Nonlinear parabolic problems with a very general quadratic gradient term

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December 16, 2005

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Keywords: Nonlinear parabolic problems, gradient term with quadratic growth, convection-diffusion problems, superlinear reaction term, existence and regularity, bounded and unbounded solutions, a priori estimates, logarithmic Sobolev inequalities.

MSC2000: 35B33, 35B45, 35K20, 35K55, 35K57.

Abstract.

We study existence and regularity of distributional solutions for a class of nonlinear parabolic problems. The equations we consider have a quasi-linear diffusion operator and a lower order term, which may grow quadratically in the gradient and may have a very fast growth (for instance, exponential) with respect to the solution. The model problem we refer to is the following

$$\begin{cases} u_t - \Delta u = \beta(u)|\nabla u|^2 + f(x,t), & \text{in} \quad \Omega\times]0,T[; \\ u(x,t) = 0, & \text{on} \quad \partial\Omega\times]0,T[; \\ u(x,0) = u_0(x), & \text{in} \quad \Omega; \end{cases} \tag{1}$$

with $\Omega \subset \mathbb{R}^N$ a bounded open set, $T > 0$, and $\beta(u) \sim e^{|u|}$; as far as the data are concerned, we assume $\exp(\exp(|u_0|)) \in L^2(\Omega)$, and $f \in X(0,T;Y(\Omega))$, where X, Y are Orlicz spaces of logarithmic and exponential type, respectively. We also study a semilinear problem having a superlinear reaction term, problem that is linked with problem (1) by a change of unknown (see (5) below). Likewise, we deal with some other related problems, which include a gradient term and a reaction term together.

1 Introduction.

The present paper is devoted to prove existence results for some quasilinear parabolic problems with lower order terms, whose model is (1). This kind of problems has been extensively studied in the last years (see for instance [19], [7], [4], [5], [8], [12], [13], [15], [17], [18], [21], [23], [24] and references therein). In those works the hypotheses on the function $\beta(s)$ imply *grosso modo* that $\beta(s)$ is bounded, with some exceptions as in [7] and [19]. In [7] the existence of subsolutions and supersolutions is assumed, while no growth assumption is made on the continuous function β; then it is proved that there exists a distributional solution to problem (1). This solution turns out to be Hölder continuous by Theorem 1.1 in §1, Chapter V of [19].

More recently, in [14], assuming that $\beta(u) = |u|^\lambda$, $\lambda > 0$, the authors are able to prove the existence of solutions to (1) without using sub/supersolutions. In that paper, the assumption on $f(x,t)$ is that

$$f(x,t) \in L^r(0,T;L^q(\Omega)), \qquad r>1, \quad q > \frac{N}{2}\max\left\{\lambda+1, \frac{r}{r-1}\right\},$$

while on the initial datum $u_0(x)$ the assumption is essentially that

$$\int_\Omega e^{2|u_0|^\delta} < +\infty \qquad \text{for some } \delta > \lambda+1.$$

For a slightly improved result for this case, see Section 4. Note that, when λ goes to infinity, the exponent q also goes to infinity . Therefore one could expect that the case where $\beta(s) = e^s$ will require the use of exponential summability in the spatial variable for $f(x,t)$ and a condition like $\int_\Omega e^{2e^{|u_0|}} < +\infty$, for the initial datum. On the other hand, this strong assumption in the x variable allows to require just a slightly superlinear integrability in the time variable (see also Section 7 for some comments on these assumptions).

The aim of this paper, which improves and generalises the results of [14], is, in fact, to deal with very general growth for $\beta(u)$ like, for example, $\beta(u) = e^u$ or $\beta(u) = \exp_{(k)}(s) = \underbrace{\exp(\ldots(\exp}_{k}(s))\ldots)$. The assumptions on $f(x,t)$ and $u_0(x)$, when $\beta(s) = e^s$, are

$$\int_0^T \|f(.,t)\|_\phi \left(\log^* \|f(.,t)\|_\phi\right)\left(\log^*\log^* \|f(.,t)\|_\phi\right) dt < +\infty, \tag{2}$$

where $\log^* s = \max\{1, \log|s|\}$, $\|\ .\ \|_\phi$ denotes an Orlicz norm (see the definition in Section 2) with N-function $\phi(s) = \exp(\exp(s)) - e(s+1)$, and

$$\int_\Omega e^{2e^{|u_0|}} < +\infty\,; \tag{3}$$

this is exactly in the same spirit of the previous considerations.

In order to explain the existence and regularity results, and the assumptions on the data, let us consider problem (1) with $f, u_0 \geq 0$, for a general continuous function $\beta : [0,+\infty[\to [0,+\infty[$ satisfying $\lim_{s\to+\infty}\beta(s) \in]0,+\infty]$. If one performs the change of variable

$$v = \Psi(u) = \int_0^u \exp\Big(\int_0^s \beta(\sigma)\,d\sigma\Big)\,ds, \tag{4}$$

one obtains the semilinear problem

$$\begin{cases} v_t - \Delta v = f(x,t)g(v), & \text{in} \quad \Omega\times]0,T[; \\ v(x,t) = 0, & \text{on} \quad \partial\Omega\times]0,T[; \\ v(x,0) = v_0(x) := \Psi(u_0), & \text{in} \quad \Omega\,, \end{cases} \tag{5}$$

where $g(v) = \exp\Big(\int_0^u \beta(s)\,ds\Big) = \Psi'(\Psi^{-1}(v))$ has a linear or slightly superlinear growth in the sense that

$$\lim_{s\to+\infty}\frac{g(s)}{s} = \lim_{s\to+\infty}\beta(s) \in]0,+\infty]$$

and

$$\int^{+\infty} \frac{1}{g(s)}\, ds = +\infty, \tag{6}$$

as can easily be checked by a change of variable. For instance, if we start from equation

$$u_t - \Delta u = (e^u + 1)|\nabla u|^2 + f$$

and apply the change of variable $v = \exp\big(\exp(s) - 1\big) - 1$, then we obtain the equation

$$v_t - \Delta v = f(v+1)\big(\log(v+1) + 1\big).$$

Similarly, if $\beta(s) = e^{e^s}$, we have that $g(s)$ grows at infinity like $s\,(\log s)\,(\log\log s)$, and so on.

It is well known (see [10]) that if condition (6) does not hold, that is if $\displaystyle\int^{+\infty} \frac{1}{g(s)}\, ds < +\infty$, then there is no global solution to (5) for large f. On the other hand, if condition (6) is satisfied and f and u_0 are bounded, it is easy to obtain a priori estimates using a supersolution of the semilinear problem (5) depending only on the variable t. However, this method is useless if one deals with unbounded f and u_0. So one of the crucial points of the present paper is to find a priori estimates for the problem with an equation like (5) for unbounded data. This has been done by the authors in [14] for an equation whose model is

$$v_t - \Delta v = f(v+1)\big(\log(v+1) + 1\big)^\theta, \qquad 0 < \theta < 1.$$

In order to find results of global existence for the general problem (5), with assumption (6), we use more general techniques which make use of a generalized logarithmic Sobolev's inequality similar to those proved in [2] and [11] leading to the assumptions on f described above. Moreover, the change of variable from quasilinear problem (1) to the semilinear one (5) is not possible in general if one considers operators endowed with more general structure like

$$u_t - \operatorname{div} a(x,t,u,\nabla u) = b(x,t,u,\nabla u), \tag{7}$$

but with similar growths. To overcome this difficulty, one is led to use, as in several other papers on elliptic and parabolic equations with quadratic gradient terms, test functions of exponential type instead of changing the unknown variable. The role of these tests functions in dealing with gradient term is put in evidence in Proposition 3.3 below. Actually, we will consider general equations of the form (7).

The plan of this paper is as follows. The main existence results, Theorems 2.1 and 2.2 below, respectively for problems (5) and (1), are stated in Section 2 together with a result on bounded solutions. In Theorem 2.1 we prove the existence of at least one distributional solution of (5), assuming (2) and $v_0 \in L^2$, while in Theorem 2.2 the hypotheses on the data are (2) and (3) (see (18) below for the exact statement). In both theorems the differential operators and the lower order terms which appear in (5) and (1) may be replaced by fairly general nonlinear terms having similar growth, as in (7).

The proofs of the main results are given in Section 3.

Then Section 4 is devoted to comparison with some previous results, improving those in [14]. Section 5 deals with the case of higher growth for the function β in (1), which corresponds to slightly higher growth for the function g in (5). For instance, if $\beta(s) = \exp_{(2)}(s) = e^{e^s}$, then $g(s)$ grows like $s\,(\log s)\,(\log\log s)$ at infinity; if $\beta(s) = \exp_{(3)}(s)$, then $g(s) \sim s\,(\log s)\,(\log\log s)\,(\log\log\log s)$, and so on. We can prove existence of distributional solutions to these problems under suitable (stronger) assumptions on the data f and u_0, depending on the growth of the functions g and β. The precise results are stated in Theorems 5.1 and 5.2.

In Section 6 we consider the case where both lower order terms appear together in the right-hand side, that is, we consider equations of the form

$$u_t - \Delta u = \beta(u)|\nabla u|^2 + f_1(x,t)g(u) + f_2(x,t)\,,$$

β and g being as above. In this case the nonlinear behaviour of the terms implies that the assumptions on f_1 must be stronger than the one required for f in (5) (that is, when $\beta(s) \equiv 0$), while the hypothesis on f_2 remains the same as the one for f in problem (1).

The last section is devoted to further extensions, a summary and several remarks. We discuss our method of proving the a priori estimates, showing the possibility of using different Orlicz spaces in their proofs and we point out the lack of uniqueness by analyzing the change of unknown function (4).

2 Assumptions and main results.

Let Ω be a bounded open subset in $\mathbb{R}^N$, $N \geq 1$. For $T > 0$, we define the cylinder $Q_T = \Omega\times]0,T[$. The symbols $L^q(\Omega)$, $L^r(0,T;L^q(\Omega))$, and so forth, denote the usual Lebesgue spaces, see for instance [9] or [16]. Moreover we will sometimes use the shorter notations $\|f\|_q$, $\|f\|_{r,q}$ instead of $\|f\|_{L^q(\Omega)}$, $\|f\|_{L^r(0,T;L^q(\Omega))}$, respectively. The simbol $H_0^1(\Omega)$ denotes the Sobolev space of functions with distributional derivatives in $L^2(\Omega)$ which have zero trace on $\partial\Omega$. $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$. The spaces $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$ have obvious meanings, see again [9] or [16]. Let us recall that a function $\varphi(s) : [0,+\infty[\to [0,+\infty[$ is called an N-function if it admits the representation

$$\varphi(s) = \int_0^s p(t)\, dt$$

where $p(t)$ is right continuous for $t \geq 0$, positive for $t > 0$, nondecreasing and satisfying $p(0) = 0$ and $p(\infty) = \infty$. If φ is an N-function, we call Orlicz space associated to φ, denoted by $L_\varphi(\Omega)$, the class of those measurable real functions u, defined on Ω, for which the norm

$$\|u\|_{L_\varphi(\Omega)} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \varphi\Big(\frac{|u|}{\lambda}\Big)\, dx \leq 1\right\}$$

is finite. If the function φ satisfies the so-called Δ_2-condition, i.e.

$$\varphi(2t) \leq k\,\varphi(t)\,,$$

at least for all t large enough, then this space is the same as

$$\left\{u \ : \Omega \to \mathbb{R} \text{ measurable s.t. } \int_\Omega \varphi(|u(x)|)\, dx < +\infty\right\},$$

otherwise this last set is not a vector space. It is clear that N-functions which are asymptotically equivalent near infinity generate the same Orlicz spaces. The following inequality always holds true:

$$\|u\|_{L_\varphi(\Omega)} \leq 1 + \int_\Omega \varphi(|u(x)|)\, dx \tag{8}$$

We will sometimes write $\|u\|_\varphi$ instead of $\|u\|_{L_\varphi(\Omega)}$.

Let φ and $\tilde\varphi$ be two N-functions of class C^1. We say that they are conjugate if $\varphi' = (\tilde\varphi')^{-1}$. For instance, the functions $\varphi(s) = s^p/p$ and $\tilde\varphi(s) = s^{p'}/p'$, with $p, p' > 1$ and $1/p + 1/p' = 1$, are conjugate N-functions. Moreover, as in the case of Lebesgue's spaces, if φ and $\tilde\varphi$ are two conjugate N-functions, the following Hölder inequality holds:

$$\int_\Omega uv\, dx \leq 2\|u\|_{L_\varphi(\Omega)}\|v\|_{L_{\tilde\varphi}(\Omega)}\,,$$

for all $u \in L_\varphi(\Omega)$, $v \in L_{\tilde\varphi}(\Omega)$. Evolution Orlicz spaces $L_\psi(0,T;L_{\tilde\varphi}(\Omega))$ can be defined in an obvious way.

We will consider the following two parabolic problems:

$$\begin{cases} v_t - \operatorname{div} a(x,t,v,\nabla v) = F(x,t,v), & \text{in} \quad Q_T; \\ v(x,t) = 0, & \text{on} \quad \partial\Omega\times]0,T[; \\ v(x,0) = v_0(x), & \text{in} \quad \Omega; \end{cases} \tag{9}$$

$$\begin{cases} u_t - \operatorname{div} a(x,t,u,\nabla u) = b(x,t,u,\nabla u), & \text{in} \quad Q_T; \\ u(x,t) = 0, & \text{on} \quad \partial\Omega\times]0,T[; \\ u(x,0) = u_0(x), & \text{in} \quad \Omega; \end{cases} \tag{10}$$

In both problems, the vector-valued function

$$a(x,t,s,\xi) \; : \; \Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$$

is a Carathéodory function, that is, it is measurable with respect to (x,t) for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) for almost every $(x,t)\in Q_T$. Moreover we assume there exist positive constant Λ and α satisfying:

$$|a(x,t,s,\xi)| \le \Lambda|\xi|\,, \tag{11}$$

$$a(x,t,s,\xi)\cdot\xi \ge \alpha|\xi|^2\,, \tag{12}$$

$$\big[a(x,t,s,\xi) - a(x,t,s,\eta)\big]\cdot(\xi-\eta) > 0\,, \qquad \xi\neq\eta\,. \tag{13}$$

We first consider problem (9), and state the assumptions and the results for this problem.

The function $F : Q_T\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory conditions; moreover there exists a positive measurable function $f(x,t)$ satisfying

$$\int_0^T \|f(.,t)\|_\phi \left(\log^* \|f(.,t)\|_\phi\right)\left(\log^*\log^* \|f(.,t)\|_\phi\right) dt < +\infty, \tag{14}$$

where $\log^* s = \max\{\log s, 1\}$ and $\| \, . \, \|_\phi$ denotes the Orlicz norm associated to the N-function $\phi(s) = \exp(\exp(s)) - e(s+1)$ (see also Remark 2.2 below), such that

$$|F(x,t,s)| \le \big(1+|s|\,\log^*|s|\big)f(x,t)\,; \tag{15}$$

and

$$v_0(x)\in L^2(\Omega). \tag{16}$$

Theorem 2.1 *Under the assumptions* (11)–(16)*, there exists at least one distributional solution* v *to problem* (9) *such that*

$$v\in L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))\,, \qquad F(x,t,v)\in L^1(Q_T)\,.$$

We now turn our attention to problem (10). We will assume that $b(x,t,s,\xi) : Q_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function satisfying

$$|b(x,t,s,\xi)| \le (\mu e^{|s|}+1)|\xi|^2 + f(x,t) =: \beta(s)|\xi|^2 + f(x,t)\,, \tag{17}$$

where μ is a positive constant, and we will assume that f satisfies (14), while u_0 satisfies

$$\exp_{(2)}(|u_0|)\in L^{2\mu}(\Omega)\,. \tag{18}$$

The main result for problem (10) is the following:

Theorem 2.2 *Under the assumptions* (11)–(13)*,* (17) *and* (18)*, there exists at least one distributional solution* u *to problem* (10) *such that*

$$u\in L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))\,, \qquad b(x,t,u,\nabla u)\in L^1(Q_T)\,,$$

and moreover

$$e^{\mu e^{|u|}}\in L^2(0,T;H^1(\Omega))\cap C^0([0,T];L^2(\Omega))\,, \tag{19}$$

Remark 2.1 The assumptions on the initial data in the last two theorems can be weakened, as far as the existence is concerned. For instance, in Theorem 2.1, one can assume $v_0 \in L^d(\Omega)$, for some $d > 1$, but in this case one only obtains that $(1+v)^{d/2} - 1$, rather than v, belongs to $L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$.

Similarly, the assumption on u_0 in Theorem 2.2 can be weakened to $\exp_{(2)}(|u_0|) \in L^{d\mu}(\Omega)$, with $d > 1$, instead of (18), but in this case one obtains $e^{\frac{d}{2}\mu e^{|u|}} \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$ instead of (19). See Remark 3.1 below.

Remark 2.2 We also point out that the assumption (14) on the datum $f(x,t)$ can be weakened too. Indeed, as will be clear in the proof of the main a priori estimates (Propositions 3.1 and 3.4), it is enough to assume that (14) is satisfied with an N-function $\phi(s)$ which grows at infinity like $\int_0^s \exp(\exp(s))\,ds$ instead of $\phi(s) \sim \exp(\exp(s))$.

Furthermore, if the data are slightly more regular, we can prove that the solutions found in Theorems 2.1 and 2.2 are bounded:

Theorem 2.3 *Let* (11)–(13) *be assumed and consider*

$$f \in L^r(0,T;L^q(\Omega))\,, \qquad r,q > 1, \quad \frac{N}{2q} + \frac{1}{r} < 1\,. \tag{20}$$

i) Under hypotheses (15) *and* $v_0 \in L^\infty(\Omega)$*, the solution obtained in Theorem 2.1 is bounded in* Q_T*.*

ii) Under hypotheses (17) *and* $u_0 \in L^\infty(\Omega)$*, the solution obtained in Theorem 2.2 is bounded in* Q_T*.*

Remark 2.3 Condition (20) on (r,q) is known as the Aronson-Serrin condition (see [3]). The assumptions (14), and (20) are all satisfied, for instance, if

$$f \in L^r(0,T;L_\phi(\Omega))\,.$$

with $r > 1$, where ϕ is the same as in (14).

Remark 2.4 We do not know whether the assumptions (14) and $v_0 \in L^\infty(\Omega)$ are sufficient, without the hypothesis (20), to guarantee the existence of bounded solutions of problem (9). A similar remark applies to problem (10).

An essential tool for proving the a priori estimates is given by the following Proposition, which is a generalisation of a logarithmic Sobolev inequality of the same form as in [2] and [11].

Proposition 2.1 *Let* $A : [0,+\infty[\to [0,+\infty[$ *be a nonnegative, nondecreasing function satisfying the* Δ_2*-condition, that is,*

$$A(2t) \le K A(t)\,,$$

for all t *and for some positive* K*. Then there exists a positive constant* $\overline{C}$ *(depending on* N*,* K*,* $|\Omega|$*,* A*) such that, for every* $\varepsilon > 0$ *and for every* $u \in H_0^1(\Omega)$*,*

$$\int_\Omega |u(x)|^2 A(\log^* |u(x)|)\,dx \le \overline{C}\left[\varepsilon \int_\Omega |\nabla u(x)|^2\,dx + \|u\|_2^2 A\left(\log^* 1/\varepsilon\right) + \|u\|_2^2 A(\log^* \|u\|_2)\right].$$

In the proofs of the next Sections we will often use the letter c to denote different constants appearing in the calculations, depending only on the data of the considered problem. The value of c may change from line to line.

3 Proof of the main results.

We begin this section by proving Proposition 2.1.
Proof of Proposition 2.1: It is enough to prove the inequality for $\varepsilon \le 1$. Let us first suppose $\|u\|_2 = 1$. Then

$$\int_\Omega u^2 A(\log^* |u|) \le \int_{\{|u|>e\}} u^2 A(\log |u|) + A(1)\|u\|_2^2$$
$$= \int_{\{|u|>e\}} u^2 A\Big(\log(\varepsilon^{N/2}|u|) + \frac{N}{2}\log(1/\varepsilon)\Big) + c\,.$$

Note that the Δ_2-condition implies

$$A(r+s) \le k\big(A(r) + A(s)\big) \qquad \text{for every } r, s \in [0, +\infty[,$$

so that

$$\int_{\{|u|>e\}} u^2 A\Big(\log(\varepsilon^{N/2}|u|) + \frac{N}{2}\log(1/\varepsilon)\Big) \le k\int_\Omega u^2 A\big(\big(\log(\varepsilon^{N/2}|u|)\big)_+\big) + ckA(\log 1/\varepsilon)\,\|u\|_2^2\,,$$

where $s_+ = \max\{s, 0\}$ denotes the positive part of s. Applying Hölder's inequality to the first term of the right-hand side, we get

$$\int_\Omega u^2 A\big(\big(\log(\varepsilon^{N/2}|u|)\big)_+\big) \le \Big(\int_\Omega |u|^{2N/(N-2)}\Big)^{\frac{N-2}{N}} \Big(\int_\Omega \Big(A\big(\big(\log(\varepsilon^{N/2}|u|)\big)_+\big)\Big)^{N/2}\Big)^{\frac{2}{N}}$$
$$\le c\,\varepsilon \int_\Omega |\nabla u|^2\,,$$

where we have used Sobolev's inequality and the fact that a function satisfying the Δ_2-condition has at most a polynomial growth, so that

$$A\big(\big(\log(\varepsilon^{N/2}|u|)\big)_+\big) \le c\,\varepsilon |u|^{2/N}\,,$$

and, as a consequence,

$$\Big[\int_\Omega \Big(A\big(\big(\log(\varepsilon^{N/2}|u|)\big)_+\big)\Big)^{N/2}\Big]^{2/N} \le c\,\varepsilon\Big[\int_\Omega |u|\Big]^{2/N} \le c\,\varepsilon \|u\|_2^{2/N} \le c\,\varepsilon\,.$$

Hence we have proved that, if $\|u\|_2 = 1$,

$$\int_\Omega u^2 A(\log^* |u|) \le c\,\Big(\varepsilon\|\nabla u\|_2^2 + A(\log 1/\varepsilon) + 1\Big) \le c\,\Big(\varepsilon\|\nabla u\|_2^2 + A(\log^* 1/\varepsilon)\Big)\,.$$

When $\|u\|_2 \ne 1$, we may write $u = \|u\|_2 v$ and apply the last estimate to v to obtain the result, after using the inequality

$$A\big(\log^*(\|u\|_2|v|)\big) \le k\,\Big(A\big(\log^* \|u\|_2\big) + A\big(\log^* |v|\big)\Big)\,.$$

■

In order to prove Theorem 2.1, we introduce the following approximating problems.

$$\begin{cases} v_t - \operatorname{div} a(x,t,v,\nabla v) = T_n\Big(F(x,t,v)\Big), & \text{in} \quad \Omega\times]0,T[; \\ v(x,t) = 0, & \text{on} \quad \partial\Omega\times]0,T[; \\ v(x,0) = T_n(v_0(x)), & \text{in} \quad \Omega, \end{cases} \tag{21}$$

where $T_n(s) = \min\{n\,,\, \max\{s, -n\}\}$ is the usual truncation at levels $\pm n$.

It is well known that there exists (see [19] or [22]) at least a bounded weak solution

$$v_n \in L^2\big(0,T; H^1_0(\Omega)\big) \cap C^0\big([0,T]; L^2(\Omega)\big).$$

The next a priori estimate is the main tool for proving the existence result:

Proposition 3.1 *Let v_n be a solution of problem* (21). *Then there exists a positive constant C such that, for all $n \in \mathbb{N}$,*

$$\sup_{t\in[0,T]} \int_\Omega |v_n|^2 + \int_{Q_T} |\nabla v_n|^2 \le C \tag{22}$$

$$\int_{Q_T} f\,|v_n|^2\, \log^* |v_n| \le C\,. \tag{23}$$

Moreover, the sequence $\Big\{ \int_\Omega |v_n|^2 dx \Big\}_n$ *is relatively compact in* $C([0,T])$.

Proof: Using v_n as test function in (21), we obtain, fixed $t \in]0,T[$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_n^2\, dx + \alpha \int_\Omega |\nabla v_n|^2\, dx \le \int_\Omega f(x,t)\, |v_n|(1+|v_n|\log^*|v_n|)\, dx = I\,. \tag{24}$$

Note that

$$I \le c\Bigg(\int_\Omega f\, dx + \int_\Omega f\, v_n^2\, \log^*|v_n|\, dx\Bigg).$$

We now use the generalized Hölder-Orlicz inequality, with the pair

$$\varphi(s) = \int_0^s \log\big(1+\log(1+\sigma)\big)\, d\sigma \qquad \tilde\varphi(s) = \int_0^s \big(e^{(e^\sigma - 1)} - 1\big)\, d\sigma$$

of conjugate N-functions. Note that, for $s \to +\infty$,

$$\varphi(s) \sim s\log\log s\,, \qquad \tilde\varphi(s) \sim \int_0^s \exp(\exp(\sigma))\, d\sigma\,, \qquad \frac{\tilde\varphi(s)}{\phi(s)} \to 0\,, \tag{25}$$

where we denote $\phi(s) = \exp(\exp(s)) - e(s+1)$. Then we obtain

$$\int_\Omega f\, v_n^2\, \log^*|v_n|\, dx \le 2\, \|v_n^2\, \log^*|v_n|\|_\varphi \|f\|_{\tilde\varphi} \le c\, \|v_n^2\, \log^*|v_n|\|_\varphi \|f\|_\phi\,.$$

Now, by (8) and (25), one has

$$\|v_n^2\, \log^*|v_n|\|_\varphi \le 1 + \int_\Omega \varphi(v_n^2\, \log^*|v_n|)\, dx \le c\Bigg(1 + \int_\Omega v_n^2\, \big(\log^*|v_n|\big)\,\big(\log^*\log^*|v_n|\big)\, dx\Bigg).$$

Now, using Proposition 2.1 with $A(s) = s\log^* s$ and $\varepsilon = \alpha/(2\,c\,\overline{C}\,\|f\|_\phi)$, we then obtain

$$I \le c\,\big(\|f\|_1 + \|f\|_\phi\big) + \frac{\alpha}{2}\int_\Omega |\nabla v_n|^2\, dx + c\,\|f\|_\phi \|v_n\|_2^2 \Big(A\big(\log^*\|v_n\|_2^2\big) + A\big(\log^*\|f\|_\phi\big)\Big)\,. \tag{26}$$

Putting (24) and (26) together, one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v_n^2\, dx + \frac{\alpha}{2}\int_\Omega |\nabla v_n|^2\, dx &\le c\,\big(\|f\|_1 + \|f\|_\phi\big) + \\ &+ c\,\|f\|_\phi \|v_n\|_2^2 \Big(A\big(\log^*\|v_n\|_2^2\big) + A\big(\log^*\|f\|_\phi\big)\Big)\,.\end{aligned} \tag{27}$$

Norms of f appearing in the right-hand side of (27) depend on t, and are integrable functions on $]0,T[$, by our assumptions on f. Therefore, if we set

$$\xi_n(t) = \int_\Omega v_n^2(t)\, dx\,,$$

from (27) we obtain an inequality of the form

$$\xi_n'(t) \le \Upsilon(t)\,[1 + H(\xi_n(t))]\ , \tag{28}$$

where $\Upsilon(t) \in L^1(0,T)$, while $H(s)$ is a positive function such that

$$\int_0^{+\infty} \frac{ds}{1+H(s)} = +\infty\,. \tag{29}$$

Therefore, if we define

$$G(s) = \int_0^s \frac{d\sigma}{1+H(\sigma)}\,,$$

from (28) we obtain

$$G(\xi_n(t)) - G(\xi_n(0)) \le c\,,$$

which implies a global estimate on $\xi_n(t)$ by (29). Going back to (27), this yields the first desired estimate (22). Observe that the integral I in (24) is bounded by an integrable function of t, so that the second estimate follows.

Let us turn to see the sequence $\left\{ \int_\Omega |v_n|^2 dx \right\}_n$ is relatively compact in $C([0,T])$. It follows from

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v_n|^2 dx = \int_\Omega T_n\big(F(x,t,v_n)\big)v_n - \int_\Omega a(x,t,v_n,\nabla v_n)\cdot\nabla v_n$$

that

$$|\xi_n'(t)| = \Big|\frac{d}{dt}\int_\Omega |v_n|^2 dx\Big| \le 2\int_\Omega |T_n\big(F(x,t,v_n)\big)v_n| + 2\Lambda\int_\Omega |\nabla v_n|^2 \le \rho(t)\,,$$

where the right hand side is a function belonging to $L^1(0,T)$. Hence,

$$|\xi_n(t) - \xi_n(s)| \le \Big|\int_s^t \rho(t)\, dt\Big|$$

and so we obtain the required equicontinuity on $\{\xi_n\}_n$. ∎

Proof of Theorem 2.1 By Proposition 3.1, the sequence $\{v_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and in $L^2(0,T;H_0^1(\Omega))$. Moreover, by the equation, $\{(v_n)_t\}$ is bounded in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$. Using standard compactness results for evolution spaces (see for instance [25]), we can extract a subsequence (still denoted by $\{v_n\}$) which converges to some function v strongly in $L^2(Q_T)$ and weakly in $L^2(0,T;H_0^1(\Omega))$. Moreover, the right-hand side of the equation is equi-integrable:

$$\int_E f\left(1 + |v_n|\log^*|v_n|\right) \le \int_{E\cap\{|v_n|\le k\}} f\left(1 + |v_n|\log^*|v_n|\right) + \frac{1}{k}\int_{\{|v_n|\ge k\}} f\,|v_n|\left(1 + |v_n|\log^*|v_n|\right),$$

thus, the equi-integrability follows from (23).

Applying [6], it is easy to see that the gradients converge strongly in $L^2(Q_T)$, and so pass to the limit in the weak formulation of (21), thus showing that v solves (9) in the sense of distributions. As far as the initial datum is concerned, one can use Proposition 3.2 below (which can be proved using the same argument as in Proposition 6.4 of [13]), which also shows that $v \in C([0,T];L^1(\Omega))$.

Finally, we will prove that our solution belongs to $C\big([0,T];L^2(\Omega)\big)$. Defining $\xi(t) = \int_\Omega |v(x,t)|^2\, dx$, we deduce from Proposition 3.1 that ξ belongs to $C([0,T])$. Since we already know that $v \in C\big([0,T];L^1(\Omega)\big)$, it follows that it actually belongs to $C\big([0,T];L^2(\Omega)\big)$. ∎

Proposition 3.2 *Let* $v_n \in L^2\big(0,T;H_0^1(\Omega)\big) \cap C\big([0,T];L^2(\Omega)\big)$ *be a sequence of solutions of problems*

$$\begin{cases} (v_n)_t - \operatorname{div} a(x,t,v_n,\nabla v_n) = g_n, & in \quad \Omega\times]0,T[; \\ v_n(x,0) = v_{0,n}, & in \quad \Omega, \end{cases}$$

such that

$$g_n \in L^2(Q_T)\,, \quad g_n \to g \text{ in } L^1(Q_T),$$
$$v_{0,n} \to v_0 \text{ in } L^1(\Omega),$$
$$\nabla T_k(v_n) \to \nabla T_k(v) \text{ in } L^2(Q_T;\mathbb{R}^N), \text{ for every } k>0,$$
$$\nabla v_n \text{ bounded in } L^2(Q_T;\mathbb{R}^N).$$

Then $v_n \to v$ *in* $C\big([0,T];L^1(\Omega)\big)$.

Remark 3.1 Now we can explain how to obtain the result which has been stated in Remark 2.1 in the case where the initial datum v_0 is less integrable. Indeed one can use $\big((1+|v_n|)^{d-1}-1\big)$ sign v_n, instead of v_n, as test function in (21), and then follow the outline of the previous proof. Note that this choice of test function does not affect the assumptions on the datum f, but provides different regularity of solutions. A similar consideration applies to the proof of the next proposition concerning the quasi-linear quadratic problem.

Let us turn our attention to problem (10), under hypothesis (17). We recall that $\beta(s) = \mu e^{|s|}+1$. Then we define

$$\gamma(s) = \int_0^s \beta(\sigma)\,d\sigma = \mu(e^{|s|}-1)\text{ sign } s + s\,, \qquad \Psi(s) = \mu^{-1}\big(e^{\mu(e^{|s|}-1)}-1\big)\text{ sign } s\,. \tag{30}$$

For $n \in \mathbb{N}$, we introduce the following approximate problem:

$$\begin{cases} (u_n)_t - \operatorname{div} a(x,t,u_n,\nabla u_n) = T_n(b(x,t,u_n,\nabla u_n)), & \text{in} \quad \Omega\times]0,T[; \\ u_n(x,t) = 0, & \text{on} \quad \partial\Omega\times]0,T[; \\ u_n(x,0) = u_{0,n}(x), & \text{in} \quad \Omega, \end{cases} \tag{31}$$

where $u_{0,n}$ are in $L^\infty(\Omega)\cap H_0^1(\Omega)$ and satisfy the same requirements as in [13] with a view to pass to the limit, namely:

$$\frac{1}{n}\|u_{0,n}\|_{H_0^1(\Omega)} \to 0 \qquad \text{as} \quad n\to\infty\,,$$
$$\Psi(u_{0,n}) \to \Psi(u_0) \qquad \text{a.e. and strongly in } L^2(\Omega)\,,$$

(see also [13] for an easy proof of the existence of such a sequence). Note that there exists at least a bounded solution $u_n \in L^2(0,T;H_0^1(\Omega))$ of (31) by [22].

We will need the following cancellation result (see [13]):

Proposition 3.3 *Assume that* u_n *is a bounded weak solution of* (31)*. If* ψ *is a locally Lipschitz continuous and increasing function such that* $\psi(0)=0$, *then for a.e.* $t\in]0,T[$ *one has*

$$\frac{d}{dt}\int_\Omega \phi(u_n(\cdot,t))\,dx + \alpha\int_\Omega e^{|\gamma(u_n(\cdot,t))|}\psi'(u_n(\cdot,t))|\nabla u_n(\cdot,t)|^2\,dx \le \int_\Omega f(\cdot,t)e^{|\gamma(u_n(\cdot,t))|}|\psi(u_n(\cdot,t))|\,dx\,,$$

and therefore

$$\sup_{\tau\in[0,T]}\int_\Omega \phi(u_n)(\tau))\,dx + \alpha\int_{Q_T} e^{|\gamma(u_n)|}\psi'(u_n)|\nabla u_n|^2 \le 2\int_{Q_T} fe^{|\gamma(u_n)|}|\psi(u_n)| + 2\int_\Omega \phi(u_{0,n})\,dx\,,$$

where $\phi(s) = \int_0^s e^{|\gamma(\sigma)|}\psi(\sigma)\,d\sigma$.

The next a priori estimate is the main tool for proving the existence result:

Proposition 3.4 *Let u_n be a solution of problem* (31). *Then there exists a positive constant C such that*

$$\sup_{t\in[0,T]} \int_\Omega |\Psi(u_n)|^2\,dx + \int_{Q_T} |\nabla \Psi(u_n)|^2 \le C \tag{32}$$

and

$$\int_{Q_T} f\,\Psi(u_n)^2\,\log^*|\Psi(u_n)| \le C\,.$$

for all $n \in \mathbb{N}$. Moreover:

$$\text{the sequence } \Big\{ \int_\Omega |\Psi(u_n)|^2 dx \Big\}_n \text{ is relatively compact in } C([0,T])\,, \tag{33}$$

$$\text{the sequence } \Big\{ \int_\Omega u_n^2 dx \Big\}_n \text{ is relatively compact in } C([0,T])\,. \tag{34}$$

Proof: It is easy to show that

$$e^{|\gamma(s)|} \le c\Big(1 + |\Psi(s)|\log^*|\Psi(s)|\Big)\,. \tag{35}$$

Using Proposition 3.3 with $\psi = \Psi$, so that $\phi(s) = \Psi(s)^2/2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \Psi(u_n)^2\,dx + \alpha\int_\Omega |\nabla\Psi(u_n)|^2\,dx &= \int_\Omega f(x,t)\,e^{|\gamma(u_n)|}|\Psi(u_n)|\,dx\,.\\ &\le c\int_\Omega f(x,t)\,|\Psi(u_n)|\Big(1 + |\Psi(u_n)|\log^*|\Psi(u_n)|\Big)\,.\end{aligned}$$

This inequality is formally identical to that in (24), whith $v_n = \Psi(u_n)$. Therefore proceeding as in the proof of Proposition 3.1 we get the desired estimates and the equicontinuity (33). In order to show (34) one can use the same method, multiplying problem (31) by u_n and using the a priori estimates. ∎

Remark 3.2 Note that, since Ψ has a superlinear growth, estimate (32) implies that the sequence $\{u_n\}$ is bounded in $L^2\big(0,T;H_0^1(\Omega)\big)$.

Proposition 3.5 *1) There exists a positive constant C such that*

$$\int_{Q_T} T_n(b(x,t,u_n,\nabla u_n)) \le C\,, \qquad \textit{for every } n\in\mathbb{N}.$$

2)

$$\lim_{m\to+\infty} \int_{\{|u_n|>m\}} |T_n(b(x,t,u_n,\nabla u_n))| = 0\,, \qquad \textit{uniformly in } n\in\mathbb{N}.$$

Proof: 1) By (17)

$$\int_{Q_T} T_n(b(x,t,u_n,\nabla u_n)) \le \int_{Q_T} (\mu e^{|u_n|}+1)|\nabla u_n|^2 + \int_{Q_T} f = I_1 + I_2\,.$$

Since (see (30))

$$\mu e^{|s|} + 1 \le c\Psi'(s) \le c\Psi'(s)^2\,, \tag{36}$$

we have

$$I_1 \le c \int_{Q_T} |\nabla \Psi(u_n)|^2 \,,$$

which is bounded by Proposition 3.3, while the estimate of the term I_2 is straightforward.

2)

$$\int_{\{|u_n|>m\}} |T_n(b(x,t,u_n,\nabla u_n))| \le \int_{\{|u_n|>m\}} (\mu e^{|u_n|}+1)|\nabla u_n|^2 + \int_{\{|u_n|>m\}} f = J_1 + J_2 \,.$$

By (36) and estimate (32) one has

$$J_1 \le \frac{c}{\Psi'(m)} \int_{Q_T} |\nabla \Psi(u_n)|^2 \le \frac{c}{\Psi'(m)} \,.$$

The estimate of J_2 is trivial. ∎

Proof of Theorem 2.2. We consider a sequence $\{u_n\}$ of solutions of the approximate problems (31). By Proposition 3.4 and Remark 3.2, the sequence $\{u_n\}$ is bounded in $L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$, which, together with Proposition 3.5, implies that $(u_n)_t$ is bounded in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$. Therefore, using a standard compactness result (see for instance [25]), we can deduce that there exists $u \in L^2(0,T;H_0^1(\Omega))$ such that, up to a subsequence,

$$\begin{aligned} u_n &\to u && \text{almost everywhere in } Q_T, \\ u_n &\to u && \text{strongly in } L^2(Q_T), \\ u_n &\to u && \text{weakly in } L^2(0,T;H_0^1(\Omega)). \end{aligned}$$

The strong convergence in $L^2(\Omega;\mathbb{R}^N)$ of $\{\nabla T_k(u_n)\}_n$, for every $k>0$, can be proved as in Proposition 6.2 in [13], with minor modifications needed in Step 2 of the proof. Thus the gradients ∇u_n converge almost everywhere in Q_T. This fact and the previous convergences imply on the one hand that

$$\begin{gathered} a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u) \qquad \text{strongly in } L^q(Q_T;\mathbb{R}^N) \text{ for every } q<2, \\ a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u) \qquad \text{weakly in } L^2(Q_T;\mathbb{R}^N) \end{gathered}$$

(recall assumption (11)) and on the other hand that

$$T_n\big(b(x,t,u_n,\nabla u_n)\big) \to b(x,t,u,\nabla u_n) \qquad \text{strongly in } L^1(Q_T) \,,$$

by Proposition 3.5. Hence, one can pass to the limit in each term of the distributional formulation of (31). Therefore u is a distributional solution of the equation in (10). Using Proposition 3.2, we obtain that $u_n \to u$ in $C([0,T];L^1(\Omega))$, which shows that the initial datum is attained. By (33) and (34), one easily obtains that $u, \Psi(u) \in C([0,T];L^2(\Omega))$. Recalling the definition of Ψ, this gives (19). ∎

Proof of Theorem 2.3. As far as the semilinear problem (9) is concerned, the result is a consequence of Theorem 2.1, p.425, of [19] and of the a priori estimates obtained in Proposition 3.1.

In the case of the quadratic problem, we wish to use exponential test functions in order to apply again the techniques by Ladyzenskaja, Solonnikov and Ural'ceva in [19]: first of all we observe that one can assume that the initial data of the approximating problems (31) also satisfy

$$\|u_{0n}\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \,;$$

Then we take $\psi(s) = G_m(\Psi(s))$, $m \ge \|\Psi(u_0)\|_{L^\infty(\Omega)}$, in Proposition 3.3, where Ψ is defined by (30), and

$$G_m(s) := s - T_k(s) = (|s|-k)_+ \operatorname{sign} s \,.$$

We obtain

$$\sup_{t\in[0,T]}\int_\Omega G_m(\Psi(u_n))^2\,dx+\int_{Q_T}|\nabla G_m(\Psi(u_n))|^2\le c_\varepsilon\int_{Q_T\cap\{|\Psi(u_n)|>m\}} f\,|\Psi(u_n)|^{2+\varepsilon}\,,$$

where we have used (35) and the fact that $\log^* s$ grows less than any power s^ε at infinity ($\varepsilon>0$ to be determined). Now, by Young's inequality,

$$f\,|\Psi(u_n)|^{2+\varepsilon}\le c\,\left(f^{p'}\Psi(u_n)^2+|\Psi(u_n)|^{2+\varepsilon p}\right)\,,$$

with $p>1$ such that

$$\varphi(x,t)=f^{p'}\in L^\rho(0,T;L^\sigma(\Omega))\,,\qquad \rho,\sigma>1,\quad \frac{N}{2\sigma}+\frac{1}{\rho}<1\,,$$

Once p is fixed, ε is chosen in such a way that $\delta:=2+\varepsilon p$ is close enough to 2 (for instance, $\varepsilon<4/(pN)$ will suffice). Then one has

$$\sup_{t\in[0,T]}\int_\Omega G_m(\Psi(u_n))^2\,dx+\int_{Q_T}|\nabla G_m(\Psi(u_n))|^2\le c\int_{Q_T\cap\{|\Psi(u_n)|>m\}}\varphi\,|\Psi(u_n)|^2+c\int_{Q_T\cap\{|\Psi(u_n)|>m\}}|\Psi(u_n)|^\delta\,,$$

which is essentially inequality (2.8) on page 425 of [19] for the function $\Psi(u_n)$, under the same hypotheses. Therefore, this yields an L^∞-estimate of the sequence $\{\Psi(u_n)\}_n$ and so, by the definition of Ψ (30), also an estimate on $(u_n)_n$. Once an estimate in $L^\infty(Q_T)$ is obtained, it is clear that this also holds for the limit function u. ■

4 Comparison with earlier results

4.1 Comparison with the case $\beta\equiv 1$.

In this subsection we will deal with the most classical case in which

$$|b(x,t,u,\nabla u)|\le|\nabla u|^2+f \tag{37}$$

(while the operator on the left-hand side of the equations continues to verify assumptions (11)–(13)). This corresponds to our problem (10) when $\mu=0$ in (17). Observe that this situation falls outside the framework studied in the previous Section, since there we have assumed $\mu>0$ in (17). One may wonder what happens if we apply our schema to this borderline problem. Then we would obtain the following result.

Theorem 4.1 *If $f(x,t)\in L^1\log L^1(0,T;L_{\exp}(\Omega))$, that is,*

$$\int_0^T\|f(.,t)\|_{L_\varphi(\Omega)}\log^*\|f(.,t)\|_{L_\varphi(\Omega)}\,dt<\infty\,,$$

where $\varphi(s)=e^s-s-1$, and $e^{|u_0|}\in L^2(\Omega)$, then there exists at least one distributional solution u of problem (10) with right-hand side satisfying (37) such that the functions u and $e^{|u|}-1$ belong to $L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))$.

This result is proved in a similar way to Theorem 2.1: if one sets $\gamma(s)=s$, after defining the approximate problems (31), one can check that Proposition 3.3 hold. Then, applying it with $\psi(s)=\Psi(s)=\left(e^{|s|}-1\right)\operatorname{sign}s$, one easily obtain an a priori estimate of the same type as Proposition 3.4. The conclusion is straightforward.

Remark 4.1 We point out that, in the situation of problem (10) with right hand side (37), the standard hypothesis (see [19] and [22]) on the datum f is

$$f(x,t) \in L^r(0,T;L^q(\Omega)), \qquad r>1, \quad \frac{1}{r}+\frac{N}{2q} \le 1.$$

Following our scheme, we cannot deduce an a priori estimate from this assumption and Proposition 2.1. Indeed, we have to estimate the term containing f in the proof of Proposition 3.4:

$$\begin{aligned} I &= \int_\Omega f(x,t)\, e^{|u|} |\Psi(u)|\, dx = \int_\Omega f\,(1+|\Psi(u)|)\, |\Psi(u)|\, dx \\ &\le \frac{1}{2}\int_\Omega f\, dx + \frac{3}{2}\int_\Omega f\,\Psi(u)^2\, dx \le \frac{1}{2}\|f(.,t)\|_1 + \frac{3}{2}\,\|f(.,t)\|_q \Big(\int_\Omega |\Psi(u)|^{2q'}\, dx\Big)^{1/q'}, \end{aligned}$$

where $\Psi(s) = \big(e^{|s|}-1\big)$ sign s. Since

$$\int_\Omega |\Psi(u)|^{2q'}\, dx = \int_\Omega \Psi(u)^2 A(\log^* |\Psi(u)|)\, dx$$

with $A(s) = e^{2s(q'-1)}$ for s large, it is obvious that A does not satisfies the Δ_2-condition, therefore we cannot apply Proposition 2.1. The conclusion, in this situation, is that the inequality of Proposition 2.1 can only be applied to treat source functions belonging to Orlicz' spaces slightly superlinear on t and of exponential type on x.

4.2 Comparison with the case $\beta(s) = |s|^\lambda$

We continue by comparing our results with those in [14]. In that paper two main problems are studied, the first one considers problem (9) in the case where the source term $F(x,t,u)$ satisfies

$$|F(x,t,s)| \le f(x,t)\big(1+|s|(\log^*|s|)^\theta\big), \qquad \text{with} \quad 0<\theta<1, \tag{38}$$

instead of (15). We will next improve a little bit those results by applying Proposition 2.1.

Theorem 4.2 *Assume $f \in L^r(\log L)^{r\theta}\big(0,T;L^q(\Omega)\big)$; that is, $\|f(\cdot,t)\|_q\big(\log^*\|f(\cdot,t)\|_q\big)^\theta \in L^r(0,T)$; with*

$$q \ge \frac{N}{2}\max\left\{\frac{r}{r-1}, \frac{1}{1-\theta}\right\}.$$

If $v_0 \in L^2(\Omega)$, then there exists at least one distributional solution v of problem (9) *such that $v \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$.*

Proof: As usual, we begin the proof by considering the "truncated" problems (21), and let (v_n) denote the corresponding sequence of bounded solutions.

All we need to prove is a priori estimates similar to those of Proposition 3.1. Then the convergence of approximate solutions follows the proof of Theorem 2.1, and will be omitted.

We point out that $2q = Nr'$ may be assumed; if it is not the case, just replace r by a smaller value satisfying the equality and apply the usual inclusions between Lebesgue's spaces. So we will assume $2q = Nr'$ which implies $\dfrac{1}{1-\theta} \le \dfrac{r}{r-1}$, that is, $\theta r \le 1$.

Now, for t fixed, multiply the equation in (21) by $v_n(.,t)$ and integrate on Ω. Using the assumption (38), this gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega v_n^2\, dx + \alpha\int_\Omega |\nabla v_n|^2\, dx &\le \int_\Omega f\,|v_n|\,\big(1+|v_n|(\log^*|v_n|)^\theta\big)\, dx \\ &\le c\int_\Omega f\, dx + c\int_\Omega f\,|v_n|^2(\log^*|v_n|)^\theta\, dx\,. \end{aligned} \tag{39}$$

We then proceed to estimate the last integral. Observing that $2q = Nr'$ implies

$$\frac{1}{q} + \frac{2}{2^* r'} + \frac{1}{r} = 1\,,$$

it follows from the Hölder, Young and Sobolev inequalities that, for every $\delta > 0$,

$$\begin{aligned}\int_\Omega f\,|v_n|^2 (\log^* |v_n|)^\theta\,dx &= \int_\Omega f\,|v_n|^{2/r'} |v_n|^{2/r} (\log^* |v_n|)^\theta\,dx \\ &\le \|f(.,t)\|_q\,\|v_n(.,t)\|_{2^*}^{2/r'} \Big(\int_\Omega v_n^2 (\log^* |v_n|)^{\theta r}\,dx\Big)^{1/r} \\ &\le \delta\,\|v_n(.,t)\|_{2^*}^2 + c(\delta)\|f(.,t)\|_q^r \int_\Omega v_n^2 (\log^* |v_n|)^{\theta r}\,dx \\ &\le \delta\,c \int_\Omega |\nabla v_n|^2 dx + c(\delta)\|f(.,t)\|_q^r \int_\Omega v_n^2 (\log^* |v_n|)^{\theta r}\,dx\,.\end{aligned}$$

Hence, (39) becomes

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_n^2\,dx + \alpha \int_\Omega |\nabla v_n|^2\,dx \le \delta\,c \int_\Omega |\nabla v_n|^2 dx + c\,\|f(.,t)\|_1 + c(\delta)\|f(.,t)\|_q^r \int_\Omega v_n^2 (\log^* |v_n|)^{\theta r}\,dx$$

and taking $\delta = \alpha/(2c)$, it yields

$$\frac{d}{dt}\int_\Omega v_n^2\,dx + \int_\Omega |\nabla v_n|^2\,dx \le c\,\|f(.,t)\|_1 + c\,\|f(.,t)\|_q^r \int_\Omega v_n^2 (\log^* |v_n|)^{\theta r}\,dx\,. \tag{40}$$

By applying Proposition 2.1 with $A(s) = s^{\theta r}$, we deduce that

$$\int_\Omega v_n^2 \big(\log^* |v_n|\big)^{\theta r}\,dx \le \varepsilon\,c \int_\Omega |\nabla v_n|^2\,dx + c\left[(\log^* 1/\varepsilon)^{\theta r} + \Big(\log^* \int_\Omega v_n^2\,dx\Big)^{\theta r}\right] \int_\Omega v_n^2\,dx\,,$$

and so, choosing $\epsilon = \frac{\alpha}{2c\|f(t)\|_q^r}$, it follows from (40) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega v_n^2\,dx + \int_\Omega |\nabla v_n|^2\,dx \le c\,\|f(.,t)\|_1 + c\,\|f(.,t)\|_q^r \big(\log^* \|f(.,t)\|_q\big)^{\theta r} \int_\Omega v_n^2\,dx \\ + c\,\|f(.,t)\|_q^r \int_\Omega v_n^2\,dx \Big(\log^* \int_\Omega v_n^2\,dx\Big)^{\theta r}\,.\end{aligned} \tag{41}$$

Then, denoting

$$\xi_n(t) = \int_\Omega v_n^2(x,t)\,dx\,,$$

we have the following differential inequality

$$\xi_n'(t) \le c\Big(\|f(.,t)\|_1 + \|f(.,t)\|_q^r \big(\log^* \|f(.,t)\|_q\big)^{\theta r}\Big)\Big(1 + \xi_n(t)\big(\log^* \xi_n(t)\big)^{\theta r}\Big) = \Upsilon(t)\,(1 + H(\xi_n(t)))\,,$$

where $\Upsilon \in L^1(0,T)$ and, taking into account $\theta r \le 1$, the function $H(s) = s(\log^* s)^{\theta r}$ satisfies

$$\int^{+\infty} \frac{ds}{1 + H(s)} = +\infty\,.$$

Thus, the function $\xi_n(t)$ is bounded in $[0,T]$, uniformly in n. Going back to (41), we deduce the desired estimates. ■

Once Theorem 4.2 has been proved and having in mind the proof of [14] Theorem 2.2, we study the second example in [14], which is of type (10) with

$$|b(x,t,s,\xi)| \le |s|^{\lambda}|\xi|^2 + f(x,t)\,, \tag{42}$$

with $\lambda > 0$. A few preliminaries are in order. We define the function

$$\Psi(s) = \int_0^s \exp\left(\frac{|\sigma|^{\lambda+1}}{\lambda+1}\right) d\sigma\,. \tag{43}$$

An easy application of De L'Hôpital's rule yields

$$|s|^{\lambda+1} \sim \log|\Psi(s)| \qquad \text{and} \qquad \exp\left(\frac{|s|^{\lambda+1}}{\lambda+1}\right) \sim |\Psi(s)|\big(\log|\Psi(s)|\big)^{\frac{\lambda}{\lambda+1}} \qquad \text{as } s \to \infty\,. \tag{44}$$

Theorem 4.3 *If $f \in L^r \log L^{r\lambda/(\lambda+1)}\big(0,T; L^q(\Omega)\big)$, with $q \ge \frac{N}{2}\max\{\frac{r}{r-1}, \lambda+1\}$, and $\Psi(u_0) \in L^2(\Omega)$, then there exists at least one distributional solution u to problem* (10) *with right-hand side satisfying* (42)*, such that*

$$u, \Psi(u) \in L^2(0,T; H_0^1(\Omega)) \cap C^0([0,T]; L^2(\Omega))\,, \qquad |u|^{\lambda}|\nabla u|^2 \in L^1(Q_T)\,.$$

Proof: We consider the truncated problems (31). Applying Proposition 3.3 with $\psi = \Psi$, one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Psi(u_n)^2\,dx + \alpha\int_\Omega |\nabla\Psi(u_n)|^2\,dx \le \int_\Omega f\,\exp\left(\frac{|u_n|^{\frac{\lambda}{\lambda+1}}}{\lambda+1}\right)|\Psi(u_n)|\,dx\,.$$

Taking (44) into account, it yields

$$\frac{d}{dt}\int_\Omega \Psi(u_n)^2\,dx + \int_\Omega |\nabla\Psi(u_n)|^2\,dx \le c\int_\Omega f\,\left(1 + \Psi(u_n)^2\big(\log^*|\Psi(u_n)|\big)\big)^{\frac{\lambda}{\lambda+1}}\right)\,dx\,.$$

This is the same inequality as in (39), so that the same arguments may be followed to obtain a priori estimates. The convergence of approximate solutions may be proved as for Theorem 2.2.

∎

5 Extension to higher growth

We are now interested in higher growths for the continuous function β in problem (1); more precisely, we will consider problem (10) with

$$b(x,t,s,\xi)\;:\;\Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$$

satisfying

$$|b(x,t,s,\xi)| \le \beta(|s|)|\xi|^2 + f(x,t)\,, \tag{45}$$

while the functions a satisfies the usual assumptions (11)–(13). Let us introduce the hypotheses satisfied by β and f. To begin with, the function $\beta(s) : [0,+\infty[\to [0,+\infty[$ satisfies

$$0 \le \beta(s) \le c_1\exp_{(k)}(s) + c_2\,, \tag{46}$$

for some $k \in \mathbb{N}$, $k \ge 2$, where, as above,

$$\exp_{(k)}(s) = \underbrace{\exp(\dots(\exp}_{k}(s))\dots)\,.$$

Let us also define the functions

$$\log^*_{(i)} s = \underbrace{\log^*(\log^*(\dots(\log^*}_{i} s)\dots))\,, \qquad \text{where } \log^* s = \max\{1, \log|s|\}\,,$$

$$A_m(s) = |s| \log^* |s| \log^*_{(2)} |s| \dots \log^*_{(m)} |s| \,. \tag{47}$$

Regarding the function $f(x,t)$ which appear in (45), we assume that

$$\int_0^T A_{k+1}\big(\|f(\cdot,t)\|_\phi\big)\, dt < +\infty\,, \quad \text{where } \phi(s) \sim \exp_{(k+1)}(s)\,. \tag{48}$$

Remark 5.1 Indeed, as we shall see in the proof of Theorem 5.1, a more general hypothesis on ϕ in (48) should be $\phi(s) \sim \int_0^s \exp_{(k+1)}(\sigma)\, d\sigma$, which is weaker than the previous one, but less simple to handle. Note also that assumption (48) is satisfied if $\exp_{(k+1)}(f) \in L^r(0,T;L^1(\Omega))$, with $r > 1$.

On the initial datum u_0, we require that:

$$\exp_{(k+1)}(|u_0|) \in L^2(\Omega)\,. \tag{49}$$

We now state the main result of this section:

Theorem 5.1 *Under the above assumptions* (45)–(49)*, there exists at least one distributional solution u of problem* (10) *such that*

$$u \in L^2(0,T;H^1_0(\Omega)) \cap C^0([0,T];L^2(\Omega))\,, \qquad b(x,t,u,\nabla u) \in L^1(Q_T)\,,$$
$$\exp_{(k+1)}(u) \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))\,.$$

In order to prove this result, the only part that must be modified with respect to Theorem 2.2 is the a priori estimate, which is stated in Proposition 5.1 below (the other changes being straightforward). To this aim, we introduce the following families of functions defined on the real line

$$\beta_0(s) = 1\,, \qquad \gamma_0(s) = s\,,$$
$$\beta_i(s) = e^{|\gamma_{i-1}(s)|} + \beta_{i-1}(s)\,, \qquad \gamma_i(s) = \int_0^s \beta_i(\sigma)\, d\sigma\,, \qquad \Psi_i(s) = \int_0^s e^{|\gamma_i(\sigma)|}\, d\sigma\,, \qquad i = 0,1,2,\dots$$

Note that the functions $\beta_i(s)$ are even, while the functions $\gamma_i(s)$ and $\Psi_i(s)$ are odd. For positive s, one has

$$\beta_0(s) = 1\,, \qquad \gamma_0(s) = s\,, \qquad \Psi_0(s) = e^s - 1\,;$$
$$\beta_1(s) = e^s + 1\,, \qquad \gamma_1(s) = e^s - 1 + s\,, \qquad \Psi_1(s) = e^{e^s - 1} - 1\,;$$
$$\beta_2(s) = e^{e^s - 1 + s} + e^s + 1\,, \quad \gamma_2(s) = (e^{e^s-1} - 1) + (e^s - 1) + s\,, \quad \Psi_2(s) = e^{e^{e^s-1}-1} - 1\,;$$

and so on. It is easy to prove that one has, for $i \geq 1$,

$$\exp_{(i)}(|s|) \leq \beta_i(s) + c_i\,, \qquad \gamma_i(s) = \Psi_{i-1}(s) + \gamma_{i-1}(s)\,, \qquad |\Psi_i(s)| = e^{|\Psi_{i-1}(s)|} - 1\,. \tag{50}$$

We define the sequence of auxiliary functions

$$L_0(s) = 1 + |s|\,, \quad L_1(s) = 1 + \log(1 + |s|)\,, \quad L_2(s) = 1 + \log(1 + \log(1 + |s|))\,,$$
$$L_i(s) = L_0(\log(L_{i-1}(s))) = L_{i-1}(\log(L_0(s)))\,, \qquad i = 1,2,\dots\,. \tag{51}$$

Note that

$$L_i(s) \sim \log^*_{(i)} |s| \sim \log^*_{(i+1+k)} |\Psi_k(s)| \qquad \text{for } s \to \infty, \tag{52}$$
$$L_{k+1}(s) = 1 + \Psi_k^{-1}(s) \qquad \text{for } s \geq 0, \tag{53}$$
$$A_m(s) \leq L_0(s) L_1(s) \dots L_m(s)\,.$$

The following Lemma will be useful for the a priori estimates:

Lemma 5.1 *For every $j = 0, 1, 2, \dots,$ one has*

$$e^{|\gamma_j(s)|} = L_0(\Psi_j(s))L_1(\Psi_j(s)) \dots L_j(\Psi_j(s)) . \tag{54}$$

Proof: We prove the assertion by induction on j. Indeed, (54) is obviously true for $j = 0$. Assume now that it holds for some j, and let us prove it for $j + 1$. Indeed, using (50) and (51), we obtain

$$\begin{aligned} e^{|\gamma_{j+1}(s)|} &= e^{|\Psi_j(s)|} e^{|\gamma_j(s)|} \\ &= e^{|\Psi_j(s)|} L_0(\Psi_j(s)) L_1(\Psi_j(s)) \dots L_j(\Psi_j(s)) \\ &= (1 + |\Psi_{j+1}(s)|) L_0(\log(1 + |\Psi_{j+1}(s)|)) L_1(\log(1 + |\Psi_{j+1}(s)|)) \dots L_j(\log(1 + |\Psi_{j+1}(s)|)) \\ &= L_0(\Psi_{j+1}(s)) L_1(\Psi_{j+1}(s)) \dots L_{j+1}(\Psi_{j+1}(s)) . \end{aligned}$$

■

Proposition 5.1 *Let u_n be a solution of problem* (31) *with the function b satisfying* (45) *and assume the hypotheses* (46)*,* (48) *and* (49)*. Then there exists a positive constant C such that*

$$\sup_{t \in [0,T]} \int_\Omega |\Psi_k(u_n)|^2 + \int_{Q_T} |\nabla \Psi_k(u_n)|^2 + \int_{Q_T} f \, |\Psi_k(u_n)| \, A_k(\Psi_k(u_n)) \le C$$

for all $n \in \mathbb{N}$.

Proof: Let us first observe that (46) and (50) imply

$$\beta(s) \le \beta_k(s) + \delta \qquad \text{for some } \delta \ge 0. \tag{55}$$

For sake of simplicity, all the proofs in the present section will be written assuming $\delta = 0$. For the general case, we will point out the necessary changes in Remark 5.2. Using Proposition 3.3 with $\psi = \Psi_k$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \Psi_k(u_n)^2 \, dx + \alpha \int_\Omega |\nabla \Psi_k(u_n)|^2 \, dx \le \int_\Omega f(x,t) \, e^{|\gamma_k(u_n)|} |\Psi_k(u_n)| \, dx = I . \tag{56}$$

Let us estimate the integral I. We set, for brevity,

$$v = |\Psi_k(u_n)| .$$

Using (52) and Lemma 5.1, we have

$$I \le \int_\Omega f \, |v| \, L_0(v) \, L_1(v) \dots L_k(v) \, dx \le c \int_\Omega f \, dx + c \int_\Omega f \, v \, A_k(v) \, dx , \tag{57}$$

where $A_k(s)$ has been defined in (47). To estimate the last integral, we use the generalised Hölder-Orlicz inequality, with the pair

$$\varphi(s) = \int_0^s (L_{k+1}(\sigma) - 1) \, d\sigma , \qquad \tilde{\varphi}(s) = \int_0^s (\varphi')^{-1}(\sigma) \, d\sigma = \int_0^s \Psi_k(\sigma) \, d\sigma$$

of conjugate N-functions. Note that, for $s \to +\infty$,

$$\varphi(s) \sim s \log^*_{(k+1)} s \sim s \, L_{k+1}(s) , \qquad \frac{\tilde{\varphi}(s)}{\phi(s)} = \frac{\tilde{\varphi}(s)}{\exp_{(k+1)}(s)} \to 0 , \tag{58}$$

where ϕ is the N-function appearing in the assumption (48) on f. In particular we obtain

$$\int_\Omega f \, v \; A_k(v) \, dx \le c \left\| v \; A_k(v) \right\|_\varphi \left\| f \right\|_\phi .$$

Now, by (8), (58), for every nonnegative function w one has

$$\|w\|_{\varphi} \le 1 + \int_\Omega \varphi(w)\,dx \le c\Big(1 + \int_\Omega w\,\log^*_{(k+1)} w\,dx\Big),$$

therefore

$$\int_\Omega f\,v\,A_k(v)\,dx \le c\|f\|_{\phi}\Big(1 + \int_\Omega v\,A_{k+1}(v)\,dx\Big), \tag{59}$$

since $A_k(s)\log^*_{(k+1)}(s) = A_{k+1}(s)$. From (57) and (59), using Proposition 2.1 with $A = A_k$ and $\varepsilon = \alpha/(2c\overline{C}\|f\|_{\phi})$, we then obtain

$$\begin{aligned} I \le c\left(\|f\|_1 + \|f\|_{\phi}\right) + \frac{\alpha}{2}\int_\Omega |\nabla v|^2\,dx + \\ + c\,\|f\|_{\phi} A_{k+1}\left(\|v\|_2^2\right) + c\,\|v\|_2^2\|f\|_{\phi} A_k\Big(\log^*\left(c\|f\|_{\phi}\right)\Big). \end{aligned} \tag{60}$$

Putting (56) and (60) together and having in mind $A_k\Big(\log^*\left(c\|f\|_{\phi}\right)\Big) \le C\Big(A_k\left(\log^*\left(\|f\|_{\phi}\right)\right) + 1\Big)$, one obtains

$$\begin{aligned} \frac{d}{dt}\int_\Omega v^2\,dx + \int_\Omega |\nabla v|^2\,dx \\ \le c\Big(\|f\|_1 + \|f\|_{\phi} + \|f\|_{\phi} A_{k+1}\left(\|v\|_2^2\right) + \|v\|_2^2 A_{k+1}\left(\|f\|_{\phi}\right) + \|f\|_{\phi}\|v\|_2^2\Big). \end{aligned} \tag{61}$$

All the norms appearing in the right-hand side of (61) depend on t, and are integrable functions on $]0,T[$, by our assumptions on f. Therefore, if we set $\xi(t) = \int_\Omega v(t)^2\,dx$, we can conclude the proof of the a priori estimate just as in the proof of Proposition 3.4. ∎

Remark 5.2 In the previous proof we have assumed that (55) holds with $\delta = 0$. In the general case, some modifications are needed. We define

$$\gamma(s) = \gamma_k(s) + \delta s, \qquad \Psi(s) = \int_0^s e^{|\gamma(\sigma)|}d\sigma,$$

and we observe that, by De L'Hôpital's rule,

$$\Psi(s) \sim e^{\delta|s|}\Psi_k(s) \quad \text{for } s \to \pm\infty, \tag{62}$$

and, as a consequence, by Lemma 5.1,

$$e^{|\gamma(s)|} \le e^{\delta|s|}\,|\Psi_k(s)|\,A_{k-1}(\log^*|\Psi_k(s)|) \le c\,|\Psi(s)|\,A_{k-1}(\log^*|\Psi(s)|).$$

Similarly, from (52) and (62) it follows that

$$L_i(s) \sim \log_{i+k+1}|\Psi(s)| \quad \text{for } s \to \pm\infty.$$

The proof proceeds by replacing Ψ_k by Ψ and γ_k by γ everywhere (including Proposition 3.3).

It is clear that, using the same technique, one can prove an existence result for problem (9) under slightly weaker growth assumptions on the reaction term $F(x,t,u)$, that is:

$$|F(x,t,s)| \le f(x,t)\,\left(1 + A_k(s)\right), \tag{63}$$

where $f(x,t)$ satisfies assumption (48) and $A_k(s)$ is defined as in (47), while

$$v_0 \in L^2(\Omega). \tag{64}$$

The following theorem holds:

Theorem 5.2 *Under the above assumptions* (63), (48), (64)*, there exists at least one distributional solution u of problem* (9) *such that*

$$u \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega)), \qquad F(x,t,u) \in L^1(Q_T).$$

6 A nonlinear phenomenon.

6.1 Problems with a gradient term and a reaction term together

It is possible to consider problems which feature both a quadratic gradient term as in (10) and a linear or slightly superlinear reaction term as in (9). For simplicity, one could consider a very simple case:

$$u_t - \Delta u = |\nabla u|^2 + f_1(x,t)\, u + f_2(x,t)\,. \tag{65}$$

One could guess that the assumptions on the functions f_1 and f_2 should be the same as in the following situations where only one of the terms depending on u is taken:

$$u_t - \Delta u = |\nabla u|^2 + f_2(x,t)\,.$$

$$u_t - \Delta u = f_1(x,t)\, u\,.$$

In these cases one is lead to assume $f_1, f_2 \in L^r(0,T;L^q(\Omega))$, with

$$r, q > 1\,, \qquad \frac{N}{2q} + \frac{1}{r} \leq 1\,.$$

We will try to provide evidence that this is a sufficient assumption on f_2, but not on f_1, so that one needs stronger hypotheses on f_1 due to the presence of the quadratic term in (65). Indeed, if one makes the change of unknown $v = e^u - 1$ in order to get rid of the quadratic term, equation (65) becomes

$$v_t - \Delta v = f_1(x,t)(1+v)\log(1+v) + f_2(x,t)(1+v)\,.$$

Then the summability of f_1 should be higher than before in order to get a priori estimates and existence, while the hypotheses on f_2 may be the same.

The same kind of remark can be extended to more general equations of the form (10), with

$$|b(x,t,s,\xi)| \leq c_1\beta_k(s)|\xi|^2 + f_1(x,t)(1 + A_h(s)) + f_2(x,t)\,. \tag{66}$$

In this case we can still obtain a priori estimate and an existence result with a similar choice of test functions with respect to Theorem 2.2, provided suitable assumptions are made on the data.

More precisely, as far as the functions $f_1(x,t)$ and $f_2(x,t)$ which appear in (66) are concerned, we assume that

$$\int_0^T A_{h+k+2}\big(\|f_1(\cdot,t)\|_{\phi_1}\big)\; dt < +\infty\,, \tag{67}$$

where $\phi_1(s)$ is an N-function which is equivalent to $\exp_{(h+k+2)}(s)$ near infinity (actually one can write a weaker assumption as in Remark 5.1), and k, h are the same as in (66). Similarly, we require

$$\int_0^T A_{k+1}\big(\|f_2(\cdot,t)\|_{\phi_2}\big)\; dt < +\infty\,, \tag{68}$$

where $\phi_2(s)$ is now equivalent to $\exp_{(k+1)}(s)$. On the initial datum u_0, we require that:

$$\exp_{(k+1)}(|u_0|) \in L^2(\Omega)\,. \tag{69}$$

We now state the existence result under these generalized hypotheses:

Theorem 6.1 *Under the assumptions* (66)–(69)*, there exists at least one distributional solution u to problem* (10) *such that*

$$u \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))\,, \qquad b(x,t,u,\nabla u) \in L^1(Q_T)\,,$$

and moreover

$$\exp_{(k+1)}(u) \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))\,,$$

The proof of this result is a straightforward modification of the proof of Theorem 2.2 and we will not write it. In order to obtain an a priori estimate for the approximate problems (31), one again has to use $\Psi_k(u_n)$ in Proposition 3.3. The only different term is now the integral

$$\int_\Omega f_1(x,t)\,(1+A_h(u_n))\,e^{|\gamma_k(u_n)|}|\Psi_k(u_n)|\,dx\,,$$

which, using (52), may be handled as follows:

$$\begin{aligned}\int_\Omega & f_1(x,t)\,(1+A_h(u_n))\,e^{|\gamma_k(u_n)|}|\Psi_k(u_n)|\,dx \\ &\le c\int_\Omega f_1\,(1+A_h(u_n))\,|\Psi_k(u_n)|\,L_0(\Psi_k(u_n))\,L_1(\Psi_k(u_n))\dots L_k(\Psi_k(u_n))\,dx \\ &\le c\left(\int_\Omega f_1\,dx+\int_\Omega f_1\,|\Psi_k(u_n)|\,A_{k+h+1}(\Psi_k(u_n))\,dx\right).\end{aligned}$$

From now on the proof follows the same course as the one of Proposition 5.1. ■

6.2 Related problems

Let us recall that in [14] the authors study separately the semilinear problem (9) with F satisfying (38) and the quasilinear problem (10) with b satisfying (42). In the spirit of the considerations at the beginning of this Section, one is led to consider the problem in which the right hand side of the previous equations occur together, that is, problem (10) with

$$b(x,t,u,\nabla u)=|u|^\lambda|\nabla u|^2+f_1(x,t)\,(1+|u|)\,(\log^*|u|)^\theta+f_2(x,t)\,,\qquad \text{with}\quad 0<\theta<1\,. \tag{70}$$

To deal with this problem, we consider the function Ψ defined in (43): $\Psi(s)=\int_0^s\exp\left(\frac{|\sigma|^{\lambda+1}}{\lambda+1}\right)d\sigma$. The existence of a distributional solution may be proved under suitable assumptions on the data.

Theorem 6.2 *Let $\varphi(s)$, $\tilde\varphi(s)$ be a pair of conjugate N-functions such that $\varphi(s)\sim s(\log^*_{(2)}s)^{1-\theta}$ at $+\infty$, so that $\tilde\varphi(s)\sim\int_0^s\exp_{(2)}((1-\theta)\sigma^{1/(1-\theta)})\,d\sigma$, and assume*

$$\int_0^T A_2\big(\|f_1(\cdot,t)\|_{\tilde\varphi}\big)\;dt<\infty\,, \tag{71}$$

$$\int_0^T \|f_2(\cdot,t)\|_q^r\big(\log^*\|f_2(\cdot,t)\|_q\big)^{\theta r}\;dt<\infty\qquad\textit{and}\qquad \Psi(|u_0|)\in L^2(\Omega)\,.$$

Then there exists at least one distributional solution u to problem (10) *with right-hand side* (70) *such that*

$$u\,,\ \Psi(u)\in L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))\,,\quad u^\lambda|\nabla u|^2\,;\ f_1(x,t)\,(1+|u|)\,(\log^*|u|)^\theta\in L^1(Q_T)\,.$$

Remark 6.1 Once again a nonlinear phenomenon appears, namely the hypothesis on f_2 is the same as in Theorem 4.3, but on f_1 must be more restrictive than in Theorem 4.2. Of course, this is due to the fact that looking for an estimate similar to that of Proposition 3.1, the term containing f_1; that is

$$I_1=\int_\Omega f_1(x,t)\,(1+|u_n|)\,(\log^*|u_n|)^\theta\Psi(u_n)^2\,\big(\log^*|\Psi(u_n)|\big)^{\lambda/(\lambda+1)}\,dx$$

must be estimated in a different way. Indeed, denoting as usual $v=\Psi(u_n)$, and taking (44) into account, we get

$$I_1\le\int_\Omega f_1(x,t)\,\big(1+v^2\,\log^* v\,(\log^*_{(2)}v)^\theta\big)\,dx\,.$$

Applying the Hölder inequality for Orlicz' spaces with conjugate N-functions φ and $\tilde\varphi$, it yields

$$I_1 \le \|f_1(\cdot,t)\|_{\tilde\varphi}\left(1+\int_\Omega v^2\,\log^* v\,\log^*_{(2)} v\,dx\right) = \|f_1(\cdot,t)\|_{\tilde\varphi}\left(1+\int_\Omega v^2 A(\log^* v)\,dx\right),$$

where $A(s) = s\log^* s$. Finally, by Proposition 2.1, there appears a term of the form

$$\|v\|_2^2\,\|f_1(\cdot,t)\|_{\tilde\varphi}\,\log^*\|f_1(\cdot,t)\|_{\tilde\varphi}\,\log^*_{(2)}\|f_1(\cdot,t)\|_{\tilde\varphi}\,,$$

and hence the hypothesis (71) must be required.

Remark 6.2 Let us consider the situation in the borderline cases, namely when θ goes to 1 and to 0. To obtain a distributional solution of problem (10) with right hand side

$$b(x,t,u,\nabla u) = |u|^\lambda|\nabla u|^2 + f_1(x,t)\,(1+|u|)\,\log^*|u| + f_2(x,t)\,,$$

we consider an N-function such that $\varphi(s) \sim s\log\log\log s$ and denote by $\tilde\varphi$ its conjugate, which satisfies $\tilde\varphi(s) \sim \int_0^s \exp_{(3)}(\sigma)\,d\sigma$. Then we prove an estimate on the solution under the assumption

$$\int_0^T A_3\big(\|f_1(\cdot,t)\|_{\tilde\varphi}\big)\,dt < \infty\,.$$

Thus a little discontinuity appears, since the assumption on f_1 is not the same one would obtain when $\theta \to 1$ in (71).

On the other hand, in the case where

$$b(x,t,u,\nabla u) = |u|^\lambda|\nabla u|^2 + f_1(x,t)\,(1+|u|) + f_2(x,t)\,,$$

the conjugate N-functions φ and $\tilde\varphi$, where $\varphi(s) \sim s\log\log s$ and $\tilde\varphi(s) \sim \int_0^s \exp_{(2)}(\sigma)\,d\sigma$, will be considered. We may get a distributional solution if the following assumption on f_1 is done:

$$\int_0^T A_2\big(\|f_1(\cdot,t)\|_{\tilde\varphi}\big)\,dt < \infty\,.$$

We point out that now there is continuity with the previous hypothesis (71) as $\theta \to 0$.

As a final example, we turn to consider what happens when a sublinear growth on u is considered. Let us consider problem (10) with right hand side

$$b(x,t,u,\nabla u) = |u|^\lambda|\nabla u|^2 + f_1(x,t)\,(1+|u|)^\theta + f_2(x,t)\,,\qquad \text{where}\quad 0<\theta<1\,.$$

To study this problem, we may follow the proof of Theorem 4.2 requiring as assumption on f_1 that it belongs to $L^r(\log L)^{r(\lambda+\theta)/(\lambda+1)}\big(0,T;L^q(\Omega)\big)$, with $q \ge \frac{N}{2}\max\{\frac{r}{r-1},\frac{\lambda+1}{1-\theta}\}$, while we assume on f_2 the same hypothesis we made in Theorem 4.3.

7 Comments and remarks

Remark 7.1 In Section 5 we dealt with functions $\beta(s)$ in (45) which grow like $\exp_{(k)}(s)$ for an arbitrary positive integer k. Continuous functions $\beta(s)$ with completely free growth will induce a function $e^{\gamma(s)}$ which may be written as $\Psi(s)A(\log\Psi(s))$ with $A(s) = e^{\gamma(\Psi^{-1}(e^s))-s}$ (remember that $\gamma(s) = \int_0^s \beta(\sigma)d\sigma$, $\Psi(s) = \int_0^s e^{|\gamma(\sigma)|}d\sigma$). The problem to deal with such a kind of function $\beta(s)$ is that actually we do not know if the corresponding function A satisfies always the Δ_2-condition which we need in order to prove the logarithmic Sobolev inequality. In fact the case of totally general growth for $\beta(s)$ is an open question.

Let us summarize in the following first table the situation for different classes of functions β in (45) and for the related involved functions. In the second table we focus our attention on different classes of admissible data $f(x,t)$ depending on different situations for β. We do not consider the case $f(x,t) \in L^1(0,T;L^\infty(\Omega))$, since, as already pointed out, in this case for every function β we have always existence of solutions via a sub-supersolution method. For simplicity we only consider $s > 0$.

Function $\beta(s)$	Function $\Psi(s) = \int_0^s e^{\gamma(\sigma)} d\sigma$	$e^{\gamma(s)}$ as a function of $\Psi(s)$
$\beta(s)$ bounded	$= e^{cs} - 1$	$\sim \Psi(s)$
$\beta(s) = s^\lambda$	$= \int_0^s \exp\left(\frac{\sigma^{\lambda+1}}{\lambda+1}\right) d\sigma$	$\sim \Psi(s)\big(\log \Psi(s)\big)^{\lambda/(1+\lambda)}$
$\beta(s) = e^s + 1$	$= e^{e^s - 1} - 1$	$\sim \Psi(s) \log \Psi(s)$
$\beta(s) = \exp_{(k)} s$	$= \underbrace{\exp(\exp(\ldots(\exp}_{k+1}(s) - 1)\ldots) - 1) - 1$	$\sim \Psi(s)\ \log^*_{(1)} \Psi(s) \ldots \log^*_{(k)} \Psi(s)$
Any $\beta(s)$ fixed	$= \int_0^s e^{\gamma(\sigma)}\, d\sigma$	$\sim \Psi(s) A\big(\log \Psi(s)\big)$, $A(s) = \exp\big(\gamma(\Psi^{-1}(e^s)) - s\big)$

Table 1. Relations between the functions β, γ, Ψ in some model cases. We recall that $\gamma(s) = \int_0^s \beta(\sigma)\, d\sigma$.

Function $\beta(s)$	Condition on source function $f(x,t)$
$\beta(s)$ bounded	$\int_0^T \\|f(.,t)\\|_q^r \, dt < \infty\,, \qquad 2q/N \geq r' > 1$
$\beta(s) = s^\lambda$	$\int_0^T \\|f(.,t)\\|_q^r (\log^* \\|f(.,t)\\|_q)^{\theta r}\, dt < \infty\ , \quad 2q \geq N \max\{\lambda+1, r'\}$
$\beta(s) = e^s$	$\int_0^T A_2\big(\\|f(.,t)\\|_\phi\big)\, dt < \infty\ , \quad \phi(s) \sim \int_0^s \exp_{(2)}(\sigma)\, d\sigma$
$\beta(s) = \exp_{(k)}(s)$	$\int_0^T A_{k+1}\big(\\|f(\cdot,t)\\|_\phi\big)\, dt < \infty\ , \quad \phi(s) \sim \int_0^s \exp_{(k+1)}(\sigma)\, d\sigma$
Any $\beta(s)$ fixed	?

Table 2. Assumptions on the datum $f(x,t)$ as a function of $\beta(s)$, in order to obtain existence for problem (10). We recall that $A_k(s) = s \log^* s \log^*_{(2)} s \ldots \log^*_{(k)} s$.

For what refers to the initial datum u_0, a sufficient assumption for the existence is, in all cases, $\Psi(u_0) \in L^2(\Omega)$ (actually, it suffices $\Psi(u_0) \in L^\delta(\Omega)$, $\delta > 1$, see Remark 3.1).

Remark 7.2 We point out that the functions φ and $\tilde{\varphi}$ which appear in the proof of Proposition 3.1 are not uniquely determined by β and several choices are possible. This implies that assumptions on $f(x,t)$, different that those above, can be made in order to get existence. For instance, for our exponential model example (1), that is when $\beta(s) = e^{|s|} + 1$, we have chosen in Proposition 3.1

$$\varphi(s) = \int_0^s \big(L_2(\sigma) - 1\big)\, d\sigma \sim s \log\log s \qquad \text{and} \qquad \tilde{\varphi}(s) = \int_0^s \Big(e^{e^{\sigma}-1} - 1\Big)\, d\sigma\,.$$

As a consequence, one is led to choose $A(s) = |s| \log^* |s|$ in Proposition 2.1, and the assumption on the datum f has to be

$$\int_0^T \|f(.,t)\|_{\tilde{\varphi}} \log^* \|f(.,t)\|_{\tilde{\varphi}} \log^*_{(2)} \|f(.,t)\|_{\tilde{\varphi}}\, dt < +\infty\,.$$

Besides this, other possibilities on φ are:

$$\text{(i)} \qquad \varphi(s) = \int_0^s \big(L_2(\sigma) - 1\big)^{\theta}\, d\sigma \qquad \text{with} \quad 0 < \theta < 1\,,$$

$$\text{(ii)} \qquad \varphi(s) = \int_0^s \big(L_3(\sigma) - 1\big)\, d\sigma\,,$$

$$\text{(iii)} \qquad \varphi(s) = \int_0^s \big(L_2(\sigma) - 1\big)\big(L_3(\sigma) - 1\big)\, d\sigma\,.$$

In each of the first two cases, its conjugate N-function may easily be written (this is not the case in the third one), namely:

$$\text{(i)} \qquad \tilde{\varphi}(s) = \int_0^s \big(e^{e^{\sigma^{1/\theta}}-1} - 1\big)\, d\sigma\,,$$

$$\text{(ii)} \qquad \tilde{\varphi}(s) = \int_0^s \big(e^{e^{e^{\sigma}-1}-1} - 1\big)\, d\sigma\,,$$

respectively. Each choice implies a different function A and a different assumption on the datum f. On the one hand, in the case (i), one takes $A(s) = |s|\,(\log^* |s|)^{\theta}$, and f should satisfy

$$\int_0^T \|f(.,t)\|_{\tilde{\varphi}} \log^* \|f(.,t)\|_{\tilde{\varphi}} \big(\log^*_{(2)} \|f(.,t)\|_{\tilde{\varphi}}\big)^{\theta}\, dt < +\infty\,.$$

On the other hand, when (ii) is considered, $A(s) = |s| \log^*_{(2)} |s|$ and the assumption on f should be

$$\int_0^T \|f(.,t)\|_{\tilde{\varphi}} \log^* \|f(.,t)\|_{\tilde{\varphi}} \log^*_{(3)} \|f(.,t)\|_{\tilde{\varphi}}\, dt < +\infty\,.$$

It is clear that we may take functions φ growing even more slowly which implies that A grows also more slowly, and so the corresponding space where f lies is smaller in x and larger in t. Hence, the datum f belongs to a space of the type $L_{\phi}(0,T; L_{\tilde{\varphi}}(\Omega))$ in such a way that the smaller the space is with respect to the variable x, the larger it is with respect to the variable t. It seems that Aronson-Serrin's curve has been extended through the Orlicz spaces. Note, however, that if we choose the smallest space in x, namely $L^\infty(\Omega)$, we will have $L^1\log L^1(0,T;L^\infty(\Omega))$. This space, obviously, is not the largest one we may obtain, because one can always take $f\in L^1(0,T;L^\infty(\Omega))$ using sub/supersolutions.

The choice that we have done throughout this paper is to consider spaces in the variable x for which explicit calculations are possible, and notations are quite intuitive. Thus, we have chosen $\varphi(s)=\int_0^s\big(L_2(\sigma)-1\big)\,d\sigma$ and not $\varphi(s)=\int_0^s\big(L_2(\sigma)-1\big)\big(L_3(\sigma)-1\big)\,d\sigma$ since the last one is not as easy as the previous one to handle.

Remark 7.3 The fact that different choices of Orlicz spaces are possible, allows another way of comparing the conditions on f for several related problems. Consider, for instance, the problems appearing in Section 4 jointly with our exponential model problem. Starting from the N-functions $\varphi(s)\sim s\log^*\log^* s$ and $\phi(s)\sim\int_0^s\exp_{(2)}(\sigma)\,d\sigma$, and applying our procedure, one obtains the situation showed in the following table.

Function $\beta(s)$	Condition on source function $f(x,t)$
$\beta(s)$ bounded	$\displaystyle\int_0^T \|f(.,t)\|_\phi \log\log\|f(.,t)\|_\phi\,dt<\infty$
$\beta(s)=s^\lambda$	$\displaystyle\int_0^T \|f(.,t)\|_\phi (\log\|f(.,t)\|_\phi)^{\lambda/(\lambda+1)}\log\log\|f(.,t)\|_\phi\,dt<\infty$
$\beta(s)=e^s$	$\displaystyle\int_0^T \|f(.,t)\|_\phi \log\|f(.,t)\|_\phi\log\log\|f(.,t)\|_\phi\,dt<\infty$

Table 3. Comparison among conditions on the datum $f(x,t)$ taking the N-functions $\varphi(s)\sim s\log^*\log^* s$ and $\phi(s)\sim\int_0^s\exp_{(2)}(\sigma)\,d\sigma$ as starting point.

Remark 7.4 In Section 5 we have considered a family of functions defined by recurrence. Its main advantage is that these functions can easily be handle. Unfortunately, steps between two consecutive functions belonging to this family are very big. In other words, fixed a function $\beta(s)$ and supposed we may find the smallest $k\in\mathbb{N}$ such that $\exp_{(k)}(s)$ grows faster than $\beta(s)$; then the fact is that $\exp_{(k)}(s)$ may exceed $\beta(s)$ overmuch.

We point out that other possible families can be considered to obtain a function closer to a given $\beta(s)$. For instance, fixed $\nu>0$, we could have introduced

$$\beta_0(s)=1\,,\qquad \gamma_0(s)=s\,,$$

$$\beta_i(s)=\beta_{i-1}(s)+\nu e^{|\gamma_{i-1}(s)|}\,,\qquad \gamma_i(s)=\int_0^s\beta_i(\sigma)\,d\sigma\,,\qquad \Psi_i(s)=\int_0^s e^{|\gamma_i(\sigma)|}\,d\sigma\,,\qquad i=0,1,2,\dots$$

Thus, $\beta_1(s)=\nu e^{|s|}+1$, $\gamma_1(s)=(\nu e^{|s|}+|s|-\nu)\operatorname{sign} s$, $\Psi_1(s)=\frac{e^{\nu e^{|s|}}-e^\nu}{\nu e^\nu}\operatorname{sign} s$, and so on.

Remark 7.5 As far as uniqueness is concerned, let us consider the model case, where, for sake of simplicity, we assume that all data are positive.

$$\begin{cases} u_t-\Delta u=|\nabla u|^2+f(x,t), & \text{in}\quad \Omega\times]0,T[;\\ u(x,0)=u_0(x), & \text{in}\quad \Omega,\\ u\in L^2(0,T;H_0^1(\Omega))\,. \end{cases}\tag{72}$$

If u is a solution of (72) such that $v = \Psi(u) = e^u - 1 \in L^2(0,T;H_0^1(\Omega))$, (for instance this happens if u is bounded), then v satisfies the linear problem

$$\begin{cases} v_t - \Delta v = f\,(v+1), & \text{in} \quad \Omega\times]0,T[; \\ v(x,0) = \Psi(u_0(x)), & \text{in} \quad \Omega, \\ v \in L^2(0,T;H_0^1(\Omega))\,, \end{cases}$$

which has a unique solution. However the solution of (72) is not necessarily unique, and it is indeed possible to show that problem (72) admits infinitely many unbounded solutions. Every solution of (72) corresponds, via the same change of unknown function, to a solution of problem

$$\begin{cases} v_t - \Delta v = f\,(v+1) + \mu, & \text{in } Q_T; \\ v(x,0) = \Psi(u_0(x)), & \text{in} \quad \Omega, \\ v \in L^q(0,T;W_0^{1,q}(\Omega))\,, \quad \text{for every } q < (N+2)/(N+1)\,, \end{cases} \tag{73}$$

where μ is a positive "singular" Radon measure on Q_T (i.e., concentrated on a set of null capacity). Viceversa, for every such singular measure μ one can solve problem (73). Then $u = \log(1+v)$ can be shown to be a solution of problem (72). For these results, see the paper [1].

In the case of a more general operator as in problem (10), the question of uniqueness is open even in the class of functions such that $\Psi(u) \in L^2(0,T;H_0^1(\Omega))$ (where $\Psi(s)$ is defined as in (4)).

8 Acknowledgements

This work was begun when S. Segura de León was visiting professor at the Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", and he wish to thank for the very warm hospitality enjoyed there. He has been partially supported by the Spanish PNPGC, proyecto BFM2002-01145.

References

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral: Existence and nonuniqueness results for nonlinear parabolic problems having natural growth in the gradient, *in preparation.*

[2] R.A. Adams: General logarithmic Sobolev inequalities and Orlicz imbeddings, *J. Funct. Anal.* **34** (1979), 292–303.

[3] D.G. Aronson, J. Serrin: Local behavior of solutions of quasilinear parabolic equations, *Arch. Rat. Mech. Anal.* **25** (1967), 81–122.

[4] A. Ben-Artzi, P. Souplet, F.B. Weissler: The local theory for the viscous Hamilton-Jacobi equations in Lebesgue spaces, *J. Math. Pure. Appl.* **9** (2002), 343–378.

[5] D. Blanchard, A. Porretta: Nonlinear parabolic equations with natural growth terms and measure initial data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30** (2001), 583–622.

[6] L. Boccardo, F. Murat: Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, *Nonlinear Anal.* **19**, (1992), no. 6, 581–597.

[7] L. Boccardo, F. Murat, J.P. Puel: Existence results for some quasilinear parabolic equations, *Nonlinear Analysis T.M.A.* **13** (1989), 373–392.

[8] L. Boccardo, M.M. Porzio: Bounded solutions for a class of quasi-linear parabolic problems with a quadratic gradient term, *Progress Nonlinear Diff. Eq.* **50** (2002), 39–48.

[9] H. Brezis: *Analyse Fonctionnelle. Théorie et applications,*Collection Mathématiques Appliquées pour la maitrise, Masson, Paris (1983)

[10] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa: Blow up for $u_t - \Delta u = g(u)$ revisited, *Adv. Diff. Eq.* **1** (1996), 73–90

[11] F. Cipriani, G. Grillo: Uniform bounds for solutions to quasilinear parabolic equations, *J. Diff. Eq.* **177** (2001), 209–234.

[12] A. Dall'Aglio, D. Giachetti, J.P. Puel: Nonlinear parabolic equations with natural growth in general domains, *Boll. Un. Mat. Ital.*, to appear.

[13] A. Dall'Aglio, D. Giachetti, C. Leone, S. Segura de León: Quasi-linear parabolic equations with degenerate coercivity having a quadratic gradient term, *Ann. Inst. H. Poincaré. Ann. non Linéaire.*, to appear

[14] A. Dall'Aglio, D. Giachetti, S. Segura de León: Semilinear parabolic equations with superlinear reaction terms, and application to some convection-diffusion problems, *Ukr. Math. Bull.*, to appear.

[15] A. Dall'Aglio, L. Orsina: Nonlinear parabolic equations with natural growth conditions and L^1 data, *Nonlinear Anal. T.M.A.* **27** (1996), 59–73.

[16] L.C. Evans: *Partial differential equations.* Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI (1998).

[17] V. Ferone, M.R. Posteraro, J.M. Rakotoson: Nonlinear parabolic problems with critical growth and unbounded data, *Indiana Univ. Math. J.* **50** (2001), no. 3, 1201–1215.

[18] B. Gilding, M. Guedda, R. Kersner: The Cauchy problem for $u_t - \Delta u = |\nabla u|^q$. *J. Math. Anal. Appl.* **284** (2003), 733–755.

[19] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva: *Linear and quasi-linear equations of parabolic type,* Translations of Math. Monographs, vol 23, Providence, 1968.

[20] R. Landes: On the existence of weak solutions for quasilinear parabolic boundary value problems, *Proc. Royal Soc. Edinburgh Sect. A* **89** (1981), 217–237.

[21] R. Landes, V. Mustonen: On parabolic initial-boundary value problems with critical growth for the gradient, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **11** (1994), 135–158.

[22] J.-L. Lions: *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[23] A. Mokrane: Existence of bounded solutions of some nonlinear parabolic equations, *Proc. Roy. Soc. Edinburgh Sect. A* **107** (1987), nos. 3-4, 313–326.

[24] L. Orsina, M.M. Porzio: $L^\infty(Q)$-estimate and existence of solutions for some nonlinear parabolic equations, *Boll. Un. Mat. Ital.* **6-B** (1992), 631–647.

[25] J. Simon: Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **146** (1987), 65–96.