(-u_n^-+h-k\eta_{\nu j}(u^+)\big).\end{aligned}\tag{52}$$

Observing that the first term in the right hand side is equal to $\int_{Q_T} |\nabla T_k(u_n^-)|^2 \eta_{\nu j}(u^+)$, to prove our claim all we have to see is that the other terms in (52) tend (in a suitable way) to 0.

We begin with the second and the third terms. Since $T_k(u_n)^- \rightharpoonup T_k(u)^-$ weakly in $L^2\big(0,T;H_0^1(\Omega)\big)$, we have that

$$\int_{\{-k<u_n<0\}} u_n \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) = \int_{Q_T} T_k(u_n)^- \nabla T_k(u_n)^- \cdot \nabla \eta_{\nu j}(u^+)$$

tends to $\int_{Q_T} T_k(u)^- \nabla T_k(u)^- \cdot \nabla T_k(u)^+ = 0$. Similarly,

$$-\int_{\{-h<u_n<-k\}} \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) = \int_{Q_T} \Big(\nabla T_h(u_n^-) - \nabla T_k(u_n^-)\Big) \cdot \nabla \eta_{\nu j}(u^+)$$

tends to $\int_{Q_T} \Big(\nabla T_h(u)^- - \nabla T_k(u)^-\Big) \cdot \nabla T_k(u)^+ = 0$. Consequently,

$$\int_{\{-k<u_n<0\}} u_n \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) - k \int_{\{-h<u_n<-k\}} \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) = \omega(n,\nu). \tag{53}$$

Next, in order to analyze the last term in (52), we use the following notation; we set $M = k^2 + h$,

$$E_n^+ = \{-M + k\eta_{\nu j}(u^+) < u_n < -h\} \cap \{u \geq 0\}$$

and

$$E_n^- = \{-M + k\eta_{\nu j}(u^+) < u_n < -h\} \cap \{u < 0\}.$$

Then

$$\begin{aligned}
&\textstyle\int_{\{u_n<-h\}} \nabla u_n^- \cdot \nabla T_{k^2}\big(-u_n^- + h - k\eta_{\nu j}(u^+)\big) = \\
&= \textstyle\int_{\{-k^2-h+k\eta_{\nu j}(u^+)<u_n<-h\}} \nabla u_n^- \cdot \Big(-\nabla u_n^- - k\nabla \eta_{\nu j}(u^+)\Big) \leq \\
&\leq -k \textstyle\int_{\{-k^2-h+k\eta_{\nu j}(u^+)<u_n<-h\}} \nabla u_n^- \cdot \nabla \eta_{\nu j}(u^+) = \\
&= -k \textstyle\int_{Q_T} \chi_{E_n^+} \nabla T_M(u_n)^- \cdot \nabla \eta_{\nu j}(u^+) - k \int_{Q_T} \chi_{E_n^-} \nabla T_M(u_n)^- \cdot \nabla \eta_{\nu j}(u^+).
\end{aligned} \tag{54}$$

Since $\chi_{E_n^+}(x,t) \to 0$ a.e. and $T_M(u_n)^- \rightharpoonup T_M(u)^-$ weakly in $L^2\big(0,T;H_0^1(\Omega)\big)$, it follows that

$$\lim_{n\to\infty} \int_{Q_T} \chi_{E_n^+} \nabla T_M(u_n)^- \cdot \nabla \eta_{\nu j}(u^+) = 0.$$

With respect to the last term in (54), we apply Cauchy-Schwarz' inequality to get

$$\begin{aligned}
&\Big| \textstyle\int_{Q_T} \chi_{E_n^-} \nabla T_M(u_n)^- \cdot \nabla \eta_{\nu j}(u^+) \Big| \leq \\
&\leq \Big(\textstyle\int_{Q_T} |\nabla T_M(u_n)^-|^2 \Big)^{1/2} \cdot \Big(\int_{E_n^-} |\nabla \eta_{\nu j}(u^+)|^2 \Big)^{1/2},
\end{aligned}$$

which is equal to $\omega(n,\nu)$ just noting the integrals $\int_{Q_T} |\nabla T_M(u_n)^-|^2$ are bounded by a constant only depending on M and

$$\lim_{\nu\to\infty}\lim_{n\to\infty}\int_{E_n^-} |\nabla \eta_{\nu j}(u^+)|^2 = \int_{\{-M+k\eta_{\nu j}(u^+)<u<-h\}} |\nabla T_k(u^+)|^2 = 0.$$

Going back to (54), it yields

$$\int_{\{u_n<-h\}} \nabla u_n^- \cdot \nabla T_{k^2}\big(-u_n^- + h - k\eta_{\nu j}(u^+)\big) \le \omega(n,\nu).$$

From that last inequality, taking into account (52) and (53), we obtain

$$\int_{Q_T} \nabla u_n \cdot \nabla \theta_n \ge \int_{Q_T} |\nabla T_k(u_n^-)|^2 \eta_{\nu j}(u^+) + \omega(n,\nu).$$

Therefore, claim 2 follows from (51).

Claim 3:

$$\int_{Q_T} \big|\nabla\big(T_k(u_n^+) - \eta_{\nu j}(u^+)\big)^-\big|^2 \le \omega(n,\nu,j,h)$$

Proof: Let $\varphi : I\!R \to I\!R$ be an increasing locally Lipschitz-continuous function such that $\varphi(0)=0$ and consider the following functions: $\Phi(s) = \int_0^s \varphi(\tau)^- \, d\tau$ and $S(s) = \frac{1}{k}\int_0^s (k-\tau^-)^+ \, d\tau$; note that

$$S(s) = \begin{cases} s, & \text{if } \; s \ge 0; \\ s + \frac{s^2}{2k}, & \text{if } \; -k \le s \le 0; \\ -\frac{k}{2}, & \text{if } \; s \le -k. \end{cases}$$

Observe that we may multiply problem (8) by the function

$$\xi_n = \Big[\varphi\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-\Big] S'(u_n)$$

and integrate to get

$$\int_0^T \langle \xi_n, (u_n)_t\rangle + \int_{Q_T} \nabla u_n \cdot \nabla \xi_n + \int_{Q_T} |u_n|^{\beta-2} u_n |\nabla u_n|^q \xi_n =$$

$$= \int_{Q_T} |u_n|^{\alpha-2} u_n |\nabla u_n|^p \xi_n.$$

Performing obvious manipulations, from the above equality and the following computations

$$
\begin{aligned}
&\int_{Q_T} \nabla u_n \cdot \nabla \xi_n = \int_{Q_T} \nabla u_n \cdot \nabla \big(S(u_n) - \eta_{\nu j}(u^+)\big)^- \varphi'\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- S'(u_n) - \\
&- \int_{Q_T} \nabla u_n \cdot \nabla S(u_n)^- \varphi'\big(S(u_n)\big)^- S'(u_n) + \\
&+ \int_{Q_T} |\nabla u_n|^2 \Big(\varphi\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-\Big) S''(u_n) = \\
&= \int_{\{0 \le u_n \le \eta_{\nu j}(u^+)\}} \nabla u_n \cdot \nabla \big(u_n - \eta_{\nu j}(u^+)\big)^- \varphi'\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- S'(u_n) + \\
&+ \int_{\{-k \le u_n \le 0\}} \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) \varphi'\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- S'(u_n) + \\
&+ \int_{\{-k \le u_n \le 0\}} \nabla u_n \cdot \nabla \big(S(u_n)\big) \Big[\varphi'\big(S(u_n)\big)^- - \varphi'\big(S(u_n) - \eta_{\nu j}(u^+)\big)^-\Big] S'(u_n) + \\
&+ \tfrac{1}{k} \int_{\{-k \le u_n \le 0\}} |\nabla u_n|^2 \Big(\varphi\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-\Big),
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
&\int_{Q_T} |\nabla \big(T_k(u_n^+) - \eta_{\nu j}(u^+)\big)^-|^2 \varphi'\big(u_n^+ - \eta_{\nu j}(u^+)\big)^- = \\
&= - \int_{\{0 \le u_n \le \eta_{\nu j}(u^+)\}} \nabla \big(u_n - \eta_{\nu j}(u^+)\big) \cdot \nabla \big(u_n^+ - \eta_{\nu j}(u^+)\big)^- \varphi'\big(u_n^+ - \eta_{\nu j}(u^+)\big)^- = \\
&= - \int_{Q_T} \nabla u_n \cdot \nabla \xi_n + \\
&+ \int_{\{-k \le u_n \le 0\}} \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) \varphi'\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- S'(u_n) + \\
&+ \int_{\{-k \le u_n \le 0\}} \nabla u_n \cdot \nabla \big(S(u_n)\big) \Big[\varphi'\big(S(u_n)\big)^- - \varphi'\big(S(u_n) - \eta_{\nu j}(u^+)\big)^-\Big] S'(u_n) + \\
&+ \tfrac{1}{k} \int_{\{-k \le u_n \le 0\}} |\nabla u_n|^2 \Big(\varphi\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-\Big) + \\
&+ \int_{Q_T} \nabla \eta_{\nu j}(u^+) \cdot \nabla \big(T_k(u_n^+) - \eta_{\nu j}(u^+)\big)^- \varphi'\big(u_n^+ - \eta_{\nu j}(u^+)\big) \le \\
&\le I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,
\end{aligned}
\tag{55}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^T \langle \xi_n, (u_n)_t \rangle, \\
I_2 &= \int_{Q_T} |u_n|^{\beta - 1} |\nabla u_n|^q |\xi_n|, \\
I_3 &= \int_{Q_T} |u_n|^{\alpha - 1} |\nabla u_n|^p |\xi_n|, \\
I_4 &= \int_{\{-k \le u_n \le 0\}} \nabla u_n \cdot \nabla \eta_{\nu j}(u^+) \varphi'\big(S(u_n) - \eta_{\nu j}(u^+)\big)^- S'(u_n),
\end{aligned}
$$

$$I_5 = \int_{\{-k\le u_n\le 0\}} \nabla u_n \cdot \nabla\big(S(u_n)\big)\Big[\varphi'\big(S(u_n)\big)^- - \varphi'\big(S(u_n)-\eta_{\nu j}(u^+)\big)^-\Big]S'(u_n),$$
$$I_6 = \frac{1}{k}\int_{\{-k\le u_n\le 0\}} |\nabla u_n|^2\Big(\varphi\big(S(u_n)-\eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-\Big),$$
$$I_7 = \int_{Q_T} \nabla\eta_{\nu j}(u^+)\cdot\nabla\big(T_k(u_n^+)-\eta_{\nu j}(u^+)\big)^-\varphi'\big(u_n^+-\eta_{\nu j}(u^+)\big).$$

We are going to study each of these terms; let us begin with the first one,

$$I_1 = \int_0^T\langle \xi_n,(u_n)_t\rangle = \int_0^T\langle\varphi\big(S(u_n)-\eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-, \big(S(u_n)\big)_t\rangle =$$
$$= \int_\Omega \Phi\Big(S(u_n(T))-\eta_{\nu j}(u^+)(T)\Big) - \int_\Omega\Phi\Big(S(u_{0n})-T_k(\psi_j^+)\Big)+$$
$$+\int_{Q_T}\big(\eta_{\nu j}(u^+)\big)_t\Big(\varphi\big(S(u_n)-\eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-\Big)$$
$$-\int_\Omega\Phi\Big(S(u_n(T))\Big)+\int_\Omega\Phi\Big(S(u_{0n})\Big).$$

Taking limits when n goes to infinity, we get

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T\langle\xi_n,(u_n)_t\rangle = \\
&= \int_\Omega\Phi\Big(S(u(T))-\eta_{\nu j}(u^+)(T)\Big)-\int_\Omega\Phi\Big(S(u(T))\Big)+ \\
&+\int_\Omega\Phi\Big(S(u_0)\Big)-\int_\Omega\Phi\Big(S(u_0)-T_k(\psi_j^+)\Big)+ \\
&+\int_{Q_T}\big(\eta_{\nu j}(u^+)\big)_t\Big(\varphi\big(S(u)-\eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u)\big)^-\Big).
\end{aligned}\tag{56}$$

Since Φ is increasing and $\eta_{\nu j}(u^+)\ge 0$, it yields

$$\Phi\big(S(u(T))-\eta_{\nu j}(u^+)(T)\big)-\Phi\big(S(u(T))\big)\le 0.$$

On the other hand,

$$\lim_{j\to\infty}\int_\Omega\Phi\big(S(u_0)\big)-\int_\Omega\Phi\big(S(u_0)-T_k(\psi_j^+)\big) =$$
$$\int_\Omega\Big(\Phi\big(S(u_0)\big)-\Phi\big(S(u_0)-T_k(u_0^+)\big)\Big) =$$
$$= \int_{\{u_0\ge 0\}}\Big(\Phi\big(S(u_0)\big)-\Phi\big(S(u_0)-T_k(u_0^+)\big)\Big)+$$
$$+\int_{\{u_0\le 0\}}\Big(\Phi\big(S(u_0)\big)-\Phi\big(S(u_0)-T_k(u_0^+)\big)\Big)=0.$$

Finally, since $S(u)\le\eta_{\nu j}(u^+)$ implies $T_k(u^+)\le\eta_{\nu j}(u^+)$, we get

$$\int_{Q_T}\big(\eta_{\nu j}(u^+)\big)_t\varphi\big(S(u)-\eta_{\nu j}(u^+)\big)^- =$$
$$\nu\int_{Q_T}\big(T_k(u^+)-\eta_{\nu j}(u^+)\big)\varphi\big(S(u)-\eta_{\nu j}(u^+)\big)^-\le 0.$$

Hence, we deduce

$$I_1 \le \omega(n,\nu,j). \tag{57}$$

Let us turn to analyze I_2.

$$\begin{aligned} I_2 &\le \tfrac{q}{2}\int_{Q_T} |\nabla u_n|^2 \varphi(u_n-\eta_{\nu j}(u))^- + \\ &+\tfrac{2-q}{2}\int_{Q_T} (T_k u_n)^{\frac{2(\beta-1)}{2-q}} \varphi(u_n-\eta_{\nu j}(u))^- \le \\ &\le \tfrac{q}{2}\int_{Q_T} \nabla u_n\cdot\nabla(u_n-\eta_{\nu j}(u))\varphi(u_n-\eta_{\nu j}(u))^- + \\ &+\tfrac{q}{2}\int_{Q_T} \nabla u_n\cdot\nabla\eta_{\nu j}(u)\varphi(u_n-\eta_{\nu j}(u))^- + \\ &+\tfrac{2-q}{2}k^{\frac{2(\beta-1)}{2-q}}\int_{Q_T} \varphi(u_n-\eta_{\nu j}(u))^- \le \\ &\le \tfrac{q}{2}\int_{Q_T} |\nabla(u_n-\eta_{\nu j}(u))|^2\varphi(u_n-\eta_{\nu j}(u))^- + \\ &+\tfrac{q}{2}\int_{Q_T} \nabla\eta_{\nu j}(u)\cdot\nabla(u_n-\eta_{\nu j}(u))\varphi(u_n-\eta_{\nu j}(u))^- + \\ &+\tfrac{q}{2}\int_{Q_T} \nabla u_n\cdot\nabla\eta_{\nu j}(u)\varphi(u_n-\eta_{\nu j}(u))^- + \omega(n,\nu), \end{aligned}$$

which implies

$$I_2 \le \frac{q}{2}\int_{Q_T} |\nabla(u_n-\eta_{\nu j}(u))|^2\varphi(u_n-\eta_{\nu j}(u))^- + \omega(n,\nu). \tag{58}$$

Similarly,

$$I_3 \le \frac{p}{2}\int_{Q_T} |\nabla(u_n-\eta_{\nu j}(u))|^2\varphi(u_n-\eta_{\nu j}(u))^- + \omega(n,\nu). \tag{59}$$

The study of the term I_4 is very easy since

$$\lim_{\nu\to\infty}\lim_{n\to\infty} I_4 = -\int_{Q_T} \nabla T_k(u^-)\cdot\nabla T_k(u^+)\varphi'\big(S(u)-T_k(u^+)\big)^- S'(u) = 0. \tag{60}$$

With respect to the following term, on account of

$$\varphi'\big(S(u_n)\big)^- \le \varphi'\big(S(u_n)-\eta_{\nu j}(u^+)\big)^-$$

on the set $\{-k<u_n<0\}$, we have

$$I_5 = \int_{\{-k<u_n<0\}} |\nabla u_n|^2\Big[\varphi'\big(S(u_n)\big)^- - \varphi'\big(S(u_n)-\eta_{\nu j}(u^+)\big)^-\Big]S'(u_n)^2 \le 0. \tag{61}$$

We now pass to handle with I_6. Having in mind that the function φ is Lipschitz-continuous on $[-k,0]$, we get that

$$\Big|\varphi\big(S(u_n)-\eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-\Big| \le M\eta_{\nu j}(u^+).$$

Thus,

$$I_6 = \tfrac{1}{k}\int_{\{-k\le u_n\le 0\}} |\nabla u_n|^2\Big(\varphi\big(S(u_n)-\eta_{\nu j}(u^+)\big)^- - \varphi\big(S(u_n)\big)^-\Big) \le \\ \le \tfrac{M}{k}\int_{Q_T} |\nabla T_k(u_n)|^2 \eta_{\nu j}(u^+)\big)^- \le \omega(n,\nu,j,h), \tag{62}$$

by claim 2.

Finally, it is straightforward that

$$\lim_{\nu\to\infty}\lim_{n\to\infty} I_7 = \int_{Q_T} \nabla T_k(u^+)\cdot\nabla\big(T_k(u^+)-T_k(u^+)\big)^-\varphi'\big(u^+-T_k(u^+)\big) = 0. \tag{63}$$

Therefore, having in mind (55), the estimates (57), (58), (59), (60), (61), (62) and (63) imply

$$\int_{Q_T} |\nabla\big(T_k(u_n^+)-\eta_{\nu j}(u^+)\big)^-|^2\Big[\varphi'\big(u_n^+-\eta_{\nu j}(u^+)\big)^- - \tfrac{q+p}{2}\varphi'\big(u_n^+-\eta_{\nu j}(u^+)\big)^-\Big] \le \\ \le \omega(n,\nu,j,h).$$

Choosing $\varphi(s) = se^{\lambda s^2}$ with λ large such that $\varphi'(s) - \frac{q+p}{2}\varphi(s) \ge \frac{1}{2}$, it follows that

$$\int_{Q_T} |\nabla\big(T_k(u_n^+)-\nabla\eta_{\nu j}(u^+)\big)^-|^2 \le \omega(n,\nu,j,h)$$

and so claim 3 is proved.

Now, it follows from claim 1 and claim 3 that

$$\int_{Q_T} \left|\nabla\big(T_k(u_n^+)-\eta_{\nu j}(u^+)\right|^2 \le \omega(n,\nu,j,h),$$

so that, by the convergence $\eta_{\nu j}(u^+)\to T_k(u^+)$ in $L^2\big(0,T;H_0^1(\Omega)\big)$, we obtain

$$\lim_{n\to\infty} \nabla T_k(u_n^+) = \nabla T_k(u^+) \quad \text{in } L^2(Q_T). \tag{64}$$

The corresponding result for the negative part of truncations may be obtained by similar arguments, or by using the fact that $-u_n$ is a solution of

$$\begin{cases} v_t - \Delta v + v|v|^{\beta-2}|\nabla v|^q = |v|^{\alpha-2}v|\nabla v|^p & \text{in } Q_T;\\ v = 0 & \text{on } S_T;\\ v(x,0) = -u_{0n}(x) & \text{in } \Omega \end{cases}$$

and so we deduce from

$$\lim_{n\to\infty} \nabla T_k(-u_n)^+ = \nabla T_k(-u)^+ \quad \text{in } L^2(Q_T)$$

that

$$\lim_{n\to\infty} \nabla T_k(u_n^-) = \nabla T_k(u^-) \quad \text{in } L^2(Q_T). \tag{65}$$

Therefore, from (64) and (65), we conclude that (37) holds true.

3.- Convergence of gradient terms in $L^1(Q_T)$

Our aim in this step is to show (12) and (13); as a consequence, we also prove (10).

We begin with the proof of (12); since almost everywhere convergence is guaranteed by (34) and (38), on account of Vitali's convergence theorem, we only need to show that the sequence $\Big(|u_n|^{\beta-1}|\nabla u_n|^q\Big)_{n=1}^{\infty}$ is equi-integrable. This fact is a consequence of

$$\lim_{h\to\infty}\int_{\{|u_n|\ge h\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q=0 \qquad \text{uniformly on} \quad n\in I\!N. \tag{66}$$

To see (66), we take $T_k\big(u_n-T_h(u_n)\big)$ as test function obtaining

$$\int_{\{|u_n(T)|\ge h\}\cap\Omega}J_k(|u_n(T)|-h)+\int_{\{h<|u_n|<k+h\}\cap Q_T}|\nabla u_n|^2+$$

$$+\int_{Q_T}|u_n|^{\beta-2}u_nT_k(u_n-T_h(u_n))|\nabla u_n|^q\le$$

$$\le\int_{Q_T}|u_n|^{\alpha-2}u_nT_k(u_n-T_h(u_n))|\nabla u_n|^p+\int_{\{|u_{0n}|\ge h\}\cap\Omega}J_k(|u_{0n}|-h).$$

Disregarding non negative terms, dividing by k and letting k goes to 0, it yields

$$\int_{\{|u_n|\ge h\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q\le\int_{\{|u_n|\ge h\}\cap Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p+\int_{\{|u_{0n}|\ge h\}\cap\Omega}(|u_{0n}|-h)\le$$

$$\le\int_{\{|u_n|\ge h\}\cap Q_T}|u_n|^{\alpha-1}|\nabla u_n|^p+\int_{\{|u_{0n}|\ge h\}\cap\Omega}|u_{0n}|.$$

Applying Young's inequality we get

$$\int_{\{|u_n|\ge h\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q\le\int_{\{|u_n|\ge h\}\cap Q_T}|u_n|^{\frac{\alpha q-\beta p}{q-p}-1}+C\int_{\{|u_{0n}|\ge h\}\cap\Omega}|u_{0n}|.$$

When $(\alpha-1)q>(\beta-1)p$, it follows from (9) and (36) that (66) holds. In the other cases, when $(\alpha-1)q\le(\beta-1)p$, we also obtain it in a straightforward way.

Now we are ready to see that the sequence $\Big(|u_n|^{\beta-1}|\nabla u_n|^q\Big)_{n=1}^{\infty}$ is equi-integrable. Indeed, if E is a measurable subset of Q_T, then

$$\begin{aligned}&\textstyle\int_E|u_n|^{\beta-1}|\nabla u_n|^q=\int_{E\cap\{|u_n|<k\}}|u_n|^{\beta-1}|\nabla u_n|^q+\int_{E\cap\{|u_n|\ge k\}}|u_n|^{\beta-1}|\nabla u_n|^q\le\\&\textstyle\le k^{\beta-1}\int_E|\nabla T_k(u_n)|^q+\int_{\{|u_n|\ge k\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q.\end{aligned}\tag{67}$$

Let $\epsilon>0$. By (66), we may choose $k>0$ such that

$$\int_{\{|u_n|\ge k\}\cap Q_T}|u_n|^{\beta-1}|\nabla u_n|^q<\frac{\epsilon}{2}$$

for all $n\in I\!N$. Fixed $k>0$, as a consequence of (37), we have that the sequence $\Big(|\nabla T_ku_n|^q\Big)_{n=1}^{\infty}$ is equi-integrable. So we may find $\delta>0$ such that $|E|<\delta$ implies

$$\int_E|\nabla T_k(u_n)|^q<\frac{\epsilon}{2k^{\beta-1}}$$

for all $n \in I\!N$. Hence, it follows from (67) that $|E| < \delta$ implies $\int_E |u_n|^{\beta-1}|\nabla u_n|^q < \epsilon$ for all $n \in I\!N$.

In order to see (13), apply Young's inequality to obtain

$$|u_n|^{\alpha-1}|\nabla u_n|^p \le |\nabla T_1 u_n|^p + \frac{p}{q}|u_n|^{\frac{\alpha q-\beta p}{q-p}-1}\chi_{\{|u_n|>1\}} + \frac{q-p}{q}|u_n|^{\beta-1}|\nabla u_n|^q,$$

distinguish the cases $(\alpha-1)q > (\beta-1)p$ and $(\alpha-1)q \le (\beta-1)p$, and then use the convergences (34), (36), (37), (38) and (12).

Finally, we prove (10) by showing that

$$|\nabla u_n| \to |\nabla u| \qquad \text{in} \qquad L^q(Q_T).$$

To do that we only need to apply Vitali's theorem again. The pointwise convergence follows from (38), while the equi-integrability is a consequence of (11), (12) and the following inequality

$$\begin{aligned}\int_E |\nabla u_n|^q &= \int_E |\nabla T_1(u_n)|^q + \int_E \big|\nabla\big(u_n - T_1(u_n)\big)\big|^q \le \\ &\le |E|^{(2-q)/2}\Big(\int_E |\nabla T_1(u_n)|^2\Big)^{q/2} + \int_E |u_n|^{\beta-1}|\nabla(u_n)|^q.\end{aligned}$$

4.- Convergence in $C\big([0,T];L^1(\Omega)\big)$

In this step we prove (14). To do this fix $t \in [0,T]$, and $m,n \in I\!N$. Take u_m as test function in the generalized formulation of (8) corresponding to u_n, and u_n in that of u_m; adding up both identities we deduce that

$$\begin{gathered}\int_\Omega J_k\big(u_n(t)-u_m(t)\big) + \int_{Q_t}\nabla(u_n-u_m)\cdot\nabla T_k(u_n-u_m) + \\ + \int_{Q_t}\Big(|u_n|^{\beta-2}u_n|\nabla u_n|^q - |u_m|^{\beta-2}u_m|\nabla u_m|^q\Big)T_k(u_n-u_m) = \\ = \int_{Q_t}\Big(|u_n|^{\alpha-2}u_n|\nabla u_n|^p - |u_m|^{\alpha-2}u_m|\nabla u_m|^{p_m}\Big)T_k(u_n-u_m) + \int_\Omega J_k\big(u_{0n}-u_{0m}\big).\end{gathered}$$

From here, we obtain a suitable inequality by taking into account that for every $r \in I\!R$, $J_k(r)/k \uparrow |r|$ as $k \downarrow 0$. Indeed, we get, disregarding the nonnegative second term, that

$$\begin{gathered}\int_\Omega J_k\big(u_n(t)-u_m(t)\big) \le k\int_{Q_T}\Big||u_n|^{\beta-2}u_n|\nabla u_n|^q - |u_m|^{\beta-2}u_m|\nabla u_m|^q\Big| + \\ +k\int_{Q_T}\Big||u_n|^{\alpha-2}u_n|\nabla u_n|^p - |u_m|^{\alpha-2}u_m|\nabla u_m|^{p_m}\Big| + k\int_\Omega |u_{0n}-u_{0m}|.\end{gathered}$$

Next, dividing this inequality by k and letting k go to 0 we obtain

$$\begin{gathered}\int_\Omega \big|u_n(t)-u_m(t)\big| \le \int_{Q_T}\Big||u_n|^{\beta-2}u_n|\nabla u_n|^q - |u_m|^{\beta-2}u_m|\nabla u_m|^q\Big| + \\ + \int_{Q_T}\Big||u_n|^{\alpha-2}u_n|\nabla u_n|^p - |u_m|^{\alpha-2}u_m|\nabla u_m|^{p_m}\Big| + \int_\Omega |u_{0n}-u_{0m}|.\end{gathered}$$

Hence,

$$\sup_{t\in[0,T]}\int_\Omega \big|u_n(t)-u_m(t)\big| \le \int_{Q_T}\Big||u_n|^{\alpha-2}u_n|\nabla u_n|^q-|u_m|^{\alpha-2}u_m|\nabla u_m|^q\Big|+$$

$$+\int_{Q_T}\Big||u_n|^{\alpha-2}u_n|\nabla u_n|^p-|u_m|^{\alpha-2}u_m|\nabla u_m|^{p_m}\Big|+\int_\Omega|u_{0n}-u_{0m}|.$$

Thus, it follows from (9), (12) and (13), that $(u_n)_{n=1}^\infty$ is a Cauchy sequence in $C\big([0,T];L^1(\Omega)\big)$ and consequently (14) holds.

5.- u is a generalized solution

To finish the proof, we consider $\phi\in L^2\big(0,T;H_0^1(\Omega)\big)\cap L^\infty(Q_T)$ such that $\phi_t\in L^2\big(0,T;H^{-1}(\Omega)\big)+L^1(Q_T)$. Taking ϕ as test function in the approximating problem (8) and letting n go to ∞, having in mind (11), (12), (13) and (14), we deduce the generalized formulation of problem (1) and so the proof of Theorem 3.1 is concluded.

Theorem 3.2 *Assume that $\alpha,\beta\ge 1$, $1\le q\le 2$, $p+\alpha<q+\beta$, $0\le p<q$. Then, for every $u_0\in L^1(\Omega)$, there exists a generalized solution of problem (1). Moreover, there exists a nonnegative generalized solution when $u_0\ge 0$, and if $u_0\in L^\infty(\Omega)$, then there exists a generalized solution such that $u\in L^2\big(0,T;H_0^1(\Omega)\big)\cap L^\infty(Q_T)$.*

Proof: We will prove this result using the previous Theorem. To this end, take an approximating sequence $u_{0n}\in L^\infty(\Omega)$ which converges to u_0 in $L^1(\Omega)$ and consider the corresponding problems with these initial data. Next, we will apply [8] to solve these approximating problems; so that, we need supersolutions and subsolutions of them. Since Ω is bounded, we have $R>0$ such that $|x_1|\le R-1$ for all $x\in\Omega$. Thus, fixed $n\in I\!N$, there is $K>0$ such that the function defined by $u^*(x)=K(x_1+R)$ is a supersolution of our approximating problem; indeed,

$$(u^*)_t-\Delta u^*+|u^*|^{\beta-2}u^*|\nabla u^*|^q-|u^*|^{\alpha-2}u^*|\nabla u^*|^p=$$

$$=K^{p+\alpha-1}(x_1+R)\Big(K^{(q+\beta)-(p+\alpha)}(|x_1|+R)^{\beta-2}-(|x_1|+R)^{\alpha-2}\Big)\ge 0 \qquad \text{in } Q,$$

$$(u^*)(x,t)\ge K>0 \qquad \text{on } S,$$

$$u^*(x,0)\ge K\ge \|u_{0n}\|_\infty \qquad \text{in } \Omega,$$

for K big enough. Likewise, the function defined by $u_*(x)=-K(x_1+R)$ is a subsolution.

Hence, by [8], we get a bounded distributional solution u_n of each approximating problem. Since $u_n\in L^2\big(0,T;H_0^1(\Omega)\big)\cap L^\infty(Q_T)$, $(u_n)_t\in L^2\big(0,T;H^{-1}(\Omega)\big)+L^1(Q_T)$, and $|u_n|^{\beta-1}|\nabla u_n|^q$ and $|u_n|^{\alpha-1}|\nabla u_n|^p$ belong to $L^1(Q_T)$; by Proposition 2.2 they are generalized solutions. Moreover, these solutions are nonnegative if $u_{0n}\ge 0$. Now, by Theorem 3.1, we obtain a generalized solution of our problem.

4 Example

If our hypothesis $\alpha + p < \beta + q$ is changed by $\alpha + p > \beta + q$, when $p = 0$, it is known that the solution, with $u_0 \in L^\infty(\Omega)$, blows up in finite time. Nevertheless, this fact can not occur in the case $p > 0$ for initial datum $u_0 \in L^\infty(\Omega)$, since then $u^*(x,t) = \|u_0\|_\infty$ is a supersolution and $u_*(x,t) = -\|u_0\|_\infty$ is a subsolution of our problem, and consequently there is a global solution on account of the main result in [8].

In this last section, we will show that our condition $\alpha + p < \beta + q$ is not arbitrary. In fact, we are going to construct an one dimensional example, for the parameters $p = q = 1$ and $\alpha > \beta \geq 2$, where our stability result does not work. So, in this case, we are not able to deduce an existence result.

Let us consider the following problem,

$$\begin{cases} u_t - u_{xx} + |u|^{\beta-2}u|u_x| = |u|^{\alpha-2}u|u_x|, & \text{in } Q :=]-1,1[\times]0,T[; \\ u(x,t) = 0, & \text{on } S; \\ u(x,0) = |x|^{-\gamma} - 1, & \text{in }]-1,1[; \end{cases} \tag{68}$$

where $0 < \gamma < 1$. We are going to see that, for a suitable γ, there exists a sequence of approximate solutions for which our stability result does not apply.

Let

$$L(u) = u_t - u_{xx} + u^{\beta-1}|u_x| - u^{\alpha-1}|u_x|$$

and let

$$u(x,t) = e^{-k^\delta t} h(|x|)$$

where $k > 1$, $\delta > 0$ and

$$h(x) = \begin{cases} -\frac{\gamma}{2} k^{\frac{\gamma+2}{\gamma}} x^2 + \frac{\gamma+2}{2} k - 1 & \text{if } 0 \leq x < k^{-\frac{1}{\gamma}}; \\ x^{-\gamma} - 1 & \text{if } k^{-\frac{1}{\gamma}} \leq x \leq 1. \end{cases}$$

We remark that $h(|x|)$ belongs to C^1 in $[-1,1]$.

Let us see that choosing $\delta > \max\{\frac{2}{\gamma}, \beta - 1 + \frac{1}{\gamma}\}$ and k large enough, it yields

$$L(u) \leq 0 \quad \text{pointwise in } [-1,1] \times [0,T]. \tag{69}$$

Indeed, on the one hand, in $[0, k^{-\frac{1}{\gamma}}] \times [0,T]$,

$$L(u) \leq -k^\delta e^{-k^\delta t} h(x) + e^{-k^\delta t} \gamma k^{\frac{\gamma+2}{\gamma}} + e^{-k^\delta t \beta} h^{\beta-1}(x) \gamma k^{\frac{\gamma+2}{\gamma}} x \leq 0,$$

since $-\frac{1}{2} k^\delta e^{-k^\delta t} h(x) + e^{-k^\delta t} \gamma k^{\frac{\gamma+2}{\gamma}} \leq 0$ and $-\frac{1}{2} k^\delta e^{-k^\delta t} h(x) + e^{-k^\delta t \beta} h^{\beta-1}(x) \gamma k^{\frac{\gamma+2}{\gamma}} x \leq 0$. On the other hand, in $[k^{-\frac{1}{\gamma}}, 1] \times [0,T]$,

$$L(u) \leq -k^\delta e^{-k^\delta t} h(x) + e^{-k^\delta t \beta} h^{\beta-1}(x) \gamma x^{-\gamma-1} \leq 0.$$

Finally, by a symmetric argument, we conclude that (69) holds true. Therefore, we get that u is a subsolution of problem

$$\begin{cases} u_t - u_{xx} + |u|^{\beta-2}u|u_x| = |u|^{\alpha-2}u|u_x|, & \text{in } Q :=]-1,1[\times]0,T[; \\ u(x,t) = 0, & \text{on } S; \\ u(x,0) = h(|x|), & \text{in }]-1,1[. \end{cases} \tag{70}$$

A supersolution of this problem can be found in a straightforward way, only a constant function equal to $\|h(|x|)\|_\infty$ is needed. By [8], then a bounded distributional solution v of the above problem exists, with $u \leq v \leq \|h(|x|)\|_\infty$.

Since $h(|x|) \to |x|^{-\gamma} - 1$ in L^1 as $k \to \infty$, if Theorem (3.1) holds for our parameters, then there exists a solution w of (68) such that

$$\int_0^T \int_{-1}^1 v^{\alpha-1}|v_x|^p \to \int_0^T \int_{-1}^1 w^{\alpha-1}|w_x|^p \quad \text{as } k \to \infty.$$

Now, it follows from $0 \leq u \leq v$ that

$$\int_0^T \int_0^1 v^{\alpha-1}|v_x| = \frac{1}{\alpha}\int_0^T \int_0^1 |(v^\alpha)_x| \geq \frac{1}{\alpha}\int_0^T \int_0^1 -(v^\alpha)_x$$

$$= \frac{1}{\alpha}\int_0^T v^\alpha(0) \geq \frac{1}{\alpha}\int_0^T u^\alpha(0) = \frac{1}{\alpha}\int_0^T \left(e^{-k^\delta t}h(0)\right)^\alpha$$

$$= \frac{1}{\alpha}\int_0^T e^{-k^\delta t\alpha}h^\alpha(0) = \frac{1}{\alpha^2}\left(\frac{\gamma+2}{2}k - 1\right)^\alpha k^{-\delta}(1 - e^{-k^\delta T\alpha}).$$

So that, if we may take $\delta < \alpha$, then this last term goes to infinity as k goes to infinity, which contradicts that the first term is bounded.

To get this δ, since $\alpha > \beta \geq 2$, consider $0 < \gamma < 1$ such that

$$\max\{\frac{2}{\gamma}, \beta - 1 + \frac{1}{\gamma}\} < \alpha$$

and consequently take $\delta > 0$ satisfying

$$\max\left\{\frac{2}{\gamma}, \beta - 1 + \frac{1}{\gamma}\right\} < \delta < \alpha.$$

5 Acknowledgements

We wish to thank Alessio Porretta for his useful suggestions. This article has partially been supported by the Spanish DGICYT, Proyecto PB98-1442.

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