ELLIPTIC EQUATIONS HAVING A SINGULAR QUADRATIC GRADIENT TERM AND A CHANGING SIGN DATUM

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Abstract. In this paper we study a singular elliptic problem whose model is

$$\begin{cases} -\Delta u = \frac{|\nabla u|^2}{|u|^\theta} + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,; \end{cases}$$

where $\theta \in (0,1)$ and $f \in L^m(\Omega)$, with $m \geq \frac{N}{2}$. We do not assume any sign condition on the lower order term, nor assume the datum f has a constant sign.

We carefully define the meaning of solution to this problem giving sense to the gradient term where $u = 0$, and prove the existence of such a solution. We also discuss related questions as the existence of solutions when the datum f is less regular or the boundedness of the solutions when the datum $f \in L^m(\Omega)$ with $m > \frac{N}{2}$.

1. Introduction

The systematic study of second order equations having a gradient term with natural growth was initiated by Boccardo, Murat and Puel in the 80's of last century (see [11], [12] and [13]). This gradient term also depends on the solution, for instance it can be written as $g(u)|\nabla u|^2$, but always in a continuous way. Recently existence of solutions of problems whose model is

$$\begin{cases} -\Delta u = \frac{|\nabla u|^2}{|u|^\theta} + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,; \end{cases} \tag{1.1}$$

where $\theta > 0$ and Ω is a bounded open set in $\mathbb{R}^N$, has attracted the attention of several authors (see for example [2], [4], [5], [6], [7], [8], [16], [3], [1]; other related problems are studied in [10] and [17].) The problem presents a lower order term which is singular in the u-variable and has a natural (quadratic) growth in the ∇u-variable. The interest in studying this kind of problems relies, first of all, on the fact that the equation looks like a simplified version of the formal Euler's equation for a functional of the type

$$I[u] = \int_\Omega |u|^{\alpha-1} u |\nabla u|^2 - \int_\Omega fu$$

with $\alpha \in (0,1)$.

Another motivation occurs by considering equations of the type

$$u_t - \Delta(|u|^{m-1}u) = |\nabla u|^2 + f\,,$$

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with $m>1$, which represents a model of gas flow in porous media. If we consider steady states solutions and we perform a change of unknown $|u|^{m-1}u=v$, we get an equation with singular behaviour in v, with quadratic growth in the ∇v-variable.

The papers we quoted before deal with different situations depending on the exponent θ of the singularity, on the sign and size of the lower order term. Existence and nonexistence of solutions in $H_0^1(\Omega)$ or $H_{loc}^1(\Omega)$, depending on the regularity of the datum $f(x)$ (which can induce bounded or unbounded solutions) and other related questions are considered. Anyway, all the previous known results are strictly confined to the case of nonnegative data $f(x)$, since they are mainly based on the strong maximum principle. In other words, the sign of the datum guarantees that the possible solutions do not cross the singularity; this is due to the fact that $u\equiv 0$ is, in a certain sense, a subsolution to the problem.

Dealing with data that do not have constant sign adds then some new extra difficulties to the study of this kind of equations. First of all, since the method of sub/supersolutions does not apply in this case, we need both to obtain new a priori estimates and to perform a deeper analysis near the singularity $u=0$ to study the singular quotient $\frac{|\nabla u|^2}{|u|^\theta}$.

Moreover, a very basic remark on the meaning of the solution is in order. Referring again to the model problem (1.1) and to the case $f\geq 0$, we observe that the definition of solution is completely clear if $u>0$ in Ω. In our situation, where f can change its sign, the solution u can vanishes inside Ω. This fact is not only a possibility, it really occurs as shown in Proposition 4.2 below. If we look for $H_0^1(\Omega)$–solutions, an indeterminate quotient appears since, by Stampacchia's theorem, $|\nabla u|=0$ on the set $\{u=0\}$. Therefore, we have to carefully define the meaning of solution and it is done in Definition 2.1 and Lemma 2.2 below. There, we introduce a suitable notion of solution that ensures us that $u\in H_0^1(\Omega)$ and $\frac{|\nabla u|^2}{|u|^\theta}\in L^1(\Omega)$.

In the present paper, we present a complete account on the existence of finite energy solutions for problems modelled by (1.1) with general, possibly changing-sign, data $f\in L^m(\Omega)$, $m\geq\frac{N}{2}$ and $\theta\in(0,1)$. We will obtain a priori estimates by means of a generalized Cole–Hopf change of unknown. Recall that, if a lower order term appears in the form $g(u)|\nabla u|^2$, test functions involving terms like $\exp(\gamma(u))$, where $\gamma(s)$ is a primitive function of $g(s)$, are often used in order to get a priori estimates (see [14], [19] and [18]). Observe that, if $\theta\in(0,1)$, then the function $g(s)=\frac{1}{|s|^\theta}$ is an L^1–function near the singularity $s=0$ so that $\exp(\gamma(s))$ is well–defined. Obviously, this fact does not occur if $\theta\geq 1$.

Nevertheless, we point out that our restriction on θ is not technical: indeed, even if $\theta=1$ and $f\geq 0$, solutions do not belong, in general, to $H_0^1(\Omega)$ anymore, nor the gradient term to $L^1(\Omega)$, as shown in [3]. In other words, if the singularity is too strong (e.g. $\frac{1}{|s|^\theta}$, with $\theta\geq 1$), then there is no room for a solution of finite energy to satisfy the boundary condition and the solution must loose its regularity.

The paper is organized as follows. Section 2 is devoted to the hypotheses and the statements of the results. Section 3 deals with the proof of the main theorem. Section 4 contains further results on boundedness of solutions in the case $f\in L^m(\Omega)$, $m>\frac{N}{2}$ and on stability with respect to the lower order term; it also provides examples and possible extensions.

2. Hypotheses and statements of results

Let us state our main assumptions. Let Ω be an open bounded set in $\mathbb{R}^N (N \geq 3)$. We will deal with the following problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = b(x,u,\nabla u) + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{2.2}$$

The function

$$a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

satisfies the Carathéodory conditions (i.e. $a(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$ and $a(\cdot,s,\xi)$ is measurable for any $s \in \mathbb{R}, \xi \in \mathbb{R}^N$) and there exist some constants $\alpha > 0$ and $\nu > 0$ such that

$$a(x,s,\xi)\cdot\xi \geq \alpha|\xi|^2, \tag{2.3}$$

$$|a(x,s,\xi)| \leq \nu|\xi|, \tag{2.4}$$

$$(a(x,s,\xi) - a(x,s,\eta))\cdot(\xi-\eta) > 0\,; \tag{2.5}$$

for all $\xi,\eta \in \mathbb{R}^N$, with $\xi \neq \eta$, for all $s \in \mathbb{R}$ and for almost all $x \in \Omega$.

The function

$$b(x,s,\xi) : \Omega \times \mathbb{R}\backslash\{0\} \times \mathbb{R}^N \to \mathbb{R}$$

also satisfies the Carathéodory conditions and there exists a nonnegative continuous function $g : \mathbb{R}\backslash\{0\} \to [0,+\infty)$ such that

$$|b(x,s,\xi)| \leq g(s)|\xi|^2; \tag{2.6}$$

for all $\xi \in \mathbb{R}^N$, for all $s \in \mathbb{R}\backslash\{0\}$ and for almost all $x \in \Omega$. Moreover,

$$\lim_{|s|\to\infty} g(s) = 0 \tag{2.7}$$

and there exist constants $\Lambda, s_0 > 0$ and $\theta \in (0,1)$ such that $g(s) = \frac{\Lambda}{|s|^\theta}$ for all $|s| \leq s_0$.

Remark 2.1. We explicitly observe that, without loss of generality, we can choose g to be nonincreasing in $[0,+\infty[$ and to be nondecreasing in $]-\infty,0]$. Indeed, changing the value of s_0 if necessary, it is not difficult to define a continuous $\overline{g} : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ satisfying the same hypotheses of g and moreover

- $\overline{g}(s) \geq g(s)$ for all $s \in \mathbb{R}$.
- $\overline{g}$ is nonincreasing in $[0,+\infty[$ and nondecreasing in $]-\infty,0]$.

As far as the datum f is concerned, it satisfies

$$f(x) \in L^m(\Omega), \quad m \geq \frac{N}{2}, \tag{2.8}$$

while no sign condition is assumed (cfr. with [4], [8], [16] and references therein).

Let us point out that, under the general assumption (2.7), the summability requested to f is optimal as showed in [18]. In Section 4 we will show how this assumption can be relaxed depending on the behaviour of the lower order term.

We remark that, as we look for solutions $u \in H_0^1(\Omega)$, the equation in (2.2) involves an indeterminate quotient on $\{u=0\}$, since $|b(x,u,\nabla u)|\chi_{\{|u|\leq s_0\}} \leq \dfrac{\Lambda}{|u|^\theta}|\nabla u|^2$ and $|\nabla u| = 0$ on the set $\{u=0\}$, by Stampacchia's Theorem. To clarify this situation, we define

Definition 2.2. *If u and $|u|^{1-\frac{\theta}{2}}$ belong to $H_0^1(\Omega)$, we define*

$$\frac{|\nabla u|^2}{|u|^\theta} = \frac{4}{(2-\theta)^2}|\nabla\big(|u|^{1-\frac{\theta}{2}}\big)|^2 .$$

Observe that, by definition, $\dfrac{|\nabla u|^2}{|u|^\theta}$ always belongs to $L^1(\Omega)$. Moreover, as a consequence of Stampacchia's Theorem, we obtain

$$\frac{|\nabla u|^2}{|u|^\theta} = 0 \quad \text{a.e. in } \{u=0\} .$$

As a consequence of (2.6), we may extend $b(x,s,\xi)$ to $s=0$ (only when $s=u$ and $\xi = \nabla u$) and define

$$b(x,u,\nabla u) = 0 \quad \text{a.e. in } \{u=0\} . \tag{2.9}$$

Hence, $b(x,u,\nabla u) \in L^1(\Omega)$.

Remark 2.3. We would like to explicitly stress that solutions satisfying $|\{u = 0\}| > 0$ can actually occur. For instance consider the function defined in $B_2(0)$, the ball of radius 2 of $\mathbb{R}^N$, by

$$w(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} , & \text{if } |x| \le 1 ; \\ 0 , & \text{if } 1 < |x| \le 2 . \end{cases}$$

An easy computation (using that $\theta < 1$) shows that there exists $\overline{f} \in C^\infty(\overline{B}_2(0))$ such that w solves

$$\begin{cases} -\Delta w = \frac{|\nabla w|^2}{|w|^\theta} + \overline{f} , & \text{in } B_2(0) ; \\ w = 0 , & \text{on } \partial B_2(0) . \end{cases}$$

Definition 2.4. *A weak solution to problem (2.2) is a function $u \in H_0^1(\Omega)$ satisfying $|u|^{1-\frac{\theta}{2}} \in H_0^1(\Omega)$ (so that $b(x,u,\nabla u) \in L^1(\Omega)$) and*

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla v = \int_\Omega b(x,u,\nabla u)v + \int_\Omega fv,$$

for any $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

In order to check that a function $u \in H_0^1(\Omega)$ is actually solution to problem (2.2), we will have to see $|u|^{1-\frac{\theta}{2}} \in H_0^1(\Omega)$. To this aim the following simple claim will be applied. Although its proof is similar to that of Lemma 2.1 in [17], we sketch it for the sake of completeness. Here and below we will use the following auxiliary functions: for any $s \in \mathbb{R}$ we consider the standard truncation function defined by $T_k(s) = \max(-k, \min(s,k))$, while we denote $G_k(s) = s - T_k(s)$.

Lemma 2.5. *Let $u \in H_0^1(\Omega)$. If $g(u)|\nabla u|^2$ is integrable on $\{u \neq 0\}$, then*

$$|u|^{1-\frac{\theta}{2}} \in H_0^1(\Omega) .$$

Moreover, $b(x,u,\nabla u)$ is integrable on Ω, and

$$\int_\Omega b(x,u,\nabla u) = \int_{\{u\neq 0\}} b(x,u,\nabla u) .$$

Proof. Observe that
$$\int_{\{0<|u|\le s_0\}} \frac{|\nabla u|^2}{(\frac{1}{n}+|u|)^\theta} \le \frac{1}{\Lambda}\int_{\{0<|u|\le s_0\}} g(u)|\nabla u|^2 \le C\,,$$
for all $n\in\mathbb{N}$. In other words,
$$\int_\Omega \Big|\nabla\Big(\Big(\frac{1}{n}+T_{s_0}(|u|)\Big)^{1-\frac{\theta}{2}} - \Big(\frac{1}{n}\Big)^{1-\frac{\theta}{2}}\Big)\Big|^2 \le C\,,$$
for all $n\in\mathbb{N}$. Hence, $\Big(\frac{1}{n}+T_{s_0}(|u|)\Big)^{1-\frac{\theta}{2}} - \Big(\frac{1}{n}\Big)^{1-\frac{\theta}{2}}$ is bounded in $H_0^1(\Omega)$ and, up to subsequences, there exists $v\in H_0^1(\Omega)$ such that
$$\Big(\frac{1}{n}+T_{s_0}(|u|)\Big)^{1-\frac{\theta}{2}} - \Big(\frac{1}{n}\Big)^{1-\frac{\theta}{2}} \rightharpoonup v$$
weakly in $H_0^1(\Omega)$. Obviously, passing to a subsequence if necessary, we get
$$v(x) = \lim_{n\to\infty}\Big(\frac{1}{n}+T_{s_0}(|u(x)|)\Big)^{1-\frac{\theta}{2}} - \Big(\frac{1}{n}\Big)^{1-\frac{\theta}{2}} = \big(T_{s_0}(|u(x)|)\big)^{1-\frac{\theta}{2}} \quad \text{a.e. in } \Omega\,,$$
so that $\big(T_{s_0}(|u|)\big)^{1-\frac{\theta}{2}}\in H_0^1(\Omega)$. In particular, since $0<\theta<1$, using the chain rule for Sobolev spaces we have
$$\nabla T_{s_0}(|u|) = \nabla\Big(\big(T_{s_0}(|u|)\big)^{1-\frac{\theta}{2}}\Big)^{\frac{2}{2-\theta}} = \frac{2}{2-\theta}\big(T_{s_0}(|u|)\big)^{\frac{\theta}{2}}\nabla\big(T_{s_0}(|u|)\big)^{1-\frac{\theta}{2}},$$
a.e. on Ω.
Note that, on account of Stampacchia's theorem, the two functions which appear at the right-hand side and at the left-hand side are both a.e. zero on the set $\{u=0\}$ and so we get
$$\nabla\big(T_{s_0}(|u|)\big)^{1-\frac{\theta}{2}} = \frac{2-\theta}{2}\frac{\nabla T_{s_0}(|u|)}{\big(T_{s_0}(|u|)\big)^{\frac{\theta}{2}}} \quad \text{a.e. on} \quad \Omega.$$

Therefore, denoting $k = s_0^{1-\frac{\theta}{2}}$, we have just seen that $T_k\big(|u|^{1-\frac{\theta}{2}}\big)\in H_0^1(\Omega)$. Moreover,
$$\nabla T_k\big(|u|^{1-\frac{\theta}{2}}\big) = \Big(1-\frac{\theta}{2}\Big)\frac{\nabla|u|}{|u|^{\frac{\theta}{2}}}\chi_{\{|u|<s_0\}}, \quad \text{a.e. on } \Omega.$$

Since $G_k\big(|s|^{1-\frac{\theta}{2}}\big)$ defines a Lipschitz continuous function, it follows from $u\in H_0^1(\Omega)$ that $G_k\big(|u|^{1-\frac{\theta}{2}}\big)\in H_0^1(\Omega)$ and
$$\nabla G_k\big(|u|^{1-\frac{\theta}{2}}\big) = \Big(1-\frac{\theta}{2}\Big)\frac{\nabla|u|}{|u|^{\frac{\theta}{2}}}\chi_{\{|u|\ge s_0\}}, \quad \text{a.e. on } \Omega.$$
Therefore, $|u|^{1-\frac{\theta}{2}} = T_k\big(|u|^{1-\frac{\theta}{2}}\big) + G_k\big(|u|^{1-\frac{\theta}{2}}\big)\in H_0^1(\Omega)$ and
$$\big|\nabla|u|^{1-\frac{\theta}{2}}\big|^2 = \Big(1-\frac{\theta}{2}\Big)^2\frac{|\nabla u|^2}{|u|^\theta}\,.$$
Hence, the term $\dfrac{|\nabla u|^2}{|u|^\theta}$ is well–defined and belongs to $L^1(\Omega)$. As a consequence, $g(u)|\nabla u|^2$ is well–defined in $\{u=0\}$, where $g(u)|\nabla u|^2=0$ a.e., and $g(u)|\nabla u|^2\in L^1(\Omega)$. Thus, by assumption (2.6), recalling (2.9), we deduce the second assertion of our Lemma. □

Our main result is the following

Theorem 2.1. *There exist a weak solution $u \in H_0^1(\Omega)$ to problem (2.2)*

3. Proof of Theorem 2.1

3.1. **Approximating Problems.** We shall take approximating problems without singularities. To this end, we will consider truncating continuous functions b_n of b. Since b is not assumed to be an even function with respect to s, our truncation will not be standard. So, for any $n \in \mathbb{N}$, we define the following bounded sequence of functions

$$b_n(x,s,\xi) := \begin{cases} \frac{1+t}{2} b(x, \frac{1}{n}, \xi) + \frac{1-t}{2} b(x, \frac{-1}{n}, \xi)\,, & \text{if } s = \frac{t}{n}\,,\ |t| \le 1\,; \\ b(x,s,\xi)\,, & \text{if } |s| > \frac{1}{n}\,; \end{cases} \tag{3.10}$$

for any $\xi \in \mathbb{R}^N$, and a.e. $x \in \mathbb{R}^N$. Moreover, let $f_n := T_n(f)$ and consider

$$\begin{cases} -\operatorname{div}(a(x,u_n,\nabla u_n)) = b_n(x,u_n,\nabla u_n) + f_n(x)\,, & \text{in } \Omega\,; \\ u_n = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.11}$$

A bounded weak solution to problem (3.11) does exist as proved in [18]. That is there exists $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v = \int_\Omega b_n(x,u_n,\nabla u_n)v + \int_\Omega f_n v, \tag{3.12}$$

for any $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

For $n \ge 1/s_0$, we define the following auxiliary functions:

$$g_n(s) := \begin{cases} \Lambda n^\theta\,, & \text{if } |s| \le \frac{1}{n} \\ g(s)\,, & \text{otherwise,} \end{cases} \tag{3.13}$$

$$\gamma_n(s) = \frac{1}{\alpha}\int_0^s g_n(\sigma)\,d\sigma, \quad \text{and} \quad \Psi_n(s) = \int_0^s e^{|\gamma_n(\sigma)|} d\sigma. \tag{3.14}$$

Observe that $\gamma_n(s)$ is Lipschitz continuous, while $\Psi_n(s)$ is locally Lipschitz continuous and it satisfies

$$|\Psi_n(s)| \ge |s|\,, \quad \text{for all } s \in \mathbb{R}\,. \tag{3.15}$$

Moreover, thanks to (2.6),

$$|b_n(x,s,\xi)| \le g_n(s)|\xi|^2. \tag{3.16}$$

Of course, there is some connection among all these functions, which we want to highlight. Let

$$\gamma(s) = \frac{1}{\alpha}\int_0^s g(\sigma)\,d\sigma, \quad \text{and} \quad \Psi(s) = \int_0^s e^{|\gamma(\sigma)|} d\sigma\,,$$

so that it also holds

$$|\Psi(s)| \ge |s|\,, \quad \text{for all } s \in \mathbb{R}\,. \tag{3.17}$$

Observe that

$$0 \le \gamma_n(s)\operatorname{sign}(s) \le \gamma(s)\operatorname{sign}(s)\,, \quad \text{for all } s \in \mathbb{R} \text{ and all } n \in \mathbb{N}\,, \tag{3.18}$$

and that we also have

$$0 \le |\gamma(s)| - |\gamma_n(s)| \le \frac{\Lambda}{\alpha n^{1-\theta}} \frac{\theta}{1-\theta}\,,$$

for all $s \in \mathbb{R}$. It follows from $\lim_{n\to\infty} \frac{\Lambda}{\alpha n^{1-\theta}} \frac{\theta}{1-\theta} = 0$ that

$$\begin{aligned} &e^{|\gamma_n(s)|} \le e^{|\gamma(s)|} \le C_n e^{|\gamma_n(s)|}\,, \quad \forall\, s \in \mathbb{R}\,, \\ &|\Psi_n(s)| \le |\Psi(s)| \le C_n |\Psi_n(s)|\,, \quad \forall\, s \in \mathbb{R}\,, \end{aligned} \tag{3.19}$$

where C_n satisfies $\lim_{n\to\infty} C_n = 1$. On the other hand, since g vanishes at infinity, by L'Hôpital's rule we have

$$\lim_{|s|\to\infty} \frac{e^{|\gamma(s)|}}{|\Psi(s)|} = 0\,, \tag{3.20}$$

so that for any $\varepsilon > 0$ there exists a constant C such that

$$e^{|\gamma(s)|} \le \varepsilon |\Psi(s)| + C\,, \quad \forall\, s \in \mathbb{R}\,.$$

Thanks to (3.19), we deduce that, given $\varepsilon > 0$, there is an only constant C satisfying

$$e^{|\gamma_n(s)|} \le \varepsilon |\Psi_n(s)| + C\,, \quad \forall\, n \in \mathbb{N}, s \in \mathbb{R}\,; \tag{3.21}$$

we will use this kind of bound in what follows.

Let us specify some useful notation we will use from now on. If not differently stated, the symbol C will indicate a positive constant, only dependent on the data, whose value may change line by line. Moreover, the symbol $\omega(\varepsilon)$, $\omega(n)$ will denote any quantity that vanishes as the argument goes to its natural limit (that is $\varepsilon \to 0$, $n \to \infty$).

3.2. **Estimate on both $\Psi_n(u_n)$ and u_n in $H_0^1(\Omega)$.** We take $e^{|\gamma_n(u_n)|}\Psi_n(u_n)$ as test in (3.12) to obtain, using (3.16) and the fact that both $\gamma_n(u_n)$ and $\Psi_n(u_n)$ have the same sign as u_n

$$\begin{aligned} &\int_\Omega g_n(u_n) e^{|\gamma_n(u_n)|} |\Psi_n(u_n)| |\nabla u_n|^2 + \alpha \int_\Omega e^{2|\gamma_n(u_n)|} |\nabla u_n|^2 \\ &\le \int_\Omega g_n(u_n) e^{|\gamma_n(u_n)|} |\Psi_n(u_n)| |\nabla u_n|^2 + \int_\Omega |f| e^{|\gamma_n(u_n)|} |\Psi_n(u_n)|\,, \end{aligned}$$

that is,

$$\alpha \int_\Omega |\nabla \Psi_n(u_n)|^2 \le \int_\Omega |f| e^{|\gamma_n(u_n)|} |\Psi_n(u_n)|\,. \tag{3.22}$$

Using first (3.21) and then Young's inequality, we get

$$\begin{aligned} \int_\Omega |f| e^{|\gamma_n(u_n)|} |\Psi_n(u_n)| &\le \varepsilon \int_\Omega |f| |\Psi_n(u_n)|^2 + C \int_\Omega |f| |\Psi_n(u_n)| \\ &\le 2\varepsilon \int_\Omega |f| |\Psi_n(u_n)|^2 + C \int_\Omega |f|\,. \end{aligned} \tag{3.23}$$

Now, by Hölder's inequality, the summability of f and Sobolev's inequality, we obtain (choosing a suitable ε)

$$2\varepsilon \int_\Omega |f| |\Psi_n(u_n)|^2 \le 2\varepsilon \|f\|_{N/2} \Big(\int_\Omega |\Psi_n(u_n)|^{2N(N-2)} \Big)^{(N-2)/N} \le \frac{\alpha}{2} \int_\Omega |\nabla \Psi_n(u_n)|^2\,,$$

which, by (3.23), implies

$$\int_\Omega |f| e^{|\gamma_n(u_n)|} |\Psi_n(u_n)| \le \frac{\alpha}{2} \int_\Omega |\nabla \Psi_n(u_n)|^2 + C \int_\Omega |f| .$$

Going back to (3.22) we deduce

$$\int_\Omega |\nabla \Psi_n(u_n)|^2 \le C , \quad \text{for all } n \in \mathbb{N} , \tag{3.24}$$

and

$$\int_\Omega |f| e^{|\gamma_n(u_n)|} |\Psi_n(u_n)| \le C , \quad \text{for all } n \in \mathbb{N} . \tag{3.25}$$

Moreover, Young's inequality implies

$$\int_\Omega |f| e^{|\gamma_n(u_n)|} \le \frac{1}{2} \int_\Omega |f| e^{2|\gamma_n(u_n)|} + \frac{1}{2} \int_\Omega |f| ,$$

which, due to (3.21), becomes

$$\int_\Omega |f| e^{|\gamma_n(u_n)|} \le C , \quad \text{for all } n \in \mathbb{N} . \tag{3.26}$$

On the other hand, notice that, by (3.21) again,

$$\text{the sequence } e^{|\gamma_n(u_n)|} \text{ is bounded in } L^{\frac{2N}{N-2}}(\Omega) . \tag{3.27}$$

Thus, since

$$|\nabla u_n|^2 \le e^{2|\gamma_n(u_n)|} |\nabla u_n|^2 = |\nabla \Psi_n(u_n)|^2,$$

we also have

$$\|u_n\|_{H_0^1(\Omega)} \le C , \quad \text{for all } n \in \mathbb{N} .$$

Therefore, up to subsequences, there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $u_n \to u$ strongly in $L^2(\Omega)$ and a.e. on Ω.

3.3. **Estimate of $b_n(x, u_n, \nabla u_n)$ in $L^1(\Omega)$.** Here we want to prove an L^1–bound for the lower order term $b_n(x, u_n, \nabla u_n)$. We take $(e^{|\gamma_n(u_n)|} - 1)\text{sign}\,(u_n)$ as test function in (3.12) and we use (2.3) to get

$$\begin{aligned} &\int_\Omega g_n(u_n) e^{|\gamma_n(u_n)|} |\nabla u_n|^2 \\ &\le \int_\Omega |b_n(x, u_n, \nabla u_n)| (e^{|\gamma_n(u_n)|} - 1) + \int_\Omega |f| (e^{|\gamma_n(u_n)|} - 1) \\ &\le \int_\Omega g_n(u_n) |\nabla u_n|^2 (e^{|\gamma_n(u_n)|} - 1) + \int_\Omega |f| (e^{|\gamma_n(u_n)|} - 1) , \end{aligned}$$

that implies, using (3.16) and (3.26),

$$\int_\Omega |b_n(x, u_n, \nabla u_n)| \le \int_\Omega g_n(u_n) |\nabla u_n|^2 \le \int_\Omega |f| e^{|\gamma_n(u_n)|} \le C . \tag{3.28}$$

3.4. **Near the singularity.** Here we want to prove that, for any $\varepsilon > 0$

$$\lim_{\varepsilon\to 0}\sup_n \int_{\{|u_n|\le\varepsilon\}} |b_n(x,u_n,\nabla u_n)| = 0\,. \tag{3.29}$$

To this end, consider the function

$$v = \begin{cases} \big(e^{|\gamma_n(u_n)|}-1\big)\operatorname{sign} u_n\,, & \text{if} \quad |u_n|\le\varepsilon\,;\\ e^{|\gamma_n(\varepsilon)|}-1\,, & \text{if} \quad u_n>\varepsilon\,;\\ 1-e^{|\gamma_n(-\varepsilon)|}\,, & \text{if} \quad u_n<-\varepsilon\,; \end{cases}$$

and observe that, by (3.19),

$$|v| \le \max\{e^{|\gamma_n(\varepsilon)|}-1, e^{|\gamma_n(-\varepsilon)|}-1\} \le \max\{e^{|\gamma(\varepsilon)|}-1, e^{|\gamma(-\varepsilon)|}-1\}\,,$$

that is $|v|\le\omega(\varepsilon)$ uniformly in n. Choosing v as test function in (3.12) and applying (2.3), we obtain

$$\begin{aligned}\int_{\{|u_n|\le\varepsilon\}} & e^{|\gamma_n(u_n)|} g_n(u_n)|\nabla u_n|^2\\ &\le \frac{1}{\alpha}\int_{\{|u_n|\le\varepsilon\}} g_n(u_n)e^{|\gamma_n(u_n)|}a(x,u_n,\nabla u_n)\cdot\nabla u_n\\ &\le \int_\Omega |b_n(x,u_n,\nabla u_n)|\,|v| + \int_\Omega |f|\,|v|\,.\end{aligned}$$

Hence, by (3.16),

$$\begin{aligned}\int_{\{|u_n|\le\varepsilon\}} |b_n(x,u_n,\nabla u_n)| &\le \int_{\{|u_n|\le\varepsilon\}} e^{|\gamma_n(u_n)|}g_n(u_n)|\nabla u_n|^2\\ &\le \omega(\varepsilon)\Big[\int_\Omega |b_n(x,u_n,\nabla u_n)| + \int_\Omega |f|\Big]\,.\end{aligned} \tag{3.30}$$

Since the terms in brackets are uniformly bounded, by the previous step, it yields (3.29).

3.5. **Far from the singularity.** Here we want to prove

$$\lim_{k\to\infty}\sup_n \int_{\{|u_n|>k\}} |b_n(x,u_n,\nabla u_n)| = 0. \tag{3.31}$$

We consider

$$\big(e^{|\gamma_n(G_k(u_n))|}-1\big)\operatorname{sign} u_n$$

as test function in (3.12); applying (2.3) and (2.6) we obtain

$$\begin{aligned}\int_{\{|u_n|>k\}} & g_n(G_k(u_n))e^{|\gamma_n(G_k(u_n))|}|\nabla u_n|^2\\ &\le \int_{\{|u_n|>k\}} |b_n(x,u_n,\nabla u_n)|\big(e^{|\gamma_n(G_k(u_n))|}-1\big) + \int_{\{|u_n|>k\}} |f|\big(e^{|\gamma_n(G_k(u_n))|}-1\big)\\ &\le \int_{\{|u_n|>k\}} g_n(u_n)e^{|\gamma_n(G_k(u_n))|}|\nabla u_n|^2 - \int_{\{|u_n|>k\}} |b_n(x,u_n,\nabla u_n)|\\ &\qquad + \int_{\{|u_n|>k\}} |f|\big(e^{|\gamma_n(G_k(u_n))|}-1\big)\end{aligned}$$

As we said in Remark 2.1, we may assume that g is nondecreasing on $]-\infty,0]$ and nonincreasing on $[0,+\infty[$. It follows from the inequality $g(u_n)\le g(G_k(u_n))$ on $\{|u_n|>k\}$ that we may cancel two terms, and so

$$\int_{\{|u_n|>k\}}|b_n(x,u_n,\nabla u_n)|\le\int_{\{|u_n|>k\}}|f|\,(e^{|\gamma_n(G_k(u_n))|}-1)\,. \tag{3.32}$$

Having in mind (3.25), we set $C=\sup_n\int_\Omega|f|\,|\Psi_n(u_n)|e^{|\gamma_n(u_n)|}$. Then, due to (3.15),

$$\int_{\{|u_n|>k\}}|b_n(x,u_n,\nabla u_n)|\le\int_{\{|u_n|>k\}}|f|\,e^{|\gamma_n(u_n)|}$$
$$\le\frac{1}{\min\{|\Psi_n(k)|,|\Psi_n(-k)|\}}\int_{\{|u_n|>k\}}|f|\,|\Psi_n(u_n)|\,e^{|\gamma_n(u_n)|}\le\frac{C}{k}\,,$$

which gives (3.31).

3.6. **Strong convergence of truncations.** Here we want to prove that, for each $k>0$, $\nabla T_k(u_n)$ strongly converges to $\nabla T_k(u)$ in $L^p(\Omega;\mathbb{R}^N)$.

First we take $e^{\gamma_n(u_n)}(T_k(u_n)-T_k(u))^+$ as test function in (3.12), to get

$$\frac{1}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\;g_n(u_n)e^{\gamma_n(u_n)}(T_k(u_n)-T_k(u))^+$$
$$+\int_\Omega e^{\gamma_n(u_n)}a(x,u_n,\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))^+$$
$$\le\int_\Omega g_n(u_n)|\nabla u_n|^2e^{\gamma_n(u_n)}(T_k(u_n)-T_k(u))^++\int_\Omega|f|e^{\gamma_n(u_n)}(T_k(u_n)-T_k(u))^+\,,$$

that is, using (2.3) and simplifying,

$$\int_\Omega e^{\gamma_n(u_n)}a(x,u_n,\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))^+\le\int_\Omega|f|e^{\gamma_n(u_n)}(T_k(u_n)-T_k(u))^+. \tag{3.33}$$

The right hand side of the previous inequality goes to zero as n diverges since $f\in L^{\frac{N}{2}}(\Omega)$, the sequence $e^{\gamma_n(u_n)}$ is bounded in $L^{\frac{2N}{N-2}}(\Omega)$, by (3.27), and $T_k(u_n)$ converges to $T_k(u)$ strongly in $L^{\frac{2N}{N-2}}(\Omega)$, due to the pointwise convergence. So that we can write

$$\int_{\{|u_n|\le k\}}e^{\gamma_n(u_n)}a(x,u_n,\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))^+$$
$$\le\omega(n)+\int_{\{u_n>k\}}e^{\gamma_n(u_n)}a(x,u_n,\nabla u_n)\cdot\nabla T_k(u)\,.$$

Now observe that

$$|e^{\gamma_n(u_n)}a(x,u_n,\nabla u_n)|\le\nu e^{|\gamma_n(u_n)|}|\nabla u_n|=\nu|\nabla\Psi_n(u_n)|\,,$$

it implies, thanks to the estimate on $\Psi_n(u_n)$,

$$\int_{\{u_n>k\}}e^{\gamma_n(u_n)}a(x,u_n,\nabla u_n)\cdot\nabla T_k(u)=\omega(n).$$

Therefore, gathering together the previous estimates we have

$$\int_{\{|u_n|\le k\}} e^{\gamma_n(u_n)} a(x,u_n,\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))^+ \le \omega(n)\,. \tag{3.34}$$

On the other hand, since $|a(x,u_n,\nabla T_k(u))|\le \nu|\nabla T_k(u)|\in L^2(\Omega)$, the sequence $e^{\gamma_n(u_n)}\chi_{\{|u_n|\le k\}}$ is uniformly bounded in $L^\infty(\Omega)$ and $T_k(u_n)\rightharpoonup T_k(u)$ weakly in $H_0^1(\Omega)$, it follows that

$$\int_{\{|u_n|\le k\}} e^{\gamma_n(u_n)} a(x,u_n,\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))^+ = \omega(n)\,. \tag{3.35}$$

Now we can subtract (3.34) and (3.35) to obtain

$$\int_{\{|u_n|\le k\}} e^{\gamma_n(u_n)}(a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)))\cdot\nabla(T_k(u_n)-T_k(u))^+ \le \omega(n)\,.$$

Recall that, by (3.18), we have that $|\gamma_n(s)|\le \max\{\gamma(k),-\gamma(-k)\}$ for all $s\in[-k,k]$ and consequently $\inf\limits_{\{|s|\le k\}} e^{\gamma_n(s)}\ge \min\{e^{-\gamma(k)},e^{\gamma(-k)}\}>0$. Applying this fact and the monotonicity condition (2.5), we deduce

$$\int_{\{|u_n|\le k\}} (a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)))\cdot\nabla(T_k(u_n)-T_k(u))^+ \le \omega(n)\,.$$

Hence, we get

$$\begin{aligned}\int_\Omega &(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot\nabla(T_k(u_n)-T_k(u))^+\\ &\le \omega(n)+\int_{\{u_n>k\}} a(x,k,\nabla T_k(u))\cdot\nabla T_k(u)=\omega(n)\,,\end{aligned}$$

the last equality is due to Lebesgue's Theorem and the following inequalities

$$0\le \int_{\{u_n>k\}} a(x,k,\nabla T_k(u))\cdot\nabla T_k(u)\le \nu\int_{\{u_n>k\}}|\nabla T_k(u)|^2\,.$$

To the deal with the negative part, we may follow a similar argument, using now $-e^{-\gamma_n(u_n)}(T_k(u_n)-T_k(u))^-$ as test function in (3.12). Adding both, the positive and the negative part, we obtain that

$$\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot\nabla(T_k(u_n)-T_k(u)) \tag{3.36}$$

tends to 0 as n goes to ∞. A result by Browder (see [15] or [13]), implies that

$$\nabla T_k(u_n)\to\nabla T_k(u)\,,\qquad \text{strongly in } L^2(\Omega;\mathbb{R}^N)\,.$$

A diagonal argument now supplies us the pointwise convergence of the gradients

$$\nabla u_n(x)\to\nabla u(x)\,,\qquad \text{a.e. in } \Omega\,. \tag{3.37}$$

Three important consequences of this fact are

$$a(x,u_n(x),\nabla u_n(x))\to a(x,u(x),\nabla u(x))\,,\quad \text{a.e. in } \Omega\,, \tag{3.38}$$

$$b_n(x,u_n(x),\nabla u_n(x))\to b(x,u(x),\nabla u(x))\,,\quad \text{a.e. in } \{u\ne 0\}\,, \tag{3.39}$$

$$g_n(u_n(x))|\nabla u_n(x)|^2\to g(u(x))|\nabla u(x)|^2\,,\quad \text{a.e. in } \{u\ne 0\}\,. \tag{3.40}$$

It follows from this last convergence, (3.28) and Fatou's Lemma, that

$$g(u(x))|\nabla u(x)|^2\in L^1(\{u\ne 0\})\,,$$

from where, thanks to Lemma 2.5, we obtain $|u|^{1-\frac{\theta}{2}} \in H_0^1(\Omega)$, $b(x,u,\nabla u) \in L^1(\Omega)$ and

$$\int_{\{u\neq 0\}} b(x,u,\nabla u) = \int_\Omega b(x,u,\nabla u)\,. \tag{3.41}$$

3.7. **Equi–integrability of $b_n(x,u_n,\nabla u_n)$.** Consider a measurable set $E \subset \Omega$ and $\delta > 0$. Applying (3.29) and (3.31), given $\delta > 0$, we may find $\varepsilon, k > 0$ satisfying

$$\int_{\{|u_n|<\varepsilon\}} |b_n(x,u_n,\nabla u_n)| + \int_{\{|u_n|>k\}} |b_n(x,u_n,\nabla u_n)| \le \frac{\delta}{2}\,.$$

Thus, it yields

$$\int_E |b_n(x,u_n,\nabla u_n)| \le \frac{\delta}{2} + \int_{E\cap\{\varepsilon\le|u_n|\le k\}} |b_n(x,u_n,\nabla u_n)|$$
$$\le \frac{\delta}{2} + \int_{E\cap\{\varepsilon\le|u_n|\le k\}} g(u_n)|\nabla u_n|^2 \le \frac{\delta}{2} + \sup_{\varepsilon\le|s|\le k} g(s) \int_E |\nabla T_k(u_n)|^2\,,$$

and, when $|E|$ is small enough, the last term becomes less than $\frac{\delta}{2}$ since $\nabla T_k(u_n)$ converges strongly in $L^2(\Omega;\mathbb{R}^N)$. Therefore, the sequence $b_n(x,u_n,\nabla u_n)$ is equi–integrable.

3.8. **Passage to the limit.** In order to prove that u is a weak solution to (2.2), we fix $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and consider it as test function in (3.12). Then

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v = \int_\Omega b_n(x,u_n,\nabla u_n)v + \int_\Omega f_n v\,. \tag{3.42}$$

It is easy to pass to the limit in the last term, but two facts are needed to handle the other terms. On the one hand,

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u)$$

weakly in $L^2(\Omega;\mathbb{R}^N)$. This is due to our estimate of u_n in $H_0^1(\Omega)$, (2.4) and (3.38). So that we may pass to the limit in the second order term.

On the other hand, the previous step and (3.39) imply

$$b_n(x,u_n,\nabla u_n) \to b(x,u,\nabla u)\,, \quad \text{strongly in } L^1(\{u\neq 0\})\,.$$

This fact has as consequence that we may pass to the limit in the gradient term; indeed, given $\delta > 0$ and using (3.29), we may find $\varepsilon > 0$ satisfying

$$\|v\|_\infty \int_{\{|u_n|\le\varepsilon\}} |b_n(x,u_n,\nabla u_n)| < \delta/2\,,$$

for all $n \in \mathbb{N}$. Then, it follows from Fatou's Lemma that

$$\|v\|_\infty \int_{\{|u|\le\varepsilon\}\cap\{u\neq 0\}} |b(x,u,\nabla u)| < \delta/2\,.$$

Hence, applying the previous estimates,

$$\Big|\int_{\{|u_n|\le\varepsilon\}} b_n(x,u_n,\nabla u_n)v\Big| + \Big|\int_{\{|u|\le\varepsilon\}\cap\{u\neq 0\}} b(x,u,\nabla u)v\Big| < \delta\,,$$

from where it yields

$$\Big|\int_\Omega b_n(x,u_n,\nabla u_n)v-\int_{\{u\neq 0\}} b(x,u,\nabla u)v\Big|$$
$$\leq \Big|\int_{\{|u_n|\geq\varepsilon\}} b_n(x,u_n,\nabla u_n)v-\int_{\{|u|\geq\varepsilon\}} b(x,u,\nabla u)v\Big|+\delta\,.$$

Since $v\in L^\infty(\Omega)$ and the sequence $b_n(x,u_n,\nabla u_n)\chi_{\{|u_n|\geq\varepsilon\}}$ is equi–integrable, to see that the absolute value of the right hand side tends to 0, we only have to check the pointwise convergence. We split

$$b_n(x,u_n,\nabla u_n)\chi_{\{|u_n|\geq\varepsilon\}}=b_n(x,u_n,\nabla u_n)\chi_{\{|u_n|\geq\varepsilon\}\cap\{|u|>\varepsilon\}}$$
$$+\,b_n(x,u_n,\nabla u_n)\chi_{\{|u_n|\geq\varepsilon\}\cap\{|u|<\varepsilon\}}+b_n(x,u_n,\nabla u_n)\chi_{\{|u_n|\geq\varepsilon\}\cap\{|u|=\varepsilon\}}\,,$$

the first term converges pointwise to $b(x,u,\nabla u)\chi_{\{|u|>\varepsilon\}}$ (observe that is equal to $b(x,u,\nabla u)\chi_{\{|u|\geq\varepsilon\}}$ by (2.6) and Stampacchia's Theorem), while the second one tends to 0. Regarding the third one we have

$$|b_n(x,u_n,\nabla u_n)\chi_{\{|u_n|\geq\varepsilon\}\cap\{|u|=\varepsilon\}}|\leq g_n(u_n)|\nabla u_n|^2\chi_{\{|u|=\varepsilon\}}\to g(u)|\nabla u|^2\chi_{\{|u|=\varepsilon\}}$$

that vanishes by Stampacchia's Theorem. Thus, we deduce that

$$\lim_{n\to\infty}\int_{\{|u_n|\geq\varepsilon\}} b_n(x,u_n,\nabla u_n)v=\int_{\{|u|\geq\varepsilon\}} b(x,u,\nabla u)v\,,$$

and, therefore,

$$\limsup_{n\to\infty}\Big|\int_\Omega b_n(x,u_n,\nabla u_n)v-\int_{\{u\neq 0\}} b(x,u,\nabla u)v\Big|\leq\delta\,.$$

Since $\delta>0$ is arbitrary and having (3.41) in mind, we obtain

$$\lim_{n\to\infty}\int_\Omega b_n(x,u_n,\nabla u_n)v=\int_{\{u\neq 0\}} b(x,u,\nabla u)v=\int_\Omega b(x,u,\nabla u)v.$$

Passing to the limit in (3.42), we have proved that u is a solution to problem (2.2).

4. Further remarks, extensions and examples

4.1. **Remarks on the estimates satisfied by u.** We explicitly point out that the solution we have found satisfies many of the estimates proved to u_n in the proof of Theorem 2.1. For instance, it is easy to see that

$$\int_\Omega |b(x,u,\nabla u)|\leq\int_\Omega |f|e^{|\gamma(u)|} \tag{4.43}$$

holds. Indeed, observe that in (3.28) we have proved

$$\int_\Omega |b_n(x,u_n,\nabla u_n)|\leq\int_\Omega |f|e^{|\gamma_n(u_n)|}$$

for all $n \in \mathbb{N}$. Taking into account that u_n and ∇u_n pointwise converge to u and ∇u, respectively, we apply in the left hand side Fatou's Lemma to obtain

$$\int_\Omega |b(x,u,\nabla u)| = \int_{\{u\neq 0\}} |b(x,u,\nabla u)| \\ \leq \lim_{n\to\infty} \int_{\{u\neq 0\}} |b_n(x,u_n,\nabla u_n)| \leq \lim_{n\to\infty} \int_\Omega |b_n(x,u_n,\nabla u_n)|\,.$$

On the other hand, it follows from (3.27), Hölder's inequality and the pointwise convergence that $e^{|\gamma_n(u_n)|} \to e^{|\gamma(u)|}$ strongly in $L^{N/(N-2)}(\Omega)$. Thus, it follows from $f \in L^{N/2}(\Omega)$ that

$$\int_\Omega |f| e^{|\gamma(u)|} = \lim_{n\to\infty} \int_\Omega |f| e^{|\gamma_n(u_n)|}\,.$$

Other inequalities that also hold true are

$$\int_{\{|u|\leq\varepsilon\}} |b(x,u,\nabla u)| \leq \omega(\varepsilon)\Big[\int_\Omega |b(x,u,\nabla u)| + \int_\Omega |f|\Big] \tag{4.44}$$

$$\int_{\{|u|\geq k\}} |b(x,u,\nabla u)| \leq \int_{\{|u|\geq k\}} |f|\, e^{|\gamma(u)|}\,, \tag{4.45}$$

letting n go to infinity in (3.30) and (3.32), respectively. There are other type of estimates that can be adapted, namely, those which appear in the proof of the strong convergence of truncations. For instance, it follows from (3.33) that

$$\int_\Omega e^{\gamma(u)} a(x,u,\nabla u)\cdot\nabla (T_k(u)-T_k(w))^+ \\ \leq \int_\Omega |f| e^{\gamma(u)} (T_k(u)-T_k(w))^+ \tag{4.46}$$

holds for every $w \in H_0^1(\Omega)$.

4.2. **Bounded solutions.** Throughout this paper, we have assumed that f belongs to $L^{N/2}(\Omega)$; if the datum has a greater summability, the boundedness of the solution is guaranteed.

Proposition 4.1. *Assume that $f \in L^m(\Omega)$, with $m > \frac{N}{2}$. Then there exists a bounded weak solution to problem* (2.2)*.*

To prove it, consider again the function given by $G_k(s) = s - T_k(s)$ and take

$$e^{|\gamma_n(u_n)|} G_k\big(\Psi_n(u_n)\big)$$

as test function in (3.12). Since this function lives far from the singularity, we may now follow the proof of Theorem 3.1 in [18] and deduce that $\|\Psi_n(u_n)\|_\infty$ is bounded by a constant that only depends on the function g and the parameters m, $\|f\|_m$, N, and $|\Omega|$. Hence, $\Psi(u) \in L^\infty(\Omega)$ and, by (3.17), $u \in L^\infty(\Omega)$.

4.3. **Stability with respect to the lower order term.** In this subsection we provide a stability result with respect to perturbations of the lower order term. The result is important by his own; moreover, in the next subsection we show, as a consequence of this result, that there always exist solutions with no constant sign.

Let

$$b_\rho(x,s,\xi), b(x,s,\xi) : \Omega\times\mathbb{R}\backslash\{0\}\times\mathbb{R}^N \to \mathbb{R}$$

be Carathéodory functions satisfying

$$\lim_{\rho\to 0} b_\rho(x,s,\xi) = b(x,s,\xi)$$

for all $\xi \in \mathbb{R}^N$, for all $s \in \mathbb{R}\backslash\{0\}$ and for almost all $x \in \Omega$. Moreover, for fixed $\rho > 0$, there exist nonnegative functions $g_\rho, g : \mathbb{R}\backslash\{0\} \to [0,+\infty)$ such that

$$|b_\rho(x,s,\xi)| \le g_\rho(s)|\xi|^2, \quad |b(x,s,\xi)| \le g(s)|\xi|^2; \tag{4.47}$$

for all $\xi \in \mathbb{R}^N$, for all $s \in \mathbb{R}\backslash\{0\}$ and for almost all $x \in \Omega$; and there exist constants $\Lambda_\rho, \Lambda \ge 0$, $s_0 > 0$ and $\theta_\rho, \theta \in (0,1)$ such that $g_\rho(s) = \frac{\Lambda_\rho}{|s|^{\theta_\rho}}$ and $g(s) = \frac{\Lambda}{|s|^\theta}$ for all $|s| \le s_0$. We assume that, as $\rho \to 0$, $\theta_\rho \to \theta$ and $\Lambda_\rho \to \Lambda$.

These hypotheses imply that $\gamma_\rho(s) \to \gamma(s)$ and $\Psi_\rho(s) \to \Psi(s)$ uniformly on $[-s_0, s_0]$, where γ_ρ and Ψ_ρ are the auxiliary functions associated with each g_ρ. We also assume that

(1) $g_\rho(s) \to g(s)$ local uniformly on $(-\infty, -s_0] \cup [s_0, +\infty)$ as ρ goes to ∞.

(2) $$\lim_{|s|\to+\infty} \frac{e^{|\gamma_\rho(s)|}}{\Psi_\rho(s)} = 0\,, \quad \text{uniformly with respect to } \rho\,.$$

The last condition seems to be a little cumbersome. A simple case where it is certainly satisfied is when $g_\rho(s) = g(s)$ for $|s|$ large enough. We have essentially applied in this way in the proof of Theorem 2.1, and so will be used in the example of the following subsection.

Due to our assumptions, we can derive that, for every $\epsilon > 0$ there exists $C > 0$, not depending on ρ, satisfying

$$e^{|\gamma_\rho(s)|} \le \epsilon\Psi_\rho(s) + C\,, \quad \text{for all } s \in \mathbb{R}\,. \tag{4.48}$$

Finally, consider $f \in L^m(\Omega)$ with $m \ge \frac{N}{2}$ and u_ρ as the solution to problem

$$\begin{cases} -\mathrm{div}(a(x,u_\rho,\nabla u_\rho)) = b_\rho(x,u_\rho,\nabla u_\rho) + f(x)\,, & \text{in } \Omega\,; \\ u_\rho = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{4.49}$$

given in Theorem 2.1.

Theorem 4.1. *There exists $u \in H_0^1(\Omega)$ such that (up to subsequences)*

$$u_\rho \rightharpoonup u\,, \quad \textit{weakly in } H_0^1(\Omega)\,,$$
$$u_\rho \to u\,, \quad \textit{a.e. in } \Omega\,,$$
$$\nabla u_\rho \to \nabla u\,, \quad \textit{a.e. in } \Omega\,,$$

and u is a weak solution of problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) = b(x,u,\nabla u) + f(x)\,, & \textit{in } \Omega\,; \\ u = 0\,, & \textit{on } \partial\Omega\,, \end{cases}$$

Moreover, $b_\rho(x,u_\rho,\nabla u_\rho)$ strongly converges to $b(x,u,\nabla u)$ in $L^1(\Omega)$.

Sketch of the Proof. The proof of this result is based on a careful adaptation of the same steps in the proof of Theorem 2.1. The key point is that, thanks to (4.48), the estimate on $\Psi_n(u_n)$ in the proof of Theorem 2.1 does depend on α, $|\Omega|$, $\|f\|_{L^{\frac{N}{2}}(\Omega)}$ and $\mathcal{S}_N$, but it does not depend on ρ.

Now a remark concerning the test functions used in the proof is in order. It is not clear that, in each step, we may take the corresponding test function. The reason lies in the singularity at 0 of functions g_ρ that does not hold in the approximating functions g_n. To overcome this difficulty, we can apply the estimates deduced in Subsection 4.1.

So that, by arguing as in the proof of Theorem 2.1 we easily obtain the bound in $H_0^1(\Omega)$ for u_ρ and so, up to subsequences, a weak limit $u \in H_0^1(\Omega)$ is found. Moreover, we also obtain, as in (3.25) and (3.26), that

$$\int_\Omega |f| e^{|\gamma_\rho(u_\rho)|} |\Psi_\rho(u_\rho)| \le C$$
$$\int_\Omega |f| e^{|\gamma_\rho(u_\rho)|} \le C\,,$$

C being a positive constant not depending on ρ. From this last fact and (4.43), we derive the estimate of $b_\rho(x, u_\rho, \nabla u_\rho)$ in $L^1(\Omega)$. It follows from the estimate (4.44) that, for $\epsilon > 0$,

$$\int_{\{|u_\rho| \le \epsilon\}} b_\rho(x, u_\rho, \nabla u_\rho) \le \omega(\epsilon)\,.$$

The lower order term can be studied far from the singularity by using (4.45) and so, for $k > 0$, we get

$$\int_{\{|u_\rho| \ge k\}} b_\rho(x, u_\rho, \nabla u_\rho) \le \frac{C}{k},$$

where C is a positive constant non depending on ρ. We can also apply estimates like (4.46) to prove the strong convergence of $T_k(u_\rho)$ to $T_k(u)$ and, by a diagonal argument, deduce that ∇u_ρ tends to ∇u pointwise. The only actual difference relies in proving the equi–integrability of the lower order term where we use again the local uniform convergence of g_ρ to prove that

$$\sup_{\epsilon \le s \le k} g_\rho(s) \int_E |\nabla T_k(u_\rho)|^2 \le C_{\epsilon,k}\ \omega(|E|).$$

This way we get the equi–integrability of the lower order term and this allow us to pass to the limit in the weak formulation for u_ρ and to conclude the proof. □

4.4. **Example of a sign-changing solution.** It is worth to give an example of a solution which changes his sign. For the sake of simplicity we take as a model the problem

$$\begin{cases} -\Delta u = g(u)|\nabla u|^2 + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{4.50}$$

with a nonnegative g satisfying the same assumptions as in (2.6) and $f \in L^\infty(\Omega)$. The proof is based on the maximum principle together with the stability result given in Theorem 4.1.

Proposition 4.2. *There exist g and f such that the solution of problem (4.50) has no constant sign.*

Proof. Let us fix a g satisfying our assumptions and such that $g(s) = 0$ for $|s| \ge s_1$ for some $s_1 > s_0$. Consider $v \in C_0^2(\Omega)$ such that v changes his sign. Then, by the maximum principle, the function

$$f := -\Delta v$$

changes his sign. Now consider u as the solution, given by Theorem 2.1, of problem (4.50). Since g is nonnegative, it follows that u turns out to satisfy

$$-\Delta u \geq f, \quad u \in H_0^1(\Omega),$$

in $\mathcal{D}'(\Omega)$. So that by comparison, $u \geq v$. In particular there exists a set $E \subset \Omega$ of positive measure such that $u > 0$ on E. Now, suppose by contradiction that $u \geq 0$ on Ω and, for any fixed ρ, consider the family of problems

$$\begin{cases} -\Delta u_\rho = \rho g(u_\rho)|\nabla u_\rho|^2 + f(x)\,, & \text{in } \ \Omega\,; \\ u_\rho = 0\,, & \text{on } \ \partial\Omega\,. \end{cases} \tag{4.51}$$

Reasoning as before we deduce that, for any ρ, $u_\rho \geq v$ on Ω. In particular we can assume $u_\rho \geq 0$ on Ω since, if this is not the case, the proof is concluded with f and $\rho g(s)$ as data.

Therefore, applying Theorem 4.1 we can deduce that

$$\rho g(u_\rho)|\nabla u_\rho|^2 \to 0, \quad \text{in} \quad L^1(\Omega),$$

as ρ goes to zero, and, since the solution to the limit problem is unique, we get

$$0 \leq u_\rho \longrightarrow v \ \text{ a.e. on } \ \Omega$$

which is a contradiction since v changes his sign.

□

4.5. **Weakening the hypotheses on g.** Throughout this paper we have assumed that $g(s) \to 0$ as $|s| \to +\infty$. However, this hypothesis can be changed by being g bounded, if $\|f\|_{N/2}$ is small enough. We remark that we only apply that $g(s) \to 0$ to obtain (3.21) and it is just used (in an essential way) to deduce an estimate of $\Psi_n(u_n)$ in $H_0^1(\Omega)$.

Proposition 4.3. *Assume, instead of* (2.7)*, that there exists $M > 0$ satisfying $\limsup_{|s|\to\infty} g(s) \leq M$ and, besides* (2.8)*, that $\|f\|_{N/2} < \frac{\alpha}{MS_N^2}$, S_N denoting the Sobolev constant. Then there exists a weak solution to problem* (2.2)*.*

Proof. Consider the same approximating problems (3.11). To check the estimate of $\Psi_n(u_n)$ in $H_0^1(\Omega)$, first observe that condition $\limsup_{|s|\to\infty} g(s) \leq M$ implies that there exists a constant $C > 0$ such that

$$e^{|\gamma_n(s)|} \leq M|\Psi_n(s)| + C\,, \quad \forall\, n \in \mathbb{N}\,,\ s \in \mathbb{R}\,;$$

to see it, just recall the argument used to derive (3.21).

Taking $e^{|\gamma_n(u_n)|}\Psi_n(u_n)$ as test function in (3.12) and dropping nonnegative terms we also obtain

$$\alpha \int_\Omega |\nabla \Psi_n(u_n)|^2 \leq \int_\Omega |f| e^{|\gamma_n(u_n)|} |\Psi_n(u_n)|\,. \tag{4.52}$$

Then we reason as follows. Hölder's and Sobolev's inequalities imply

$$M \int_\Omega |f||\Psi_n(u_n)|^2 \leq MS_N^2 \|f\|_{N/2} \int_\Omega |\nabla \Psi_n(u_n)|^2\,.$$

Thus, (4.52) becomes

$$\alpha\int_\Omega |\nabla\Psi_n(u_n)|^2 \le M\int_\Omega |f||\Psi_n(u_n)|^2 + C\int_\Omega |f||\Psi_n(u_n)|$$
$$\le MS_N^2\|f\|_{N/2}\int_\Omega |\nabla\Psi_n(u_n)|^2 + C\int_\Omega |f||\Psi_n(u_n)|$$

and it yields

$$(\alpha - MS_N^2\|f\|_{N/2})\int_\Omega |\nabla\Psi(u_n)|^2 \le C\int_\Omega |f||\Psi_n(u_n)|\,.$$

It easily follows the estimate of $\Psi_n(u_n)$ in $H_0^1(\Omega)$.

Next we may follow the same proof that the one of Theorem 2.1. □

4.6. **Taking less regular data.** In this subsection, we will assume extra hypotheses on g that allow us to consider less regular data. In the following result, we will assume

- There exists

$$\lim_{|s|\to\infty} g(s)|s|\,. \tag{4.53}$$

- There exist constants $\lambda > 0$ and $M \ge 0$ satisfying

$$\lim_{|s|\to\infty} \frac{e^{|\gamma(s)|}}{|s|^\lambda} = M\,. \tag{4.54}$$

Remark 4.4. Condition (4.54) seems a little bit strange, since it is not a direct assumption on g. Let us see what is the behaviour of g to satisfy this condition.

(1) Conditions (4.53) and (4.54) imply that $\lim_{|s|\to\infty} g(s)|s| = \alpha\lambda$.
(2) If $g(s) = \frac{\lambda}{|s|}$ for all $s \ge s_0$, then (4.53) and (4.54) hold, since $\frac{e^{|\gamma(s)|}}{|s|^{\lambda/\alpha}}$ is constant.
(3) One could think that condition (4.54) holds for every function g satisfying $\lim_{|s|\to\infty} g(s)|s| = \lambda\alpha$. As the function given by $g(s) = \frac{\lambda\alpha}{|s|} + \frac{1}{|s|\log|s|}$ (for s large enough) shows, it is not true.
(4) The limit occurring in (4.54) vanishes, when it exists, for every function g such that $\lim_{|s|\to\infty} g(s)|s| < \alpha\lambda$.
(5) In some cases function g satisfies condition (4.54) for all λ (and so $M = 0$). Obviously, this is the case when g is summable at infinity. An instance of a non summable function satisfying condition (4.54) for all λ is the function given by $g(s) = \frac{1}{|s|\log|s|}$, for s large enough.

Proposition 4.5. *Assume, instead of* (2.7)*, that* (4.53) *and* (4.54) *hold and, instead of* (2.8)*, that* $f \in L^m(\Omega)$*, with* $m = \left(\frac{2^*(\lambda+1)}{2\lambda+1}\right)' = \frac{2N(\lambda+1)}{N+2(2\lambda+1)}$*. Then there exists a weak solution to problem* (2.2).

Proof. In this case, we have to change (3.20) by

$$\lim_{|s|\to\infty} \frac{e^{|\gamma_n(s)|}}{|\Psi_n(s)|^{\lambda/(\lambda+1)}} = M^{1/(\lambda+1)}(\lambda+1)^{\lambda/(\lambda+1)}$$

and so (3.21) becomes

$$e^{|\gamma_n(s)|} \le C|\Psi_n(s)|^{\lambda/(\lambda+1)} + C \quad \text{for all } s \in \mathbb{R}\,. \tag{4.55}$$

This inequality is used to estimate $\int_\Omega f e^{|\gamma_n(u_n)|}\Psi_n(u_n)$ as follows. By Hölder's inequality, we obtain

$$\int_\Omega |f| e^{|\gamma_n(u_n)|}|\Psi_n(u_n)| \le C\int_\Omega |f||\Psi_n(u_n)|^{(2\lambda+1)/(\lambda+1)} + C\int_\Omega |f||\Psi_n(u_n)|$$
$$\le C\|f\|_m \|\Psi_n(u_n)\|_{2^*}^{(2\lambda+1)/(\lambda+1)} + C\int_\Omega |f||\Psi_n(u_n)|\,.$$

Since $\frac{2\lambda+1}{\lambda+1} < 2$, we may apply Young's inequality to get

$$C\|f\|_m \|\Psi_n(u_n)\|_{2^*}^{(2\lambda+1)/(\lambda+1)} \le \epsilon\|\Psi_n(u_n)\|_{2^*}^2 + C(\epsilon)\|f\|_m^{2(\lambda+1)}.$$

Then, taking $e^{|\gamma_n(u_n)|}\Psi_n(u_n)$ as test function in (3.12), we deduce

$$\alpha\int_\Omega |\nabla\Psi_n(u_n)|^2 \le \epsilon\|\Psi_n(u_n)\|_{2^*}^2 + C(\epsilon)\|f\|_m^{2(\lambda+1)} + C\int_\Omega |f||\Psi_n(u_n)|\,,$$

from where estimates on both $\Psi_n(u_n)$ and u_n in $H_0^1(\Omega)$ are obtained. Moreover, the sequence $e^{|\gamma_n(u_n)|}$ is bounded in $L^{2^*(\lambda+1)/\lambda}(\Omega)$, due to (4.55). Next we may follow the same proof that the one of Theorem 2.1. □

Observe that $\frac{2N(\lambda+1)}{N+2(2\lambda+1)}$ goes to $N/2$ as λ goes to $+\infty$, while it converges to $\frac{2N}{N+2}$ as λ goes to 0 that correspond to the case of an integrable g. Thus, the previous Proposition along with Theorem 2.1 and the following result show that there is continuity with respect to the summability of the datum.

Proposition 4.6. *Assume, instead of* (2.7)*, that* $g \in L^1(\mathbb{R})$ *and, instead of* (2.8)*, that* $f \in L^m(\Omega)$*, with* $m = (2^*)' = \frac{2N}{N+2}$*. Then there exists a weak solution to problem* (2.2)*.*

We may easily obtain estimates on both $\Psi_n(u_n)$ and u_n in $H_0^1(\Omega)$, having in mind that we now have $e^{|\gamma_n(s)|} \le C$ for all $s\in\mathbb{R}$ and this implies, taking $e^{|\gamma_n(u_n)|}\Psi_n(u_n)$ as test function in (3.12), that

$$\alpha\int_\Omega |\nabla\Psi_n(u_n)|^2 \le C\int_\Omega |f||\Psi_n(u_n)|\,.$$

The proof now follows the same steps that the one of Theorem 2.1.

Let us finally remark that, in this case in which $g\in L^1(\mathbb{R})$, we may want to take less regular data up to $m = 1$ by readapting the arguments in [19]. This is certainly possible, but this would bring us out of our framework of finite energy solutions.

4.7. **Lower order terms satisfying a sign condition.** In this last subsection, we deal with a lower order term having the sign condition. Our aim is to show how the behavior of these type of lower order terms allow us to choose an even less regular datum f.

For the sake of simplicity, we will consider the model problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + g(u)|\nabla u|^2 = f(x)\,, & \text{in } \Omega\,;\\ u = 0\,, & \text{on } \partial\Omega\,,\end{cases} \tag{4.56}$$

where g satisfies

$$\lambda|s|^{1-\theta} \le g(s)s \le \Lambda|s|^{1-\theta} \qquad \text{for all } s\in\mathbb{R}\,. \tag{4.57}$$

For a more general lower order term, we would change this condition by

$$\lambda|s|^{1-\theta}|\xi|^2 \le -b(x,s,\xi)s \le \Lambda|s|^{1-\theta}|\xi|^2 \qquad \text{for all } s\in\mathbb{R}\,.$$

Proposition 4.7. *Assume that* (4.57) *holds. If $f\in L^m(\Omega)$, with $m=\left(\frac{2^*}{\theta}\right)'$, then there exists a weak solution to problem* (4.56).

Proof. For fixed n we define the continuous functions

$$g_n(s) := \begin{cases} g(s)\,, & \text{if } |s|\ge \frac{1}{n}\,;\\ ng(1/n)s\,, & \text{if } 0\le s<\frac{1}{n}\,;\\ -ng(-1/n)s\,, & \text{if } -\frac{1}{n}<s\le 0\,; \end{cases} \tag{4.58}$$

note that these functions satisfy the same sign condition of g, namely, $g_n(s)s\ge 0$. We consider the approximating problems

$$\begin{cases} -\mathrm{div}(a(x,u_n,\nabla u_n)) + g_n(u_n)|\nabla u_n|^2 = T_n(f(x))\,, & \text{in } \Omega\,;\\ u_n = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{4.59}$$

By [18] (or, alternatively, by applying Theorem 2.1 and Proposition 4.1), we may find a bounded weak solution u_n to problem (4.59).

To obtain an estimate on u_n in $H_0^1(\Omega)$, we first take $T_1(u_n)$ as test function. Dropping nonnegative terms, we get

$$\alpha\int_\Omega |\nabla T_1(u_n)|^2 \le \int_\Omega T_n(f)T_1(u_n) \le \int_\Omega |f|\,.$$

Hence,

$$\int_{\{|u_n|\le 1\}} |\nabla u_n|^2 \le \frac{1}{\alpha}\int_\Omega |f|\,. \tag{4.60}$$

Now we take $(\epsilon+|u_n|)^\theta \frac{T_k(u_n)}{k}$ as test function. Disregarding nonnegative terms, it yields

$$\int_\Omega (\epsilon+|u_n|)^\theta \frac{T_k(u_n)}{k} g_n(u_n)|\nabla u_n|^2 \le \int_\Omega |f|(\epsilon+|u_n|)^\theta \frac{T_k(u_n)}{k}\,.$$

Letting ε and k go to 0, we obtain

$$\int_\Omega |u_n|^\theta |g_n(u_n)||\nabla u_n|^2 \le \int_\Omega |f||u_n|^\theta\,,$$

from here, using (4.57) and the definition of g_n, we deduce

$$\int_{\{|u_n|>1\}} |\nabla u_n|^2 \le \frac{1}{\lambda}\int_\Omega |f||u_n|^\theta\,. \tag{4.61}$$

Putting together (4.60) and (4.61), it yields

$$\int_\Omega |\nabla u_n|^2 \le C\int_\Omega |f||u_n|^\theta + C\,,$$

from where, using first the Hölder inequality and then the Sobolev one, an estimate of u_n in $H_0^1(\Omega)$ can be obtained.

From now on, the proof runs as that of Theorem 2.1 with a suitable simplification. In order to reproduce the Steps 3.3, 3.4, 3.5 and 3.6 in the proof of Theorem 2.1, we argue as follows. Consider the following auxiliary function:

$$\gamma_n(s) = \frac{1}{\alpha}\int_0^s g_n(\sigma)\,d\sigma\,,$$

and observe that $\gamma_n(s) \geq 0$ for all $s \in \mathbb{R}$.

(1) We take $\frac{T_k(u_n)}{k}$ as test function and then let k tend to 0 to prove the L^1–estimate on the lower order term.

(2) We consider

$$v = \begin{cases} \big(1 - e^{-\gamma_n(u_n)}\big)\operatorname{sign} u_n\,, & \text{if} \quad -\varepsilon \leq u_n \leq \varepsilon\,; \\ 1 - e^{-\gamma_n(\varepsilon)}\,, & \text{if} \quad u_n > \varepsilon\,; \\ e^{-\gamma_n(-\varepsilon)} - 1\,, & \text{if} \quad u_n < -\varepsilon \end{cases}$$

as test function to control the singularity on the set $\{|u_n| < \varepsilon\}$.

(3) We choose $T_1(G_k(u_n))$ (with $G_k(s) = s - T_k(s)$ as before) as test function to handle the set where u_n is large. This way we obtain

$$\int_{\{|u_n|>k+1\}} |g_n(u_n)||\nabla u_n|^2 \leq \int_{\{|u_n|>k\}} |f|\,.$$

(4) We consider $e^{\gamma_n(u_n)}(T_k(u_n) - T_k(u))^+$ and $-e^{-\gamma_n(u_n)}(T_k(u_n) - T_k(u))^-$ as test functions to check the strong convergence of $\nabla T_k(u_n)$ in $L^2(\Omega;\mathbb{R}^N)$.

This is enough to prove that the limit u is a weak solution to (4.56). □

Let us observe that $m = (\frac{2^*}{\theta})'$ converges to 1 as θ goes to 0. That is, in the limit, we recover the classical nonsingular result of [9].

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