Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term

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Abstract - Our aim in this article is to study the following nonlinear elliptic Dirichlet problem:

$$\begin{cases} -\operatorname{div}\big[a(x,u)\cdot\nabla u\big] + b(x,u,\nabla u) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 2$, $f \in L^m(\Omega)$. Under wide conditions on functions a and b, we prove that there exists a type of solution for this problem; this is a bounded weak solution for $m > N/2$, and an unbounded entropy solution for $N/2 > m \geq 2N/(N+2)$. Moreover, we show when this entropy solution is a weak one and when can be taken as test function in the weak formulation. We also study the summability of the solutions.

0.- Introduction and Assumptions

This paper is devoted to study existence and regularity of solutions of an elliptic problem whose model example is the following:

$$\begin{cases} -\mathrm{div}\big(\alpha(u)\nabla u\big) = \beta(u)|\nabla u|^2 + f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{0.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 2$, $f \in L^m(\Omega)$, m being bigger than $2N/(N+2)$, and α and β are two positive continuous functions satisfying $\alpha \notin L^1([0,+\infty[) \cup L^1(]-\infty,0])$ and $\beta/\alpha \in L^1(\mathbb{R})$. For instance, $\alpha(s) = 1/\sqrt{1+s^2}$ and $\beta(s) = 1/\sqrt{(1+s^2)^3}$, or $\alpha(s) = e^{|s|}$ and $\beta(s) = e^{|s|}/(1+s^2)$.

In some recent papers problems similar to (0.1) have been considered. Without lower order terms and $\alpha(s) = (1+|s|)^{-\theta}$, with $0 \le \theta < 1$, a priori estimates can be found in [1, 5, and 6], existence and regularity results in [5 and 6] and in a limit case in [24], and uniqueness is shown in [19] . On the other hand, problem (0.1) may be seen as the Euler equation of a real functional when $\alpha' = -\beta$; from this point of view has been studied in [12].

The existence of weak solutions of quasilinear elliptic equations with lower order terms having quadratic growth with respect to the gradient has been studied in some papers in the last years. We point out that, in those papers, the L^∞ estimates on the solutions are proved thanks to the presence of a zero order term ([10 and 11]) or thanks to a sign condition (easy situation). Also the previous existence results of unbounded solutions depends on a sign condition on the quadratic term ([3, 4, 7 and 9]). In [14] a limit case is studied. In [25] the existence of bounded solutions is obtained when the datum f is "small" and the assumptions on α and β are different. In this paper we no dot use the presence of a zero order term nor a sign condition in order to show the existence of bounded or unbounded solutions (it depends on the summability of the data). On the other hand, we shall use the assumptions (H4) and (H5) below.

Finally, when the function α is bounded but the equation has the same lower order term, this problem is dealt in [20], where existence and uniqueness for L^1-data are studied. Moreover the L^∞-estimate of section 2 below is applied in that paper to obtain a kind of uniqueness.

Two remarks concerning the difficulties in dealing with this problem are in order. First of all, observe that no bounds are assumed on function α, and so classical methods do not apply. Indeed, the operator defined by $-\mathrm{div}\big(\alpha(u)\nabla u\big)$ does not satisfy the Leray-Lions conditions (see [18]) since, on the one hand, we suppose no growth limitation on function α so that it can happen $-\mathrm{div}\big(\alpha(u)\nabla u\big) \notin$

$H^{-1}(\Omega)$ for all $u \in H_0^1(\Omega)$ and, on the other hand, it is not coercive; see [19] for an explicit example. To deal with that equation we will obtain the solution by approximation, getting a sequence of approximated solutions $(u_n)_n$ which converge to the solution u. To avoid some troubles in this convergence process, we will consider the convergence of another sequence $\big(A(u_n)\big)_n$, function A being the primitive of α such that $A(0) = 0$. The other remark is about the lower order term: it does not satisfy the "right" sign condition, since β is positive. Thus, it appears the problem of getting the a priori estimates. This hindrance is overcome by considering test functions of exponential type as in [20]; see lemma 2.2 bellow.

We next state our assumptions more precisely. We shall study the following nonlinear elliptic problem:

$$\begin{cases} -\mathrm{div}\big[a(x,u)\cdot\nabla u\big] + b(x,u,\nabla u) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.2}$$

Here Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 2$, and $a : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$ and $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions satisfying the following hipotheses

$$\begin{cases} \sup_{|s|\le k} |a(x,s)| \in L^\infty(\Omega), \quad \text{for all} \quad k > 0, & (H1) \\ [a(x,s)\cdot\xi]\cdot\xi \ge \alpha(s)|\xi|^2, & (H2) \\ |b(x,s,\xi)| \le \beta(s)|\xi|^2, & (H3) \end{cases}$$

α and β being positive continuous functions such that

$$\alpha \notin L^1([0,+\infty[) \cup L^1(]-\infty,0]), \tag{H4}$$

and

$$\frac{\beta}{\alpha} \in L^1(\mathbb{R}). \tag{H5}$$

So that, defining

$$\gamma(s) = \int_0^s \frac{\beta(\tau)}{\alpha(\tau)}\, d\tau$$

and

$$A(s) = \int_0^s \alpha(\tau)\, d\tau,$$

we have that the function γ is bounded, while

$$\lim_{s\to\pm\infty} |A(s)| = +\infty.$$

On the other hand, $f \in L^m(\Omega)$, with $m \ge 2N/(N+2)$; however, we will not deal with the limit case $m = N/2$. Recall that in the classical case, that is, when the function $a(x,s)\cdot\xi$ satisfies the Leray-Lions conditions; if $f \in L^{N/2}(\Omega)$, then the solution belongs to an exponencial Orlicz space.

The plan of this article is as follows. The next section is on notations. In section 2, problem (0.2) is studied when $f \in L^m(\Omega)$, with $m > N/2$. We define weak solution for (0.2) and get an L^∞-estimate which allows us to obtain a weak solution of (0.2). Section 3 is devoted to prove existence of an entropy solution of (0.2) when $f \in L^m(\Omega)$, with $2N/(N+2) \le m < N/2$. In this section we also see some conditions to guarantee that that entropy solution is actually a weak solution, and to obtain an energy type equality. Moreover, we end this section studying the summability of a solution. Finally, in Section 4, we will present an alternative approach in finding a priori estimates for solutions of problem (0.2), based on the symmetrization techniques.

1.- Notations

Some notations are used throughout this paper. $\Omega \subset \mathbb{R}^N$ will denote an open bounded set, $|A|$ Lebesgue measure of $A \subset \Omega$ and c_i positive constants which only depend on the parameters of our problem.

For $k > 0$ we define the truncature at level $\pm k$ as $T_k(s) = (-k) \vee [k \wedge s]$; we also consider $G_k(s) = s - T_k(s) = (|s| - k)^+\text{sign}(s)$.

Following [2], we introduce $\mathcal{T}_0^{1,p}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k u \in W_0^{1,p}(\Omega)$ for all $k > 0$. We point out that $\mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega) = W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

For a measurable function u belonging to $\mathcal{T}_0^{1,p}(\Omega)$, a gradient can be defined: it is a measurable function which is also denoted by ∇u and satisfies $\nabla T_k u = (\nabla u)\chi_{\{|u|<k\}}$ for all $k > 0$ (see [2, lemma 2.1]).

2.- Existence of bounded solutions

In this section we begin by defining weak solution of problem (0.2), then a new type of L^∞-estimates is obtained in theorem (2.3) (we remark again that we do not suppose the sign condition $b(x,s,\xi)s \ge 0$ for all $s \in \mathbb{R}$). As a consequence we have an existence result for (0.2).

The interest for getting L^∞-estimates for a problem as (0.2) is well known. The simplest examples deal with Euler's equations of functionals defined on $H_0^1(\Omega)$ or assume $b(x,s,\xi)s \ge 0$ for all $s \in \mathbb{R}$. A different approach (without that sign condition) was introduced by [11] for studying problem (0.2) when, for instance, $b(x,s,\xi) = \lambda s - |\xi|^2$, with $\lambda > 0$.

Next, let us define weak solution for (0.2).

Definition (2.1). We will say that a function $u \in \mathcal{T}_0^{1,2}(\Omega)$ is a weak solution of (0.2) if $a(x,u)\cdot\nabla u \in L^2(\Omega)$, $b(x,u,\nabla u)\in L^1(\Omega)$ and

$$\int_\Omega [a(x,u)\cdot\nabla u]\cdot\nabla\varphi + \int_\Omega b(x,u,\nabla u)\varphi = \int_\Omega f\varphi \tag{2.1}$$

holds for all $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$.

We will also say that φ may be taken as test function in (2.1) if the functions $[a(x,u)\cdot\nabla u]\cdot\nabla\varphi$, $b(x,u,\nabla u)\varphi$ and $f\varphi$ belong to $L^1(\Omega)$, and (2.1) holds true.

From now on in this section we assume

$$f \in L^m(\Omega), \quad \text{with } m > N/2.$$

Lemma (2.2). *Let u be a weak solution of (0.2) and let $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$.*

(i) If $v \geq 0$, then $e^{\gamma(u)}v$ can be taken as test function in (2.1) and

$$\int_\Omega e^{\gamma(u)}[a(x,u)\cdot\nabla u]\cdot\nabla v \leq \int_\Omega e^{\gamma(u)} f^+ v.$$

(ii) If $v \leq 0$, then $e^{-\gamma(u)}v$ may be taken as test function in (2.1) and

$$\int_\Omega e^{-\gamma(u)}[a(x,u)\cdot\nabla u]\cdot\nabla v \leq \int_\Omega e^{-\gamma(u)} f^-(-v).$$

(iii) If Φ is a locally Lipschitz continuous and increasing real function such that $\Phi(0)=0$ and $\Phi(u)$ may be taken as test function in (2.1), then there exists $c_1>0$ satisfying

$$\int_\Omega \alpha(u)\Phi'(u)|\nabla u|^2 \leq c_1\int_\Omega |f\ \Phi(u)|. \tag{2.2}$$

Proof. (i) Let $v\geq 0$. First observe that we have $e^{\gamma(T_k u)}\in H^1(\Omega)\cap L^\infty(\Omega)$, for every $k>0$. Thus, $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$ implies $e^{\gamma(T_k u)}v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and so it is an admissible test function in (2.1). Taking it so, it yields

$$\int_\Omega \frac{\beta(T_k u)}{\alpha(T_k u)} e^{\gamma(T_k u)} v[a(x,u)\cdot\nabla u]\cdot\nabla T_k u + \int_\Omega e^{\gamma(T_k u)}[a(x,u)\cdot\nabla u]\cdot\nabla v +$$

$$+\int_\Omega b(x,u,\nabla u)e^{\gamma(T_k u)}v = \int_\Omega f e^{\gamma(T_k u)} v. \tag{2.3}$$

Now we are going to study these integrals. Note that

$$\int_\Omega \frac{\beta(T_k u)}{\alpha(T_k u)} e^{\gamma(T_k u)} v[a(x,u)\cdot\nabla u]\cdot\nabla T_k u = \int_{\{|u|<k\}} \frac{\beta(u)}{\alpha(u)} e^{\gamma(u)} v[a(x,u)\cdot\nabla u]\cdot\nabla u;$$

so that, by the positivity of α and β, and by (H2), the integrand function is nonnegative. Hence, applying the monotone convergence theorem, we have

$$\lim_{k\to\infty}\int_\Omega \frac{\beta(T_k u)}{\alpha(T_k u)} e^{\gamma(T_k u)} v[a(x,u)\cdot\nabla u]\cdot\nabla T_k u = \int_\Omega \frac{\beta(u)}{\alpha(u)} e^{\gamma(u)} v[a(x,u)\cdot\nabla u]\cdot\nabla u.$$

On the other hand, the functions $[a(x,u)\cdot\nabla u]\cdot\nabla v$, $b(x,u,\nabla u)v$ and fv are summable, and the functions $e^{\gamma(T_k u)}$ are bounded in $L^\infty(\Omega)$; so Lebesgue's dominated convergence theorem may be applied in the remaining integrals. Thus, letting k tend to ∞ in (2.3), we obtain

$$\int_\Omega \frac{\beta(u)}{\alpha(u)} e^{\gamma(u)} v[a(x,u)\cdot\nabla u]\cdot\nabla u + \int_\Omega e^{\gamma(u)}[a(x,u)\cdot\nabla u]\cdot\nabla v +$$
$$+\int_\Omega b(x,u,\nabla u)e^{\gamma(u)}v = \int_\Omega f e^{\gamma(u)} v$$

and so $e^{\gamma(u)}v$ may be taken as test function in (2.1).

Finally, since

$$\int_\Omega \frac{\beta(u)}{\alpha(u)} e^{\gamma(u)} v[a(x,u)\cdot\nabla u]\cdot\nabla u + \int_\Omega b(x,u,\nabla u)e^{\gamma(u)}v \geq 0,$$

by (H2) and (H3); it follows that

$$\int_\Omega e^{\gamma(u)}[a(x,u)\cdot\nabla u]\cdot\nabla v \leq \int_\Omega f e^{\gamma(u)} v \leq \int_\Omega f^+ e^{\gamma(u)} v.$$

(ii) Let $v \leq 0$ and consider now $e^{-\gamma(T_k u)}v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ as a test function in (2.1) and reason in the same way as above.

(iii) Let Φ be a locally Lipschitz continuous and increasing real function such that $\Phi(0)=0$. Assume also that $\Phi(u)$ can be taken as test function in (2.1) and let $k>0$. Observe first that $\Phi(T_k u) = T_{\max\{\Phi(k),-\Phi(-k)\}}\Phi(T_k u) \in H_0^1(\Omega)\cap L^\infty(\Omega)$. Thus, taking $v=\Phi(T_k u^+)$ in (i) and $v=-\Phi(T_k u^-)$ in (ii), it yields

$$\int_{\{0\leq u<k\}} e^{\gamma(u)}\Phi'(u)[a(x,u)\cdot\nabla u]\cdot\nabla u \leq \int_\Omega e^{\gamma(u)} f^+ \Phi(T_k u^+)$$

and

$$\int_{\{0\geq u>-k\}} e^{-\gamma(u)}\Phi'(u)[a(x,u)\cdot\nabla u]\cdot\nabla u \leq \int_\Omega e^{-\gamma(u)} f^- \Phi(T_k u^-).$$

Having in mind (H2) and that $e^{\pm\gamma(u)}$ are bounded, we add up both formulae obtaning $c_1>0$ such that

$$\int_{\{|u|<k\}} \alpha(u)\Phi'(u)|\nabla u|^2 \leq c_1\int_\Omega |f\ \Phi(T_k u)| \leq c_1 \int_\Omega |f\ \Phi(u)|$$

holds true for all $k>0$. Therefore, we get (2.2) from Fatou's lemma. Observe that, since $\Phi(u)$ may be taken as test function in (2.1), $f\ \Phi(u)\in L^1(\Omega)$.

Theorem (2.3). *(i) Suppose that* $\alpha(s) \geq \lambda$ *for some* $\lambda > 0$. *If* u *is a weak solution of (0.2) which may be taken as test function, then* $\|u\|_\infty \leq c_2$, *where* $c_2 > 0$ *only depends on* λ, *on* m, *on the norm of* f *in* $L^m(\Omega)$ *and on the parameters of (0.2): that is, on* N, *on Lebesgue measure of* Ω, *and on the function* γ.

(ii) If u *is a weak solution of (0.2) such that* $A(u)$ *may be taken as test function, then* $\|A(u)\|_\infty \leq c_3$, *where* $c_3 > 0$ *only depends on* m, *on the norm of* f *in* $L^m(\Omega)$ *and on the parameters of (0.2); thus,* $\|u\|_\infty \leq \max\{A^{-1}(c_3), -A^{-1}(-c_3)\}$.

Proof. (i) Taking $\Phi(s) = G_k(s)$ in (2.2), it yields

$$\lambda \int_{\{|u|>k\}} |\nabla u|^2 \leq \int_{\{|u|>k\}} \alpha(u)|\nabla u|^2 \leq c_4 \int_\Omega |fG_k(u)|$$

for some $c_4 > 0$ and consequently

$$\int_\Omega |\nabla G_k(u)|^2 \leq \frac{c_4}{\lambda} \int_\Omega |fG_k(u)|.$$

By applying Stampacchia's L^∞-regularity procedure (see [21]), it follows that there exists $c_5 > 0$ satisfying $\|u\|_\infty \leq c_5$.

(ii) The proof is similar to (i), we only have to take $\Phi(s) = G_k(A(s))$ in (2.2) and reason in the same way. The last assertion is a consequence of being A strictly increasing and $\lim_{s\to\pm\infty} |A(s)| = +\infty$.

Theorem (2.4). *There exists* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *which is a weak solution of (0.2).*

Proof. We obtain the solution u by approximation: consider the following sequence of problems.

$$\begin{cases} -\mathrm{div}\big[a_n(x,u_n)\cdot\nabla u_n\big] + b_n(x,u_n,\nabla u_n) = f, & \text{in } \Omega; \\ u_n = 0, & \text{on } \partial\Omega; \end{cases} \tag{2.4}$$

where $a_n(x,s) = a(x, T_n s)$, $\alpha_n(s) = \alpha(T_n s)$, $\beta_n(s) = \alpha_n(s)\frac{\beta(s)}{\alpha(s)}$ and

$$b_n(x,s,\xi) = \min\Big[T_n\big(\beta_n(s)|\xi|^2\big), \max\big[-T_n\big(\beta_n(s)|\xi|^2\big), b(x,s,\xi)\big]\Big].$$

Let us see some simple properties of these functions. Note that, by (H1), the function $|a_n|$ is bounded from above; thus there exists $\Lambda_n > 0$ such that $|a_n(x,s)| \leq \Lambda_n$. Moreover, since α is continuous, there exists $\lambda_n > 0$ such that $\alpha_n(s) \geq \lambda_n$ and so $\alpha_n \notin L^1([0,+\infty[) \cup L^1(]-\infty,0])$. Then

$$\lambda_n|\xi|^2 \leq \alpha_n(s)|\xi|^2 \leq [a_n(x,s)\cdot\xi]\cdot\xi \leq \Lambda_n|\xi|^2.$$

On the other hand, the function b_n is bounded and

$$|b_n(x,s,\xi)| \le T_n\Big(\beta_n(s)|\xi|^2\Big) \le \beta_n(s)|\xi|^2.$$

We also observe that $\beta_n/\alpha_n = \beta/\alpha \in L^1(\Omega)$ and

$$\beta_n \le \max\{\alpha(s) : |s| \le n\}\frac{\beta}{\alpha},$$

so that $\beta_n \in L^1(\Omega)$. Furthermore, consider $A_n(s) = \int_0^s \alpha_n(\tau)\,d\tau$, and note that the functions defined by these formulae are continuous and strictly increasing. We finally point out that if $|s| \le n$, then $a_n(x,s) = a(x,s)$, $\alpha_n(s) = \alpha(s)$, $\beta_n(s) = \beta(s)$ and $A_n(s) = A(s)$.

Applying the classical result by Leray-Lions [18], for each $n \in \mathbb{N}$, there exists a weak solution $u_n \in H_0^1(\Omega)$, which is an admissible test functions in the weak formulation of (2.4). Hence, lemma (2.2) and theorem (2.3) can be applied. By theorem (2.3) (i), we have $u_n \in L^\infty(\Omega)$ and so $A_n(u_n) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for all $n \in \mathbb{N}$. Applying now theorem (2.3) (ii), there is $c_6 > 0$ such that $\|u_n\|_\infty \le \max\{A_n^{-1}(c_6), -A_n^{-1}(-c_6)\}$ and, since the sequences $\big(\pm A_n^{-1}(\pm c_6)\big)_n$ converge to $\pm A^{-1}(\pm c_6)$, it follows that $(u_n)_n$ is bounded in $L^\infty(\Omega)$. An easy consequence is that $\alpha_n(u_n) = \alpha(u_n)$ for n big enough, so that the sequence $\big(\alpha_n(u_n)\big)_n$ is bounded from below by a positive number, say μ. Likewise, the sequence $\big(a_n(x,u_n)\big)_n$ is bounded from above, by (H1); we also point out that

$$\mu|\xi|^2 \le \alpha_n(u_n)|\xi|^2 \le [a_n(x,u_n)\cdot\xi]\cdot\xi$$

for all $\xi \in \mathbb{R}^N$.

Taking $\Phi(s) = s$ in lemma (2.2) (iii), there is $c_7 > 0$ such that

$$\int_\Omega \alpha_n(u_n)|\nabla u_n|^2 \le c_7 \int_\Omega |f u_n|$$

and so

$$\mu \int_\Omega |\nabla u_n|^2 \le \int_\Omega \alpha_n(u_n)|\nabla u_n|^2 \le c_8 \int_\Omega |f| = c_9.$$

This estimate proves that $(u_n)_n$ is bounded in $H_0^1(\Omega)$. Hence, up to subsequences, $(u_n)_n$ converges weakly; moreover, Rellich-Kondrachov's theorem implies that we may also assume that converges almost everywhere in Ω. Let u be that limit; then $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \tag{2.5}$$

and

$$u_n \to u \text{ almost everywhere in } \Omega. \tag{2.6}$$

Next, we will see that

$$u_n \to u \text{ in } H_0^1(\Omega). \tag{2.7}$$

To prove this, take $v = (u_n - u)^+$ in lemma (2.2) (i) and $v = -(u_n - u)^-$ in lemma (2.2) (ii) to get

$$\int_\Omega e^{\gamma(u_n)}[a_n(x,u_n)\cdot\nabla u_n]\cdot\nabla(u_n-u)^+ \le \int_\Omega e^{\gamma(u_n)}|f|(u_n-u)^+ \tag{2.8}$$

and

$$-\int_\Omega e^{-\gamma(u_n)}[a_n(x,u_n)\cdot\nabla u_n]\cdot\nabla(u_n-u)^- \le \int_\Omega e^{-\gamma(u_n)}|f|(u_n-u)^-. \tag{2.9}$$

If we denote $\delta = \mu e^{-\sup|\gamma|}$, then

$$\delta\int_\Omega |\nabla(u_n-u)|^2 =$$

$$= \delta\int_{\{u_n-u\ge 0\}} |\nabla(u_n-u)|^2 + \delta\int_{\{u_n-u<0\}} |\nabla(u_n-u)|^2 \le$$

$$\le \int_\Omega e^{\gamma(u_n)}[a_n(x,u_n)\cdot\nabla(u_n-u)]\cdot\nabla(u_n-u)^+ -$$

$$-\int_\Omega e^{-\gamma(u_n)}[a_n(x,u_n)\cdot\nabla(u_n-u)]\cdot\nabla(u_n-u)^- =$$

$$= \int_\Omega e^{\gamma(u_n)}[a_n(x,u_n)\cdot\nabla u_n]\cdot\nabla(u_n-u)^+ - \int_\Omega e^{\gamma(u_n)}[a_n(x,u_n)\cdot\nabla u]\cdot\nabla(u_n-u)^+ -$$

$$-\int_\Omega e^{-\gamma(u_n)}[a_n(x,u_n)\cdot\nabla u_n]\cdot\nabla(u_n-u)^- + \int_\Omega e^{-\gamma(u_n)}[a_n(x,u_n)\cdot\nabla u]\cdot\nabla(u_n-u)^-.$$

Applying (2.8) and (2.9), it yields

$$\int_\Omega |\nabla(u_n-u)|^2 \le$$

$$\le \frac{1}{\delta}\int_\Omega e^{\gamma(u_n)}|f|(u_n-u)^+ - \frac{1}{\delta}\int_\Omega e^{\gamma(u_n)}[a_n(x,u_n)\cdot\nabla u]\cdot\nabla(u_n-u)^+ -$$

$$-\frac{1}{\delta}\int_\Omega e^{-\gamma(u_n)}|f|(u_n-u)^- + \frac{1}{\delta}\int_\Omega e^{-\gamma(u_n)}[a_n(x,u_n)\cdot\nabla u]\cdot\nabla(u_n-u)^-.$$

Now, taking into account that u_n, $a_n(x,u_n)$ and $e^{\pm\gamma(u_n)}$ define bounded sequences, it follows from (2.5), (2.6) and Lebesgue's convergence theorem that the right hand side converges to 0. Therefore, (2.7) is proved and, as a consequence, there exists a subsequence (still denoted by $(u_n)_n$) such that

$$\nabla u_n \to \nabla u \text{ almost everywhere in } \Omega. \tag{2.10}$$

In order to take limits in the weak formulation of (2.4), we next show that

$$b_n(x, u_n, \nabla u_n) \to b(x, u, \nabla u) \text{ in } L^1(\Omega). \tag{2.11}$$

Since, by (2.6) and (2.10), we already know that $b_n(x, u_n, \nabla u_n) \to b(x, u, \nabla u)$ almost everywhere in Ω, it is enough to see the equi-integrability of this sequence and then apply Vitali's convergence theorem. Observe that $\beta_n(u_n) = \beta(u_n)$ for n big enough, so that the sequence $\big(\beta_n(u_n)\big)_n$ is bounded, that is, there is $c_{10} > 0$ such that $\|\beta_n(u_n)\|_\infty \leq c_{10}$. Thus, $\beta_n(u_n)|\nabla u_n|^2 \leq c_{10}|\nabla u_n|^2$ and so $|b_n(x, u_n, \nabla u_n)| \leq c_{10}|\nabla u_n|^2$. Finally, the equi-integrability of $\big(|\nabla u_n|^2\big)_n$, which follows from (2.7), implies that of $\big(b_n(x, u_n, \nabla u_n)\big)_n$. Hence, (2.11) is proved.

Let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then

$$\int_\Omega [a_n(x, u_n) \cdot \nabla u_n] \cdot \nabla\varphi + \int_\Omega b_n(x, u_n, \nabla u_n)\varphi = \int_\Omega f\varphi \tag{2.12}$$

for all $n \in \mathbb{N}$, by the weak formulation of (2.4). Having in mind that the sequence $\big(a_n(x, u_n)\big)_n$ is bounded in $L^\infty(\Omega)$, it follows from (2.5), (2.6) and (2.11) that we may pass to the limit in (2.12) obtaining that u is a weak solution of (0.2).

3.- Existence of unbounded solutions

In this section, we assume $f \in L^m(\Omega)$, with $2N/(N+2) \leq m < N/2$. We first define entropy solution and then prove that there is an entropy solution of problem (0.2) through a convergence process which involves the sequence $\big(A_n(u_n)\big)_n$. Furthermore, we will see when this entropy solution is a weak solution and when we may take u as a test function in the weak formulation of the Dirichlet problem (0.2).

Definition (3.1). We will say that a function $u \in \mathcal{T}_0^{1,2}(\Omega)$ is an entropy solution of (0.2) if $b(x, u, \nabla u) \in L^1(\Omega)$ and

$$\int_\Omega [a(x, u) \cdot \nabla u] \cdot \nabla T_k[u - \varphi] + \int_\Omega b(x, u, \nabla u) T_k[u - \varphi] = \int_\Omega f T_k[u - \varphi]$$

holds for all $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Theorem (3.2). *There exists* $u \in \mathcal{T}_0^{1,2}(\Omega)$ *such that* $A(u) \in H_0^1(\Omega) \cap L^{m^{**}}(\Omega)$, $b(x,u,\nabla u) \in L^1(\Omega)$ *and it is an entropy solution of (0.2).*

Proof. Consider, for instance, the following approximating problems

$$\begin{cases} -\mathrm{div}\big[a(x,u_n)\cdot\nabla u_n\big] + b(x,u_n,\nabla u_n) = T_n f, & \text{in } \Omega; \\ \qquad\qquad\qquad\qquad\qquad u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We know, by theorem (2.4), that there exists $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$, which is a weak solution of (3.1). We shall see that this sequence $(u_n)_n$ converges, in some sense, to the entropy solution. The proof will be based on four lemmata.

Lemma (3.3). *There exists* $c_1 > 0$ *such that*

$$\int_{\{|A(u_n)|>k\}} |\nabla A(u_n)|^2 \le c_1 \left(\int_{\{|A(u_n)|>k\}} |f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{N}} \tag{3.2}$$

for all $k \ge 0$ *and all* $n \in \mathbb{N}$.

In particular, taking $k = 0$, *the sequence* $\big(A(u_n)\big)_n$ *is bounded in* $H_0^1(\Omega)$.

Proof of lemma (3.3). Taking $\Phi(s) = G_k(A(s))$ in lemma (2.2) (iii), one deduces

$$\int_\Omega |\nabla G_k(A(u_n))|^2 \le c_2 \int_\Omega |(T_n f)\; G_k(A(u_n))|.$$

It follows from this, and Hölder's and Sobolev's inequalities, that

$$\int_\Omega |\nabla G_k(A(u_n))|^2 \le c_2 \int_\Omega |T_n f| \cdot |G_k(A(u_n))| \le$$

$$\le c_2 \left(\int_\Omega |G_k(A(u_n))|^{2^*} \right)^{\frac{1}{2^*}} \cdot \left(\int_{\{|A(u_n)|>k\}} |T_n f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}} \le$$

$$\le c_3 \left(\int_\Omega |\nabla G_k(A(u_n))|^2 \right)^{\frac{1}{2}} \cdot \left(\int_{\{|A(u_n)|>k\}} |f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}}$$

so that lemma (3.3) is proved.

As a consequence of (3.2), we can extract a subsequence (still denoted by $(u_n)_n$), such that $A(u_n) \rightharpoonup w$ weakly in $H_0^1(\Omega)$. Moreover, by Rellich-Kondrachov's theorem, we may assume that $A(u_n) \to w$ almost everywhere in Ω. Since A is strictly increasing, defining $u = A^{-1}(w)$ we get a measurable function u such that

$$u_n \to u \text{ almost everywhere in } \Omega \tag{3.3}$$

and

$$A(u_n) \rightharpoonup A(u) \text{ weakly in } H_0^1(\Omega). \tag{3.4}$$

Using the results of [8], from inequality (3.2), it follows that $A(u) \in L^{m^{**}}(\Omega)$.

Lemma (3.4).

$$A(u_n) \to A(u) \quad in \quad H_0^1(\Omega). \tag{3.5}$$

In other words,

$$\alpha(u_n)\nabla u_n \to \alpha(u)\nabla u \quad in \quad L^2(\Omega). \tag{3.6}$$

Proof of lemma (3.4). To begin with we have to see two facts. The first one is to prove that

$$\int_\Omega |\nabla T_k\big(A(u_n) - T_h A(u)\big)|^2 \leq \omega_{n,h} + \eta_n(h,k), \tag{3.7}$$

where $\lim_{n\to\infty} \eta_n(h,k) = 0$ for any $h,k > 0$ and $\lim_{h\to\infty}\lim_{n\to\infty}\omega_{n,h} = 0$.

To see it, we apply lemma (2.2) (i) with $v = T_k\big(A(u_n) - T_h(A(u_n))\big)^+$ to obtain

$$\int_\Omega e^{\gamma(u_n)}[a(x,u_n)\cdot\nabla u_n]\cdot\nabla T_k\big(A(u_n) - T_h A(u)\big)^+ \leq$$

$$\leq \int_\Omega e^{\gamma(u_n)}|T_n f|\; T_k\big(A(u_n) - T_h A(u)\big)^+ \tag{3.8}$$

and lemma (2.2) (ii) with $v = T_k\big(A(u_n) - T_h(A(u_n))\big)^-$ to get

$$-\int_\Omega e^{-\gamma(u_n)}[a(x,u_n)\cdot\nabla u_n]\cdot\nabla T_k\big(A(u_n) - T_h A(u)\big)^- \leq$$

$$\leq \int_\Omega e^{-\gamma(u_n)}|T_n f|\; T_k\big(A(u_n) - T_h A(u)\big)^-. \tag{3.9}$$

Now, let $\delta = e^{-\sup|\gamma|}$ and observe that (H2) implies $|\xi|^2 \leq \frac{1}{\alpha(s)}[a(x,s)\cdot\xi]\cdot\xi$, so that

$$\delta\int_\Omega |\nabla T_k\big(A(u_n) - T_h A(u)\big)|^2 \leq$$

$$\leq \delta\int_{\{A(u_n)-T_hA(u)\geq 0\}}\Big[\frac{a(x,u_n)}{\alpha(u_n)}\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)\Big]\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big) +$$

$$+\delta\int_{\{A(u_n)-T_hA(u)<0\}}\Big[\frac{a(x,u_n)}{\alpha(u_n)}\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)\Big]\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big) \leq$$

$$\leq \int_\Omega e^{\gamma(u_n)}\Big[\frac{a(x,u_n)}{\alpha(u_n)}\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)\Big]\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)^+ -$$

$$-\int_\Omega e^{-\gamma(u_n)}\Big[\frac{a(x,u_n)}{\alpha(u_n)}\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)\Big]\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)^-.$$

Then

$$\int_\Omega |\nabla T_k\big(A(u_n) - T_h A(u)\big)|^2 \leq$$

$$\leq \frac{1}{\delta}\int_\Omega e^{\gamma(u_n)}[a(x,u_n)\cdot\nabla u_n]\cdot\nabla T_k\big(A(u_n) - T_h A(u)\big)^+ -$$

$$-\frac{1}{\delta}\int_\Omega e^{\gamma(u_n)}\Big[\frac{a(x,u_n)}{\alpha(u_n)}\cdot\nabla T_hA(u)\Big]\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)^+-$$

$$-\frac{1}{\delta}\int_\Omega e^{-\gamma(u_n)}[a(x,u_n)\cdot\nabla u_n]\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)^-+$$

$$+\frac{1}{\delta}\int_\Omega e^{-\gamma(u_n)}\Big[\frac{a(x,u_n)}{\alpha(u_n)}\cdot\nabla T_hA(u)\Big]\cdot\nabla T_k\big(A(u_n)-T_hA(u)\big)^-.$$

Thus, taking into account (3.8) and (3.9), there exists $c_4>0$ such that

$$\int_\Omega|\nabla T_k\big(A(u_n)-T_hA(u)\big)|^2\le$$

$$\le c_4\int_\Omega|T_nf|\cdot|T_k\big(A(u_n)-T_hA(u)\big)|+$$

$$+c_4\int_\Omega\frac{|a(x,u_n)|}{\alpha(u_n)}|\nabla T_hA(u)|\cdot|\nabla T_k\big(A(u_n)-T_hA(u)\big)|\le$$

$$\le c_4\int_\Omega|f|\cdot|A(u_n)-T_hA(u)|+c_4\int_\Omega\frac{|a(x,u_n)|}{\alpha(u_n)}|\nabla T_hA(u)|\cdot|\nabla T_k\big(A(u_n)-T_hA(u)\big)|=$$

$$=\omega_{n,h}+\eta_n(h,k).$$

We next go on estimating these integrals. On the one hand, it follows from (3.4) and the Sobolev imbedding that $A(u_n)\rightharpoonup A(u)$ weakly in $L^{\frac{2N}{N-2}}(\Omega)$ and from (3.3) that $A(u_n)\to A(u)$ almost everywhere in Ω. Thus,

$$\big|A(u_n)-T_h\big(A(u)\big)\big|\rightharpoonup\big|A(u)-T_h\big(A(u)\big)\big|\quad\text{weakly in}\quad L^{\frac{2N}{N-2}}(\Omega).$$

Since we also have $f\in L^{\frac{2N}{N+2}}(\Omega)$, it yields

$$\lim_{n\to\infty}\int_\Omega|f|\cdot|A(u_n)-T_h\big(A(u)\big)|=\int_\Omega|f|\cdot|A(u)-T_h\big(A(u)\big)|,$$

so that $\lim_{h\to\infty}\lim_{n\to\infty}\int_\Omega|f|\cdot|A(u_n)-T_h\big(A(u)\big)|=0.$

On the other hand, note that

$$\int_\Omega\frac{|a(x,u_n)|}{\alpha(u_n)}|\nabla T_hA(u)|\cdot|\nabla T_k\big(A(u_n)-T_hA(u)\big)|=$$

$$=\int_{\{|A(u_n)-T_hA(u)|<k\}}\frac{|a(x,u_n)|}{\alpha(u_n)}|\nabla T_hA(u)|\cdot|\nabla T_k\big(A(u_n)-T_hA(u)\big)|=$$

$$=\int_{\{|u_n|<A^{-1}(k+h)\}}\frac{|a(x,u_n)|}{\alpha(u_n)}|\nabla T_hA(u)|\cdot|\nabla T_k\big(A(u_n)-T_hA(u)\big)|,$$

and, in this integration set, the function $\frac{|a(x,u_n)|}{\alpha(u_n)}$ is bounded, by (H1). Hence, we also deduce from (3.4) that

$$\lim_{n\to\infty}\int_\Omega \frac{|a(x,u_n)|}{\alpha(u_n)}|\nabla T_h A(u)|\cdot|\nabla T_k(A(u_n)-T_hA(u))| =$$

$$= \int_\Omega \frac{|a(x,u)|}{\alpha(u)}|\nabla T_h A(u)|\cdot|\nabla T_k(A(u)-T_hA(u))| = 0.$$

Therefore, (3.7) is proved.

The second claim is easier. We will see that, for each $h>0$,

$$\int_\Omega |\nabla G_k(A(u_n)-T_hA(u))|^2 \le \epsilon_k(h), \tag{3.10}$$

where $\lim_{k\to\infty}\epsilon_k(h)=0$.

To do this, fix $h>0$ and observe that

$$|A(u_n)-T_h(A(u))|>k \quad \text{implies} \quad |A(u_n)|>k-h.$$

Then

$$\int_\Omega |\nabla G_k(A(u_n)-T_hA(u))|^2 = \int_{\{|A(u_n)-T_h(A(u))|>k\}} |\nabla(A(u_n)-T_hA(u))|^2 \le$$

$$\le \int_{\{|A(u_n)|>k-h\}} |\nabla(A(u_n)-T_hA(u))|^2 \le$$

$$\le 2\int_{\{|A(u_n)|>k-h\}} |\nabla A(u_n)|^2 + 2\int_{\{|A(u_n)|>k-h\}} |\nabla A(u)|^2.$$

Hence, by (3.2), we obtain

$$\int_\Omega |\nabla G_k(A(u_n)-T_hA(u))|^2 \le$$

$$\le c_5\left(\int_{\{|A(u_n)|>k-h\}} |f|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}} + 2\int_{\{|A(u_n)|>k-h\}} |\nabla A(u)|^2$$

and so it converges to 0 when k goes to infinity, uniformly on n.

Now, having in mind (3.2), (3.7) and (3.10); we will proceed to prove (3.5). Let $\epsilon>0$ be fixed. Consider $h>0$ satisfying

$$\|\nabla A(u)-\nabla T_h(A(u))\|_2^2 = \int_{\{|A(u)|>h\}} |\nabla A(u)|^2 < \frac{\epsilon^2}{16} \tag{3.11}$$

and $\lim_{n\to\infty}\omega_{n,h} < \epsilon^2/8$, $\omega_{n,h}$ being that of (3.7). Since h have already been choosen, claim (3.10) implies that there is $k > 0$ such that

$$\|\nabla G_k(A(u_n) - T_h A(u))\|_2^2 = \int_\Omega |\nabla G_k(A(u_n) - T_h A(u))|^2 < \frac{\epsilon^2}{16} \tag{3.12}$$

for all $n \in \mathbb{N}$. Recall that $\lim_{n\to\infty}\omega_{n,h} < \epsilon^2/8$ and , also by claim (3.7), we have $\lim_{n\to\infty}\eta_n(h,k) = 0$; so $n_0 \in \mathbb{N}$ can be found such that if $n \geq n_0$, then $\omega_{n,h} < \epsilon^2/8$ and $\eta_n(h,k) < \epsilon^2/8$. Hence,

$$\|\nabla T_k(A(u_n) - T_h A(u))\|_2^2 = \int_\Omega |\nabla T_k(A(u_n) - T_h A(u))|^2 < \frac{\epsilon^2}{4} \tag{3.13}$$

for all $n \geq n_0$. Finally, since

$$A(u_n) - A(u) = T_k(A(u_n) - T_h A(u)) + G_k(A(u_n) - T_h A(u)) - (A(u) - T_h A(u)),$$

it follows from (3.11), (3.12) and (3.13) that

$$\|\nabla A(u_n) - \nabla A(u)\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$$

for all $n \geq n_0$. Therefore, $A(u_n) \to A(u)$ strongly in $H_0^1(\Omega)$ and lemma (3.4) is completely proved.

Lemma (3.5). *The sequence $(u_n)_n$ has the following properties:*

(i) $\nabla u_n \to \nabla u$ in measure in Ω so that, up to a subsequence,

$$\nabla u_n \to \nabla u \quad \textit{almost everywhere in } \Omega. \tag{3.14}$$

(ii) For every $k > 0$,

$$T_k u_n \to T_k u \quad \textit{in } H_0^1(\Omega). \tag{3.15}$$

As a consequence, $T_k u \in H_0^1(\Omega)$ for all $k > 0$ and so $u \in \mathcal{T}_0^{1,2}(\Omega)$.

Proof of lemma (3.5). (i) It follows from (3.6) that $\alpha(u_n)\nabla u_n \to \alpha(u)\nabla u$ in measure in Ω. Since (3.3) implies $1/\alpha(u_n) \to 1/\alpha(u)$ in measure in Ω, we conclude that $\nabla u_n \to \nabla u$ in measure.

(ii) Let $k > 0$ and, taking into account the continuity of α, denote $\alpha_k = \min\{\alpha(s) : |s| \leq k\}$. Note that (3.6) may be written as

$$\frac{\alpha(u_n)}{\alpha_k}\nabla u_n \to \frac{\alpha(u)}{\alpha_k}\nabla u \text{ in } L^2(\Omega).$$

From this observation and the inequalities

$$|\nabla T_k u_n|^2 \leq \frac{\alpha(u_n)^2}{\alpha_k^2}|\nabla T_k u_n|^2 \leq \frac{\alpha(u_n)^2}{\alpha_k^2}|\nabla u_n|^2,$$

we deduce that the sequence $(|\nabla T_k u_n|^2)_n$ is equi-integrable. Hence, it follows from (3.14) and Vitali's convergence theorem that (3.15) holds true.

Lemma (3.6).

$$\beta(u_n)|\nabla u_n|^2 \to \beta(u)|\nabla u|^2 \quad in \quad L^1(\Omega) \tag{3.16}$$

and it follows from (H3), (3.3) and (3.4), that

$$b(x, u_n, \nabla u_n) \to b(x, u, \nabla u) \quad in \quad L^1(\Omega) \tag{3.17}$$

Proof of lemma (3.6). Observe that, by (3.3) and (3.14), $\beta(u_n)|\nabla u_n|^2 \to \beta(u)|\nabla u|^2$ almost everywhere in Ω. Thus, to get (3.16) we only have to check the equi-integrability of the sequence and apply Vitali's convergence theorem.

To proof the equi-integrability, consider the function $\Phi(s) = \gamma(G_k(s)+k) - \gamma(k)$ and observe that, if $|s| \geq k$, then $G_k(s)+k = s$ and so $\alpha(s)\Phi'(s) = \beta(s)$. Thus, taking Φ in lemma (2.1) (iii), there is $c_6 > 0$ such that

$$\int_{\{|u_n|\geq k\}} \beta(u_n)|\nabla u_n|^2 \leq c_6 \int_{\{|u_n|\geq k\}} |f|.$$

Let $\epsilon > 0$ and fix $k > 0$ satisfying

$$\int_{\{|u_n|\geq k\}} \beta(u_n)|\nabla u_n|^2 < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$. Letting $\beta_k > \max\{\beta(s) : |s| \leq k\}$, by (3.15), there exists $\delta > 0$ such that if $n \in \mathbb{N}$ and $|A| < \delta$, then $\int_A |\nabla T_k u_n|^2 < \frac{\epsilon}{2\beta_k}$. Therefore,

$$\int_A \beta(u_n)|\nabla u_n|^2 = \int_{A\cap\{|u_n|<k\}} \beta(u_n)|\nabla u_n|^2 + \int_{A\cap\{|u_n|\geq k\}} \beta(u_n)|\nabla u_n|^2 \leq$$

$$\leq \beta_k \int_A |\nabla T_k u_n|^2 + \int_{\{|u_n|\geq k\}} \beta(u_n)|\nabla u_n|^2 < \beta_k \frac{\epsilon}{2\beta_k} + \frac{\epsilon}{2} = \epsilon,$$

as required.

End of the proof of theorem (3.2). We only have to show that u is an entropy solution of (0.2). Note first that $u \in \mathcal{T}_0^{1,2}(\Omega)$, by lemma (3.5), and that $b(x, u, \nabla u) \in L^1(\Omega)$, by lemma (3.6).

Let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and consider $v = T_k[u_n - \varphi]$ as test function in the weak formulation of (3.1). Then

$$\int_\Omega [a(x, u_n) \cdot \nabla u_n] \cdot \nabla T_k[u_n - \varphi] + \int_\Omega b(x, u_n, \nabla u_n) T_k[u_n - \varphi] = \int_\Omega (T_n f)\, T_k[u_n - \varphi].$$

We point out that, in the first integral, we may consider the integration set as $\{|u_n| < \|\varphi\|_\infty + k\}$ without loss of generality; observe that, in this integration set, the functions $|a(x, u_n)|$ are bounded in $L^\infty(\Omega)$, by (H1). Now, it follows from (3.3), (3.14), (3.15) and (3.17) that we may take limit as n tends to $+\infty$ obtaining

$$\int_\Omega [a(x, u) \cdot \nabla u] \cdot \nabla T_k[u - \varphi] + \int_\Omega b(x, u, \nabla u) T_k[u - \varphi] = \int_\Omega f T_k[u - \varphi]$$

and proving that u is an entropy solution of (0.2).

Proposition (3.7). *Let u be an entropy solution of (0.2). If $a(x,u)\cdot\nabla u \in L^2(\Omega)$, then u is a weak solution.*

As a consequence, u is a weak solution if $|a(x,s)| \le \lambda\alpha(s)$ for some $\lambda > 0$.

Proof. Let $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and take $\varphi = T_h u - v$ in the formulation of entropy solution of (0.2). Then

$$\int_\Omega [a(x,u)\cdot\nabla u]\cdot\nabla T_k[u - T_h u + v] + \int_\Omega b(x,u,\nabla u) T_k[u - T_h u + v] = \int_\Omega f T_k[u - T_h u + v].$$

Now, as h goes to ∞, Lebesgue's dominated convergence theorem implies

$$\int_\Omega [a(x,u)\cdot\nabla u]\cdot\nabla T_k v + \int_\Omega b(x,u,\nabla u) T_k v = \int_\Omega f T_k v.$$

Finally, it is enough to take $k > \|v\|_\infty$ to obtain

$$\int_\Omega [a(x,u)\cdot\nabla u]\cdot\nabla v + \int_\Omega b(x,u,\nabla u) v = \int_\Omega f v.$$

Hence, u is a weak solution. The second part follows from $\nabla A(u) = \alpha(u)\nabla u \in L^2(\Omega)$.

Proposition (3.8). *Suppose that $\beta(s)|s| \le \lambda\alpha(s)$ for some $\lambda > 0$. Let u be an entropy solution of (0.2). If $fu \in L^1(\Omega)$, then u may be taken as test function in the weak formulation of (0.2) obtaining the following energy type equality:*

$$\int_\Omega [a(x,u)\cdot\nabla u]\cdot\nabla u + \int_\Omega b(x,u,\nabla u) u = \int_\Omega f u.$$

Proof. First of all, we will prove that $[a(x,u)\cdot\nabla u]\cdot\nabla u \in L^1(\Omega)$. Let $h > k > j > 0$. Observe that it follows from $\beta(s)/\alpha(s) \le \lambda/|s|$ and the continuity of β/α that $\gamma' = \beta/\alpha \in L^\infty(\Omega)$; consequently the functions $e^{\gamma(u)}T_j u^+$ and $e^{-\gamma(u)}T_j u^-$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Taking $\varphi = T_h(u) - e^{\gamma(u)}T_j(u^+)$ in the entropy formulation of (0.2), we get

$$\int_{\{u\le 0\}} [a(x,u)\cdot\nabla u]\cdot\nabla T_k[u - T_h u] + \int_{\{u>0\}} [a(x,u)\cdot\nabla u]\cdot\nabla T_k[u - T_h u + e^{\gamma(u)}T_j u^+] +$$

$$+ \int_\Omega b(x,u,\nabla u) T_k[u - T_h u + e^{\gamma(u)}T_j u^+] = \int_\Omega f T_k[u - T_h u + e^{\gamma(u)}T_j u^+].$$

The first two integrals are nonnegative: so that we may drop the first one and apply Fatou's lemma to the other as h tend to infinity. In the others two integrals we may apply Lebesgue's dominated convergence theorem as h goes to infinity. Thus, it yields

$$\int_{\{u>0\}} [a(x,u)\cdot\nabla u]\cdot\nabla T_k[e^{\gamma(u)}T_j u^+] + \int_\Omega b(x,u,\nabla u)T_k[e^{\gamma(u)}T_j u^+] \le$$

$$\le \int_\Omega f T_k[e^{\gamma(u)}T_j u^+].$$

Since $e^{\gamma(u)}T_j u^+$ is a bounded function, when k is big enough, we obtain

$$\int_\Omega e^{\gamma(u)}\gamma'(u)T_j u^+[a(x,u)\cdot\nabla u]\cdot\nabla u + \int_\Omega e^{\gamma(u)}[a(x,u)\cdot\nabla u]\cdot\nabla T_j u^+ +$$

$$+ \int_\Omega b(x,u,\nabla u)e^{\gamma(u)}T_j u^+ \le \int_\Omega f e^{\gamma(u)}T_j u^+.$$

Reasoning in the same way as lemma (2.2) (i), we deduce that there is $c_7>0$ such that

$$\int_\Omega [a(x,u)\cdot\nabla T_j u^+]\cdot\nabla T_j u^+ \le c_7\int_\Omega |fT_j u^+|. \tag{3.18}$$

On the other hand, taking $\varphi = T_h(u)+e^{-\gamma(u)}T_j(u^-)$ in the entropy formulation of (0.2) and having now in mind lemma (2.2) (ii), arguments similar to that above imply that there exists $c_8>0$ such that

$$\int_\Omega [a(x,u)\cdot\nabla T_j u^-]\cdot\nabla T_j u^- \le c_8\int_\Omega |fT_j u^-|. \tag{3.19}$$

Adding up (3.18) and (3.19), we conclude that there exists $c_9>0$ such that

$$\int_\Omega [a(x,u)\cdot\nabla T_j u]\cdot\nabla T_j u \le c_9\int_\Omega |fT_j u|,$$

and so, letting j tend to infinity, by Fatou's lemma in the left hand side and Lebesgue's theorem in the right one, we deduce that

$$\int_\Omega [a(x,u)\cdot\nabla u]\cdot\nabla u \le c_9\int_\Omega |fu|.$$

Hence, it follows from $fu\in L^1(\Omega)$ that $[a(x,u)\cdot\nabla u]\cdot\nabla u\in L^1(\Omega)$.

Taking $\varphi=0$ in the entropy formulation of (0.2), it yields

$$\int_\Omega [a(x,u)\cdot\nabla T_k u]\cdot\nabla T_k u + \int_\Omega b(x,u,\nabla u)T_k u = \int_\Omega fT_k u. \tag{3.20}$$

Now, observe that the following inequalities hold:

$$|fT_k u| \le |fu| \in L^1(\Omega),$$

$$[a(x,u)\cdot\nabla T_k u]\cdot\nabla T_k u \le [a(x,u)\cdot\nabla u]\cdot\nabla u \in L^1(\Omega)$$

and

$$|b(x,u,\nabla u)u| \le \beta(u)|u|\cdot|\nabla u|^2 \le \lambda\alpha(u)|\nabla u|^2 \le \lambda[a(x,u)\cdot\nabla u]\cdot\nabla u \in L^1(\Omega).$$

Therefore, by Lebesgue's dominated convergence theorem, it follows from (3.20) that

$$\int_\Omega [a(x,u)\cdot\nabla u]\cdot\nabla u + \int_\Omega b(x,u,\nabla u)u = \int_\Omega fu,$$

as desired.

Up to now, we have not really seen the summability of the solution u, but that of $A(u)$. In the last result, we study this regularity of the solution when α and β are concrete functions.

Proposition (3.9). *Let $\theta > -1$ and suppose that $\alpha(s) = (1+|s|)^\theta$ and $\beta(s) = (1+|s|)^{\theta-1-\epsilon}$, with $\epsilon > 0$.*

If u is an entropy solution of (0.2), then $u \in L^r(\Omega)$ and $|\nabla u| \in L^q(\Omega)$, where $r = Nm(1+\theta)/(N-2m)$ and $q = \min\left[2, Nm(1+\theta)/\big(N-m(1-\theta)\big)\right]$; thus, u belongs to a Sobolev space if $m \ge N/\big(N+1+\theta(N-1)\big)$.

Moreover, u can be taken as test function in the weak formulation of problem (0.2) if $m \ge N(2+\theta)/\big(N(1+\theta)+2\big)$.

Proof. Obviously, $\alpha(s) = (1+|s|)^\theta$ implies $|A(s)| = \frac{1}{1+\theta}(1+|s|)^{1+\theta}$. Since $A(u) \in L^{m^{**}}(\Omega)$, by theorem (3.2), we have $(1+|u|)^{1+\theta} \in L^{Nm/(N-2m)}(\Omega)$ and then $u \in L^{Nm(1+\theta)/(N-2m)}(\Omega)$.

We now pass to see the regularity of the gradient ∇u. On the one hand, $\theta \ge 0$ implies $\alpha(s) \ge 1$ and so $|\nabla u| \le \alpha(u)|\nabla u|$. Moreover, since $\alpha(u)|\nabla u| = |\nabla\big(A(u)\big)| \in L^2(\Omega)$, by theorem (3.2), it follows that $|\nabla u| \in L^2(\Omega)$. We point out that then $Nm(1+\theta)/\big(N-m(1-\theta)\big) \le 2$. On the other hand, if $-1 < \theta < 0$, then we may follow the arguments of [6, lemma 2.3] to obtain that

$$q = \min\left[2, \frac{Nm(1+\theta)}{N-m(1-\theta)}\right].$$

Next, considering $m \ge N(2+\theta)/\big(N(1+\theta)+2\big)$, we have

$$\frac{m}{m-1} \le \frac{Nm(1+\theta)}{N-2m},$$

so that $u \in L^{m'}(\Omega)$ and $fu \in L^1(\Omega)$. By proposition (3.8), we deduce that u may be taken as test function. Observe also that when $\theta \ge 0$,

$$\frac{N(2+\theta)}{N(1+\theta)+2} \le \frac{2N}{N+2}$$

and hence, in this case, u can always be taken as test function.

4.- A PRIORI ESTIMATES BY SYMMETRIZATION.

An alternative approach in finding a priori estimates for solutions of problem (0.2), like the ones obtained in Theorem 2.3 and in Lemma 3.3, is based on the so called symmetrization techniques which go back to the papers by Talenti ([22], [23]) and which have been widely used in similar contexts (see for example [1], [15]).

In this section we show how it is possible to use such an approach pointing out that in this way we can obtain estimates also when f belongs to intermediate spaces (see Remarks 4.5 and 4.8).

We first recall the definition of decreasing rearrangement of a function u. If u is a measurable function in an open bounded set $\Omega \subset \mathbb{R}^N$, we denote by μ_u the distribution function of u:

$$\mu_u(t) = |\{x \in \Omega : |u| > t\}|, \qquad t \geq 0.$$

The decreasing rearrangement $u^*(s)$ of u is defined by

$$u^*(s) = \sup\{t > 0 : \mu_u(t) > s\}, \qquad s \in [0, |\Omega|]$$

and the spherically decreasing rearrangement $u^\#(x)$ of u is defined by

$$u^\#(x) = u^*(C_N|x|^N), \qquad x \in \Omega^\#,$$

where $\Omega^\#$ is the ball centered at the origin having the same measure of Ω, and C_N denotes the measure of the unit ball of $\mathbb{R}^N$.

We recall also that for every $1 \leq p < +\infty$ it results

$$\int_\Omega |u(x)|^p dx = \int_0^{|\Omega|} (u^*(s))^p ds$$

and if $u \in L^\infty(\Omega)$

$$||u||_{L^\infty} = ||u^*||_{L^\infty} = u^*(0).$$

For an exhaustive treatment of rearrangements we refer for example to [17 or 23].

We recall Bliss inequalities, (see [22]), which are used to obtain estimates of L^p-norms of solutions in terms of L^q-norms of the datum f.

Lemma (4.1). . *If $\varphi(r)$ is positive for $0 < r < +\infty$ and $1 < p < q$, then*

$$\int_0^{+\infty} \left(\frac{1}{r}\int_0^r \varphi(s)ds\right)^q r^{-1+q/p} dr \leq A(N,p,q)\left(\int_0^{+\infty} \varphi(r)^p dr\right)^{q/p}$$

$$\int_0^{+\infty} \left(\int_r^{+\infty} \varphi(s)ds\right)^q dr \leq B(N,p,q)\left(\int_0^{+\infty} \varphi(r)^p r^{-1+p+p/q} dr\right)^{q/p}$$

Now we prove an a priori estimate for a solution u of (0.2).

Proposition (4.2). . *Let $u \in L^\infty(\Omega)$ be a weak solution of (0.2) in the sense of (2.1), then there exists a constant $M_1 > 0$ such that:*

$$[A(u)]^*(s) \leq M_1 \int_s^{|\Omega|} \frac{1}{N^2 C_N^{2/N} r^{2-2/N}} \left(\int_0^r f^*(\sigma) d\sigma \right) dr, \quad s \in (0, |\Omega|). \qquad (4.1)$$

Proof. Applying Lemma (2.2)(iii) to the function $\Phi(s) = T_h\big[A(s) - T_t\big(A(s)\big)\big]$ we get:

$$\int_{\{t<|A(u)|\leq t+h\}} \alpha^2(u)|\nabla u|^2 \leq M_1 \int_\Omega |f T_h\Big(A(s) - T_t\big(A(s)\big)\Big)| \leq M_1 h \int_{\{|A(u)|>t\}} |f|$$

where M_1 depends only on the data ($M_1 = e^{2\sup|\gamma|}$).

Setting $w = A(u)$ we have:

$$\frac{1}{h} \int_{\{t<|w|\leq t+h\}} |\nabla w|^2 \leq M_1 \int_{\{|w|>t\}} |f| \qquad (4.2)$$

Schwartz inequality, Fleming-Rishel formula (see [16]) and the isoperimetric inequality give (see [13])

$$1 \leq M_1 \frac{-\mu_w'(t)}{N^2 C_N^{2/N} \mu_w(t)^{2-2/N}} \int_0^t f^*(r) dr$$

from which, using the properties of rearrangement (see [22]) we get inequality (4.1).

Remark (4.3) If we denote by v the solution of the problem

$$\begin{cases} -\Delta v = f^\# & \text{in } \Omega^\#; \\ v = 0 & \text{on } \partial\Omega^\#; \end{cases} \qquad (4.3)$$

the function v is given by

$$v(x) = N^{-2} C_N^{-2/N} \int_{C_N|x|^N}^{|\Omega|} s^{2/N-2} \left(\int_0^s f^*(\sigma) d\sigma \right) ds. \qquad (4.4)$$

From Lemma 4.2 we deduce that

$$[A(u)]^*(s) \leq M_1 v^*(s), \qquad s \in (0, |\Omega|),$$

where v^* is the decreasing rearrangment of the solution v of (4.3).

Clearly if $f \in L^m(\Omega)$ with $m > N/2$ then v is bounded and its maximum value is $v(0)$.

The following two Corollaries give uniform estimates on the L^p-norms of solutions of (0.2).

Corollary (4.4). . *Let $u \in L^\infty(\Omega)$ be a solution of (0.2) in the sense of (2.1), then if $m > \frac{N}{2}$ there exists a constant $M_2 > 0$ such that*

$$||u||_{L^\infty} \leq \max\{A^{-1}(M_2), |A^{-1}(-M_2)|\},$$

where $M_2 = \left(\frac{M_1|\Omega|^{2/N-1/m}}{NC_N^{2/N}}\frac{m}{2m-N}||f||_{L^m}\right)$, A^{-1} is the inverse function of A and M_1 is the constant defined in Proposition 4.2.

Proof. Setting $w = A(u)$, inequality (4.1) and Hölder inequality imply:

$$||A(u)||_{L^\infty(\Omega)} \leq [A(u)]^*(0) \leq M_1 \int_0^{|\Omega|} \frac{1}{N^2C_N^{2/N} r^{2-2/N}} \int_0^r f^*(\sigma)d\sigma \leq \quad (4.5)$$

$$\leq \frac{M_1}{N^2C_N^{2/N}} \int_0^{|\Omega|} \frac{1}{r^{2-2/N}} \left(\int_0^r f^*(\sigma)^m d\sigma\right)^{1/m} r^{1/m'} \leq$$

$$\leq \frac{M_1|\Omega|^{2/N-1/m}}{NC_N^{2/N}} \frac{m}{2m-N}||f||_{L^m} = M_2.$$

Remark (4.5). We observe that by inequality (4.5) the estimate on the L^∞ norm of $A(u)$, and then on the L^∞ norm of u, holds true if

$$\int_0^{|\Omega|} \frac{1}{N^2C_N^{2/N} r^{2-2/N}} \int_0^r f^*(\sigma)d\sigma < +\infty.$$

The above condition is satisfied if f belongs to the Lorentz space $L(N/2, 1)$.

Corollary (4.6). . *Let $u \in L^\infty(\Omega)$ be a solution of (0.2) in the sense of definition (2.1) then, if $\frac{2N}{N+2} \leq m < \frac{N}{2}$, there exists a constant $M_3 > 0$ such that:*

$$||A(u)||_{L^q} \leq M_3||f||_{L^m},$$

where $q = m^{**}$.

Proof. From inequality 4.1 we have:

$$||A(u)||_{L^q} = ||[A(u)]^*||_{L^q} \leq$$

$$\leq M_1 \frac{1}{N^2C_N^{2/N}} \left(\int_0^{|\Omega|} \left(\int_s^{|\Omega|} \left(\int_0^r f^*(\sigma)d\sigma\right) r^{-2+2/N} dr\right)^q ds\right)^{1/q}.$$

Using Bliss and Hardy inequality (see [22]) we get

$$||A(u)||_{L^q} \leq M_3||f||_{L^m}.$$

The following result gives an estimate of $A(u)$ in $H_0^1(\Omega)$.

Proposition (4.7). . *Let $u \in L^\infty(\Omega)$ be a solution of (0.2) in the sense of definition (2.1), then $A(u)$ belongs to $H_0^1(\Omega)$ and there exists a constant M_4 such that*

$$||A(u)||_{H_0^1} \leq M_4 ||f||_{L^{2N/(N+2)}}$$

Proof. Setting $w = A(u)$, from estimate (4.2) we get :

$$-\frac{d}{dt} \int_{\{|w|>t\}} |\nabla w|^2 \leq M_1 \int_0^{\mu_w(t)} f^*(\sigma) d\sigma$$

Moreover it results that (see [22])

$$\int_\Omega |\nabla w|^2 \leq \int_0^{+\infty} \left(\frac{-\mu_w(t)^{-1+1/N}}{N C_N^{1/N}} \frac{d}{dt} \int_{\{|w|>t\}} |\nabla w|^2 \right)^2 (-d\mu_w(t)).$$

Setting $\bar{f}(t) = \frac{1}{t} \int_0^t f^*(s) ds \quad \forall t > 0$ using Bliss inequality we get:

$$\int_\Omega |\nabla w|^2 \leq M_1^2 \int_0^{+\infty} \left(\frac{-\mu_w(t)^{-1+1/N}}{N C_N^{1/N}} \int_0^{\mu_w(t)} f^*(\sigma) d\sigma \right)^2 (-d\mu_w(t)) =$$

$$= \frac{M_1^2}{N^2 C_N^{2/N}} \int_0^{|\Omega|} \left(r^{1/N-1} \int_0^r f^*(\sigma) d\sigma \right)^2 dr =$$

$$= \frac{M_1^2}{N^2 C_N^{2/N}} \int_0^{|\Omega|} (r^{(2+N)/2N} \bar{f}(r))^2 \frac{dr}{r} \leq M_4 ||f||^2_{2N/(N+2)}$$

and the thesis follows.

Remark (4.8). We observe that the estimate on the H_0^1 norm of $A(u)$ holds if

$$\int_0^{|\Omega|} (r^{(2+N)/2N} \bar{f}(r))^2 \frac{dr}{r} < +\infty.$$

The above condition is satisfied if f belongs to the Lorentz space $L(2N/(N+2), 2)$.

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