SINGULAR ELLIPTIC EQUATIONS HAVING A GRADIENT TERM WITH NATURAL GROWTH

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Abstract. We study a class of Dirichlet boundary value problems whose prototype is

$$\begin{cases} -\Delta_p u = h(u)|\nabla u|^p + u^{q-1} + f(x) & \text{in } \Omega\,, \\ u \geq 0\,, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{0.1}$$

where Ω an open bounded subset of $\mathbb{R}^N$, $0 < q < 1$, $1 < p < N$, h is a continuous function and f belongs to a suitable Lebesgue space. The main features of this problem are the presence of a singular term and a first order term with natural growth in the gradient. A priori estimates and existence results are proved depending on the summability of the datum f.

1. Introduction

In the present paper we study the existence of a nonnegative solution u for a nonlinear elliptic equation having both a zero order term which tends to infinity at $u = 0$ and a first order term which has a natural growth in the gradient of u. More precisely, this paper concerns with problems of the type

$$\begin{cases} -\mathrm{div}\,(\boldsymbol{a}(x,u,\nabla u)) = b(x,u,\nabla u) + g(x,u) + f(x)\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

Here Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, $-\mathrm{div}\,(\boldsymbol{a}(x,u,\nabla u))$ is a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$, $b(x,u,\nabla u)$ is a nonlinear term which grows like $|\nabla u|^p$ and more precisely satisfies

$$|b(x,s,z)| \leq h(s)|z|^p\,,$$

for a continuous function $h : \mathbb{R} \to \mathbb{R}$. Moreover, $g(x,u)$ is a singular term at $u = 0$, that is

$$0 \leq g(x,s) \leq \Lambda s^{q-1}\,, \quad \Lambda > 0\,, \quad 0 < q < 1\,.$$

Finally the datum f belongs to a suitable Lebesgue space.

The existence of a weak solution to problem (1.1) when the singular term $g(x,u)$ does not appear has been investigated by many authors starting by the 80s. In the papers [7, 9] the existence of bounded solutions is proved when the problem (1.1) has also a (non singular) zero-order absorption term, while in papers [6] a more general class of equations, which have a gradient term satisfying suitable sign conditions, is considered and existence of unbounded solutions has been proved.

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Existence and nonexistence of unbounded solutions to problem (1.1) having a reaction gradient terms with natural quadratic growth satisfying a further suitable condition have been faced in [21, 11, 32] by using test functions that simulate the Cole–Hopf transformation. The more general case, where natural growth different of the quadratic one, is treated in [22, 31]. In [31] optimal conditions on the growth of h at infinity to ensure that, given f with a certain summability, problem (1.1) admits a solution are given. Similar results in this order of ideas can be found in [30].

The existence of nonnegative solutions for semilinear second order partial differential problems singular at $u = 0$, without first order term, is also a classical problem which has been considered by several authors since 70's. In [18], it is shown the existence and uniqueness of a nonnegative solution in a case where the equation is not written in a divergence form and the solution is a classical solution, i.e. it is in $C^2(\Omega) \cap C(\overline{\Omega})$ which is strictly positive in Ω. In [17] it is considered the case $g(x, s) = 1/s^\gamma + (\lambda s)^\alpha$ with $\gamma, \lambda, \alpha > 0$ and existence and nonexistence results for classical solutions are proved (see also [34]). Looking for a nonnegative solution in a Sobolev space, the problem has been considered in [10], where it is studied the case $g(x, u) = f(x)/s^\gamma$, with $\gamma > 0$, $f \geq 0$ not identically zero and belonging to a suitable Lebesgue space. In this paper existence, uniqueness and regularity of a distributional solution, strictly positive in Ω is proved. The proofs of these results are manly based on the fact that $g(x, s)$ is a nonincreasing function in the variable s and on the use of strong maximum principle. Uniqueness and comparison results for this type of solution has been proved in [4] and, by using symmetrization techniques, in [12]. In order to avoid the use of strong maximum principle and monotonicity assumption on $g(x, \cdot)$, a new definition of nonnegative solutions have been provided and existence, stability and uniqueness results for these notions are proved in [23, 24, 25].
Further contributions to semilinear elliptic equations having this type of singularity are contained for example in [1, 2, 3, 5, 14, 15, 16, 28, 29].

A very few results are known about existence of solutions which changes its sign and a first paper in this direction is [13], where the authors show that if the "singular term" goes to infinity at zero faster than $1/|u|$ then only nonnegative solutions are possible, while in the other case nonpositive solution or even solutions changing the sign are possible. In [20] the solution is defined as a minimum point for a suitable functional and the definition of solution given by the authors uses test functions which vanish at $u = 0$ and thus the equation is satisfied in $\Omega \setminus \{u = 0\}$. It is also proved the uniqueness for nonnegative solutions when g(x, .) is decreasing.

The effects of the presence of two singularities, both in a gradient term having natural growth and in a zero order term, has been addressed in [27]. In this paper the function h is summable on $\mathbb{R}$ and the singular zero order term involves the datum: $g(x, u) = f(x)/u^\gamma$. These assumptions allow the study of the equation with L^1–data.

The novelty of this paper consists in analyzing the effects of a gradient term having natural growth and a singular term of the type $g(x, u) = 1/u^\gamma$. Following [31], the existence of nonnegative solutions to problem (1.1) is proven depending on the behaviour of h at $+\infty$ (we point out that we do not assume $h \in L^1(\mathbb{R})$) and the summability of the datum f.

Our approach is based on a priori estimates for weak solutions to a sequence of approximating problems. The proof of these a priori estimates is obtained by adapting the classical approach to study these type of equations made by a change of unknown of Cole–Hopf type given by Lemma 2.1. These a priori estimates allow to deduce the existence of a limit function u such that the approximate solutions u_n and their gradients ∇u_n converge to u and ∇u respectively. A procedure of passage to the limit permits to prove that such a function u is, indeed, a solution of problem (1.1). The main difficulties in proving firstly the a priori estimates and then in passing to limit in the approximating problems are due to the presence of the singular term $g(x,u)$ and the necessity to prove that it can really be bounded
In this paper, we consider three different types of summability for a datum $f \in L^m(\Omega)$: (a) $m > N/p$, (b) $m = N/p$ and (c) $\frac{Np}{Np-N+p} \le m < \frac{N}{p}$, which will be analyzed separately into three existence results, one for each. Our main results are given by Theorem 3.1, Theorem 4.1 and Theorem 5.1 and proven in Section 3, Section 4 and Section 5, respectively.

2. Notation, assumptions and preliminary results

Throughout this paper, Ω stands for an open bounded set of $\mathbb{R}^N$, with $N \ge 2$. The Lebesgue measure of $E \subset \Omega$ will be denoted by $|E|$.
On the other hand, the positive and negative part of a function u is denoted by u_+ and u_-, respectively. Moreover, we denote

$$\{|u| \ge \delta\} = \{x \in \Omega : \ |u(x)| \ge \delta\}\,,$$

for any $\delta > 0$.
In what follows, we will also consider two auxiliary functions. For any $s \in \mathbb{R}$ and any $k > 0$ we define

$$G_k(s) = (|s| - k)_+ \mathrm{sign}\,(s)\,, \tag{2.1}$$

$$T_k(s) = \max\{-k, \min\{s,k\}\}\,. \tag{2.2}$$

From now on, we will denote by C a positive constant that only depends on the data, not on n and that may change from line to line.

2.1. Assumptions.

The aim of this subsection is to give the hypotheses on the data of problem (1.1) which we make in the whole paper. We also introduce the notion of weak solution which we use.

As pointed out in the Introduction we study solutions to the following singular nonlinear problem

$$\begin{cases} -\mathrm{div}\,(\boldsymbol{a}(x,u,\nabla u)) = b(x,u,\nabla u) + g(x,u) + f(x)\,, & \text{in } \ \Omega \\ u \ge 0\,, & \text{in } \ \Omega \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{2.3}$$

We assume that, for some $1 < p < N$,

$$\boldsymbol{a} \,:\, \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$
$$b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$
$$g \,:\, \Omega \times (\mathbb{R} \setminus \{0\}) \longrightarrow [0,+\infty)$$

are Carathéodory functions which satisfy the growth conditions

$$|\boldsymbol{a}(x,s,z)| \le a_0|z|^{p-1} + a_1|s|^{p-1} + a_2\,, \quad a_0, a_1, a_2 > 0\,, \tag{2.4}$$

$$|b(x,s,z)| \leq h(s)|z|^p\,, \tag{2.5}$$

where $h:\mathbb{R}\to\mathbb{R}$ is a continuous function, and

$$0 \leq g(x,s) \leq \Lambda s^{q-1}\,, \quad \Lambda>0\,, \quad 0<q<1\,. \tag{2.6}$$

Moreover the function $\boldsymbol{a}$ satisfies the ellipticity condition

$$\boldsymbol{a}\,(x,s,z)\cdot z \geq \lambda\,|z|^p\,, \qquad \lambda>0\,, \tag{2.7}$$

and the monotonicity condition

$$(\boldsymbol{a}\,(x,s,z) - \mathbf{a}\,(x,s,z'))\cdot(z-z') > 0\,, \tag{2.8}$$

These hypotheses hold for every $z,z'\in\mathbb{R}^N$, with $z\neq z'$, for every $s\in\mathbb{R}$ and for almost every $x\in\Omega$.
Finally we assume

$$f\geq 0 \quad \text{and} \quad f\in L^m(\Omega)\,, \tag{2.9}$$

where m will be specified later.

Remark 2.1. It is worth remarking that no singularity occurs in the product $g(x,s)s^{1-q}$ since

$$g(x,s)\,s^{1-q} \leq \Lambda\,. \tag{2.10}$$

Obviously the product $g(x,s)s$ has also no singularities.

Now we give the definition of weak solution to problem (2.3) whose existence is proved in Sections 3 - 5.

Definition 2.1. *A function $u\in W_0^{1,p}(\Omega)$ is a* **weak solution** *to* (2.3) *if*

$$\frac{|\nabla u|^p}{|u|^q}\in L^1(\Omega)\,, \tag{2.11}$$

$$b(x,u,\nabla u)\in L^1(\Omega)\,, \tag{2.12}$$

$$\int_\Omega g(x,u)v\,dx < +\infty\,, \qquad \forall v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\,, \tag{2.13}$$

and

$$\int_\Omega \boldsymbol{a}(x,u,\nabla u)\,\cdot\,\nabla\varphi\,dx \;=\; \int_\Omega b(x,u,\nabla u)\varphi\,dx \,+\, \int_\Omega g(x,u)\varphi \,+\, \int_\Omega f\varphi\,. \tag{2.14}$$

for every $\varphi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$.

2.2. **Auxiliary functions.** In this section we recall some well-known facts which are tools to address the study of quasi-linear elliptic equations with natural growth in the gradient.
Denote

$$H(s)=\frac{1}{\lambda}\int_0^s h(\sigma)\,d\sigma\,, \tag{2.15}$$

$$\Phi(s)=\int_0^s e^{\frac{|H(\sigma)|}{p-1}}\,d\sigma\,. \tag{2.16}$$

These auxiliary functions are used throughout the whole paper. A simple remark is in order: every function u satisfies

$$|\Phi(u)| = \left|\int_0^u e^{\frac{|H(\sigma)|}{p-1}}\,d\sigma\right| \geq |u| \tag{2.17}$$

$$|\Phi(u)| \leq |u|e^{\frac{|H(u)|}{p-1}} \tag{2.18}$$

and
$$|\nabla\Phi(u)| = e^{\frac{|H(u)|}{p-1}}|\nabla u| \geq |\nabla u| \,. \tag{2.19}$$
Therefore, if $\Phi(u) \in L^r(\Omega)$ or $\Phi(u) \in W_0^{1,r}(\Omega)$ for some $r \geq 1$, then $u \in L^r(\Omega)$ or $u \in W_0^{1,r}(\Omega)$ respectively.

2.3. **Approximating problems.** We are concerned with proving that problem (2.3) has at least a weak solution. We will prove this result by approximations. To do so, we consider the following problems
$$\begin{cases} -\mathrm{div}(\boldsymbol{a}(x,u_n,\nabla u_n)) = b(x,u_n,\nabla u_n) + g_n(x,u_n) + f_n(x)\,, & \text{in } \Omega \\ u_n \geq 0\,, & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{2.20}$$
where $g_n(x,s) = T_n(g(x,s))$ and $f_n = T_n(f)$. For any fixed n, problem (2.20) exhibits at least a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as a consequence of the results of [31], it is enough to take $b_0(x) = n$ in [31, Theorem 1.1].

Actually, we may straightly consider the datum f, without truncations, since by assumptions on its summability, the datum f is always an element of the dual space of the Sobolev space $W_0^1(\Omega)$, $W^{-1,p'}(\Omega)$. Starting from the truncated data allows us to easily use a cancellation lemma. It is a consequence of a kind of change of unknown obtained multiplying the equation by a suitable exponential function of u_n (see, for example, [31]). Since $u_n \in L^\infty(\Omega)$, there is no need to worry about whether these exponential test functions can really be chosen.

Lemma 2.1. *(Cancellation lemma) Let $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to problem (2.20).*

(1) If $v \in W_0^{1,p}(\Omega)$, then
$$\int_\Omega e^{sign\,(v) H(u_n)}\boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla v\,dx \leq \int_\Omega e^{sign\,(v)H(u_n)} v g(x,u_n)\,dx + \int_\Omega e^{sign\,(v)H(u_n)} v f\,dx.$$
(2) If Ψ is a locally Lipschitz continuous and nondecreasing function such that $\Psi(0) = 0$, then
$$\lambda\int_\Omega e^{|H(u_n)|}\Psi'(u_n)|\nabla u_n|^p dx \leq \int_\Omega e^{|H(u_n)|}\Psi(u_n) g(x,u_n)\,dx + \int_\Omega e^{|H(u_n)|}\Psi(u_n) f\,dx\,.$$

We first apply this Lemma to check that the approximate solutions are nonnegative. To this end, we choose $v = -u_n^-$ in Lemma 2.1 (1) to get
$$\int_{\{u_n<0\}} e^{-H(u_n)}\,\boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla u_n \leq -\int_\Omega (e^{-H(u_n)}u_n^-)T_n(g(x,u_n)) - \int_\Omega (e^{-H(u_n)}u_n^-) f_n$$
By (2.7), since the right-hand side is nonpositive, we obtain
$$\lambda\int_{\{u_n<0\}} e^{-H(u_n)}|\nabla u_n|^p \leq 0$$

from where $u_n \geq 0$ a.e. in Ω follows.

3. Existence result for $m > N/p$

The main result of this section concerns existence of nonnegative weak solutions to problem (2.3) when the datum f is an element of the Lebesgue space $L^m(\Omega)$, with $m > N/p$. It is given by the following theorem:

Theorem 3.1. *Assume* (2.4)-(2.9) *with*

$$f \in L^m(\Omega)\,, \qquad m > \frac{N}{p}$$

and

$$\lim_{s\to\pm\infty} \frac{e^{H(s)}}{(1+\Phi(s))^{p-1}} = 0\,. \tag{3.1}$$

Then problem (2.3) *has at least a weak solution u such that $\Phi(u) \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and, consequently, $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

We prove that the sequence of *approximate solutions* $\{u_n\}_n$ to problem (2.20) satisfies some a priori estimates. By these estimates we deduce that u_n, up to subsequences, converges to a function u which we prove is the sought weak solution.

3.1. A priori estimates when $m > N/p$.

In this subsection we prove that the sequence of *approximate solutions* $\{u_n\}_n$ satisfies a priori estimates in $L^\infty(\Omega)$ and in $W_0^{1,p}(\Omega)$. As pointed out, by these estimates we deduce that u_n converges, up to subsequences, to a function u which is the sought solution.

Lemma 3.1. *(Estimates in $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$). For any fixed $n \in \mathbb{N}$, let $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to problem* (2.20)*. Under the assumptions of Theorem 3.1, the following estimates hold true:*

$$\|u_n\|_{L^\infty(\Omega)} \leq C_1\,, \tag{3.2}$$

$$\|\nabla u_n\|_{L^p(\Omega)} \leq C_2\,, \tag{3.3}$$

where C_1, C_2 are positive constants which only depend on $|\Omega|$, N, m, p, $\|f\|_{L^m}$, λ, but do not depend on n.

Proof. Most of the proof follows that of [31, Proposition 3.1]. We insert it to highlight that the presence of the singular term does not affect the result.

For any fixed $k > 0$ consider $\Psi(s) = G_k(\Phi(s))$ in Lemma 2.1 (2) with $g(x, u_n)$ replaced by $T_n(g(x, u_n))$. By (2.7), (2.6) and Hölder's inequality, we obtain

$$\lambda \int_\Omega |\nabla G_k(\Phi(u_n))|^p \, dx \tag{3.4}$$

$$\leq \int_\Omega T_n(g(x,u_n))\, e^{H(u_n)}\, |G_k(\Phi(u_n))| \, dx + \int_\Omega f_n\, e^{H(u_n)}\, |G_k(\Phi(u_n))| \, dx$$

$$\leq \Lambda \int_\Omega u_n^{q-1}\, e^{H(u_n)}\, |G_k(\Phi(u_n))| \, dx + \|f\|_m \left(\int_\Omega e^{m'H(u_n)}\, |G_k(\Phi(u_n))|^{m'} \, dx\right)^{\frac{1}{m'}}$$

Denote for any k,

$$\eta(k)=\sup_{\{\Phi(s)>k\}}\frac{e^{H(s)}}{(1+|\Phi(s)|)^{p-1}}\,. \tag{3.5}$$

Since $\lim\limits_{s\to\pm\infty}\Phi(s)=\pm\infty$, it is easy to verify from (3.1) that

$$\lim_{k\to+\infty}\eta(k)=0\,. \tag{3.6}$$

Moreover as in [31], by (3.6), we obtain

$$\begin{aligned} e^{m'H(u_n)} &\le \frac{e^{m'H(u_n)}}{(1+|\Phi(u_n)|)^{m'(p-1)}}(1+k+|G_k(\Phi(u_n))|)^{m'(p-1)} \\ &\le C\eta(k)^{m'}(k^{m'(p-1)}+|G_k(\Phi(u_n))|^{m'(p-1)}) \end{aligned} \tag{3.7}$$

for all $k\ge 1$. In analogous way we get

$$e^{H(u_n)}\le C\eta(k)(k^{p-1}+|G_k(\Phi(u_n))|^{p-1}) \tag{3.8}$$

for all $k\ge 1$. By (3.4), we deduce

$$\begin{aligned} &\lambda\int_\Omega|\nabla G_k(\Phi(u_n))|^p\,dx \\ &\qquad\le C\eta(k)\int_\Omega\frac{|G_k(\Phi(u_n))|}{u_n^{1-q}}(k^{p-1}+|G_k(\Phi(u_n))|^{p-1})\,dx \\ &+C\eta(k)\|f\|_m\left(\int_\Omega|G_k(\Phi(u_n))|^{m'}(k^{m'(p-1)}+|G_k(\Phi(u_n))|^{m'(p-1)})\,dx\right)^{\frac{1}{m'}} \end{aligned} \tag{3.9}$$

Moreover since Φ is an increasing function, we deduce

$$\begin{aligned} \lambda\int_\Omega|\nabla G_k(\Phi(u_n))|^p\,dx \;&\le\frac{C\eta(k)k^{p-1}}{[\Phi^{-1}(k)]^{1-q}}\int_\Omega|G_k(\Phi(u_n))|\,dx \\ &+\frac{C\eta(k)}{[\Phi^{-1}(k)]^{1-q}}\int_\Omega|G_k(\Phi(u_n))|^p\,dx \\ &+C\eta(k)k^{p-1}\|f\|_m\left(\int_\Omega|G_k(\Phi(u_n))|^{m'}\,dx\right)^{\frac{1}{m'}} \\ &+C\eta(k)\|f\|_m\left(\int_\Omega|G_k(\Phi(u_n))|^{pm'}\,dx\right)^{\frac{1}{m'}} \end{aligned} \tag{3.10}$$

for all $k\ge 1$. The monotonicity of Φ also implies that $\Phi^{-1}(k)\ge 1$ for k larger enough, so that we get rid of this coefficient for k larger than certain k'.
Denote $A_k=\{\Phi(u)\ge k\}$. Now we estimate each term in (3.10) by using Hölder's inequality. The following inequality holds true:

$$\int_\Omega|G_k(\Phi(u_n))|\,dx=\int_{A_k}|G_k(\Phi(u_n))|\,dx\le|A_k|^{1-\frac{1}{p^*}}\left(\int_{A_k}|G_k(\Phi(u_n))|^{p^*}\,dx\right)^{\frac{1}{p^*}}.$$

Thanks to these estimates and the Sobolev embedding theorem, (3.10) becomes

$$\lambda S\left(\int_\Omega |G_k(\Phi(u_n))|^{p^*}\,dx\right)^{\frac{p}{p^*}} \tag{3.11}$$

$$\le C\eta(k)k^{p-1}|A_k|^{\frac{1}{m'}-\frac{1}{p^*}}|\Omega|^{1-\frac{1}{m'}}\left(\int_\Omega |G_k(\Phi(u_n))|^{p^*}\,dx\right)^{\frac{1}{p^*}}$$

$$+ C\eta(k)|\Omega|^{1-\frac{p}{p^*}}\left(\int_\Omega |G_k(\Phi(u_n))|^{p^*}\,dx\right)^{\frac{p}{p^*}}$$
$$+ C\eta(k)k^{p-1}|A_k|^{\frac{1}{m'}-\frac{1}{p^*}}\|f\|_m\left(\int_\Omega |G_k(\Phi(u_n))|^{p^*}\,dx\right)^{\frac{1}{p^*}}$$
$$+ C\eta(k)\|f\|_m|\Omega|^{\frac{1}{m'}-\frac{p}{p^*}}\left(\int_\Omega |G_k(\Phi(u_n))|^{p^*}\,dx\right)^{\frac{p}{p^*}}$$

for all $k > k'$. Since $\eta(k)$ goes to 0 when k tends to $+\infty$, it yields that the second and the fourth terms on the right hand side can be absorbed by the left hand side. Then there exists $k_0 > 0$ such that, for $k > k_0$, we obtain:

$$\left(\int_\Omega |G_k(\Phi(u_n))|^{p^*}\,dx\right)^{\frac{p-1}{p^*}} \le C\eta(k)k^{p-1}|A_k|^{\frac{1}{m'}-\frac{1}{p^*}} \tag{3.12}$$

This is the same inequality as in [31, Proposition 3.1]. So, following its procedure, we deduce that $\{\Phi(u_n)\}_n$, and consequently $\{u_n\}_n$, is bounded in $L^\infty(\Omega)$. Therefore, there exists a positive constant $C_1 > 0$ satisfying $\|u_n\|_\infty \le C_1$ for all $n \in \mathbb{N}$.

Now we prove the a priori estimates in $W_0^{1,p}(\Omega)$ given by (3.3).
Consider $\Psi(s) = s$ in Lemma 2.1 (2) with $g(x, u_n)$ replaced by $T_n(g(x, u_n))$. By (2.7), (2.6), (3.2) and Hölder's inequality, we obtain

$$\lambda \int_\Omega |\nabla u_n|^p\,dx \le \int_\Omega T_n(g(x,u_n))\,e^{H(u_n)}\,u_n\,dx + \int_\Omega f_n\,e^{H(u_n)}\,u_n\,dx \tag{3.13}$$

$$\le \Lambda \int_\Omega |u_n|^q\,e^{H(u_n)}\,dx + \int_\Omega f\,e^{H(u_n)}\,u_n\,dx$$

Thus,

$$\int_\Omega |\nabla u_n|^p\,dx \le \frac{1}{\lambda}\Big[\Lambda C_1^q e^{H(C_1)}|\Omega| + e^{H(C_1)}C_1\|f\|_{L^1}\Big]$$

and estimate (3.3) is proven. ■

3.2. **Strong convergence of ∇u_n.** In this subsection we prove that the sequence of the *approximate solutions* $\{u_n\}_n$ and their gradients converge to a function u and its gradient ∇u respectively. Moreover we also prove that the different terms appearing in equation (2.20) converge.

For any fixed $n \in \mathbb{N}$, let $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to problem (2.20). As a consequence of Lemma 3.1 we deduce that there exists a nonnegative function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that, up to subsequences,

$$\nabla u_n \rightharpoonup \nabla u\,, \qquad \text{weakly in } L^p(\Omega;\mathbb{R}^N)\,, \tag{3.14}$$

$$u_n \to u\,, \qquad \text{strongly in } L^r(\Omega) \text{ for } 1 \le r < p^*\,, \tag{3.15}$$

$$u_n(x) \to u(x)\,, \qquad \text{a.e. in } \Omega\,. \tag{3.16}$$

Actually, the L^∞–estimate (3.2) implies that

$$u_n \to u\,, \qquad \text{strongly in } L^r(\Omega) \text{ for } 1 \le r < +\infty\,. \tag{3.17}$$

Lemma 3.2. *(Strong convergence of ∇u_n). Under the assumptions of Theorem 3.1,*

$$\nabla u_n \to \nabla u\,, \quad \textit{strongly in } (L^p(\Omega))^N \tag{3.18}$$

$$\boldsymbol{a}(x,u_n,\nabla u_n) \to \boldsymbol{a}(x,u,\nabla u)\,, \quad \textit{strongly in } L^{p'}(\Omega;\mathbb{R}^N)\,, \tag{3.19}$$

$$b(x,u_n,\nabla u_n) \to b(x,u,\nabla u) \qquad \textit{strongly in } L^1(\Omega)\,. \tag{3.20}$$

Proof. We proceed to check all the conditions by dividing the proof in several steps. *Step 1. Strong convergence of the gradients.* In order to prove (3.18), we check that (see [8, Lemma 5])

$$\lim_{n\to+\infty} \int_\Omega [\boldsymbol{a}(x,u_n,\nabla u_n) - \boldsymbol{a}(x,u_n,\nabla u)] \cdot \nabla(u_n-u)\,dx = 0\,. \tag{3.21}$$

Consider $v = u_n - u \in W_0^{1,p}(\Omega)$ in Lemma 2.1 (1) to obtain

$$\begin{aligned}&\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}\boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla(u_n-u)\,dx\\ &\le \int_\Omega T_n(g(x,u_n))e^{\text{sign}\,(u_n-u)H(u_n)}(u_n-u)\,dx + \int_\Omega f_n e^{\text{sign}\,(u_n-u)H(u_n)}(u_n-u)\,dx\,.\end{aligned} \tag{3.22}$$

Since $u_n - u = (u_n-u)_+ - (u_n-u)_-$ and $-(u_n-u)_- \le 0$ a.e. in Ω, we obtain

$$\begin{aligned}&\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}\boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla(u_n-u)\,dx\\ &\le \int_\Omega T_n(g(x,u_n))e^{\text{sign}\,(u_n-u)H(u_n)}(u_n-u)_+\,dx + \int_\Omega f_n e^{\text{sign}\,(u_n-u)H(u_n)}(u_n-u)\,dx\,.\end{aligned} \tag{3.23}$$

Now let us analyze the following integral

$$\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla(u_n-u)\,dx \tag{3.24}$$

$$\le -\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)\,dx$$

$$+\int_\Omega T_n(g(x,u_n))e^{\text{sign}\,(u_n-u)H(u_n)}(u_n-u)_+\,dx$$

$$+\int_\Omega f_n e^{\text{sign}\,(u_n-u)H(u_n)}(u_n-u)\,dx$$

$$= I_n^1+I_n^2+I_n^3\,.$$

Let us evaluate I_n^1. We prove

$$\lim_{n\to+\infty} I_n^1=\lim_{n\to+\infty}\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)\,dx=0 \tag{3.25}$$

Indeed, first we split

$$\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)\,dx$$
$$=\int_\Omega e^{H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)_+\,dx-\int_\Omega e^{-H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)_-\,dx$$

We remark that, owing to (3.14),

$$\nabla(u_n-u)_+\rightharpoonup 0 \quad \text{weakly in } L^p(\Omega;\mathbb{R}^N)\,. \tag{3.26}$$

In fact, if $\varphi\in C_0^\infty(\Omega)$, then

$$\int_\Omega \frac{\partial(u_n-u)_+}{\partial x_i}\varphi\,dx=-\int_\Omega (u_n-u)_+\frac{\partial\varphi}{\partial x_i}\,dx$$

tends to 0 for all $i=1,\dots,N$, by (3.15).

Since

$$e^{H(u_n)}|\boldsymbol{a}(x,u_n,\nabla u)|$$
$$\le e^{H(C_1)}(a_0|\nabla u|^{p-1}+a_1|u_n|^{p-1}+a_2)\,,\qquad \text{a.e. in }\Omega\,,$$

it follows from (3.15) that the right hand side converges in $L^{p'}(\Omega)$, so that the left hand side is equiintegrable. Therefore by (3.16) and Vitali's convergence theorem we deduce

$$e^{H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\to e^{H(u)}\boldsymbol{a}(x,u,\nabla u)\,,\qquad \text{strongly in } L^{p'}(\Omega)^N \tag{3.27}$$

Combining (3.27) and (3.26), we infer that

$$\lim_{n\to+\infty}\int_\Omega e^{H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)_+\,dx=0$$

Analogously,

$$\lim_{n\to+\infty}\int_\Omega e^{-H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)_-\,dx=0$$

and (3.25) is proven.
Let us evaluate I_n^2. By the growth condition on g (2.6), for any fixed $\delta>0$, we get:

$$I_n^2\le\Lambda\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}(u_n-u)_+u_n^{q-1}\,dx$$

$$\le\Lambda\delta^{q-1}\int_{\{u_n\ge\delta\}}e^{\text{sign}\,(u_n-u)H(u_n)}(u_n-u)_+\,dx+\Lambda\int_{\{u_n\le\delta\}\cap\{u_n\ge u\}}e^{H(u_n)}u_n^q\,dx$$

$$\le\Lambda\delta^{q-1}e^{H(C_1)}\int_\Omega|u_n-u|\,dx+\Lambda\delta^q e^{H(C_1)}|\Omega|$$

It yields

$$\limsup_{n\to\infty}I_n^2\le\Lambda\delta^q e^{H(C_1)}|\Omega| \tag{3.28}$$

for all $\delta>0$, so that $\lim_{n\to\infty}I_n^2=0$.

Finally we evaluate I_n^3. By (3.15), our assumption of summability on f and the L^∞ estimate, we have

$$\lim_{n\to+\infty}I_n^3=\lim_{n\to+\infty}\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}f(u_n-u)\,dx=0 \tag{3.29}$$

By (3.24), combining (3.25), (3.28) and (3.29), we get

$$\limsup_{n\to+\infty}\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla(u_n-u)\,dx\le 0\,.$$

Since by (2.7)

$$e^{\text{sign}\,(u_n-u)H(u_n)}[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla(u_n-u)\ge 0$$

we deduce

$$\lim_{n\to+\infty}\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla(u_n-u)\,dx=0\,. \tag{3.30}$$

Therefore by (3.30), we have

$$\lim_{n\to+\infty}\int_\Omega[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla(u_n-u)\,dx \tag{3.31}$$

$$=\lim_{n\to+\infty}\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}e^{-\text{sign}\,(u_n-u)H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)\,dx$$

$$\le e^{H(C_1)}\lim_{n\to+\infty}\int_\Omega e^{\text{sign}\,(u_n-u)H(u_n)}\boldsymbol{a}(x,u_n,\nabla u)\cdot\nabla(u_n-u)\,dx=0\,.$$

This proves (3.21) and therefore (3.18).

Step 2. Strong convergence of $\boldsymbol{a}(x,u_n,\nabla u_n)$ *and* $b(x,u_n,\nabla u_n)$ A straightforward conseguence of (3.18) is that, up to subsequences,

$$\nabla u_n\to\nabla u\,,\qquad\text{a.e. in }\Omega\,. \tag{3.32}$$

Easy consequences of the pointwise convergence of the gradients are

$$\boldsymbol{a}(x,u_n,\nabla u_n)\to \boldsymbol{a}(x,u,\nabla u)\,,\qquad \text{a.e. in } \Omega,$$

and

$$b(x,u_n,\nabla u_n)\to b(x,u,\nabla u)\,,\qquad \text{a.e. in } \Omega\,.$$

Furthermore, the strong convergence of the gradients (3.18) jointly with (3.17) imply that the sequence

$$a_0|\nabla u_n|^{p-1}+a_1|u_n|^{p-1}+a_2$$

is equi-integrable. So, our hypothesis (2.4), gives the equiintegrability of $\boldsymbol{a}(x,u_n,\nabla u_n)$ and, by Vitali's Theorem, (3.19) follows. On the other hand, the L^∞–estimate leads to the boundedness of $h(u_n)$. Hence, it follows from (2.4) and the strong convergence of the gradients that the sequence $b(x,u_n,\nabla u_n)$ is equiintegrable. Applying again Vitali's Theorem, we get (3.20). Additionally, we also obtain

$$b(x,u,\nabla u)\in L^1(\Omega). \tag{3.33}$$

■

3.3. **Existence: proof of Theorem 3.1.** In this subsection, we prove that the function u is a weak solution to problem (2.3) according to Definition 2.1.
Since we have proved (3.33), condition (2.12) is satisfied. Therefore we proceed to check the other conditions in Definition 2.1.
Step 1. u satisfies (2.11)
Let us consider the weak solution u_n to the approximate problem (2.20). For any fixed $k>0$ we take $v=G_k(u_n^{1-q})$ in Lemma 2.1 (1) and so

$$\begin{aligned}(1-q)\lambda\int_{\{u_n^{1-q}>k\}} & e^{H(u_n)}\;\boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla u_n\,u_n^{-q}\,dx\\ &\le \int_\Omega e^{H(u_n)}T_n(g(x,u_n))G_k(u_n^{1-q})\,dx+\int_\Omega e^{H(u_n)}f_n\,G_k(u_n^{1-q})\,dx\,.\end{aligned}$$

By assumption on g (2.6), since u_n is a nonnegative function, we have:

$$T_n(g(x,u_n))\,G_k(u_n^{1-q})\le g(x,u_n)\,u_n^{1-q}\le\Lambda\,,$$

a.e. in $\{u_n^{1-q}\ge k\}$.

By ellipticity condition (2.7), Remark 2.1 and estimate (3.2), we get

$$\begin{aligned}(1-q)\lambda\int_{\{u_n^{1-q}>k\}} & e^{H(u_n)}u_n^{-q}|\nabla u_n|^p\,dx\\ &\le C\int_\Omega T_n(g(x,u_n))G_k(u_n^{1-q})\,dx+C\int_\Omega f_n\,G_k(u_n^{1-q})\,dx\\ &\le C\Lambda|\Omega|+C\int_\Omega f\,G_k(u_n^{1-q})\,dx\\ &\le C\Lambda|\Omega|+C\|u_n\|_\infty\int_\Omega f\,dx\le C\,.\end{aligned}$$

Since $e^{H(u_n)}\ge 1$, we obtain

$$\int_{\{u_n^{1-q}>k\}}|u_n|^{-q}|\nabla u_n|^p\,dx\le C\,.$$

So, owing to the Fatou Lemma,

$$\int_{\{u^{1-q}>k\}} |u|^{-q}|\nabla u|^p\,dx \le C\,. \tag{3.34}$$

Now we let k go to 0 on the left-hand side, by monotone convergence Theorem,

$$\lim_{k\to 0}\int_{\{u^{1-q}>k\}} u^{-q}|\nabla u|^p dx = \int_{\{u>0\}} u^{-q}|\nabla u|^p dx < +\infty\,.$$

Applying [26, Lemma 2.5], we obtain that $u^{1-\frac{q}{p}} \in W_0^{1,p}(\Omega)$ and

$$\int_{\{u>0\}} u^{-q}|\nabla u|^p dx = \int_\Omega u^{-q}|\nabla u|^p dx\,,$$

so that $u^{-q}|\nabla u|^p \in L^1(\Omega)$.

Step 2. u satisfies (2.13)

Consider $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $v \ge 0$ a.e. in Ω as test function in (2.20), we get

$$\int_\Omega \boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla v\,dx = \int_\Omega b(x,u_n,\nabla u_n)v\,dx + \int_\Omega T_n\left(g(x,u_n)\right)v\,dx + \int_\Omega f_n v\,dx\,.$$

Therefore passing to the limit for n which goes to $+\infty$, by (3.16), (3.20) and Fatou's lemma, we get

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega \boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla v\,dx& \\ -\lim_{n\to+\infty}\left(\int_\Omega b(x,u_n,\nabla u_n)v\,dx + \int_\Omega f_n v\,dx\right)& \\ \ge \int_\Omega g(x,u)v\,dx\,.&\end{aligned} \tag{3.35}$$

that is

$$\int_\Omega \boldsymbol{a}(x,u,\nabla u)\cdot\nabla v\,dx - \int_\Omega b(x,u,\nabla u)v\,dx - \int_\Omega f v\,dx \ge \int_\Omega g(x,u)v\,dx\,. \tag{3.36}$$

This yields the conclusion for $v \ge 0$. The general case follows from the decomposition $v = v_+ - v_-$.

Step 3. Proof of (2.14) *by passing to the limit in the approximate problems.*

Let φ be any nonnegative function belonging to $W^{1,p}(\Omega)\cap L^\infty(\Omega)$. Taking $T_k(u_n)\varphi$ as test function in (2.20) and disregarding a nonnegative term, we have

$$\begin{aligned}\int_\Omega T_k(u_n)\left(\boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla\varphi\right)\,dx &\le \int_\Omega b(x,u_n,\nabla u_n)T_k(u_n)\varphi\,dx \\ &+ \int_\Omega T_n\left(g(x,u_n)\right)T_k(u_n)\varphi\,dx + \int_\Omega f_n T_k(u_n)\varphi\,dx\,.\end{aligned} \tag{3.37}$$

Now we let n go to $+\infty$ in the inequality (3.37). On the left-hand side we use the strong convergence of $\left\{\boldsymbol{a}(x,u_n,\nabla u_n)\right\}_n$ in $L^{p'}(\Omega)$, (3.19), the pointwise convergence of u_n (3.16) and Lebesgue's dominated convergence theorem in order to obtain

$$\lim_{n\to+\infty}\int_\Omega T_k(u_n)\left(\boldsymbol{a}(x,u_n,\nabla u_n)\cdot\nabla\varphi\right)\,dx = \int_\Omega T_k(u)\left(\boldsymbol{a}(x,u,\nabla u)\cdot\nabla\varphi\right)\,dx\,.$$

In analogous way, we evaluate the first term on the right-hand side (3.37). We apply the strong convergence of $\left\{b(x,u_n,\nabla u_n)\right\}_n$ in $L^1(\Omega)$ given by (3.33), the pointwise convergence of u_n (3.16) and Lebesgue's dominated convergence theorem in order to obtain

$$\lim_{n\to+\infty}\int_\Omega b(x,u_n,\nabla u_n)T_k(u_n)\varphi\,dx=\int_\Omega b(x,u,\nabla u)T_k(u)\varphi\,dx\,.$$

Concerning the second term on the right-hand side of (3.37), we observe that

$$\begin{aligned}&T_n\left(g(x,u_n)\right)T_k(u_n)\\&\qquad=T_n\left(g(x,u_n)\right)T_k(u_n)\big|_{\{u_n\le k\}}+T_n\left(g(x,u_n)\right)T_k(u_n)\big|_{\{u_n>k\}}\\&\qquad\qquad\le\Lambda u_n^{q-1}u_n\big|_{\{u_n\le k\}}+\Lambda k u_n^{q-1}\big|_{\{u_n>k\}}\le\Lambda k^q\,.\end{aligned}$$

Therefore we can apply Lebesgue's dominated convergence Theorem and we get

$$\lim_{n\to+\infty}\int_\Omega T_n\left(g(x,u_n)\right)T_k(u_n)\varphi\,dx=\int_\Omega g(x,u)T_k(u)\varphi\,dx\,.$$

Finally it is easy to verify that

$$\lim_{n\to+\infty}\int_\Omega f_nT_k(u_n)\varphi\,dx=\int_\Omega fT_k(u)\varphi\,dx.$$

Hence, passing to the limit for n which goes to $+\infty$ in (3.37), we get

$$\begin{aligned}&\int_\Omega T_k(u)\left(\boldsymbol{a}(x,u,\nabla G_k(u))\cdot\nabla\varphi\right)\,dx\\&\qquad\le\int_\Omega b(x,u,\nabla u)T_k(u)\varphi\,dx+\int_\Omega g(x,u)T_k(u)\varphi\,dx+\int_\Omega fT_k(u)\varphi\,dx\\&\qquad\qquad\le\int_\Omega b(x,u,\nabla u)T_k(u)\varphi\,dx+k\int_\Omega g(x,u)\varphi\,dx+k\int_\Omega f\varphi\,dx\,.\end{aligned}$$

Dividing by k and letting k go to 0, it follows that

$$\begin{aligned}&\int_{\{u\neq0\}}\left(\boldsymbol{a}(x,u,\nabla u)\cdot\nabla\varphi\right)\,dx\\&\le\int_{\{u\neq0\}}b(x,u,\nabla u)\varphi\,dx+\int_\Omega g(x,u)\,\varphi\,dx+\int_\Omega f\varphi\,dx\end{aligned}$$

holds true. As a consequence of Stampacchia's Theorem (cf. [33]), we obtain

$$\begin{aligned}&\int_\Omega\left(\boldsymbol{a}(x,u,\nabla u)\cdot\nabla\varphi\right)\,dx\\&\le\int_\Omega b(x,u,\nabla u)\varphi\,dx+\int_\Omega g(x,u)\,\varphi\,dx+\int_\Omega f\varphi\,dx\end{aligned}$$

Since (3.36) yields the reverse inequality, we conclude

$$\begin{aligned}&\int_\Omega\left(\boldsymbol{a}(x,u,\nabla u)\cdot\nabla\varphi\right)\,dx\\&=\int_\Omega b(x,u,\nabla u)\varphi\,dx+\int_\Omega g(x,u)\,\varphi\,dx+\int_\Omega f\varphi\,dx\end{aligned}$$

for all $\varphi\ge0$. The general case is now straightforward. ∎

4. Existence result for $m = N/p$

The main result of the section concerns existence of nonnegative weak solutions to problem (2.3) when the datum f is an element of the Lebesgue space $L^m(\Omega)$, with $m = N/p$. Its statement is the following.

Theorem 4.1. *Assume* (2.4) - (2.9) *with*

$$f \in L^{\frac{N}{p}}(\Omega)$$

and

$$\lim_{s\to\pm\infty} \frac{e^{H(s)}}{(1+\Phi(s))^{p-1}} = 0\,. \tag{4.1}$$

Then problem (2.3) *has at least a weak solution u such that $\Phi(u) \in W_0^{1,p}(\Omega)\cap L^r(\Omega)$, and hence $u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, for all $1 \le r < \infty$.*

As in the previous case, we consider the approximate problems (2.20), which for any fixed n, has at least a nonnegative bounded weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We begin by proving a priori estimates for weak solutions u_n which implies the existence of a limit function u which is proven to be a weak solution to problem (2.3).

4.1. A priori estimates when $m = N/p$.

In this subsection we prove that the sequence of *approximate solutions* $\{u_n\}_n$ satisfies a priori estimates in $L^r(\Omega)$, for any $r > 1$, and in $W_0^{1,p}(\Omega)$. By these estimates we deduce that u_n, up to subsequences, converges to a limit function u which is the sought solution.

Lemma 4.1. *(Estimates in $L^r(\Omega)$ for all $1 \le r < \infty$ and in $W_0^{1,p}(\Omega)$). For any fixed $n \in \mathbb{N}$, let $u_n \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ be a weak solution to problem* (2.20)*. Under the assumptions of Theorem 3.1, the following estimates hold true:*

$$\|u_n\|_{L^r(\Omega)} \le C_3 \qquad 1 \le r < \infty\,, \tag{4.2}$$

$$\|\nabla u_n\|_{L^p(\Omega)} \le C_4\,, \tag{4.3}$$

where C_3, C_4 are positive constants which only depend on $|\Omega|$, N, m, p, $\|f\|_{L^m}$, λ and on r, but do not depend on n.

Furthermore, it is also satisfied

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \int_\Omega |\nabla G_k(u_n)|^p\, dx = 0\,. \tag{4.4}$$

Proof. For any $\gamma \ge 1$, consider $\Psi(s) = \Phi(s)^\gamma$ in Lemma 2.1 (2). Then, since $e^{\frac{pH(u_n)}{p-1}} \ge 1$, we get

$$\begin{aligned}\lambda \int_\Omega \gamma\Phi(u_n)^{\gamma-1}|\nabla\Phi(u_n)|^p dx &\le \lambda\int_\Omega \gamma e^{\frac{pH(u_n)}{p-1}}\Phi(u_n)^{\gamma-1}|\nabla u_n|^p dx \\ &\le \int_\Omega e^{H(u_n)}\Phi(u_n)^\gamma g(x,u_n)\,dx + \int_\Omega e^{H(u_n)}\Phi(u_n)^\gamma f\,dx\end{aligned}$$

This inequality and Sobolev's imbedding Theorem lead to

$$\begin{aligned}\left(\int_\Omega \Phi(u_n)^{\frac{\gamma+p-1}{p}p^*}dx\right)^{\frac{p}{p^*}} &\le C\int_\Omega |\nabla\Phi(u_n)^{\frac{\gamma+p-1}{p}}|^p dx\\ &\le C\int_{\{\Phi(u_n)>k\}} e^{H(u_n)}\Phi(u_n)^\gamma g(x,u_n)\,dx + C\int_{\{\Phi(u_n)>k\}} e^{H(u_n)}\Phi(u_n)^\gamma f\,dx\\ &+ C\int_{\{\Phi(u_n)\le k\}} e^{H(u_n)}\Phi(u_n)^\gamma g(x,u_n)\,dx + C\int_{\{\Phi(u_n)\le k\}} e^{H(u_n)}\Phi(u_n)^\gamma f\,dx \qquad (4.5)\end{aligned}$$

where $k\ge 1$. We handle the integrals over $\{\Phi(u_n)>k\}$ employing the function η defined in (3.5). Notice that, since $k\ge 1$, it results $1+\Phi(u_n)\le 2\Phi(u_n)$ on the set $\{\Phi(u_n)>k\}$ and therefore, we deduce

$$\begin{aligned}&\int_{\{\Phi(u_n)>k\}} e^{H(u_n)}\Phi(u_n)^\gamma g(x,u_n)\,dx\\ &\qquad\le \int_{\{\Phi(u_n)>k\}} \eta(k)(1+\Phi(u_n))^{p-1}\Phi(u_n)^\gamma g(x,u_n)\,dx\\ &\qquad\le \int_{\{\Phi(u_n)>k\}} \eta(k)2^{p-1}\Phi(u_n)^{\gamma+p-1} g(x,u_n)\,dx\end{aligned}$$

Thus, we get

$$\begin{aligned}\int_{\{\Phi(u_n)>k\}} e^{H(u_n)}\Phi(u_n)^\gamma g(x,u_n)\,dx &\le C\eta(k)\int_{\{\Phi(u_n)>k\}}\Phi(u_n)^{\gamma+p-1}u_n^{q-1}\,dx \qquad (4.6)\\ &\le C\frac{\eta(k)}{[\Phi^{-1}(k)]^{1-q}}\int_{\{\Phi(u_n)>k\}}\Phi(u_n)^{\gamma+p-1}\,dx\\ &\le C\frac{\eta(k)}{[\Phi^{-1}(k)]^{1-q}}|\Omega|^{p/N}\left(\int_\Omega \Phi(u_n)^{(\gamma+p-1)\frac{p^*}{p}}\,dx\right)^{\frac{p}{p^*}}.\end{aligned}$$

Moreover by Hölder inequality, we get

$$\begin{aligned}\int_{\{\Phi(u_n)>k\}} e^{H(u_n)}\Phi(u_n)^\gamma f\,dx &\le C\eta(k)\int_{\{\Phi(u_n)>k\}}\Phi(u_n)^{\gamma+p-1}f\,dx \qquad (4.7)\\ &\le C\eta(k)\|f\|_{N/p}\left(\int_\Omega \Phi(u_n)^{(\gamma+p-1)\frac{p^*}{p}}\,dx\right)^{\frac{p}{p^*}}.\end{aligned}$$

Arguing as in Lemma 3.1, since $\eta(k)$ goes to zero when k tends to $+\infty$, the terms in the right-hand side of (4.6) and (4.7) can be absorbed by the left-hand side of (4.5). Hence, there exists k larger enough such that (4.5) becomes

$$\begin{aligned}\left(\int_\Omega \Phi(u_n)^{\frac{\gamma+p-1}{p}p^*}dx\right)^{\frac{p}{p^*}} &\le C\int_\Omega |\nabla\Phi(u_n)^{\frac{\gamma+p-1}{p}}|^p dx\\ &\le C\int_{\{\Phi(u_n)\le k\}} e^{H(u_n)}\Phi(u_n)^\gamma g(x,u_n)\,dx + C\int_{\{\Phi(u_n)\le k\}} e^{H(u_n)}\Phi(u_n)^\gamma f\,dx\,. \qquad (4.8)\end{aligned}$$

Now observe that the right hand side is bounded. Indeed, since by (2.6) and (2.18), it follows that

$$e^{H(u_n)}\Phi(u_n)^\gamma g(x,u_n) \le \Lambda u_n^{q-1} e^{H(u_n)}\Phi(u_n)^\gamma = \Lambda e^{H(u_n)} u_n^{\gamma+q-1}\left(\frac{\Phi(u_n)}{u_n}\right)^\gamma \le \Lambda e^{H(u_n)} u_n^{\gamma+q-1} e^{\gamma\frac{H(u_n)}{p-1}}\,,$$

Thus, we get

$$\int_{\{\Phi(u_n)\le k\}} e^{H(u_n)}\Phi(u_n)^\gamma g(x,u_n)\,dx \le C|\Omega| e^{H(\Phi^{-1}(k))}\Phi^{-1}(k)^{\gamma+q-1} e^{\gamma\frac{H(\Phi^{-1}(k))}{p-1}}\,.$$

The remainder term in (4.8) is obviously bounded, that is

$$\int_{\{\Phi(u_n)\le k\}} e^{H(u_n)}\Phi(u_n)^\gamma f\,dx \le C e^{H(\Phi^{-1}(k))}\Phi^{-1}(k)^\gamma \|f\|_{L^1}\,.$$

Therefore, it follows from (4.8) that

$$\left(\int_\Omega \Phi(u_n)^{\frac{\gamma+p-1}{p}p^*}dx\right)^{\frac{p}{p^*}} \le C\int_\Omega |\nabla\Phi(u_n)^{\frac{\gamma+p-1}{p}}|^p dx \le C\,,$$

for all $\gamma \ge 1$. Hence by the arbitrary of γ and therefore of $\frac{\gamma+p-1}{p}p^* \ge 1$, the sequence $\{\Phi(u_n)\}_n$ is bounded in every $L^r(\Omega)$ such that $1 \le r < \infty$ and, taking $\gamma = 1$, in $W_0^{1,p}(\Omega)$. We conclude that the same features hold for $\{u_n\}_n$.

Condition (4.4) follows from

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\int_\Omega |\nabla G_k(\Phi(u_n))|^p\,dx = 0\,.$$

and it yields from performing the following computations (with some $\gamma > 1$):

$$\int_\Omega |\nabla G_k(\Phi(u_n))|^p\,dx \le \int_\Omega \frac{\Phi(u_n)^{\gamma-1}}{k^{\gamma-1}}|\nabla G_k(\Phi(u_n))|^p\,dx \le \frac{C}{k^{\gamma-1}}\,.$$

■

Remark 4.1. In the previous proof, we have seen that the sequence $\{\Phi(u_n)\}_n$ is bounded in every $L^r(\Omega)$, with $1 \le r < +\infty$. As a consequence of assumption (4.1), we deduce that $\{e^{H(u_n)}\}_n$ is also bounded in every $L^r(\Omega)$ with $1 \le r < +\infty$.

4.2. **Strong convergence of ∇u_n.** In this subsection we prove that the sequence of the *approximate solutions* $\{u_n\}_n$ and their gradients converge to a function u and its gradient ∇u respectively. Moreover we prove that every term in equation (2.20) converges.

For any fixed $n \in \mathbb{N}$, let $u_n \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ be a weak solution to problem (2.20). By Lemma 4.1 there exists a nonnegative function $u \in W_0^{1,p}(\Omega)\cap L^r(\Omega)$ for all $1 \le r < +\infty$ such that, up to subsequences, the convergences in (3.14), (3.15) and (3.16) hold true. In this limit case, we also obtain the convergence appearing in (3.17). Furthermore, the pointwise convergence allows us to obtain the strong convergence of $e^{H(u_n)}$ to $e^{H(u)}$ in any $1 \le r < \infty$.

Lemma 4.2. *(Strong convergence of ∇u_n). Under the assumptions of Theorem 4.1,*

$$\nabla u_n \to \nabla u\,, \quad \textit{strongly in } (L^p(\Omega))^N \tag{4.9}$$

$$\boldsymbol{a}(x,u_n,\nabla u_n) \to \boldsymbol{a}(x,u,\nabla u)\,, \quad \textit{strongly in } L^{p'}(\Omega;\mathbb{R}^N)\,, \tag{4.10}$$

$$b(x,u_n,\nabla u_n) \to b(x,u,\nabla u) \qquad \textit{strongly in } L^1(\Omega)\,. \tag{4.11}$$

Proof. We proceed to check all the conditions by dividing the proof in several steps.
Step 1. Strong convergence of the gradients. We explicitly point out that we cannot apply the same proof of the previous case because now we do not have an L^∞–bound for $\{u_n\}_n$.
As in the previous case, in order to prove (4.9), we are proving that (recall [8, Lemma 5])

$$\lim_{n\to+\infty}\int_\Omega [\boldsymbol{a}(x,u_n,\nabla u_n) - \boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla(u_n-u)\,dx = 0 \tag{4.12}$$

holds true. To this aim we write

$$\nabla(u_n-u) = \nabla T_k(u_n - T_h(u)) + \nabla G_k(u_n - T_h(u)) + \nabla(T_h(u)-u)\,,$$

for certain k and h, with $k>h$, to be chosen. Hence

$$\begin{aligned}
&\int_\Omega [\boldsymbol{a}(x,u_n,\nabla u_n) - \boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla(u_n-u)\,dx = \\
&\qquad \int_\Omega [\boldsymbol{a}(x,u_n,\nabla u_n) - \boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla T_k(u_n-T_h(u))\,dx \\
&\qquad + \int_\Omega [\boldsymbol{a}(x,u_n,\nabla u_n) - \boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla (T_h(u)-u)\,dx \\
&\qquad + \int_\Omega [\boldsymbol{a}(x,u_n,\nabla u_n) - \boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla G_k(u_n-T_h(u))\,dx
\end{aligned} \tag{4.13}$$

Let us begin by proving the following equality

$$\lim_{n\to+\infty}\int_\Omega [\boldsymbol{a}(x,u_n,\nabla u_n) - \boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla T_k(u_n-T_h(u))\,dx = 0 \tag{4.14}$$

We may take $v = T_k(u_n - T_h(u)) \in W_0^{1,p}(\Omega)$ in Lemma 2.1 (1). Then we proceed as in the previous case. Actually we integrate over $\{|u_n| \le k+h\}$ and we may argue as above replacing C_1 with $k+h$. This yields (4.14).

Let us evaluate the second integral on the right-hand side of (4.13). Fix $\epsilon>0$. Taking into account that the sequence $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$, it follows from condition (2.4) that $\{\boldsymbol{a}(x,u_n,\nabla u_n)\}_n$ is bounded in $L^{p'}(\Omega)^N$. Hence, there exists $h>0$ satisfying

$$\begin{aligned}
&\int_\Omega |[\boldsymbol{a}(x,u_n,\nabla u_n) - \boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla(T_h(u)-u)|\,dx \\
&\le \Big(\|\boldsymbol{a}(x,u_n,\nabla u_n)\|_{p'} + \|\boldsymbol{a}(x,u_n,\nabla u)\|_{p'}\Big)\left(\int_{\{u>h\}}|\nabla u|^p\right)^{1/p} < \epsilon/3 \quad \forall n\in\mathbb{N}\,.
\end{aligned} \tag{4.15}$$

Having fixed h, we determine k. To this end, notice that

$$
\begin{aligned}
&\int_\Omega |[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla G_k(u_n-T_h(u))|\,dx\\
&\qquad=\int_{\{|u_n-T_h(u)|>k\}}|[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla G_k(u_n-T_h(u))|\,dx\\
&\qquad=\int_{\{u_n>k-h\}}|[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla G_k(u_n-T_h(u))|\,dx\,.
\end{aligned}\tag{4.16}
$$

Therefore by growth condition (2.4) and a priori estimates (4.3), we have

$$
\begin{aligned}
&\int_\Omega |[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla G_k(u_n-T_h(u))|\,dx\\
&\le\Big(\|\boldsymbol{a}(x,u_n,\nabla u_n)\|_{p'}+\|\boldsymbol{a}(x,u_n,\nabla u)\|_{p'}\Big)\left[\left(\int_{\{u_n>k-h\}}|\nabla u_n|^p\right)^{1/p}+\left(\int_{\{u_n>k-h\}}|\nabla u|^p\right)^{1/p}\right]\\
&\qquad\qquad\le C\left[\left(\int_{\{u_n>k-h\}}|\nabla u_n|^p\right)^{1/p}+\left(\int_{\{u_n>k-h\}}|\nabla u|^p\right)^{1/p}\right]
\end{aligned}
$$

Taking into account (4.4), we may find k such that

$$
\int_\Omega |[\boldsymbol{a}(x,u_n,\nabla u_n)-\boldsymbol{a}(x,u_n,\nabla u)]\cdot\nabla G_k(u_n-T_h(u))|\,dx<\epsilon/3\quad\forall n\in\mathbb{N}.\tag{4.17}
$$

Combining (4.13), (4.14), (4.15) and (4.17), we conclude that (4.12) holds.

Step 2. Strong convergence of $\boldsymbol{a}(x,u_n,\nabla u_n)$ A straightforward conseguence of (4.9) is that, up to subsequences,

$$
\nabla u_n\to\nabla u\,,\qquad\text{a.e. in }\Omega\,.\tag{4.18}
$$

Moreover, we also infer

$$
b(x,u_n,\nabla u_n)\to b(x,u,\nabla u)\,,\quad\text{a.e. in }\Omega
$$

and

$$
\boldsymbol{a}(x,u_n,\nabla u_n)\to\boldsymbol{a}(x,u,\nabla u)\,,\quad\text{a.e. in }\Omega\,.
$$

Now, as in the proof of Step 2 of Lemma 3.2, it follows from (2.4) and (4.9) that

$$
\boldsymbol{a}(x,u_n,\nabla u_n)\to\boldsymbol{a}(x,u,\nabla u)\,,\quad\text{strongly in }L^{p'}(\Omega;\mathbb{R}^N)\,.\tag{4.19}
$$

Step 3. Strong convergence of $b(x,u_n,\nabla u_n)$ *to* $b(x,u,\nabla u)$ Now we prove (4.11). Since no L^∞–estimate is available, we are not able to prove that $\{h(u_n)\}_n$ is bounded, so that the proof given in the previous section is not possible.
Consider the function

$$
\Xi(s)=\int_0^s h(\sigma)\chi_{\{\sigma>k\}}d\sigma\qquad(k\ge1)
$$

and note that $\Xi(s) \le H(s)\chi_{\{s>k\}}$ holds. Taking $\Psi(s) = \Xi(s)$ in Lemma 2.1 (2), we obtain

$$\begin{aligned}\lambda\int_{\{u_n>k\}} & e^{H(u_n)}h(u_n)|\nabla u_n|^p dx \\ &\le \int_\Omega e^{H(u_n)}\Xi(u_n)g(x,u_n)\,dx + \int_\Omega e^{H(u_n)}\Xi(u_n) f\,dx \\ &\le \Lambda\int_{\{u_n>k\}} e^{H(u_n)}H(u_n)k^{q-1}\,dx + \int_{\{u_n>k\}} e^{H(u_n)}H(u_n) f\,dx \end{aligned} \tag{4.20}$$

We note that $k^{q-1} \le 1$ and use the fact that $\{e^{H(u_n)}\}_n$, and hence $\{H(u_n)\}_n$, is bounded in any $L^r(\Omega)$, $1 \le r < \infty$, to get

$$\int_{\{u_n>k\}} e^{H(u_n)}h(u_n)|\nabla u_n|^p dx \le C\|(1+f)\chi_{\{u_n>k\}}\|_{L^m(\Omega)},$$

so that the right hand side tends to 0 uniformly on n. Therefore,

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>k\}} e^{H(u_n)}h(u_n)|\nabla u_n|^p\,dx = 0$$

and, since $e^{H(u_n)} \ge 1$,

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>k\}} h(u_n)|\nabla u_n|^p\,dx = 0 \tag{4.21}$$

The main consequence is that the sequence $\{h(u_n)|\nabla u_n|^p\}_n$ is equiintegrable. Indeed, if $E \subset \Omega$, then

$$\begin{aligned}\int_E & h(u_n)|\nabla u_n|^p dx \\ &= \int_{E\cap\{u_n\le k\}} h(u_n)|\nabla u_n|^p dx + \int_{E\cap\{u_n>k\}} h(u_n)|\nabla u_n|^p dx \\ &\le \big[\max_{s\in[0,k]} h(s)\big]\int_E |\nabla u_n|^p dx + \int_{\{u_n>k\}} h(u_n)|\nabla u_n|^p dx \end{aligned} \tag{4.22}$$

Now let $\epsilon > 0$. Keeping in mind (4.21) and choosing k large enough, we may obtain that

$$\int_{\{u_n>k\}} h(u_n)|\nabla u_n|^p dx < \epsilon/2$$

for all $n \in \mathbb{N}$. Fixed k, we may use the strong convergence of gradients to deduce that there exists $\delta > 0$ such that, for any set E having $|E| < \delta$,

$$\big[\max_{s\in[0,k]} h(s)\big]\int_E |\nabla u_n|^p dx < \epsilon/2$$

for all $n \in \mathbb{N}$. Therefore by (4.22), $|E| < \delta$ implies $\int_E h(u_n)|\nabla u_n|^p dx < \epsilon$ for all $n \in \mathbb{N}$, which provides the equiintegrability of the sequence $\{h(u_n)|\nabla u_n|^p\}_n$. Then by (2.5) we conclude that the sequence $\{b(x,u_n,\nabla u_n)\}_n$ is equiintegrable. Applying Vitali's Theorem, (4.11) follows. ■

4.3. **Existence: proof of Theorem 4.1.** We prove that the function u is a weak solution to problem (2.3) according to Definition 2.1. The proof of Theorem 4.1 prooceds exactly as the proof of Theorem 3.1 in subsection 3.3. We explicitely remark that the strong convergence in (4.11) obviously implies (2.12). ■

5. Existence result for $\frac{Np}{Np-N+p} \le m < \frac{N}{p}$

The main result of the section, concerning existence of nonnegative weak solutions to problem (2.3) when the datum f is an element of the Lebesgue space $L^m(\Omega)$ with $\frac{Np}{Np-N+p} \le m < \frac{N}{p}$, is stated as follows:

Theorem 5.1. *Assume* (2.7) - (2.9) *with*

$$f \in L^m(\Omega)\,, \qquad \frac{Np}{N(p-1)+p} \le m < \frac{N}{p}\,.$$

Moreover assume that there exist a constant $0 < \theta < \frac{p^*}{pm'}$ *and constants* $0 < M_1 \le M_2$ *satisfying*

$$M_1 \le \frac{e^{H(s)}}{(1+\Phi(s))^{(p-1)\theta}} \le M_2 \qquad \forall s \ge 0\,. \tag{5.1}$$

Then problem (2.3) *has at least a weak solution such that* $u \in W_0^{1,p}(\Omega) \cap L^{\frac{Nm(p-1)}{N-pm}}(\Omega)$.

As in the previous cases, we consider problems (2.20), which for any fixed n, has at least a nonnegative bounded weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and we prove a priori estimates for these approximate solutions u_n.

5.1. **A priori estimates.** In this subsection we prove that the sequence of *approximate solutions* $\{u_n\}_n$ satisfies a priori estimates in $L^{\frac{Nm(p-1)}{N-pm}}(\Omega)$ and in $W_0^{1,p}(\Omega)$. We point out that $\frac{Nm(p-1)}{N-pm}$ tends to ∞ as $m \to \frac{N}{p}$ and so there is no solution of continuity with the estimates of the previous section. Observe, in addition, that $m = \frac{Np}{Np-N+p}$ yields an estimate in $L^{p^*}(\Omega)$, as expected.

Lemma 5.1. *(Estimates in* $L^{\frac{Nm(p-1)}{N-pm}}(\Omega)$ *and* $W_0^{1,p}(\Omega)$*). For any fixed* $n \in \mathbb{N}$*, let* $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to problem* (2.20)*. Under the assumptions of Theorem 5.1, the following estimates hold true:*

$$\|u_n\|_{L^{\frac{Nm(p-1)}{N-pm}}(\Omega)} \le C_5\,, \tag{5.2}$$

$$\|\nabla u_n\|_{L^p(\Omega)} \le C_6\,, \tag{5.3}$$

where C_5*,* C_6 *are positive constants which only depend on* $|\Omega|$*,* N*,* m*,* p*,* $\|f\|_{L^m}$*,* λ*, but do not depend on* n*.*

Furthermore, it also holds

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \int_\Omega |\nabla G_k(u_n)|^p \, dx = 0\,. \tag{5.4}$$

Proof. For any $r > p-1$, consider $\Psi(s) = (1+\Phi(s)^{r-p}\Phi(s)$ in Lemma 2.1 (2). Since $\Psi'(u_n) \geq \min\{r+1-p, 1\}(1+\Phi(u_n))^{r-p}\Phi'(u_n)$, we get

$$\begin{aligned}\int_\Omega (1+\Phi(u_n))^{r-p}|\nabla\Phi(u_n)|^p dx &= C\int_\Omega e^{\frac{pH(u_n)}{p-1}}(1+\Phi(u_n))^{r-p}|\nabla u_n|^p dx\\ &\leq C\int_\Omega e^{H(u_n)}(1+\Phi(u_n))^{r-p}\Phi(u_n)g(x,u_n)\,dx\\ &\qquad + C\int_\Omega e^{H(u_n)}(1+\Phi(u_n))^{r-p}\Phi(u_n)f_n\,dx\,. \quad (5.5)\end{aligned}$$

This inequality and Sobolev's imbedding Theorem lead to

$$\left(\int_\Omega [(1+\Phi(u_n))^{\frac{r}{p}}-1]^{p^*}dx\right)^{\frac{p}{p^*}} \leq C\int_\Omega (1+\Phi(u_n))^{r-p}|\nabla\Phi(u_n)|^p dx \quad (5.6)$$

$$\leq C\int_{\{\Phi(u_n)>k\}} e^{H(u_n)}(1+\Phi(u_n))^{r-p+1}g(x,u_n)\,dx$$

$$+ C\int_{\{\Phi(u_n)\leq k\}} e^{H(u_n)}(1+\Phi(u_n))^{r-p}\Phi(u_n)g(x,u_n)\,dx$$

$$+ C\int_\Omega e^{H(u_n)}(1+\Phi(u_n))^{r-p+1}f\,dx\,,$$

where $k \geq 1$. Now we evaluate the integral over $\{\Phi(u_n) > k\}$ in (5.6). Since $k \geq 1$, by assumption (5.1), we have

$$\int_{\{\Phi(u_n)>k\}} e^{H(u_n)}(1+\Phi(u_n))^{r-p+1}g(x,u_n)\,dx \quad (5.7)$$

$$\leq M_2\Lambda\int_{\{\Phi(u_n)>k\}}(1+\Phi(u_n))^{r-(p-1)(1-\theta)}u_n^{q-1}\,dx$$

$$\leq \frac{M_2\Lambda}{[\Phi^{-1}(k)]^{1-q}}\int_{\{\Phi(u_n)>k\}}(1+\Phi(u_n))^{r-(p-1)(1-\theta)}\,dx$$

$$\leq \frac{M_2\Lambda}{[\Phi^{-1}(k)]^{1-q}}|\Omega|^{1-\frac{1}{m}}\left(\int_\Omega (1+\Phi(u_n))^{[r-(p-1)(1-\theta)]m'}\,dx\right)^{\frac{1}{m'}}.$$

Next we analyze the second integral over $\{\Phi(u_n) \leq k\}$ in (5.6). Taking into account (2.6), (2.18) and (5.1), we get

$$\begin{aligned}e^{H(u_n)}(1+\Phi(u_n))^{r-p}\Phi(u_n)g(x,u_n) &\leq \Lambda e^{H(u_n)}(1+\Phi(u_n))^{r-p}u_n^q\left(\frac{\Phi(u_n)}{u_n}\right)\\ &\leq \Lambda e^{H(u_n)}(1+\Phi(u_n))^{r-p}u_n^q e^{\frac{H(u_n)}{p-1}}\end{aligned}$$

Consequently, we get

$$\int_{\{\Phi(u_n)\le k\}} e^{H(u_n)}(1+\Phi(u_n))^{r-p}\Phi(u_n)g(x,u_n)\,dx \\ \le \Lambda|\Omega|e^{H(\Phi^{-1}(k))}(1+k)^{r-p}\Phi^{-1}(k)^q e^{\frac{H(\Phi^{-1}(k))}{p-1}}\,. \quad (5.8)$$

Regarding the third term on the right hand side of (5.6), we apply (5.1) and the H\'older inequality to obtain

$$\int_\Omega e^{H(u_n)}(1+\Phi(u_n))^{r-p+1}f\,dx \le M_2\int_\Omega(1+\Phi(u_n))^{r-(p-1)(1-\theta)}f\,dx \\ \le \|f\|_{L^m(\Omega)}\left(\int_\Omega(1+\Phi(u_n))^{[r-(p-1)(1-\theta)]m'}\,dx\right)^{\frac{1}{m'}} \quad (5.9)$$

Owing to (5.7), (5.8) and (5.9), inequality (5.6) becomes

$$\left(\int_\Omega[(1+\Phi(u_n))^{\frac{r}{p}}-1]^{p^*}dx\right)^{\frac{p}{p^*}} \\ \le C+C\left(\int_\Omega(1+\Phi(u_n))^{[r-(p-1)(1-\theta)]m'}\,dx\right)^{\frac{1}{m'}} \quad (5.10)$$

Now it follows from $m<\frac{N}{p}$ that $m'>\frac{p^*}{p}$. This fact allows us to choose r such that

$$r=\frac{(p-1)(1-\theta)m'}{m'-\frac{p^*}{p}}\,, \quad (5.11)$$

so that

$$[r-(p-1)(1-\theta)]m'=\frac{rp^*}{p}=\frac{Nm(p-1)(1-\theta)}{N-pm}\,.$$

Therefore, we infer from (5.10) that there exists a constant $C>0$ satisfying

$$\int_\Omega(1+\Phi(u_n))^{\frac{Nm(p-1)(1-\theta)}{N-pm}}dx\le C \quad (5.12)$$

and, going back to (5.5), that

$$\int_\Omega(1+\Phi(u_n))^{r-p}|\nabla\Phi(u_n)|^p dx\le C \quad (5.13)$$

for all $n\in\mathbb{N}$.

To go from these estimates on $\{\Phi(u_n)\}_n$ to estimates on $\{u_n\}_n$, we first apply (5.1) to get

$$M_1^{\frac{1}{p-1}}\le\frac{\Phi'(s)}{(1+\Phi(s))^\theta} \quad (5.14)$$

and so

$$M_1^{\frac{1}{p-1}}s\le\frac{1}{1-\theta}(1+\Phi(s))^{1-\theta}$$

holds for all $s\in\mathbb{R}$. On the one hand, this last inequality, jointly with (5.12), gives an estimate of $\{u_n\}_n$ in $L^{\frac{Nm(p-1)}{N-pm}}(\Omega)$, so that (5.2) is proven. On the other, (5.14) and (5.13) imply

$$M_1^{\frac{p}{p-1}}\int_\Omega (1+\Phi(u_n))^{r-p(1-\theta)}|\nabla u_n|^p dx \le \int_\Omega (1+\Phi(u_n))^{r-p}\Phi'(u_n)^p|\nabla u_n|^p dx$$
$$= \int_\Omega (1+\Phi(s))^{r-p}|\nabla \Phi(u_n)|^p dx \le C\,.$$

Since $r-p(1-\theta)=\dfrac{p^*-m'}{m'-\frac{p^*}{p}}(1-\theta)\ge 0$, the estimate (5.3) follows.

It remains to check (5.4). Having already determined r by (5.11), we now choose $\Psi(s)=(1+\Phi(G_k(s))^{r-p}\Phi(G_k(s))$ in Lemma 2.1 (2), with $k\ge 1$, getting

$$\int_\Omega e^{H(u_n)}e^{\frac{H(G_k(u_n))}{p-1}}(1+\Phi(G_k(u_n)))^{r-p}|\nabla G_k(u_n)|^p dx$$
$$\le \int_\Omega e^{H(u_n)}(1+\Phi(G_k(u_n)))^{r-p}\Phi(G_k(u_n))g(x,u_n)\,dx$$
$$+\int_\Omega e^{H(u_n)}(1+\Phi(G_k(u_n))^{r-p}\Phi(G_k(u_n))f_n\,dx$$
$$\le \int_{\{u_n>k\}} e^{H(u_n)}(1+\Phi(u_n))^{r-p+1}g(x,u_n)\,dx$$
$$+\int_{\{u_n>k\}} e^{H(u_n)}(1+\Phi(u_n))^{r-p+1}f\,dx\,.$$

Arguing as above, we deduce that

$$\int_\Omega |\nabla G_k(u_n)|^p dx \le C\int_{\{u_n>k\}}(k^{q-1}+f)e^{H(u_n)}(1+\Phi(u_n))^{r-p+1}\,dx$$
$$\le C\|(1+f)\chi_{\{u_n>k\}}\|_{L^m(\Omega)}\left(\int_\Omega (1+\Phi(u_n))^{[r-(p-1)(1-\theta)]m'}\,dx\right)^{\frac{1}{m'}}$$
$$\le C\|(1+f)\chi_{\{u_n>k\}}\|_{L^m(\Omega)}\,.$$

due to $k^{q-1}\le 1$. Since the right hand side tends to 0 as $k\to\infty$, condition (5.4) follows. ■

Remark 5.1. We point out that we have obtained (5.12), an estimate on $\{\Phi(u_n)\}_n$, which leads, thanks to (5.1), to an estimate on $\{e^{H(u_n)}\}_n$, namely:

$$\int_\Omega \left(e^{H(u_n)}\right)^{\frac{Nm(1-\theta)}{\theta(N-pm)}} dx \le C\,. \tag{5.15}$$

5.2. **Existence: proof of Theorem 5.1.** By Lemma 5.1 there exists a nonnegative function $u\in W_0^{1,p}(\Omega)\cap L^{\frac{Nm(p-1)}{N-pm}}(\Omega)$ such that, up to subsequences, conditions (3.14), (3.15) and (3.16) hold true. Actually, and owing to (5.2), we have that the strong convergence in (3.15) holds for every $1\le r<\frac{Nm(p-1)}{N-pm}$. Moreover, the pointwise convergence (3.16), (5.15) and

$$m'<\frac{Nm(1-\theta)}{\theta(N-pm)}$$

imply that

$$e^{H(u_n)}\to e^{H(u)}\quad \text{in } L^{m'+\epsilon}(\Omega)\,. \tag{5.16}$$

for some $\epsilon > 0$ small enough.

As in the previous case the proof of Theorem 5.1 needs

(1) the strong convergence of ∇u_n to ∇u in $L^p(\Omega)^N$
(2) the strong convergence of $b(x, u_n, \nabla u_n)$ to $b(x, u, \nabla u)$ in $L^1(\Omega)$

As far as the strong convergence of ∇u_n concerns, the proof proceeds exactly as in Step 1 of the proof of Lemma 4.2, bearing in mind that (5.4) holds.

In an analogous way the proof of the strong convergence of $b(x, u_n, \nabla u_n)$ to $b(x, u, \nabla u)$ proceeds as in Step 3 of the proof of Lemma 4.2. Just replace, on the right hand side of (4.20), the fact that $\{e^{H(u_n)}\}_n$ is bounded in any $L^r(\Omega)$, $r < \infty$ with the fact that $\{e^{H(u_n)}\}_n$ is bounded in $L^{m'+\epsilon}(\Omega)$. Then it is enough to perform the following inequalities over the set $\{u_n > k\}$:

$$e^{H(u_n)} H(u_n) \le \frac{2m'}{\epsilon} e^{(1+(\epsilon/2m'))H(u_n)} \le \frac{2m'}{\epsilon} e^{(1+\epsilon/m')H(u_n)} \frac{1}{e^{\epsilon H(k)/m'}}$$

wherewith

$$e^{m' H(u_n)} H(u_n)^{m'} \chi_{\{u_n > k\}} \le C e^{(m'+\epsilon)H(u_n)} \frac{1}{e^{\epsilon H(k)}} \chi_{\{u_n > k\}} \, .$$

Finally the conclusion of the proof of Theorem 5.1 follows the argument given in Subsection 3.3. ■

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