SEMILINEAR PARABOLIC EQUATIONS WITH SUPERLINEAR REACTION TERMS, AND APPLICATION TO SOME CONVECTION-DIFFUSION PROBLEMS

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Abstract.

We are interested in the existence of distributional solutions for two kinds of nonlinear evolution problems, whose models are (1.1) and (1.2) below. In the first one the nonlinear reaction term depends on the solution with a slightly superlinear growth. In the second one we consider a first order term depending also on the gradient of the solution in a quadratic way.

The two problems are strictly related from the point of view of the *a priori* estimates we can obtain on their solutions. We point out that no boundedness is assumed on the data of the problems. This implies that the methods involving sub/super-solutions do not apply, and we have to use some convenient test-function to prove the *a priori* estimates.

§1. Introduction

In this paper we are interested in solving nonlinear parabolic problems of the type

$$\begin{cases} v_t - \Delta v = f(x,t)\ \left[1 + |v|(\log|v|)^\alpha\right] & \text{in } \Omega \times]0,T[; \\ v(x,t) = 0 & \text{on } \partial\Omega \times]0,T[; \\ v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $0 < \alpha < 1$, $f(x,t) \in L^r(0,T;L^q(\Omega))$ for convenient r and q, and $v_0 \in L^2(\Omega)$. This type of problems is strictly related to parabolic convection-diffusion problems whose model is

$$\begin{cases} u_t - \Delta u = \beta(u)|\nabla u|^2 + g(x,t) & \text{in } \Omega \times]0,T[; \\ u(x,t) = 0 & \text{on } \partial\Omega \times]0,T[; \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

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where $\beta(s)$ is a continuous function which grows like an arbitrary power of s at $\pm\infty$.

In these model examples it is easy to see that it is possible to perform a change of unknown function in (1.2), i.e., $v = \Psi(u)$ (see (2.6) below for the definition of the function $\Psi(s)$), and reduce problem (1.2) to a problem which is similar to (1.1).

Nevertheless, we would like to consider also more general situations where one has a nonlinear pseudomonotone operator as a principal part in (1.2) and a general first order term which grows quadratically with respect to the gradient. In this case one cannot change the unknown function, but the previous remark suggests the use of convenient test functions which "simulate" this technique.

We give an existence result of distributional solutions for (1.1) via test-function method under the hypotheses that $f(x,t) \in L^r(0,T;L^q(\Omega))$, where the exponents r and q belong to a part of the so-called Aronson-Serrin region in the (r,q)-plane, part which also depends on the value of the parameter α. As far as the initial datum is concerned, we assume that $v_0 \in L^2(\Omega)$.

The *a priori* estimates that we will be able to obtain on a sequence $\{v_n\}$ of approximating solutions of (1.1) will also provide *a priori* estimates for a sequence of approximate solutions u_n for problem (1.2), and therefore an existence result of distributional solutions for this problem.

Besides this application, the result obtained for problem (1.1) seems to have an autonomous interest.

Let us recall some results concerning the existence of solutions of the nonlinear heat equation

$$\begin{cases} v_t - \Delta v = \lambda h(v) & \text{in } \Omega\times]0,T[; \\ v(x,t) = 0 & \text{on } \partial\Omega\times]0,T[; \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.3}$$

with $0 \le v_0 \in L^\infty(\Omega)$, while $h(s)$ is a positive function. In [BCMR] Brezis, Cazenave, Martel and Ramiandrisoa proved that if

$$\int^{+\infty} \frac{ds}{h(s)} < +\infty\,,$$

then there is no solution for large λ. This shows that one cannot hope to prove global existence for (1.1) if $\alpha > 1$. On the other hand, if

$$\int^{+\infty} \frac{ds}{h(s)} = +\infty\,,$$

and if the data $v_0(x)$ and $f(x,t)$ are bounded, it is easy to prove the existence of a global solution using sub/super-solutions independent on x. However this method does not work if either v_0 or f is unbounded. One of the aims of this paper is to present some results in this case.

Another interesting remark is the comparison with the case $\alpha = 0$ and $f(x,t) = f(x)$: it is well known (see, for instance, [LSU]) that if $f(x) \in L^q(\Omega)$, with $q \le N/2$, there exists a solution of the following linear heat equation with singular potential

$$\begin{cases} v_t - \Delta v = f(x)v & \text{in } \Omega\times]0,T[; \\ v(x,t) = 0 & \text{on } \partial\Omega\times]0,T[; \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

while, if $q > N/2$, one can have instantaneous and complete blow-up (for instance this happens if $f(x) = \lambda/|x|^2$, with λ large enough (see [BG], and [CM] for more general potentials $f(x)$). In this case our result (see Theorem 2.1 below) states that one can allow $\alpha > 0$, but has to pay a price by assuming the stronger condition $q > N/[2(1-\alpha)]$ (see Remark 2.1 below).

As far as the quasi-linear problem (1.2) is concerned, existence results have been given in [BMP] under the assumptions that the data $f(x,t)$ and $u_0(x)$ are bounded, via a sub/super-solution method. For more general data f and u_0, the result we obtain for (1.2) (see Theorem 2.2 in next Section) improves previous results proved in a wider framework in [DGLS]. In that paper (see also [DGP] and [FPR]) a special condition is assumed which prevents one from considering functions $\beta(s)$ which tends to $+\infty$ for $s \to \pm\infty$.

The plan of the paper is as follows: the next section is devoted to stating the assumptions and the main results of the paper. In Section 3 we will define the approximate problems and prove the related *a priori* estimates. In the final section we will study the limiting process.

§2. Assumptions and main results

Let Ω be a bounded open set in $\mathbb{R}^N$, $T > 0$, $Q_T = \Omega\times]0,T[$, $\Sigma_T = \partial\Omega\times]0,T[$. We will denote by $L^q(\Omega)$, $1 \le q \le +\infty$, the usual Lebesgue spaces. If X is a Banach space, we will denote by $L^q(0,T;X)$ the usual evolution spaces (see, for instance, [Br]). We will sometimes write $\|f\|_q$ instead of $\|f\|_{L^q(\Omega)}$, and $\|f\|_{r,q}$ instead of $\|f\|_{L^r(0,T;L^q(\Omega))}$. Moreover C will denote a positive constant which only depends on the data of the problem. Its value may be different from line to line.

We are interested in studying the following two types of nonlinear evolution problems:

$$\begin{cases} v_t - \text{div}\,(\mathbf{a}(x,t,v)\nabla v) = F(x,t,v) & \text{in } Q_T; \\ v(x,t) = 0 & \text{on } \Sigma_T; \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

and

$$\begin{cases} u_t - \text{div}\,(\mathbf{a}(x,t,u)\nabla u) = B(x,t,u,\nabla u) & \text{in } Q_T; \\ u(x,t) = 0 & \text{on } \Sigma_T; \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.2}$$

where in both cases the principal part satisfies

- The function $\mathbf{a} : Q_T \times \mathbb{R} \to \mathbb{R}^{N^2}$ is a Carathéodory function; that is, it is measurable with respect to (x,t) for all $s \in \mathbb{R}$ and continuous in s for almost all $(x,t) \in Q_T$; moreover there exist two positive constants ν and M such that

$$[\mathbf{a}(x,t,s)\cdot\xi]\cdot\xi \ge \nu|\xi|^2 \tag{2.3}$$

and

$$|\mathbf{a}(x,t,s)\cdot\xi| \le M|\xi|$$

hold for almost all $(x,t)\in Q_T$ for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

We first consider problem (2.1) and state the assumptions and our results. We will assume that:

- The function $F: Q_T\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory conditions; moreover there exist a constant α, with $0<\alpha<1$, and a positive measurable function $f(x,t)\in L^r(0,T;L^q(\Omega))$, with

$$q,r>1\ ,\qquad q>\frac{N}{2}\max\left\{\frac{1}{1-\alpha}\ ,\ \frac{r}{r-1}\right\}\ , \tag{2.4}$$

  such that

$$|F(x,t,s)| \le \Big(1+|s|\,\big|\log|s|\big|^{\alpha}\Big) f(x,t)\,;$$

- $v_0(x)\in L^2(\Omega)$.

Remark 2.1. If $f\in L^\infty(0,T;L^q(\Omega))$, with $q>N/[2(1-\alpha)]$, and in particular if $f(x,t)=f(x)\in L^q(\Omega)$, then obviously there exists $r<\infty$ such that (2.4) holds.

We can now state the first of our main existence results:

Theorem 2.1. *Under the above hypotheses, problem* (2.1) *admits at least one distributional solution* $v\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$.

Remark 2.2. The condition

$$q>\frac{Nr}{2(r-1)}$$

is the same one which ensures the local boundedness of the solutions of the equation

$$v_t-\Delta v=f(x,t)\,,\qquad f\in L^r(0,T;L^q(\Omega))\,,$$

as shown by Aronson and Serrin in [AS]. The problem whether the solution of problem (2.1) is bounded, for a bounded initial datum v_0, is open.

We now turn our attention to the second quasi-linear problem (2.2). We will assume that:

- The function $B: Q_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ satisfies the Carathéodory conditions; moreover there exist two positive constants λ and C_1, and a positive measurable function $g(x,t)\in L^r(0,T;L^q(\Omega))$, with

$$q,r>1\ ,\qquad q>\frac{N}{2}\max\left\{\lambda+1\,,\ \frac{r}{r-1}\right\}\ ,$$

  such that

$$|B(x,t,s,\xi)|\le C_1(1+|s|^\lambda)|\xi|^2+g(x,t)\,; \tag{2.5}$$

In order to state the assumption on the initial datum, we define two auxiliary functions by

$$\gamma(s) = \frac{C_1}{\nu} \int_0^s (1 + |\sigma|^\lambda)\, d\sigma = \frac{C_1}{\nu} \left(s + \frac{|s|^{\lambda+1}}{\lambda+1} \operatorname{sign} s \right),$$

and

$$\Psi(s) = \int_0^s e^{|\gamma(\sigma)|}\, d\sigma\,. \tag{2.6}$$

The assumption on u_0 reads as follows

- $\Psi(u_0(x)) \in L^2(\Omega)$.

Theorem 2.2. *Under the above hypotheses, problem* (2.2) *admits at least one distributional solution u such that*

$$\Psi(u) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))\,. \tag{2.7}$$

Remark 2.3. Recalling that $\Psi'(s) \geq 1$, the estimate (2.7) implies

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))\,.$$

Remark 2.4. It is easy to show that for every $\delta > \lambda + 1$ there exists $C_\delta > 0$ such that

$$|\Psi(s)| \leq C_\delta (1 + e^{|s|^\delta})$$

for every $s \in \mathbb{R}$. It follows that the assumption on the initial datum of problem (2.1), i.e., $\Psi(u_0) \in L^2(\Omega)$, is satisfied if

$$\int_\Omega e^{|u_0|^{2\delta}}\, dx < \infty$$

for some $\delta > \lambda + 1$, and, a fortiori, if $u_0 \in L^\infty(\Omega)$.

§3. A priori estimates

We first consider problem (2.1). For $n \in \mathbb{N}$, we define the following approximate problems

$$\begin{cases} (v_n)_t - \operatorname{div}\,(\mathbf{a}(x,t,v_n)\nabla v_n) = T_n(F(x,t,v_n)) & \text{in } Q_T; \\ v_n(x,t) = 0 & \text{on } \Sigma_T; \\ v_n(x,0) = T_n(v_0(x)) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $T_n(s) = \min\{n, \max\{s,-n\}\}$ is the usual truncation at levels $\pm n$. It is well known that problem (3.1) admits at least one weak solution $v_n \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

Proposition 3.1. *Under the assumptions of Theorem 2.1, there exists a constant C, depending only on the data of the problem, such that*

$$\|v_n\|_{L^\infty(0,T;L^2(\Omega))} + \|v_n\|_{L^2(0,T;H_0^1(\Omega))} \le C\,.$$

Proof. We first observe that, under the hypothesis (2.4), we can assume that the exponent r satisfies $r\alpha \le 1$. Indeed, if this is not the case, we can replace r by a smaller value which still satisfies (2.4), and apply the usual inclusions between Lebesgue spaces.

We multiply the equation (3.1) by v_n and integrate on Ω, for t fixed. Using the assumptions (2.3) and (2.4) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v_n^2\,dx + \nu\int_\Omega |\nabla v_n|^2\,dx &\le \int_\Omega \Big(1+|v_n|\,\big|\log|v_n|\big|^\alpha\Big)|v_n|\,f\,dx\\ &\le C\left[\int_\Omega f + \int_\Omega v_n^2\big|\log|v_n|\big|^\alpha f\,dx\right].\end{aligned}$$

The last integral can be estimated as follows. For $\delta\in]0,1[$, to be chosen hereafter, we can write

$$\begin{aligned}\int_\Omega f\ v_n^2\big|\log|v_n|\big|^\alpha\,dx &= \int_\Omega f\ |v_n|^{2\delta}|v_n|^{2(1-\delta)}\big|\log|v_n|\big|^\alpha\,dx\\ &\le \|f(t)\|_q\left[\int_\Omega |v_n|^{2^*}\,dx\right]^{\frac{2\delta}{2^*}}\left[\int_\Omega |v_n|^{2\rho(1-\delta)}\big|\log|v_n|\big|^{\alpha\rho}\,dx\right]^{\frac{1}{\rho}},\end{aligned}$$

where $\rho>1$ is defined by

$$\frac{1}{\rho}+\frac{2\delta}{2^*}+\frac{1}{q}=1\,.$$

Using Young's inequality, one obtains, for every $\varepsilon>0$,

$$\begin{aligned}&\int_\Omega f\ v_n^2\big|\log|v_n|\big|^\alpha\,dx\\ &\qquad\le \varepsilon\left[\int_\Omega |v_n|^{2^*}\,dx\right]^{\frac{2}{2^*}} + C(\varepsilon)\,\|f(t)\|_q^{\frac{1}{1-\delta}}\left[\int_\Omega |v_n|^{2\rho(1-\delta)}\big|\log|v_n|\big|^{\alpha\rho}\,dx\right]^{\frac{1}{\rho(1-\delta)}}\\ &\qquad\le \varepsilon C\int_\Omega |\nabla v_n|^2\,dx + C(\varepsilon)\,\|f(t)\|_q^{\frac{1}{1-\delta}}\left[\int_\Omega |v_n|^{2\rho(1-\delta)}\big|\log|v_n|\big|^{\alpha\rho}\,dx\right]^{\frac{1}{\rho(1-\delta)}}.\end{aligned}$$

Choosing $\varepsilon=\nu/(2C)$, and δ such that $\frac{1}{1-\delta}=r$, that is, $\delta=\frac{r-1}{r}$, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega v_n^2\,dx + \nu\int_\Omega |\nabla v_n|^2\,dx\\ &\qquad\le C\left(\|f(t)\|_1 + \|f(t)\|_q^{\frac{1}{1-\delta}}\left[\int_\Omega |v_n|^{\frac{2\rho}{r}}\big|\log|v_n|\big|^{\alpha\rho}\,dx\right]^{\frac{r}{\rho}}\right).\qquad(3.2)\end{aligned}$$

Since $q > \frac{Nr}{2(r-1)}$, from the definition of δ and ρ one can check that $\rho < r$. Therefore there exists an increasing and convex function $\eta(s) : [0, +\infty) \to [0, +\infty)$ such that $\eta(0) = 0$ and

$$\eta\left(|\Omega|\, s^{\frac{2\rho}{r}} (\log s)^{\alpha\rho}\right) \sim Cs^2 \qquad \text{for } s \to +\infty.$$

By Jensen's inequality

$$\eta\left(\int_\Omega |v_n|^{\frac{2\rho}{r}} \left|\log |v_n|\right|^{\alpha\rho} dx\right) \le \frac{1}{|\Omega|} \int_\Omega \eta(|\Omega|\, |v_n|^{\frac{2\rho}{r}} \left|\log |v_n|\right|^{\alpha\rho})\, dx \le C\left(\int_\Omega v_n^2\, dx + 1\right).$$

This implies

$$\int_\Omega |v_n|^{\frac{2\rho}{r}} \left|\log |v_n|\right|^{\alpha\rho} dx \le \eta^{-1}\left(C\left(\int_\Omega v_n^2\, dx + 1\right)\right). \tag{3.3}$$

It is easy to see that the function η satisfies

$$\eta^{-1}(s) \sim s^{\frac{\rho}{r}} (\log s)^{\alpha\rho} \qquad \text{for } s \to +\infty.$$

Therefore, (3.3) implies

$$\left[\int_\Omega |v_n|^{\frac{2\rho}{r}} \left|\log |v_n|\right|^{\alpha\rho} dx\right]^{\frac{r}{\rho}} \le CH\left(\int_\Omega v_n^2\, dx\right), \tag{3.4}$$

where

$$H(s) = 1 + s|\log s|^{\alpha r}. \tag{3.5}$$

If we define

$$\xi_n(t) = \int_\Omega [v_n(t)]^2\, dx,$$

we have proved that

$$\xi_n'(t) \le C\left[\|f(t)\|_1 + \|f(t)\|_q^r H(\xi_n(t))\right] \le C\left(\|f(t)\|_1 + \|f(t)\|_q^r\right)[1 + H(\xi_n(t))].$$

Since $\|f(t)\|_1 + \|f(t)\|_q^r$ is an integrable function on $]0, T[$, the last inequality yields

$$G(\xi_n(t)) - G(\xi_n(0)) \le C, \tag{3.6}$$

where

$$G(s) = \int_0^s \frac{d\sigma}{1 + H(\sigma)}.$$

From the assumption (2.4) it follows that $\alpha r \le 1$, therefore the function $1/(1 + H(s))$ is not integrable on $[0, +\infty)$. Since $G(\xi_n(0))$ is bounded, the estimate (3.6) provides a uniform estimate of v_n in $L^\infty(0, T; L^2(\Omega))$. Using (3.2) and (3.4), after integration with respect to time, one obtains an estimate in $L^2(0, T; H_0^1(\Omega))$. $\square$

We can now turn our attention to proving an estimate for problem (2.2). Once again we have to define a sequence of approximate problems:

$$\begin{cases} (u_n)_t - \operatorname{div}\,(\mathbf{a}(x,t,u_n)\nabla u_n) = T_n(B(x,t,u_n,\nabla u_n)) & \text{in } Q_T; \\ u_n(x,t) = 0 & \text{on } \Sigma_T; \\ u_n(x,0) = u_{0,n}(x) & \text{in } \Omega. \end{cases} \tag{3.7}$$

Notice that the prescribed datum at time $t = 0$ is not simply the truncation of u_0 as in the previous proposition, but is a function $u_{0,n} \in L^\infty(\Omega) \cap H_0^1(\Omega)$ such that

$$\frac{1}{n}\|u_{0,n}\|_{H_0^1(\Omega)} \to 0 \qquad \text{as} \quad n \to \infty\,,$$

$$\Psi(u_{0,n}) \to \Psi(u_0) \qquad \text{a.e. and strongly in } L^2(\Omega)\,.$$

The existence of such a sequence may be proved by truncation and convolution. These assumptions are required in order to prove the strong convergence of the gradients ∇u_n (see [DGP] and [DGLS]).

Proposition 3.2. *Under the assumptions above, there exists a constant C, depending only on the data of the problem, such that*

$$\|\Psi(u_n)\|_{L^\infty(0,T;L^2(\Omega))} + \|\Psi(u_n)\|_{L^2(0,T;H_0^1(\Omega))} \le C\,.$$

To prove Proposition 3.2, we need the following elementary lemma:

Lemma 3.1. *For every $\alpha > \frac{\lambda}{\lambda+1}$, there exists a constant C_α such that*

$$e^{|\gamma(s)|} \le C_\alpha\left(1 + |\Psi(s)|\,\big|\log|\Psi(s)|\big|^\alpha\right)$$

for every $s \in \mathbb{R}$.

Proof. It suffices to show that

$$\lim_{s\to+\infty} \frac{e^{\gamma(s)}}{\Psi(s)\,(\log\Psi(s))^\alpha} < +\infty\,,$$

and this follows easily from a repeated application of De L'Hôpital's rule. $\square$

Proof of Proposition 3.2. Let us take $e^{|\gamma(u_n)|}\Psi(u_n)$ as test function in (3.7). Recalling the assumptions on the terms of the equation and integrating on Ω, we obtain for every fixed t:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Psi(u_n)^2\,dx + \nu\int_\Omega |\nabla\Psi(u_n)|^2\,dx + C_1\int_\Omega |\nabla u_n|^2 e^{|\gamma(u_n)|}\Psi(u_n)(1+|u_n|^\lambda)\text{sign } u_n\,dx$$
$$\le C_1\int_\Omega |\nabla u_n|^2 e^{|\gamma(u_n)|}|\Psi(u_n)|(1+|u_n|^\lambda)\text{sign } u_n\,dx + \int_\Omega g e^{|\gamma(u_n)|}|\Psi(u_n)|\,dx\,.$$

Under our hypotheses on g, we can find α such that $\frac{\lambda}{\lambda+1} < \alpha < 1$ and such that (2.4) holds. Then, using Lemma 3.1, after cancellation we can estimate the last integral in the previous formula as follows:

$$\int_\Omega g e^{|\gamma(u_n)|} |\Psi(u_n)| \, dx \le C_\alpha \int_\Omega g |\Psi(u_n)| \left(1 + |\Psi(u_n)| \left|\log |\Psi(u_n)|\right|^\alpha\right) \, dx$$

$$\le C \left(\int_\Omega g \, dx + \int_\Omega g |\Psi(u_n)|^2 \left|\log |\Psi(u_n)|\right|^\alpha dx \right) .$$

From here, reasoning exactly as in the proof of Proposition 3.1, we obtain the differential inequality

$$\xi_n'(t) \le C \left(\|g(t)\|_1 + \|g(t)\|_q^r \, H(\xi_n(t)) \right) ,$$

where

$$\xi_n(t) = \int_\Omega \Psi(u_n(t))^2 \, dx ,$$

and $H(s)$ is defined by (3.5). In view of the assumptions on the initial datum, the desired estimate follows immediately. $\square$

Remark 3.2. From Proposition 3.2 and Gagliardo-Nirenberg's interpolation result (see, for instance, [DB], Chapter I, Proposition 3.1), it follows that $\Psi(u_n)$ is bounded (uniformly with respect to n) in $L^\rho(0, T; L^\sigma(\Omega))$ for every ρ and σ such that

$$2 \le \sigma \le \frac{2N}{N-2} , \qquad 2 \le \rho \le \infty$$

and

$$\frac{N}{\sigma} + \frac{2}{\rho} = \frac{N}{2} .$$

Proposition 3.3. *There exists a positive constant C, depending only on the data of the problem, such that*

$$\int_{\{|u_n|>k\}} \left|T_n B(x, t, u_n, \nabla u_n)\right| \le C \|g \chi_{\{|u_n|>k\}}\|_{r,q} + C \int_{\Omega \cap \{|u_{0,n}|>k\}} \Psi(u_{0,n}) \qquad (3.8)$$

for every $n \in \mathbb{N}$ and $k \ge 0$. In particular the sequence $\{T_n B(x, t, u_n, \nabla u_n)\}$ is bounded in $L^1(Q_T)$.

Proof. The estimate (3.8) follows easily from the following inequality:

$$\int_{\{|u_n|>k\}} (1 + |u_n|^\lambda) |\nabla u_n|^2 \le C \|g \chi_{\{|u_n|>k\}}\|_{r,q} + C \int_{\Omega \cap \{|u_{0,n}|>k\}} \Psi(u_{0,n}) \qquad (3.9)$$

To see this, we multiply the approximate problems (3.7) by $h_k(u_n)$, where

$$h_k(s) = \chi_{\{|s|>k\}}(s) \text{sign } (s) \left(e^{|\gamma(s)| - \gamma(k)} - 1 \right) ,$$

and integrate over Q_T. If we define

$$\phi_k(s) = \int_0^s h_k(\sigma)\, d\sigma\,,$$

we obtain, using the assumptions (2.3) and (2.5),

$$\begin{aligned}\int_{\Omega\cap\{|u_n(T)|>k\}} \phi_k(u_n(T)) - \int_{\Omega\cap\{|u_{0,n}|>k\}} \phi_k(u_{0,n}) + \\ + C_1 \int_{\{|u_n|>k\}} (1+|u_n|)^\lambda |\nabla u_n|^2 e^{|\gamma(u_n)|-\gamma(k)} \\ \le C_1 \int_{\{|u_n|>k\}} (1+|u_n|)^\lambda |\nabla u_n|^2 \left(e^{|\gamma(u_n)|-\gamma(k)} - 1\right) + \int_{\{|u_n|>k\}} g \left(e^{|\gamma(u_n)|-\gamma(k)} - 1\right).\end{aligned}$$

Dropping positive terms, this implies

$$\begin{aligned} C_1 \int_{\{|u_n|>k\}} (1+|u_n|)^\lambda |\nabla u_n|^2 &\le \int_{\{|u_n|>k\}} g e^{|\gamma(u_n)|} + \int_{\Omega\cap\{|u_{0,n}|>k\}} \phi_k(u_{0,n}) \\ &\le \left\|g\,\chi_{\{|u_n|>k\}}\right\|_{r,q} \left\|e^{|\gamma(u_n)|}\right\|_{r',q'} + \int_{\Omega\cap\{|u_{0,n}|>k\}} \phi_k(u_{0,n})\,. \end{aligned}$$

From Lemma 3.1 it follows that

$$e^{|\gamma(s)|} \le C\big(1 + [\Psi(s)]^2\big)\,,$$

so that

$$\left\|e^{|\gamma(u_n)|}\right\|_{r',q'} \le C\left(1 + \|\Psi(u_n)\|^2_{2r',2q'}\right).$$

It is easy to check that the exponent $\rho = 2r'$ and $\sigma = 2q'$ satisfy

$$\frac{N}{\sigma} + \frac{2}{\rho} < \frac{N}{2}\,.$$

Therefore, applying the usual embeddings between Lebesgue spaces and Remark 3.2, the last norm is bounded. Moreover $0 \le \phi_k(u_{0,n}) \le \Psi(u_{0,n})$, therefore (3.9) is completely proved. □

§5 Proof of main results.

Proof of Theorem 2.1. By Proposition 3.1, the sequence $\{v_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and in $L^2(0,T;H_0^1(\Omega))$. Moreover, by the equation, $\{(v_n)_t\}$ is bounded in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$. Using standard compactness results for evolution spaces (see for instance [S]), we can extract a subsequence (still denoted by $\{v_n\}$) which converges to some function v strongly in $L^2(Q_T) \cap C^0([0,T];W^{-1,s}(\Omega))$, for every $s < N/(N-1)$, and weakly in $L^2(0,T;H_0^1(\Omega))$. It is easy to pass to the limit in the weak formulation of (3.1), thus showing that v solves (2.1) in the sense of distributions. Moreover, since there is strong convergence in $C^0([0,T];W^{-1,s}(\Omega))$, the initial datum has a meaning. $\square$

Proof of Theorem 2.2. As in the previous proof, taking Propositions 3.2 and 3.3 into account, we can assume that, passing to a subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)),$$

$$u_n \to u \quad \text{strongly in } L^2(Q_T) \cap C^0([0,T];W^{-1,s}(\Omega)), \text{ for every } s < \frac{N}{N-1},$$

$$\Psi(u_n) \rightharpoonup \Psi(u) \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)) \text{ and } *\text{-weakly in } L^\infty(0,T;L^2(\Omega)).$$

Moreover it can be proved (exactly as in Propositions 6.2. and 6.3 of [DGLS], where a technique introduced in [LM] is developed) that

$$\nabla T_k u_n \to \nabla T_k u \quad \text{a.e. and in } L^2(Q_T), \text{ for every } k > 0,$$

$$\nabla u_n \to \nabla u \quad \text{a.e. and in } L^q(Q_T), \text{ for every } q \text{ such that } 1 \le q < 2,$$

and, using Proposition 3.3,

$$T_n B(x,t,u_n,\nabla u_n) \to B(x,t,u,\nabla u) \quad \text{in } L^1(Q_T).$$

The existence result follows easily. We remark that the initial datum has sense since $u \in C([0,T];W^{-1,s}(\Omega))$. $\square$

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