A PRIORI ESTIMATES FOR ELLIPTIC PROBLEMS WITH A STRONGLY SINGULAR GRADIENT TERM AND A GENERAL DATUM

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Abstract. In this paper we show approximation procedures for studying singular elliptic problems whose model is

$$\begin{cases} -\Delta u = b(u)|\nabla u|^2 + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,; \end{cases}$$

where $b(u)$ is singular in the u-variable at $u = 0$, and $f \in L^m(\Omega)$, with $m > \frac{N}{2}$, is a function that does not have a constant sign. We will give an overview of the landscape that occurs when different problems (classified according to the sign of $b(s)$) are considered. So, in each case and using different methods, we will obtain a priori estimates, prove the convergence of the approximate solutions and show some regularity properties of the limit.

1. Introduction

In this paper we deal with strongly singular problems which can be written as

$$\begin{cases} -\Delta u = b(u)|\nabla u|^2 + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

Here Ω is a bounded open set in $\mathbb{R}^N$, the function $b(s)$ is singular at $s = 0$, and the datum $f(x)$ belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$. The main hypotheses we will assume are that $f(x)$ does not have any sign property and the kind of singularity is such that $b \notin L^1(I)$ for every interval I containing 0 (for instance, $b(s) \sim \frac{1}{s}$); as a consequence, the lower order term $b(u)|\nabla u|^2$ exhibits a strong singular behaviour on the set $\{u = 0\}$ (both near and far from the boundary).

Note that, if we are dealing with data $f(x) \geq 0$, then the solution will lie above the singularity ($u \geq 0$) whatever is the sign of $b(u)$ (see [14], [3], and references therein). Indeed, if $f(x) \geq 0$ and also $b(u) \geq 0$, the strong maximum principle guarantees that u is strictly positive inside of Ω and the lower order term is completely well defined. The same holds true if $f(x) \geq 0$ and $b(u) \leq 0$, by a deeper use of the strong maximum principle (see for

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instance [5]). Other papers dealing with nonnegative data and studying existence and non existence of solutions in $H_0^1(\Omega)$ or in $H^1_{loc}(\Omega)$ are [2], [4] and [19].

In the case where $f(x)$ changes its sign, the solution u can vanish inside Ω, sometimes on a set of positive measure. This actually occurs even if $b(s)$ has a mild singular behaviour at $s = 0$ (i.e. it is a L^1–function in a neighbourgh 0) and this is shown in [15, Proposition 4.2]. The quoted paper [15] deals with problems having summable singularities of the type $|b(s)| \leq \dfrac{\Lambda}{|s|^\theta}$, with $\Lambda > 0$ and $0 < \theta < 1$. Due to the fact that the solution can be zero inside Ω, in [15] the authors had to define carefully the meaning of solution, in order to give sense to the gradient term on the singularity. The aim of this paper is to go beyond and study this problem when the singularity is not summable (we will actually focus on the case $\theta = 1$).

Let us point out the main features which are characteristic of problems having strong singularities:

(1) The possible solutions do not belong, in general, to the energy space $H_0^1(\Omega)$; as a matter of fact, in some cases, they do not belong even to $H^1_{loc}(\Omega)$.
(2) The Cole–Hopf change of unknown can transform our equation in a singular semilinear equation.
(3) The problem does not provide information about the behaviour of the datum $f(x)$ on the set $\{u = 0\}$.

Some remarks concerning these points are in order.

(1) The first fact happens also for problems with strong singularities and nonnegative data $f(x)$ (see [3] and [14]); anyway, in those papers the solutions always belong to the space $H^1_{loc}(\Omega)$ and the boundary condition is expressed through the fact that a function $\Psi(u)$ of the solution u does belong to the energy space $H_0^1(\Omega)$. Of course, the occurrence in our problems of a solution that may change its sign *inside* the domain brings new difficulties in order to prove its regularity. For instance, as we will see, in some cases not even distributional solutions can be expected.

(2) The Cole–Hopf change of unknown (see [11, 17]) is a typical tool introduced in order to, roughy speaking, *linearize* some nonlinear partial differential equations. Later on it has been pointed out how, in the case of nonlinear equations having a gradient term with natural growth, a Cole–Hopf type transformation can reduce the original problem to a semilinear one, at least in the model case (see [18] and, for further details, [1]). Although the use of test functions of exponential type had been used to handle nonlinear equations with a gradient term (see [7, 8]), the systematic application of those that simulate the Cole–Hopf transformation for obtaining a priori estimates and convergence of approximate solutions appears in [13, 10, 21] when the function $b(s)$ is continuous (for employing this method when $b(s)$ has a summable singularity see [15]). In our situation, however,

the semilinear equation can be singular. In order to show what happens, let us consider a simple example (when $b(s) = -\frac{1}{2s}$)

$$\begin{cases} -\Delta u + \frac{1}{2}\frac{|\nabla u|^2}{u} = f(x)\,, & \text{in } \Omega\,;\\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

In this case the change of unknown is given by $v = 2\frac{u}{\sqrt{|u|}}$ and then the associated semilinear problem is, formally,

$$\begin{cases} -\Delta v = 2f(x)\frac{1}{|v|}\,, & \text{in } \Omega\,;\\ v = 0\,, & \text{on } \partial\Omega\,; \end{cases}$$

which is singular on the right hand side. This drawback has important consequences because it will make much difficult to prove that approximate solutions converge suitably if the datum changes its sign. Let us remark that, in the case of nonnegative f, such a singular semilinear problems can be handled by suitable use of the strong maximum principle (see [9]).

(3) The third fact remains hidden when data are nonnegative: this is a special feature of data with changing sign and strong singularities. We next explain it using another simple example (that is $b(s) = \frac{1}{s}$)

$$\begin{cases} -\Delta u = \frac{|\nabla u|^2}{u} + f(x)\,, & \text{in } \Omega\,;\\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

The Cole–Hopf transformation is now defined by $v = \frac{1}{2}|u|u$. It is easy to check, formally, that v is solution to the non singular sublinear problem

$$\begin{cases} -\Delta v = f(x)\sqrt{2|v|}\,, & \text{in } \Omega\,;\\ v = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

We may solve this problem finding a solution $v \in H_0^1(\Omega)$; nevertheless, if we try to do the inverse change, we obtain u as a not Lipschitz continuous function of v, which explain the fact that in this case we do not reach $H^1_{loc}(\Omega)$–solutions. Formally we obtain the equation

$$-|u|\Delta u = |u|\Big[\frac{|\nabla u|^2}{u} + f(x)\Big]\,,$$

and we cannot divide by $|u|$ since it vanishes on $\{u = 0\}$. Thus, $f\chi_{\{u=0\}}$ cannot be viewed by the equation.

Having in mind all these difficulties, in this paper we propose new types of formulation of solution to deal with this kind of strongly singular problems. Nevertheless, as we already mentioned, these formulations do not imply the distributional formulation, since they do not provide information on the set $\{u = 0\}$. As a consequence, they do not discriminate between the nontrivial solution and the solution $u \equiv 0$. For this reason, the condition $u \not\equiv 0$ must be added to our concept of solution.

In summary, the presence of both a strong singularity and a changing sign datum imposes several new difficulties in the study of these type of problems. Due to this situation, the results we will present are far to be complete, and one can look at them as a first step in order to deal with this type of strongly singular problems. Our aim is just to collect some partial results one can obtain for this kind of problems (in particular, a complete picture on sharp local estimates for the approximating problems), interpret these results, provide some examples, and make several remarks. Even though we always find a *solution as a limit of approximations* (in the sense introduced in [12]), only in very special cases we are able to prove the existence of solution in the formulation we propose. As we will point out, as a by-product of our arguments we also turn out to recover, in a unified way, most of the available results for nonnegative data.

We next summarize the contents of the present paper. In Section 2 we introduce the problems we will handle. We analyze the different problems in Sections 3, 4 and 5, each developed according to its own peculiarities. In Section 3 all calculations will be written in detail, while in the others will be made references to that section (so one can better compare the various cases). Finally, in Appendix A we extend our examples to higher dimensions.

2. List of all possible cases and main properties

Throughout this paper Ω will denote an open bounded set in $\mathbb{R}^N, N \geq 3$ (though, all the results we present can be easily proved to be valid also in the case $N = 2$ with straightforward simplifications in the proofs). Moreover, C will be a positive constant only depending on the parameters of the problems; its value may change from line to line. We will also use two families of functions: for each $k > 0$, denote $T_k(s) = \max(-k, \min(s, k))$ and $G_k(s) = s - T_k(s)$.

In this paper we are going to analyze problems (1.1) by an approximation procedure. To this end, we will consider approximating smooth functions $b_n(s)$ and $f_n(x)$ in such a way that every $b_n(s)$ be continuous and $f_n \to f$ strongly in $L^m(\Omega)$ and then we will find a weak solution u_n to

$$\begin{cases} -\Delta u_n = b_n(u_n)|\nabla u_n|^2 + f_n(x)\,, & \text{in } \Omega\,; \\ u_n = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Our main purpose is, in each case, to obtain estimates on the sequence u_n and prove that it converges to a suitable function. According to the sign properties of the lower order terms, all possible cases can be classified into four model problems whose equations are similar but which present very different behaviours from the point of view of estimates on the approximate solutions and from the point of view of its convergence. It is worth remarking that any sign condition is associated to a different type of (possibly singular) semilinear equation derived from the Cole–Hopf transformation. This change of unknown for a problem having a lower order term $b(u)|\nabla u|^2$ is performed formally defining $h(s)$ as a primitive function of $b(s)$, taking $e^{h(s)}$ and then its primitive $\Psi(s)$. Now the change of unknown $v = \Psi(u)$, transform problem (1.1) in a semilinear one. We point out that the use of test functions involving this type of exponential functions is fundamental both for obtaining a priori estimates and for passing to the limit in the approximating problems.

In this section, we will introduce all model problems that will be studied and we will state the different features they show. Assuming hereinafter $\Lambda > 0$, these four problems we will consider can be written as

$$\begin{cases} -\Delta u = \Lambda \frac{|\nabla u|^2}{|u|} + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{2.2}$$

$$\begin{cases} -\Delta u = \Lambda \frac{|\nabla u|^2}{u} + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{2.3}$$

$$\begin{cases} -\Delta u + \Lambda \frac{|\nabla u|^2}{|u|} = f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{2.4}$$

$$\begin{cases} -\Delta u + \Lambda \frac{|\nabla u|^2}{u} = f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{2.5}$$

Problems (2.2) and (2.4) are quite similar due to the fact that we can perform the change $v = -u$ and get one problem from the other. For these two problems we will get $H^1_{loc}(\Omega)$–estimates on the approximate solutions u_n and $L^1_{loc}(\Omega)$–estimates on the approximating lower order terms. This will imply that the sequence u_n tends to a function u (so that it is a solution obtained as limit of approximate solutions) and this function can also be proved to be not identically zero. We collect all these results in Theorem 3.1. The main remaining open question is whether this function satisfies the limit problem in a suitable sense. We realize that if we look for a Cole–Hopf type transformation, we are immediately in trouble. In our case, in which the solution can cross the singularity, we have $h(s) = \log s^\Lambda$ on the set $\{s > 0\}$ and $h(s) = -\log|s|^\Lambda$ on the set $\{s < 0\}$, so that $e^{h(s)}$ is strongly unbounded near the singularity for $s < 0$. As we will see, the fact that $e^{h(s)}$ is unbounded near the singularity (actually on the left of the singularity $s = 0$) makes us unable to pass to the limit in the approximating problem (or in the associated semilinear one). Also note that, if $\Lambda \geq 1$, we even can not define the change of unknown $v = \Psi(u)$, since the function $e^{h(s)}$ is not an L^1–function near $s = 0$.

As far as the problem (2.3) is concerned, the possible solution does not belong, in general, to $H^1_{loc}(\Omega)$, neither the lower order term belongs to $L^1_{loc}(\Omega)$, as a counterexample shows (see Example 4.1 below). As a consequence, no distributional solution can be expected. Hence, we propose an alternative formulation of solution to problem (2.3), which we call "strong renormalized" solution (see Definition 4.2 below). To get it we will confine to the sets $\Omega_n = \{x \in \Omega : u_n(x) \geq \delta\}$ with $\delta > 0$ and we will be able to prove uniform $H^1_0(\Omega_n)$–estimates for the solutions u_n and $L^1(\Omega_n)$–estimates for the approximating lower order terms on this set. In the main result, stated in Theorem 4.1, we will prove that a "strong renormalized" solution exists.

Finally, the problem (2.5) is the simplest one from the point of view of the estimates, since we easily get global $H^1_0(\Omega)$–estimates for the approximate solutions u_n and $L^1(\Omega)$–estimates for the approximating lower order terms and this implies the existence of a limit function u in the good energy space. On the other hand, again, as in the case (2.2), we are not able to prove that it satisfies the formulation of the limit problem. When $0 < \Lambda < 1$, it is easy to see that, in this case, $e^{h(s)} = |s|^{-\Lambda}$, which is again unbounded at $s = 0$, actually unbounded from both the sides. For the same reasons as before, this makes us enable to pass to the limit, even if, in this case, we have very good global estimates on the approximate solutions u_n. We state in Theorem 5.1 the result concerning this last case.

We summarize the different features in the following table, where we also explicit the regularity one can expect for both the solutions and the lower order terms.

Function	Sign condition	Regularity of u	Regularity of l.o.t
$b(s) = 1/\lvert s\rvert$	Partial $b(-u^-)(-u^-) \leq 0$	$H^1_{loc}(\Omega)$	$L^1_{loc}(\Omega)$
$b(s) = 1/s$	For neither u^+ or u^-	$H^1(\{\lvert u\rvert \geq \delta\})$	$L^1(\{\lvert u\rvert \geq \delta\}$
$b(s) = -1/\lvert s\rvert$	Partial $b(u^+)(u^+) \leq 0$	$H^1_{loc}(\Omega)$	$L^1_{loc}(\Omega)$
$b(s) = -1/s$	For both signs $b(u)u \leq 0$	$H^1_0(\Omega)$	$L^1(\Omega)$

Table. Comparison of the features of the model equation $-\Delta u = b(u)|\nabla u|^2 + f$ for different functions $b(s)$.

We recall that, referring to problem (1.1), $b(s)$ satisfies the *sign condition* if $b(s)s \leq 0$. By *partial sign condition* we mean that the same condition is satisfied for either positive or negative values. We want to stress that a singularity appears in the associated (through Cole–Hopf transform) semilinear problem if and only if a sign condition (possibly partial) is satisfied; for instance, if the sign condition holds for the positive part, then there exists a singularity in the transformed semilinear equation for the positive part (e.g. the singularity occurs also for nonnegative data), and so with all possible cases.

3. The cases of lower order terms with constant sign

In this section we deal with the case of nonnegative singular lower order term of the type (2.2):

$$\begin{cases} -\Delta u = \frac{\Lambda}{|u|}|\nabla u|^2 + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.6}$$

As far as the datum f is concerned, it is a changing sign function satisfying

$$f(x) \in L^m(\Omega)\,, \quad m > \frac{N}{2}\,. \tag{3.7}$$

As we already mentioned, the results we present in this case have their analogous counterparts in the case (2.4) by the trivial change of variable $v = -u$.

3.1. **Statement of the main result.** As far as problem (3.6) is concerned, we will prove the following result.

Theorem 3.1. *There exists a nonzero function $u \in H^1_{loc}(\Omega) \cap L^\infty(\Omega)$ satisfying $|u|^\Lambda u \in H^1_0(\Omega)$ and $\dfrac{|\nabla u|^2}{|u|} \in L^1_{loc}(\Omega)$ which is a solution to (3.6) obtained as a limit of approximations.*

Moreover, the approximate solutions u_n converge to u, up to subsequences, in the following senses:

$$G_\delta(u_n) \to G_\delta(u) \quad \textit{strongly in } L^q(\Omega)\,, \quad 1 \le q < \frac{2N}{N-2}\,;$$
$$u_n(x) \to u(x) \quad \textit{pointwise a.e. in } \Omega\,;$$
$$\nabla G_\delta(u_n) \rightharpoonup \nabla G_\delta(u) \quad \textit{weakly in } L^2(\Omega;\mathbb{R}^N)\,;$$
$$\nabla G_\delta(u_n) \to \nabla G_\delta(u) \quad \textit{strongly in } L^q(\Omega;\mathbb{R}^N)\,, \quad 1 \le q < 2\,;$$
$$\nabla u_n(x) \to \nabla u(x) \quad \textit{pointwise a.e. in } \Omega\,.$$

In the above result, we have to give sense to the boundary condition and to the gradient term on the singularity.

Remark 3.1. Since no $H^1(\Omega)$–estimate is expected, we cannot prove $u \in H^1_0(\Omega)$. The boundary condition is satisfied through the requirement $|u|^\Lambda u \in H^1_0(\Omega)$. This fact is similar to that which holds for nonnegative data (see [3]).

Remark 3.2. We remark that, if one looks for solutions $u \in H^1_{loc}(\Omega)$, the equation in (3.6) involves an indeterminate quotient on $\{u = 0\}$, since $\dfrac{|\nabla u|^2}{|u|}$

and $|\nabla u| = 0$ on the set $\{u = 0\}$, by Stampacchia's Theorem (see [22, Lemma 1.1]). The next definition clarifies our framework.

Definition 3.3. *If u satisfies $\sqrt{|u|} \in H^1_{loc}(\Omega)$, we define*

$$\frac{|\nabla u|^2}{|u|} = 4|\nabla\big(\sqrt{|u|}\big)|^2 .$$

Observe that, by definition, $\dfrac{|\nabla u|^2}{|u|}$ always belongs to $L^1_{loc}(\Omega)$. Moreover, by Stampacchia's Theorem, we obtain

$$\frac{|\nabla u|^2}{|u|} = 0 \quad \text{a.e. in } \{u = 0\} .$$

It follows from Definition 3.3 that, in order to check that a function $u \in H^1_{loc}(\Omega)$ is actually distributional solution to problem (3.6), one should check whether $\sqrt{|u|} \in H^1_{loc}(\Omega)$. To this aim the following simple claim will be applied, whose proof is similar to the one in [16, Lemma 2.1] and [15, Lemma 2.4].

Lemma 3.4. *Let $u \in H^1_{loc}(\Omega)$. If $\dfrac{|\nabla u|^2}{|u|}$ is locally integrable on $\{u \neq 0\}$, then*

$$\sqrt{|u|} \in H^1_{loc}(\Omega) .$$

Moreover,

$$\int_\Omega \frac{|\nabla u|^2}{|u|}\varphi = \int_{\{u\neq 0\}} \frac{|\nabla u|^2}{|u|}\varphi , \qquad \forall \varphi \in C^\infty_0(\Omega) .$$

3.2. **Auxiliary functions.** Here we introduce some auxiliary functions that will be useful in what follows: we explicitly remark that, due to the strong singularity of the lower order term, the choice of these test functions is not a standard extension of the *summable case* (see e.g. [15]).

We define

$$h(s) := \Lambda \log |s| ; \tag{3.8}$$

observe that the strong singularity implies $\lim_{s\to 0^+} h(s) = -\infty = \lim_{s\to 0^-} h(s)$. Thereby, we set

$$E(s) := |s|^\Lambda = \begin{cases} e^{h(s)} , & \text{if } s \neq 0 ; \\ 0 , & \text{if } s = 0 . \end{cases} \tag{3.9}$$

Moreover, we will need the function given by

$$\Psi(s) := \int_0^s E(\sigma)\, d\sigma = \frac{1}{\Lambda + 1}|s|^\Lambda s . \tag{3.10}$$

Obviously,

$$\lim_{|s|\to\infty} \frac{E(s)}{\Psi(s)} = 0 , \tag{3.11}$$

so that, given $\epsilon > 0$, we may find a constant $K_\epsilon > 0$ such that

$$E(s) \le \epsilon|\Psi(s)| + K_\epsilon\,, \qquad \forall s \in \mathbb{R}\,. \tag{3.12}$$

3.3. **A priori estimates.** Here we provide our a priori estimates for the approximate solutions to problem (3.6).

Approximating problems. For each $n \in \mathbb{N}$, define

$$b_n(s) := \begin{cases} \Lambda n\,, & \text{if } |s| \le \frac{1}{n}\,; \\ \dfrac{\Lambda}{|s|}\,, & \text{otherwise}\,; \end{cases} \tag{3.13}$$

so that,

$$\lim_{|s|\to+\infty} b_n(s) = 0\,. \tag{3.14}$$

Take also a sequence of smooth functions $f_n \in L^m(\Omega)$ such that $f_n \to f$ strongly in $L^m(\Omega)$.

Consider the following problems

$$\begin{cases} -\Delta u_n = b_n(u_n)|\nabla u_n|^2 + f_n(x)\,, & \text{in } \Omega\,; \\ u_n = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.15}$$

A weak solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to problem (3.15) exists due to [20, Theorem 1.1 (i)].

To deal with problem (3.15), we will need auxiliary functions similar to those defined in (3.8), (3.9) and (3.10): we will denote them by h_n, E_n and Ψ_n. We begin by defining

$$h_n(s) := \begin{cases} \displaystyle\int_1^s b_n(\sigma)d\sigma\,, & \text{if } s > 0\,; \\ \displaystyle -\int_{-1}^s b_n(\sigma)d\sigma\,, & \text{if } s < 0\,. \end{cases} \tag{3.16}$$

Two facts concerning h_n are in order: the first one is that $h_n(s) = h(s)$ for all $|s| > \frac{1}{n}$ and the second is $\lim_{n\to\infty} h_n(0) = -\infty$, due to the strong singularity. We also define

$$E_n(s) = e^{h_n(s)}\,, \quad \text{for all } s \in \mathbb{R}\,, \tag{3.17}$$

and

$$\Psi_n(s) = \int_0^s E_n(\sigma)\,d\sigma\,. \tag{3.18}$$

We point out that

$$0 < E_n(0) \le 1\,,\ \forall n \in \mathbb{N}\,, \quad \text{and} \quad \lim_{n\to\infty} E_n(0) = 0\,. \tag{3.19}$$

A direct computation relates the approximate functions E_n and Ψ_n with E and Ψ respectively:

$$E(s) \le E_n(s) \le E(s) + E_n(0)\chi_{[-1/n,1/n]}(s)\,, \quad \forall n \in \mathbb{N}\,, \quad \forall s \in \mathbb{R}\,, \tag{3.20}$$

and

$$|\Psi(s)| \le |\Psi_n(s)| \le |\Psi(s)| + E_n(0)\,, \quad \forall n \in \mathbb{N}\,, \quad \forall s \in \mathbb{R}\,. \tag{3.21}$$

Furthermore, inequality (3.12) implies

$$E_n(s) \le E(s) + E_n(0)\chi_{[-1/n,1/n]}(s) < \epsilon|\Psi(s)| + K_\epsilon + 1 \le \epsilon|\Psi_n(s)| + K_\epsilon + 1\,, \tag{3.22}$$

for all $n \in \mathbb{N}$ and all $s \in \mathbb{R}$.

Estimate on $\Psi_n(u_n)$ in $H_0^1(\Omega)$. Take $E_n(u_n)\Psi_n(u_n)$ as test function in (3.15) to get

$$\begin{aligned}\int_\Omega E_n^2(u_n)|\nabla u_n|^2 &+ \int_\Omega \Psi_n(u_n)\nabla u_n \cdot \nabla E_n(u_n)\\ &= \int_\Omega E_n(u_n)\Psi_n(u_n)b_n(u_n)|\nabla u_n|^2 + \int_\Omega f_n E_n(u_n)\Psi_n(u_n)\\ &\le \int_\Omega E_n(u_n)|\Psi_n(u_n)|b_n(u_n)|\nabla u_n|^2 + \int_\Omega |f_n|E_n(u_n)|\Psi_n(u_n)|\,,\end{aligned}$$

So that, as $\nabla E_n(u_n) = E_n(u_n)b_n(u_n)\nabla u_n \operatorname{sign}(u_n)$ and $\Psi_n(u_n)$ has the same sign of u_n, then a cancelation occurs and it follows from (3.22) that

$$\begin{aligned}\int_\Omega |\nabla\Psi_n(u_n)|^2 &\le \int_\Omega |f_n|\,|E_n(u_n)|\,|\Psi_n(u_n)|\\ &\le \epsilon\int_\Omega |f_n|\,|\Psi_n(u_n)|^2 + (K_\epsilon + 1)\int_\Omega |f_n|\,|\Psi_n(u_n)|\,.\end{aligned}$$

By Sobolev's inequality, since $f_n \to f$ strongly in $L^m(\Omega)$ with $m > \frac{N}{2}$, we get that the sequence $(\Psi_n(u_n))_n$ is bounded in $H_0^1(\Omega)$. Then, up to subsequences, there exists $v \in H_0^1(\Omega)$ such that $\Psi_n(u_n) \rightharpoonup v$ weakly in $H_0^1(\Omega)$. Let us define

$$u := \Psi^{-1}(v)\,,$$

which is well–defined since Ψ is strictly increasing. Thus, it has been found u such that $\Psi(u) \in H_0^1(\Omega)$ and, up to a not relabeled subsequence, it satisfies

$$\nabla\Psi_n(u_n) \rightharpoonup \nabla\Psi(u)\,, \quad \text{weakly in } L^2(\Omega;\mathbb{R}^N)\,, \tag{3.23}$$

$$u_n(x) \to u(x)\,, \qquad \text{pointwise in } \Omega\,, \tag{3.24}$$

$$\Psi_n(u_n) \to \Psi(u)\,, \qquad \text{strongly in } L^q(\Omega)\,, \quad \text{for } 1 \le q < \frac{2N}{N-2}\,. \tag{3.25}$$

Moreover, (3.22), (3.24) and (3.25) imply

$$E_n(u_n) \to E(u)\,, \qquad \text{strongly in } L^q(\Omega)\,, \quad \text{for } 1 \le q < \frac{2N}{N-2}\,. \tag{3.26}$$

Estimate on $G_\delta(u_n)$ in $H_0^1(\Omega)$ for $\delta > 0$. Here we show that far from the set $\{u_n = 0\}$ one can obtain global estimates. First we point out that, for fixed $\delta > 0$, we have

$$0 < E(s)^2 \le E_n(s)^2\,, \quad \text{for all } |s| \ge \delta \text{ and all } n \in \mathbb{N}\,.$$

It follows from

$$E(\delta)^2\int_\Omega |\nabla G_\delta(u_n)|^2 = E(\delta)^2\int_{\{|u_n|\geq\delta\}}|\nabla u_n|^2 \\ \leq \int_\Omega E_n(u_n)^2|\nabla u_n|^2 = \int_\Omega |\nabla\Psi(u_n)|^2 \leq C$$

that the sequence $(G_\delta(u_n))_n$ is bounded in $H_0^1(\Omega)$. Hence,

$$\nabla G_\delta(u_n) \rightharpoonup \nabla G_\delta(u)\,, \quad \text{weakly in } L^2(\Omega;\mathbb{R}^N)\,, \tag{3.27}$$

$$G_\delta(u_n) \to G_\delta(u)\,, \quad \text{strongly in } L^q(\Omega)\,, \quad \text{for } 1\leq q<\frac{2N}{N-2}\,. \tag{3.28}$$

L^∞**–Estimate.** Taking

$$E_n(u_n)G_k\big(\Psi_n(u_n)\big)$$

as test function in (3.15) and canceling similar terms (as in the proof of the estimate of $\Psi_n(u_n)$), it yields

$$\int_{\{|\Psi_n(u_n)|>k\}} E_n(u_n)^2|\nabla u_n|^2 \leq \int_\Omega |f_n|\,E_n(u_n)\,|G_k\big(\Psi_n(u_n)\big)|\,.$$

It follows from (3.20) and (3.21) that

$$\int_\Omega |\nabla G_k\big(\Psi_n(u_n)\big)|^2 \leq \int_\Omega |f_n|\,E_n(u_n)\,|G_k\big(\Psi_n(u_n)\big)| \tag{3.29}$$

$$\leq \int_\Omega |f_n|\,(E(u_n)+1)\,|G_k\big(\Psi_n(u_n)\big)| \leq \eta(k)\int_\Omega |f_n|\,|\Psi_n(u_n)|\,|G_k\big(\Psi_n(u_n)\big)|\,,$$

where

$$\eta(k) = \sup_{\{|\Psi(s)|>k\}}\frac{E(s)+1}{\Psi(s)}\,.$$

Note that (3.11) implies that $\eta(k)$ tends to zero as k goes to ∞. Taking into account Hölder's inequality and the definition of $G_k(s)$, it yields

$$\int_\Omega |f_n|\,|\Psi_n(u_n)|\,|G_k\big(\Psi_n(u_n)\big)| \\ = k\int_\Omega |f_n|\,|G_k\big(\Psi_n(u_n)\big)| + \int_\Omega |f_n|\,|G_k\big(\Psi_n(u_n)\big)|^2$$

$$\leq k\|f_n\|_m\Big(\int_\Omega |G_k\big(\Psi_n(u_n)\big)|^{m'}\Big)^{1/m'} + \|f_n\|_m\Big(\int_\Omega |G_k\big(\Psi_n(u_n)\big)|^{2m'}\Big)^{1/m'}\,,$$

and so inequality (3.29) becomes

$$\int_\Omega |\nabla G_k\big(\Psi_n(u_n)\big)|^2 \leq \eta(k)k\|f_n\|_m\Big(\int_\Omega |G_k\big(\Psi_n(u_n)\big)|^{m'}\Big)^{1/m'} \\ + \eta(k)\|f_n\|_m\Big(\int_\Omega |G_k\big(\Psi_n(u_n)\big)|^{2m'}\Big)^{1/m'}\,.$$

Since $f_n \to f$ strongly in $L^m(\Omega)$, we may now follow the proof of [20, Theorem 3.1] and deduce that $\|\Psi_n(u_n)\|_\infty$ is bounded by a constant that only depends on Λ, $\|f_n\|_m$, and the parameters m, N, and $|\Omega|$. Hence, $\|u_n\|_\infty$ is bounded by a constant, say M, that only depends on the same parameters:

$$\|u_n\|_\infty \le M \qquad \text{for all } n \in \mathbb{N}\,. \tag{3.30}$$

We may (and will) assume that $M > 1$. As a consequence of (3.24), we get $u \in L^\infty(\Omega)$ and $\|u\|_\infty \le M$.

Estimate on the lower order term in $L^1_{loc}(\Omega)$. We want to prove that for every $\varphi \in C_0^\infty(\Omega)$ there exists $C > 0$ such that

$$\int_\Omega b_n(u_n)|\nabla u_n|^2\varphi^2 \le C\,. \tag{3.31}$$

First of all, consider $v_n = -(e^{\gamma_n(u_n)} - 1)^-$ as test function in (3.15) where

$$\gamma_n(s) = \int_0^s b_n(\sigma)d\sigma,$$

is a well defined locally Lipschitz function such that $\gamma_n(0) = 0$. We get,

$$\begin{aligned}
&\int_{\{u_n\le 0\}} e^{\gamma_n(u_n)}b_n(u_n)|\nabla u_n|^2 \\
&\quad= \int_{\{u_n\le 0\}} b_n(u_n)|\nabla u_n|^2(e^{\gamma_n(u_n)} - 1) + \int_{\{u_n\le 0\}} f_n(e^{\gamma_n(u_n)} - 1) \\
&\quad\le \int_{\{u_n\le 0\}} e^{\gamma_n(u_n)}b_n(u_n)|\nabla u_n|^2 - \int_{\{u_n\le 0\}} b_n(u_n)|\nabla u_n|^2 + \int_\Omega |f_n|\,,
\end{aligned}$$

that implies, dropping equal terms,

$$\int_{\{u_n\le 0\}} b_n(u_n)|\nabla u_n|^2 \le \int_\Omega |f_n|. \tag{3.32}$$

We point out that this estimate allows us to deduce

$$\int_{\{u_n\le 0\}} |\nabla u_n|^2 \le C\,, \tag{3.33}$$

since $\frac{\Lambda}{M}|\nabla u_n|^2 \le b_n(u_n)|\nabla u_n|^2$ (here and later $M > 1$ will denote the constant appearing in (3.30)).

Now we deal with the set $\{u_n \ge 0\}$. For $s \in \mathbb{R}$ let us consider

$$\overline{\gamma}_n(s) = -\int_s^M b_n(\sigma)d\sigma\,.$$

We fix a function $\varphi \in C_0^\infty(\Omega)$ and define

$$v_n := \begin{cases} (e^{\overline{\gamma}_n(u_n)} - 1)\varphi^2 & \text{if } \ u_n \ge 0, \\ (e^{\overline{\gamma}_n(0)} - 1)\varphi^2 & \text{if } \ u_n < 0. \end{cases}$$

Let us take v_n as test in (3.15): we get

$$\int_{\{u_n\geq 0\}} e^{\overline{\gamma}_n(u_n)} b_n(u_n)|\nabla u_n|^2\varphi^2 + \int_{\{u_n\geq 0\}} 2\varphi\nabla u_n\cdot\nabla\varphi(e^{\overline{\gamma}_n(u_n)}-1)$$
$$+\int_{\{u_n<0\}} 2\varphi\nabla u_n\cdot\nabla\varphi(e^{\overline{\gamma}_n(0)}-1) = \int_{\{u_n\geq 0\}} b_n(u_n)|\nabla u_n|^2(e^{\overline{\gamma}_n(u_n)}-1)\varphi^2$$
$$+\int_{\{u_n<0\}} b_n(u_n)|\nabla u_n|^2(e^{\overline{\gamma}_n(0)}-1)\varphi^2 + \int_\Omega f_n v_n\,.$$

Observe that $|v_n|\leq\varphi^2$, and that

$$\int_{\{u_n<0\}} b_n(u_n)|\nabla u_n|^2(e^{\overline{\gamma}_n(0)}-1)\varphi^2\leq 0,$$

so that, we can drop the identical terms to get, using also Young's inequality,

$$\int_{\{u_n\geq 0\}} b_n(u_n)|\nabla u_n|^2\varphi^2 \leq \frac{\varepsilon}{2}\int_{\{u_n\geq 0\}}|\nabla u_n|^2\varphi^2 + \frac{1}{2\varepsilon}\int_{\{u_n\geq 0\}}|\nabla\varphi|^2 + \int_\Omega|f_n|\varphi^2\,, \tag{3.34}$$

for a fixed $\varepsilon>0$ to be chosen later.

Moreover, since b is positive, we have

$$\int_{\{u_n\geq 0\}}|\nabla u_n|^2 b_n(u_n)\varphi^2\geq\frac{\Lambda}{M}\int_{\{u_n\geq 0\}}|\nabla u_n|^2\varphi^2.$$

So that, we can choose a suitable ε in (3.34) in order to obtain

$$\int_{\{u_n\geq 0\}}|\nabla u_n|^2\varphi^2\leq C,$$

that implies, again from (3.34),

$$\int_{\{u_n\geq 0\}} b_n(u_n)|\nabla u_n|^2\varphi^2\leq C.$$

Combining this latter estimate with (3.32) we get (3.31).

As a consequence of the above estimate we get the pointwise convergence of the gradients. Indeed, applying [6, Theorem 2.1] we deduce

$$\nabla u_n\to\nabla u\,,\qquad\text{strongly in } L^q_{loc}(\Omega;\mathbb{R}^N)\quad\text{for all } q<2.$$

A diagonal argument then leads, up to a non relabeled subsequence, to

$$\nabla u_n\to\nabla u\quad\text{a.e. in }\Omega\,. \tag{3.35}$$

Thanks to (3.24) and (3.35), we may apply Fatou's Lemma in (3.31) and obtain

$$\frac{|\nabla u|^2}{|u|}\quad\text{is locally integrable on the set } \{u\neq 0\}\,. \tag{3.36}$$

An estimate on u_n in $H^1_{loc}(\Omega)$. For any fixed $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u_n|^2 b_n(u_n)\varphi^2 \geq \frac{\Lambda}{M}\int_\Omega |\nabla u_n|^2\varphi^2.$$

So that, using (3.31) we get

$$\int_\Omega |\nabla u_n|^2\varphi^2 \leq C, \tag{3.37}$$

that is u_n is a bounded sequence in $H^1_{loc}(\Omega)$. Observe that, using (3.35) and Fatou's lemma, we have

$$\int_\Omega |\nabla u|^2\varphi^2 \leq C\,. \tag{3.38}$$

Therefore, $u \in H^1_{loc}(\Omega)$. Combining this fact with (3.36) and applying Lemma 3.4, it yields

$$\sqrt{|u|} \in H^1_{loc}(\Omega) \quad \text{and} \quad \frac{|\nabla u|^2}{|u|} \in L^1_{loc}(\Omega)\,.$$

The function u is not identically zero. It remains to prove that the solution we have found is not trivially identical to zero. To show that, consider the distributional formulation of (3.15), that implies

$$\int_\Omega \nabla u_n\cdot\nabla\varphi = \int_\Omega b_n(u_n)|\nabla u_n|^2\varphi + \int_\Omega f_n\varphi,$$

for any nonnegative $\varphi \in C_0^\infty(\Omega)$. Observe that we do not have a local $L^1(\Omega)$–compactness of the lower order term. Anyway, it is nonnegative and we may drop it. Hence, using the local boundedness of u_n in $H^1_0(\Omega)$ and the strong convergence of f_n, we deduce

$$\int_\Omega \nabla u\cdot\nabla\varphi \geq \int_\Omega f\varphi.$$

Now, suppose by contradiction that $u \equiv 0$; this implies that

$$\int_\Omega f\varphi \leq 0,$$

for any nonnegative $\varphi \in C_0^\infty(\Omega)$. It is a contradiction since f is suppose to change its sign.

3.4. **Strong convergence of the gradients.** In this subsection and in those that follow, we will study some questions related to convergence of the approximate solutions, as well as some possible notions of solution which are suitable for the problem. Some comments will also be done concerning the assumptions we used.

When dealing with lower order terms having quadratic growth, the $L^1(\Omega)$–convergence of the approximating lower order terms is a consequence of strong convergence of gradients in $L^2(\Omega;\mathbb{R}^N)$. Having in mind the singularity of the gradient term (and on account of (3.27)), one should expect, at least, the convergence

$$\nabla G_\delta(u_n) \to \nabla G_\delta(u)\,, \quad \text{strongly in } L^2(\Omega;\mathbb{R}^N)\,. \tag{3.39}$$

Nevertheless, we are not able to prove this convergence. The obstruction is essentially due to the fact that the exponential auxiliary function $e^{h(u)}$ is unbounded for $u<0$ near zero. In fact, we can prove the following partial convergence result

$$\nabla G_\delta(u_n^+) \to \nabla G_\delta(u^+)\,, \quad \text{strongly in } L^2_{loc}(\Omega;\mathbb{R}^N)\,, \tag{3.40}$$

while, due to this asymmetry in the Cole–Hopf transform, we are not able to deal with the negative parts.

Let us show that (3.40) holds. To better analyze the terms that will appear in the following computations, we will denote by $\omega(n)$ any quantity tending to 0 as n goes to ∞.

First of all, we take

$$E_n(u_n)\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+$$

as test function in (3.15), obtaining

$$\begin{aligned}\int_{\{u_n>\delta\}} & E_n(u_n) b_n(u_n)\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+ |\nabla u_n|^2 \\ &+ \int_\Omega E_n(u_n)\nabla u_n\cdot\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+ \\ &= \int_\Omega b_n(u_n)|\,\nabla u_n|^2 E_n(u_n)\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+ \\ &\qquad + \int_\Omega f_n\, E_n(u_n)\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+\,.\end{aligned} \tag{3.41}$$

Observe that the first terms on both sides may be canceled each other. On the other hand, the last term on the right hand side of (3.41) tends to 0 since $E_n(u_n)$ is bounded (a bound is $E(M)$, M being that of estimate (3.30)), $f_n \to f$ strongly in $L^m(\Omega)$ and $G_\delta(u_n)\to G_\delta(u)$ strongly in $L^{m'}(\Omega)$ (due to (3.28) since $m' < \frac{N}{N-2}$). Hence, (3.41) becomes

$$\int_\Omega E_n(u_n)\nabla u_n\cdot\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+ \le \omega(n)\,.$$

The left hand side can be split as

$$\begin{aligned}\int_{\{u_n>\delta\}} & E_n(u_n)\nabla u_n\cdot\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+ \\ &+ \int_{\{u_n\le\delta\}} E_n(u_n)\nabla u_n\cdot\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+\end{aligned}$$

and the second term is zero, because of the fact that $\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+$ vanishes on the set $\{u_n \le \delta\}$. So we get

$$\int_\Omega E_n(u_n)\nabla G_\delta(u_n^+)\cdot\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+ \le \omega(n)\,.$$

Since (3.27) implies

$$\int_\Omega E_n(u_n)\nabla G_\delta(u^+)\cdot\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+ = \omega(n)\,,$$

we conclude that

$$\int_\Omega E_n(u_n)\big|\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+\big|^2 \le \omega(n)\,.$$

Noting that we are integrating on $\{u_n > \delta\}$, we have that $E_n(u_n) > E_n(\delta) > E(\delta)$ and then the positiveness of the integrand implies that

$$\int_\Omega \big|\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^+\big|^2 \le \omega(n)\,. \tag{3.42}$$

Now consider $\varphi \in C_0^\infty(\Omega)$, with $\varphi \ge 0$, and take

$$-\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^-\varphi$$

as test function in (3.15). It yields

$$\begin{aligned}&-\int_\Omega \varphi\nabla u_n\cdot\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^- - \int_\Omega \big(G_\delta(u_n^+) - G_\delta(u^+)\big)^-\nabla u_n\cdot\nabla\varphi\\ &= -\int_\Omega b_n(u_n)|\nabla u_n|^2\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^-\varphi - \int_\Omega f_n\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^-\varphi\,.\end{aligned}$$

Since on the right hand side, the gradient term is nonpositive and the other tends to 0 as $n \to \infty$, we get

$$-\int_\Omega \varphi\nabla u_n\cdot\nabla\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^- - \int_\Omega \big(G_\delta(u_n^+)-G_\delta(u^+)\big)^-\nabla u_n\cdot\nabla\varphi \le \omega(n)\,.$$

Observe also that, by (3.37), the sequence

$$\Big(\nabla u_n\cdot\nabla\varphi\Big)_n$$

is bounded in $L^2(\Omega)$ and, by (3.28), $\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^-$ tends to 0 strongly in $L^2(\Omega)$. Therefore, we get

$$-\int_\Omega \varphi\nabla u_n\cdot\nabla\big(G_\delta(u_n^+) - G_\delta(u^+)\big)^- \le \omega(n)\,.$$

Moreover,

$$-\int_{\{u_n\le\delta\}} \varphi\nabla u_n\cdot\nabla\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^- = \int_{\{u_n\le\delta\}} \varphi\nabla u_n\cdot\nabla G_\delta(u^+) = \omega(n)\,,$$

so that, from the last two estimates, it follows

$$
\begin{aligned}
(3.43)\quad -\int_\Omega \varphi \nabla G_\delta(u_n^+)\cdot \nabla\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^- \\
= -\int_{\{u_n>\delta\}} \varphi \nabla u_n \cdot \nabla\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^- \le \omega(n)\,.
\end{aligned}
$$

Using again (3.27), it yields

$$
-\int_\Omega \varphi \nabla G_\delta(u^+)\cdot \nabla\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^- = \omega(n)\,.
$$

Adding this estimate to (3.43), we obtain

$$
\int_\Omega \varphi \big|\nabla\big(G_\delta(u_n^+)-G_\delta(u^+)\big)^-\big|^2 \le \omega(n)\,.
$$

Combining it with (3.42), we have

$$
\int_\Omega \varphi \big|\nabla\big(G_\delta(u_n^+)-G_\delta(u^+)\big)\big|^2 \ \le \omega(n)\,,
$$

for every nonnegative $\varphi \in C_0^\infty(\Omega)$. Therefore, we deduce that

$$
(3.44)\quad \nabla G_\delta(u_n^+) \to \nabla G_\delta(u^+) \quad \text{strongly in } L^2(U;\mathbb{R}^N) \quad \text{for all } U \subset\subset \Omega\,.
$$

3.5. **Possible concepts of solution.** Even if we could prove the strong convergence of the gradients (3.39), this fact would not imply that the limit function u be a distributional solution to the original problem (3.6).

Next example shows how, in some particular cases of strongly singular terms, one can pass to the limit in a *Cole–Hopf type formulation.*

Example 3.5. Consider problem

$$
(3.45)\qquad \begin{cases} -\Delta u = b(u)|\nabla u|^2 + f(x)\,, & \text{in } \ \Omega\,;\\ u=0\,, & \text{on } \ \partial\Omega\,, \end{cases}
$$

where

$$
b(s) = \begin{cases} \frac{\Lambda}{s} & \text{if} \quad s>0\\ 0 & \text{if} \quad s\le 0. \end{cases}
$$

We can reproduce the same computations of the previous section. Only, notice that in this case the auxiliary function $E_n(s)$ does not blows up near 0. We obtain the same compactness results as before and we are allowed to write down the equations solved by the Cole–Hopf transforms $v_n = \Psi_n(u_n)$. Thanks to what we proved before we can now pass to the limit in this equation in order to check that u satisfies

$$
-\Delta \Psi(u) = E(u) f(x), \quad \text{in } \ \mathcal{D}'(\Omega).
$$

Of course, due to the strong singularity of b we are not allowed to undo the change in order to prove that u is a distributional solution of (3.45).

In order to introduce a more natural and direct formulation for solutions to problem (3.6), we have to get rid of the singular set $\{u = 0\}$, we propose the following renormalized formulation.

Definition 3.6. *We say that u is a renormalized solution to problem (3.6) if $u \in H^1_{loc}(\Omega) \cap L^\infty(\Omega)$ satisfy $\sqrt{|u|} \in H^1_{loc}(\Omega)$, $\Psi(u) \in H^1_0(\Omega)$, and for every $S \in W^{1,\infty}(\mathbb{R})$ such that $S(0) = 0$ and every $\varphi \in C^\infty_0(\Omega)$*

$$\int_\Omega \varphi S'(u)|\nabla u|^2 + \int_\Omega S(u)\nabla u \cdot \nabla\varphi = \Lambda \int_\Omega \frac{|\nabla u|^2}{|u|}\varphi S(u) + \int_\Omega f\varphi S(u)\,. \tag{3.46}$$

Remark 3.7. Observe that $u \in H^1_{loc}(\Omega) \cap L^\infty(\Omega)$ implies $S(u) \in H^1_{loc}(\Omega) \cap L^\infty(\Omega)$ for every $S \in W^{1,\infty}(\mathbb{R})$ such that $S(0) = 0$. Thus, since $\nabla u \in L^1_{loc}(\Omega;\mathbb{R}^N)$ and $\frac{|\nabla u|^2}{|u|} \in L^1_{loc}(\Omega)$, every term in (3.46) has sense.

Remark 3.8. We stress the fact that, thanks to (3.40) we recover the previous known notions of solution for nonnegative data f (see [14, 3]). In fact, in this case the approximating solutions u_n are nonnegative and we can choose $S_\delta \in W^{1,\infty}(\mathbb{R})$ such that $S_\delta(s) = 0$ in $[-\delta, \delta]$ in the approximate formulation. Using (3.40), we then have, passing to the limit in n,

$$\int_\Omega \varphi S'_\delta(u)|\nabla u|^2 + \int_\Omega S_\delta(u)\nabla u \cdot \nabla\varphi = \Lambda \int_\Omega \frac{|\nabla u|^2}{|u|}\varphi S_\delta(u) + \int_\Omega f\varphi S_\delta(u)\,,$$

for any $\varphi \in C^\infty_0(\Omega)$. Since u is bounded by M, we can choose $S_\delta(s) = T_{M-\delta}(G_\delta(s))$. Thus, recalling that $\frac{|\nabla u|^2}{|u|} \in L^1_{loc}(\Omega)$, we can pass to the limit as δ goes to zero using dominate convergence theorem in order to get

$$\int_\Omega \nabla u \cdot \nabla\varphi = \Lambda \int_\Omega \frac{|\nabla u|^2}{|u|}\varphi + \int_{\{u>0\}} f\varphi\,,$$

that is u is a nontrivial distributional solution to the original problem as $u > 0$ a.e. on Ω (see for instance [3]) . Similar arguments will apply for the problems considered in the following sections.

Remark 3.9. We explicitly stress out that the constant function $u \equiv 0$ satisfies (3.46). We also point out that the boundary condition in Definition 3.6 holds through the condition $\Psi(u) \in H^1_0(\Omega)$. In Section 3.7 we will see that this condition is, in a certain sense, optimal by comparing it with the case of positive data (see for instance [3]) where the two definitions of solutions turn out to coincide.

We would like to stress an interesting property enjoyed by renormalized solutions to singular problems. It deals with a sort of *Kato's Identity* that is known to fail in the nonsingular case.

Proposition 3.10. *The following statements are equivalent:*

(1) *If u is a renormalized solution for problem (3.6)*

(2) The following equalities hold in the sense of distributions.

$$-\Delta u^+ = \Lambda\frac{|\nabla u^+|^2}{u^+} + f(x)\chi_{\{u>0\}}(x) \tag{3.47}$$

$$\Delta u^- = \Lambda\frac{|\nabla u^-|^2}{u^-} + f(x)\chi_{\{u<0\}}(x) \tag{3.48}$$

(3) The equation

$$-\Delta u = \Lambda\frac{|\nabla u|^2}{|u|} + f(x)\chi_{\{u\neq 0\}}(x) \tag{3.49}$$

holds in the sense of distributions.

Proof. (1) $\Longrightarrow$ (2) Assuming that (1) holds, we will only prove (3.47), the proof of (3.48) is similar. Given $\delta > 0$, we consider $S(t) = \frac{1}{\delta}T_\delta(t^+)$ and $\varphi \in C_0^\infty(\Omega)$. Applying (3.46), we get

$$\begin{aligned}\frac{1}{\delta}\int_{\{0<u<\delta\}}\varphi\,|\nabla u|^2 + \frac{1}{\delta}\int_\Omega T_\delta(u^+)\nabla u\cdot\nabla\varphi \\ = \frac{\Lambda}{\delta}\int_\Omega \frac{|\nabla u|^2}{|u|}T_\delta(u^+)\varphi + \frac{1}{\delta}\int_\Omega fT_\delta(u^+)\varphi\,.\end{aligned} \tag{3.50}$$

We will see that the first term in the left hand side tends to 0 as $\delta \to 0$. Indeed,

$$\begin{aligned}\Big|\frac{1}{\delta}\int_{\{0<u<\delta\}}\varphi\,|\nabla u|^2\Big| &\le \frac{\nu}{\delta}\|\varphi\|_\infty\int_{\{0<u<\delta\}\cap\,\mathrm{supp}\,(\varphi)}|\nabla u|^2 \\ &\le \nu\|\varphi\|_\infty\int_{\{0<u<\delta\}\cap\,\mathrm{supp}\,(\varphi)}\frac{|\nabla u|^2}{|u|} \\ &\le 4\nu\|\varphi\|_\infty\int_{\{0<u<\delta\}\cap\,\mathrm{supp}\,(\varphi)}|\nabla\sqrt{|u|}\,|^2 \to 0\,.\end{aligned}$$

On the other hand, we may let δ go to 0 in the remainders terms applying

$$\lim_{\delta\to 0}\frac{1}{\delta}T_\delta(t^+) = \chi_{\{t>0\}}(t)\,.$$

Having in mind $|\nabla u| \in L^1_{loc}(\Omega)$ and Stampacchia's Theorem, we obtain

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_\Omega T_\delta(u^+)\nabla u\cdot\nabla\varphi = \int_{\{u>0\}}\nabla u\cdot\nabla\varphi = \int_\Omega \nabla u^+\cdot\nabla\varphi\,.$$

Moreover, $\frac{|\nabla u|^2}{|u|} \in L^1_{loc}(\Omega)$ and Stampacchia's Theorem imply

$$\lim_{\delta\to 0}\frac{\Lambda}{\delta}\int_\Omega\frac{|\nabla u|^2}{|u|}T_\delta(u^+)\varphi = \Lambda\int_{\{u>0\}}\frac{|\nabla u|^2}{|u|}\varphi = \Lambda\int_\Omega\frac{|\nabla u^+|^2}{u^+}\varphi\,.$$

Finally,

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_\Omega fT_\delta(u^+)\varphi = \int_{\{u>0\}}f\varphi\,.$$

As a consequence, equation (3.50) becomes (3.47) as desired.

(2) $\Longrightarrow$ (3) It is straightforward.

(3) $\Longrightarrow$ (1) Consider $S \in W^{1,\infty}(\mathbb{R})$ such that $S(0) = 0$ and $\varphi \in C_0^\infty(\Omega)$. Then $S(u)\varphi \in H_0^1(\Omega)$ and there exists a sequence $(\psi_n)_n$ in $C_0^\infty(\Omega)$ satisfying $\psi_n \to S(u)\varphi$ in $H_0^1(\Omega)$. Thus, (3.49) implies

$$\int_\Omega \nabla u \cdot \nabla \psi_n = \Lambda \int_\Omega \frac{|\nabla u|^2}{|u|} \psi_n + \int_{\{u \neq 0\}} f \psi_n \,.$$

Letting n go to ∞, we get

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla (S(u)\varphi) &= \Lambda \int_\Omega \frac{|\nabla u|^2}{|u|} \varphi S(u) + \int_{\{u \neq 0\}} f \varphi S(u) \\ &= \Lambda \int_\Omega \frac{|\nabla u|^2}{|u|} \varphi S(u) + \int_\Omega f \varphi S(u) \,, \end{aligned}$$

from where (3.46) follows. □

Remark 3.11. Under the same hypotheses of Proposition 3.10, it is straightforward that if $f\chi_{\{u=0\}} \equiv 0$, then equation

$$-\Delta u = \Lambda \frac{|\nabla u|^2}{|u|} + f(x)$$

holds in the sense of distributions. We also remark that, with simple modifications in the proof the same statement of Proposition 3.10 holds true for virtually any problem that exhibits such a type of singularities (e.g. problems (2.3) and (2.5)).

3.6. **Comments on the assumptions and Generalizations.** For the sake of simplicity we have presented the results of this section in a model case. Anyway, observe that, a straightforward modification of our arguments shows that the same results also apply to a more general framework. In particular one can deal with the following problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) = b(x,u,\nabla u) + f(x) \,, & \text{in } \Omega \,; \\ u = 0 \,, & \text{on } \partial\Omega \,. \end{cases}$$

Here, the function

$$a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

satisfies the Carathéodory conditions (i.e. $a(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$ and $a(\cdot,s,\xi)$ is measurable for any $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$) and there exist some constants $\alpha > 0$ and $\nu > 0$ such that

$$a(x,s,\xi) \cdot \xi \geq \alpha |\xi|^2,$$
$$|a(x,s,\xi)| \leq \nu |\xi| \,;$$

for all $\xi \in \mathbb{R}^N$, for all $s \in \mathbb{R}$ and for almost all $x \in \Omega$.

The function

$$b(x,s,\xi) : \Omega \times \mathbb{R} \backslash \{0\} \times \mathbb{R}^N \to \mathbb{R}$$

also satisfies the Carathéodory conditions and there exist positive continuous functions $g_i : \mathbb{R}\backslash\{0\} \to (0, +\infty)$ $(i = 1, 2)$ such that

$$g_1(s)|\xi|^2 \le b(x, s, \xi) \le g_2(s)|\xi|^2;$$

for all $\xi \in \mathbb{R}^N$, for all $s \in \mathbb{R}\backslash\{0\}$ and for almost all $x \in \Omega$. Moreover, there exist constants $\Lambda_i, s_0 > 0$ such that

$$g_i(s) = \frac{\Lambda_i}{|s|}\,, \quad \text{for all } 0 < |s| \le s_0 \ (i = 1, 2), \tag{3.51}$$

and

$$\lim_{|s|\to+\infty} g_2(s) = 0\,.$$

Remark 3.12. Actually, to obtain the a priori estimates and convergences proved above, we may assume more general hypotheses as in [20, Condition (C1)]. In particular, those a priori estimates hold when $g_2 = g_2^1 + g_2^2$, with $g_2^1 \in L^1(\mathbb{R})$ and $\lim_{|s|\to+\infty} g_2^2(s) = 0$.

Remark 3.13. We explicitly observe that, without loss of generality, we can choose g_2 to be an even function. Indeed, it is not difficult to define a continuous $\overline{g}_2 : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ satisfying the same hypotheses of g_2 and moreover

- $\overline{g}_2(s) \ge g_2(s)$ for all $s \in \mathbb{R}$.
- $\overline{g}_2$ is an even function.

Remark 3.14. The auxiliary functions appearing in (3.8), (3.9) and (3.10) must now be defined by

$$h(s) := \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_1^s g_2(\sigma)d\sigma\,, & \text{if } s > 0\,; \\ -\dfrac{1}{\alpha}\displaystyle\int_{-1}^s g_2(\sigma)d\sigma\,, & \text{if } s < 0\,; \end{cases}$$

$$E(s) := \begin{cases} e^{h(s)}\,, & \text{if } s \ne 0\,; \\ 0\,, & \text{if } s = 0\,; \end{cases}$$

and

$$\Psi(s) := \int_0^s E(\sigma)\,d\sigma\,;$$

which satisfy the key limit

$$\lim_{|s|\to\infty} \frac{E(s)}{\Psi(s)} = 0\,.$$

3.7. **Higher regularity of the solution.** It is easy to see that our solution obtained as limit of approximations, satisfies the further estimate $\Phi(u) \in H_0^1(\Omega)$, where

$$\Phi(s) := \frac{2\sqrt{\Lambda}}{\Lambda\beta+1}|s|^{(\Lambda\beta-1)/2}s = \int_0^s E(\sigma)^{\beta/2}\sqrt{\frac{\Lambda}{|\sigma|}}\,d\sigma\,, \qquad \beta>1\,. \tag{3.52}$$

To prove it, we consider the function given by

$$\Phi_n(s) = \int_0^s E_n(\sigma)^{\beta/2}\sqrt{b_n(\sigma)}\,d\sigma\,. \tag{3.53}$$

and take $E_n(u_n)^\beta \mathrm{sign}\,(u_n)$, with $1<\beta<2$, as test function in (3.15). It yields

$$\beta\int_\Omega E_n(u_n)^\beta b_n(u_n)|\nabla u_n|^2 \le \int_\Omega E_n(u_n)^\beta b_n(u_n)|\nabla u_n|^2 + \int_\Omega |f_n|E_n(u_n)^\beta\,.$$

By (3.11) and (3.26) we deduce that

$$(\beta-1)\int_\Omega |\nabla\Phi_n(u_n)|^2 = (\beta-1)\int_\Omega E_n(u_n)^\beta b_n(u_n)|\nabla u_n|^2 \le C\,, \quad \forall n\in\mathbb{N}\,.$$

This fact and (3.24) imply that

$$\Phi_n(u_n) \rightharpoonup \Phi(u)\,, \quad \text{weakly in } H_0^1(\Omega)\,,$$

so that $\Phi(u)\in H_0^1(\Omega)$.

Remark 3.15. The regularity obtained above implies that

$$|u|^\alpha \in H_0^1(\Omega)\,, \quad \text{for all } \alpha > \frac{\Lambda+1}{2}\,.$$

It is the same regularity proved in [3, Theorem 3.8] for nonnegative data.

Remark 3.16. Here we want to consider the case where the lower order term has a stronger singular behaviour near $\{u=0\}$, i.e. the case where condtion (3.51) is replaced by

$$g_i(s) = \frac{\Lambda_i}{|s|^\gamma}\,, \quad \text{for all } 0<|s|\le s_0 \ (i=1,2), \quad 1<\gamma. \tag{3.54}$$

Therefore the model problem we refer to is the following

$$\begin{cases} -\Delta u = \Lambda\frac{|\nabla u|^2}{|u|^\gamma} + f(x)\,, & \text{in } \Omega\,;\\ u=0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.55}$$

The case of nonnegative data is treated in [14]. We can define the functions $h(s)$, $E(s)$ and $\Psi(s)$ as in (3.8), (3.9), (3.10) and their corresponding approximating versions $h_n(s)$, $E_n(s)$ and $\Psi_n(s)$ as before.

Let us point out that all the qualitative properties of these functions still hold.

In particular $\lim_{s\to 0^+} h(s) = -\infty = \lim_{s\to 0^-} h(s)$, which implies that $E(s)$ and $\Psi(s)$ are bounded function with $E(0)=\Psi(0)=0$.

This also implies that, with the new functions $h_n(s)$, $E_n(s)$ and $\Psi_n(s)$, we can construct the same kind of test functions in order to get the same estimates we obtained in the case $\gamma = 1$ on the sequence u_n of the approximating solutions.

In the model problem (3.55) $h(s)$ has the form

$$h(s) = \frac{1}{(1-\gamma)}\Big(\frac{1}{|s|^{\gamma-1}} - 1\Big)$$

and the functions $E(s)$ and $\Psi(s)$ are defined consequently.

4. A lower order term satisfying a sign condition on the right hand side

Here we deal with the case of a nonlinear term satisfying a sign condition, that is, the case (2.3). We will take changing sign data such that (3.7) holds.

4.1. **Guide Example.** We already notice that this case is completely different from the previous one and not even $H^1_{loc}(\Omega)$–estimates can be obtained. Indeed, we will show that, in general, neither the solution belongs to $H^1_{loc}(\Omega)$, nor the lower order term belongs to $L^1_{loc}(\Omega)$, when we consider the model problem

$$\begin{cases} -\Delta u = \frac{|\nabla u|^2}{u} + f(x)\,, & \text{in } \ \Omega\,; \\ u = 0\,, & \text{on } \ \partial\Omega\,. \end{cases}$$

In order to avoid unnecessary details and get explicit solutions, we will consider a one dimensional example; in the Appendix A we will point out how our arguments can be adapted to higher dimensions.

Example 4.1. Set $\Omega =]-\pi,\pi[$ and $f(x) = \frac{1}{\sqrt{2}}|\sin x|^{-\frac{1}{2}}\sin x$. A solution to

$$\begin{cases} -u'' = \frac{|u'|^2}{u} + f(x)\,, & \text{in } \ \Omega\,; \\ u(\pi) = u(-\pi) = 0\,, \end{cases} \tag{4.56}$$

is given by $u(x) = \sqrt{2}|\sin x|^{-\frac{1}{2}}\sin x$. Taking the auxiliary functions $E(s) = |s|$ and $\Psi(s) = \frac{1}{2}|s|s$, we may write $f(x) = \frac{\sin x}{E\big(\Psi^{-1}(\sin x)\big)}$ and $u(x) = \Psi^{-1}(\sin x)$ and so it is straightforward that $\Psi(u) \in H^1_0(\Omega)$. We remark that $|u'(x)|^2 = \frac{\cos^2 x}{2|\sin x|}$, so that $u \notin H^1_{loc}(\Omega)$ and, as a consequence, $\dfrac{|u'|^2}{u} \notin L^1_{loc}(\Omega)$. Hence, distributional solutions have no sense.

4.2. **Definition of solution and statement of the main result.** Having in mind the above example, everything we can expect is condensed in the following definition.

Definition 4.2. *We say that a nonzero function $u \in L^\infty(\Omega)$ is a strong renormalized solution to problem* (2.3) *if it satisfies $G_\delta(u) \in H^1_0(\Omega)$ for all $\delta > 0$, $|u|^\Lambda u \in H^1_0(\Omega)$ and the following condition holds: For every $S \in$*

$W^{1,\infty}(\mathbb{R})$ such that there exists $\delta > 0$ satisfying $S(s) = 0$ for all $s \in [-\delta, \delta]$, and every $\varphi \in C_0^\infty(\Omega)$

(4.57)
$$\int_\Omega \varphi S'(u)|\nabla u|^2 + \int_\Omega S(u)\nabla u \cdot \nabla \varphi \;=\; \Lambda \int_\Omega \frac{|\nabla u|^2}{u}\varphi S(u) + \int_\Omega f\varphi S(u)\,.$$

The result we are able to prove is the following.

Theorem 4.1. *There exists a function $u \in L^\infty(\Omega)$ satisfying $G_\delta(u) \in H_0^1(\Omega)$ for all $\delta > 0$ and $|u|^\Lambda u \in H_0^1(\Omega)$, which is a solution to (2.3) obtained as a limit of approximations.*

Moreover, the approximate solutions u_n converge to u, up to subsequences, in the following senses:

$$G_\delta(u_n) \to G_\delta(u) \quad \text{strongly in } L^q(\Omega)\,, \quad q = \frac{2N}{N-2}\,;$$
$$u_n(x) \to u(x) \quad \text{pointwise a.e. in } \Omega\,;$$
$$\nabla G_\delta(u_n) \to \nabla G_\delta(u) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^N)\,;$$
$$\nabla u_n(x) \to \nabla u(x) \quad \text{pointwise a.e. in } \{u \neq 0\}\,;$$
$$|b_n(u_n)|\,|\nabla u_n|^2 \chi_{\{|u_n|>\delta\}} \to \Lambda \frac{|\nabla u|^2}{|u|}\chi_{\{|u|>\delta\}}\,, \quad \text{strongly in } L^1(\Omega)\,.$$

Remark 4.3. As a consequence of the above result, we deduce that (4.57) holds for every $S \in W^{1,\infty}(\mathbb{R})$ such that there exists $\delta > 0$ satisfying $S(s) = 0$ for all $s \in [-\delta, \delta]$, and every $\varphi \in C_0^\infty(\Omega)$. Hence, the function u found in Theorem 4.1 would be a strong renormalized solution to problem (2.3) if we prove that does not vanish identically. In subsection 4.4 we will show that, in the concrete case of Example 4.1, this approximating procedure does not degenerate towards the trivial solution.

In order to deal with problem (2.3), we will consider the auxiliary functions defined in subsection 3.2.

4.3. **A priori estimates.** The approximating problems we will consider are

(4.58)
$$\begin{cases} -\Delta u_n = b_n(u_n)|\nabla u_n|^2 + f_n(x)\,, & \text{in } \Omega\,; \\ u_n = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

where $b_n(s) = \frac{\Lambda}{s}$ for $|s| \geq \frac{1}{n}$ and it is an odd linear function for $|s| \leq \frac{1}{n}$. On the other hand, f_n denotes a sequence that converges to f strongly in $L^m(\Omega)$. Appealing again to [20, Theorem 1.1 (i)] we found a weak solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to problem (4.58).

As in the preceding section, we will define auxiliary functions related to problems (4.58), they are given by

$$h_n(s) = \begin{cases} \log |s|^\Lambda \,, & \text{if } \ |s| \ge \frac{1}{n}\,; \\ \log\Big(\dfrac{1}{n^\Lambda}\Big) + \displaystyle\int_{-\frac{1}{n}}^s b_n(\sigma)\,d\sigma\,, & \text{if } \ |s| \le \frac{1}{n}\,; \end{cases}$$

$$E_n(s) = e^{h_n(s)}\,,$$

and

$$\Psi_n(s) = \int_0^s E_n(\sigma)\,d\sigma\,.$$

Estimates involving $\Psi_n(u_n)$. First of all, taking $E_n(u_n)G_k(\Psi_n(u_n))$ as test function in (4.58), we may follow the same argument of the previous section and obtain an $L^\infty(\Omega)$–estimate, so that (3.30) holds.

We next take $E_n(u_n)\Psi_n(u_n)$ as test function in (4.58) and, after canceling terms, we get

$$\int_\Omega |\nabla u_n|^2 E_n(u_n)^2 \le \int_\Omega |f_n|\,E_n(u_n)\,|\Psi_n(u_n)|\,.$$

Arguing as in the above section, Sobolev's inequality implies

$$\int_\Omega |\nabla \Psi_n(u_n)|^2 = \int_\Omega |\nabla u_n|^2 E_n(u_n)^2 \le C\,; \tag{4.59}$$

thus, we found a function u satisfying (3.23), (3.24) and (3.25); as a consequence, also (3.26) holds. Moreover, it follows from the $L^\infty(\Omega)$–estimate that $u \in L^\infty(\Omega)$.

We may also improve the convergence (3.23) deducing that

$$\nabla\Psi_n(u_n) \to \nabla\Psi(u)\,, \quad \text{strongly in } L^2(\Omega;\mathbb{R}^N)\,. \tag{4.60}$$

To see it, we take $E_n(u_n)(\Psi_n(u_n)-\Psi(u))$ as test function and cancel similar terms to get

$$\int_\Omega E_n(u_n)\nabla u_n\cdot\nabla(\Psi_n(u_n)-\Psi(u)) \le \int_\Omega |f_n|\,E_n(u_n)\,|\Psi_n(u_n)-\Psi(u)|\,. \tag{4.61}$$

Now, since f_n is bounded in $L^m(\Omega)$, $E_n(u_n)$ is bounded in $L^q(\Omega)$ for $1 \le q < \frac{2N}{N-2}$ (by (3.26)) and $\Psi_n(u_n) - \Psi(u) \to 0$ strongly in $L^q(\Omega)$ for $1 \le q < \frac{2N}{N-2}$ (by (3.25)), it yields that the right hand side of (4.61) goes to 0. Therefore,

$$\begin{aligned} &\limsup_{n\to\infty} \int_\Omega \nabla\Psi_n(u_n)\cdot\nabla(\Psi_n(u_n)-\Psi(u)) \\ &\qquad = \limsup_{n\to\infty}\int_\Omega E_n(u_n)\nabla u_n\cdot\nabla(\Psi_n(u_n)-\Psi(u)) \le 0\,. \end{aligned} \tag{4.62}$$

On the other hand, (3.23) implies

$$\lim_{n\to\infty}\int_\Omega \nabla\Psi(u)\cdot\nabla(\Psi_n(u_n)-\Psi(u)) = 0\,.$$

From here and (4.62), we get

$$\limsup_{n\to\infty}\int_\Omega |\nabla(\Psi_n(u_n)-\Psi(u))|^2 \le 0\,,$$

and so (4.60) is proved. From this result we deduce that, up to subsequences,

$$\nabla u_n \to \nabla u\,, \quad \text{pointwise a.e. on } \{u\neq 0\}\,. \tag{4.63}$$

Moreover, by it and (3.24), we also have

$$b_n(u_n)|\nabla u_n|^2 \to \Lambda\frac{|\nabla u|^2}{|u|}\,, \quad \text{pointwise a.e. on } \{u\neq 0\}\,. \tag{4.64}$$

Estimates away from the singularity. The above estimates imply estimates on $\nabla G_\delta(u_n)$ and $b_n(u_n)|\nabla u_n|^2\chi_{\{|u_n|>\delta\}}$. Indeed, it follows from

$$|\nabla G_\delta(u_n)| = |\nabla u_n|\chi_{\{|u_n|\ge\delta\}} \le |\nabla u_n|\frac{E_n(u_n)}{E_n(\delta)} = C_\delta|\nabla\Psi_n(u_n)| \tag{4.65}$$

and (4.59), that

$$\int_\Omega |\nabla G_\delta(u_n)|^2 \le C_\delta\,.$$

Hence, due to (3.24), both (3.27) and (3.28) hold.

Remark 4.4. In the particular case $0<\Lambda<1$ (in the counterexample $\Lambda=1$), we can improve this estimate. For instance, multiplying by u_n and using the fact that $b_n(s)s<1$ we get

$$\int_\Omega |\nabla u_n|^2 \le C\,.$$

Going back to our estimates, (4.65), (4.60), (4.63) and Vitali's Theorem imply

$$\nabla G_\delta(u_n) \to \nabla G_\delta(u)\,, \quad \text{strongly in } L^2(\Omega;\mathbb{R}^N)\,. \tag{4.66}$$

As far as the gradient term is concerned, one deduces from

$$|b_n(u_n)|\,|\nabla u_n|^2\chi_{\{|u_n|>\delta\}} \le \frac{\Lambda}{|u_n|}\,|\nabla u_n|^2\chi_{\{|u_n|>\delta\}} \le \frac{\Lambda}{\delta}\,|\nabla G_\delta(u_n)|^2\,,$$

(4.66), (4.63) and again Vitali's Theorem that

$$|b_n(u_n)|\,|\nabla u_n|^2\chi_{\{|u_n|>\delta\}} \to \Lambda\frac{|\nabla u|^2}{|u|}\chi_{\{|u|>\delta\}}\,, \quad \text{strongly in } L^1(\Omega)\,. \tag{4.67}$$

4.4. **Nondegeneracy of the approximation procedure.** Here we show that our approximation procedure does not degenerate, in Example 4.1, towards the trivial renormalized solution $u \equiv 0$.

EXAMPLE 4.5. Let us consider problem (4.56) in Example 4.1, of which we have found the solution $u(x) = \sqrt{2}|\sin x|^{-\frac{1}{2}} \sin x$. We can explicit the approximating procedure in order to deduce that, in this case, this solution u is a solution obtained as limit of approximations; therefore, this approximating procedure leads to a nontrivial function. To this end, we introduce the approximate auxiliary functions

$$b_n(s) = \begin{cases} n^2 s\,, & \text{if } |s| < \frac{1}{n}\,; \\ \dfrac{1}{s}\,, & \text{if } |s| \geq \frac{1}{n}\,; \end{cases}$$

and the corresponding

$$E_n(s) = \begin{cases} \dfrac{1}{n} e^{\frac{1}{2}(n^2 s^2 - 1)}\,, & \text{if } |s| < \frac{1}{n}\,; \\ |s|\,, & \text{if } |s| \geq \frac{1}{n}\,; \end{cases}$$

and $\Psi_n(s) = \displaystyle\int_0^s E_n(\sigma)\, d\sigma$. Next consider the following approximating problems

$$\begin{cases} -u_n'' = b_n(u_n)|u_n'|^2 + f_n(x)\,, & \text{in } \Omega\,; \\ u_n(\pi) = u_n(-\pi) = 0\,, \end{cases} \tag{4.68}$$

where $f_n(x) = \frac{\sin x}{E_n\left(\Psi_n^{-1}(\sin x)\right)}$. We point out that a solution to (4.68) is given by $u_n = \Psi_n^{-1}(\sin x) \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Next, we will see that

$$f_n \to f\,, \quad \text{strongly in any } L^m(\Omega)\,, \quad \text{for all } m < \infty\,. \tag{4.69}$$

Indeed, observe that f_n can be split as

$$f_n(x) = \frac{\sin x}{E_n\left(\Psi_n^{-1}(\sin x)\right)} \chi_{\{|\sin x| \geq \Psi_n(1/n)\}}(x) + \frac{\sin x}{E_n\left(\Psi_n^{-1}(\sin x)\right)} \chi_{\{|\sin x)| < \Psi_n(1/n)\}}(x)\,; \tag{4.70}$$

so that we will analyze each term separately. The first term can be written as

$$\begin{aligned} &\frac{\sin x}{E_n\left(\Psi_n^{-1}(\sin x)\right)} \chi_{\{|\sin x| \geq \Psi_n(1/n)\}}(x) \\ &= \frac{\sin x}{E\left(\Psi_n^{-1}(\sin x)\right)} \chi_{\{|\sin x| \geq \Psi_n(1/n)\}}(x) = \frac{\sin x}{E(u_n(x))} \chi_{\{|u_n(x)| \geq 1/n\}}(x)\,. \end{aligned} \tag{4.71}$$

This sequence of functions converges almost everywhere to f. Since the sequence $(E_n)_n$ is decreasing, so is the sequence $(|\Psi_n|)_n$ and it follows that $(|u_n|)_n$ is increasing. Thus,

$$\frac{|\sin x|}{E(u_n(x))} \le \frac{|\sin x|}{E(u_1(x))}\,, \qquad \forall n \in \mathbb{N}\,.$$

On the other hand, if $|s| \le \Psi_1(1)$, then $E_1(s) \le A := E_1(\Psi_1(1))$ and consequently we obtain $|\Psi_1(s)| \le A|s|$. It follows from $|s| \le A|\Psi_1^{-1}(s)|$ for all $|s| \le 1$, that $|\sin x| \le A|u_1(x)|$. Hence,

$$\frac{|\sin x|}{E(u_n(x))} \le \frac{|\sin x|}{E(u_1(x))} \le A\,, \qquad \forall n \in \mathbb{N}\,.$$

An appeal to Lebesgue's Theorem allows us to pass to the limit in (4.71) and deduce that

$$\frac{\sin x}{E_n\big(\Psi_n^{-1}(\sin x)\big)}\chi_{\{|\sin x| \ge \Psi_n(1/n)\}}(x) \to f(x) \tag{4.72}$$

strongly in any $L^m(\Omega)$, with $m < \infty$.

Let us now turn to study the second term in (4.70). It is enough to prove that $\frac{\sin x}{E_n\left(\Psi_n^{-1}(\sin x)\right)}$ is bounded in the set $\{x \in \Omega \,:\, |\sin x| < \Psi_n(1/n)\}$, since $\chi_{\{|\sin x)| < \Psi_n(1/n)\}}(x) \downarrow 0$. To see this boundedness, we remark that $|s| \le 1$ implies $E_n(s) \le E_n(\frac{1}{n}) = \frac{1}{n}$ and so

$$\Psi_n\Big(\frac{1}{n}\Big) = \int_0^{1/n} E_n(\sigma)\,d\sigma \le \frac{1}{n^2}\,.$$

It follows that if $|s| \le 1$, then

$$E_n(s) = \frac{1}{n}e^{\frac{1}{2}(n^2s^2-1)} \ge \frac{1}{n\sqrt{e}} = \frac{1}{\sqrt{e}}\sqrt{\frac{1}{n^2}} \ge \sqrt{\frac{\Psi_n(1/n)}{e}} \ge \sqrt{\frac{\Psi_n(s)}{e}}\,.$$

Hence, if $|\sin x| \le \Psi_n\Big(\frac{1}{n}\Big)$, then

$$E_n\Big(\Psi^{-1}(\sin x)\Big) \ge \sqrt{\frac{|\sin x|}{e}}$$

and so

$$\Big|\frac{\sin x}{E_n\Big(\Psi_n^{-1}(\sin x)\Big)}\Big| \le \frac{|\sin x|}{\sqrt{\frac{|\sin x|}{e}}} \le \sqrt{e|\sin x|} \le \sqrt{e}\,.$$

We conclude that the second term in (4.70) tends to 0 strongly in any $L^m(\Omega)$, with $m < \infty$. This fact and (4.72) gives (4.69).

Finally, it follows from $\Psi_n(u_n) \to \Psi(u)$ and $G_\delta(u_n) \to G_\delta(u)$ strongly in $H_0^1(\Omega)$, that u should be a solution like those we reached in the above sections. However, it is obvious we cannot prove that

$$b_n(u_n)|u_n'|^2 \to \frac{|u'|^2}{u} \qquad \text{strongly in } L^1_{loc}(\Omega)\,.$$

This fact does not contradict the results of [3]: indeed, if we consider our problem in $]0,\pi[$ (where f is nonnegative), then the restriction of u to $]0,\pi[$ is really a solution in this interval since $u \in H^1_{loc}(]0,\pi[)$ and $\dfrac{|u'|^2}{u} \in L^1_{loc}(]0,\pi[)$.

5. A lower order term satisfying a sign condition on the left hand side

Let us now study the model problem (2.5) where the datum f is a changing sign function satisfying

$$f(x) \in L^m(\Omega)\,, \quad m > \frac{N}{2}\,.$$

From the point of view of the estimates, this is the easiest case to deal with (since no auxiliary function is needed) and it is where a more regular solution can be obtained by approximation (i.e. $u, \sqrt{|u|} \in H^1_0(\Omega)$). However, as before, striking differences with respect to the study of mild singularities ([15]) occur, and, in particular, one is not able to prove the key estimate near the singularity that allows to pass to the limit in the lower order term (see [15, equation (3.29)]). The main result one can prove is the following.

Theorem 5.1. *There exists a function $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$, which is a solution to* (2.5) *obtained as a limit of approximations.*

Moreover, the approximate solutions u_n converge to u, up to subsequences, in the following senses:

$$u_n \to u \quad \textit{strongly in } L^q(\Omega)\,, \quad 1 \le q < \frac{2N}{N-2}\,;$$
$$u_n(x) \to u(x) \quad \textit{pointwise a.e. in } \Omega\,;$$
$$\nabla u_n \rightharpoonup \nabla u \quad \textit{weakly in } L^2(\Omega;\mathbb{R}^N)\,;$$
$$\nabla u_n \to \nabla u \quad \textit{strongly in } L^q(\Omega;\mathbb{R}^N)\,, \quad 1 \le q < 2\,;$$
$$\nabla u_n(x) \to \nabla u(x) \quad \textit{pointwise a.e. in } \Omega\,;$$
$$|b_n(u_n)||\nabla u_n(x)|^2 \to \Lambda \frac{|\nabla u(x)|^2}{|u(x)|} \quad \textit{pointwise a.e. in } \{u \neq 0\}$$

To prove this result, we consider, for each $n \in \mathbb{N}$, the functions given by

$$b_n(s) := \begin{cases} \Lambda n^2 s\,, & \text{if } |s| \le \frac{1}{n}\,; \\ \frac{\Lambda}{s}\,, & \text{otherwise}\,; \end{cases}$$

and the following problems

$$\begin{cases} -\Delta u_n + b_n(u_n)|\nabla u_n|^2 = f(x)\,, & \text{in } \Omega\,; \\ u_n = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{5.73}$$

As in previous cases, a weak solution $u_n \in H^1_0(\Omega) \cap L^\infty(\Omega)$ to problem (5.73) exists due to [20, Theorem 1.1 (i)].

5.1. **A priori estimates. Estimate on u_n in $H_0^1(\Omega)$.** Take u_n as test function in (5.73) to get

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega b_n(u_n)u_n|\nabla u_n|^2 = \int_\Omega f u_n\,.$$

Dropping a nonnegative term, we obtain

$$\int_\Omega |\nabla u_n|^2 \le \int_\Omega f u_n\,,$$

from which, using the Hölder and Sobolev inequalities, it follows that u_n is bounded in $H_0^1(\Omega)$.

Then there exists $u \in H_0^1(\Omega)$ and a subsequence, not relabeled, satisfying

$$\nabla u_n \rightharpoonup \nabla u\,, \quad \text{weakly in } L^2(\Omega;\mathbb{R}^N)\,, \tag{5.74}$$

$$u_n(x) \to u(x)\,, \qquad \text{pointwise in } \Omega\,, \tag{5.75}$$

$$u_n \to u\,, \qquad \text{strongly in } L^q(\Omega)\,, \quad \text{for } 1 \le q < \frac{2N}{N-2}\,. \tag{5.76}$$

L^∞–Estimate. Taking $G_k(u_n)$ as test function in (5.73) and disregarding the nonnegative lower order term, it yields

$$\int_\Omega |\nabla G_k(u_n)|^2 \le \int_\Omega |f|\,|G_k(u_n)|\,,$$

which is the starting point of the Stampacchia's procedure (see for instance [22, Theorem 4.1]). Since $f(x) \in L^m(\Omega)$, with $m > \frac{N}{2}$, it follows that $\|u_n\|_\infty$ is bounded by a constant that only depends on the parameters Λ, m, $\|f\|_m$, N, and $|\Omega|$:

$$\|u_n\|_\infty \le M \quad \text{ for all } n \in \mathbb{N}\,. \tag{5.77}$$

As a consequence of (5.76), $u \in L^\infty(\Omega)$ and $\|u\|_\infty \le M$.

Estimate on the lower order term in $L^1(\Omega)$. Now take $\frac{1}{\epsilon}T_\epsilon(u_n)$ as test function in (5.73) and drop a nonegative term to get

$$\frac{1}{\epsilon}\int_\Omega T_\epsilon(u_n)b_n(u_n)|\nabla u_n|^2 \le \frac{1}{\epsilon}\int_\Omega fT_\epsilon(u_n) \le \int_\Omega |f|\,.$$

Hence, letting ϵ go to 0, we deduce

$$\int_\Omega |b_n(u_n)|\,|\nabla u_n|^2 \le \int_\Omega |f|\,. \tag{5.78}$$

5.2. **Convergence.** The pointwise convergence of the gradients follows again from [6, Theorem 2.1]. So we have, up to subsequences,

$$\nabla u_n \to \nabla u \qquad \text{pointwise a.e. in } \Omega\,. \tag{5.79}$$

From (5.79) and (5.75) we deduce that

$$b_n(u_n)|\nabla u_n|^2 \to \Lambda\frac{|\nabla u|^2}{|u|} \qquad \text{pointwise a.e. in } \{u \neq 0\}\,. \tag{5.80}$$

Then Fatou's Lemma implies that

$$\frac{|\nabla u|^2}{|u|} \in L^1(\{u \neq 0\}) ,$$

from where, thanks to Lemma 3.4, we obtain $\sqrt{|u|} \in H_0^1(\Omega)$, $\frac{|\nabla u|^2}{|u|} \in L^1(\Omega)$ and

$$\frac{|\nabla u|^2}{|u|}\chi_{\{u\neq 0\}} = \frac{|\nabla u|^2}{|u|} .$$

Remark 5.1. All convergences we have proved are not enough to pass to the limit in the approximating problems and get a distributional solution to (2.5), since we do not prove equi–integrability of the gradient terms; to see it we would need the strong convergence of ∇u_n (or, at least, of $\nabla G_\delta(u_n)$) and an estimate similar to [15, equation (3.29)].

Appendix A. Counterexamples in higher dimension

We now show how Example 4.1 can be extended to higher dimensions. To this end we will consider a symmetric bounded open set in $\mathbb{R}^N$.

Example A.1. Given $x \in \mathbb{R}^N$, we will write $x = (x', x_N)$, where $x' = (x_1, x_2, \dots, x_{N-1})$. Set $\Omega = \{x = (x', x_N) \in \mathbb{R}^N \,:\, |x'| < 1\,,\ |x_N| < 1\}$ and $\Omega_+ = \{x \in \Omega \,:\, x_N > 0\}$.

Let w be a eigenfunction associated to the first eigenvalue of $-\Delta$ (with Dirichlet boundary condition) in Ω_+. In other words, $w \in H_0^1(\Omega_+) \cap L^\infty(\Omega_+)$ which is a positive and smooth function, and solves the problem

$$\begin{cases} -\Delta w = \lambda_1 w\,, & \text{in } \Omega_+\,;\\ w = 0\,, & \text{on } \partial\Omega_+\,;\end{cases}$$

λ_1 being the first eigenvalue. Next we extend this function to Ω by defining

$$v(x) = \begin{cases} w(x', x_N)\,, & \text{if } x_N > 0\,;\\ 0\,, & \text{if } x_N = 0\,;\\ -w(x', -x_N)\,, & \text{if } x_N < 0\,.\end{cases}$$

It is easy to see that the two parts of v stick suitably and so v is a weak solution to

$$\begin{cases} -\Delta v = \lambda_1 v(x)\,, & \text{in } \Omega\,;\\ v = 0\,, & \text{on } \partial\Omega\,.\end{cases}$$

Defining, as above, the auxiliary functions $E(s) = |s|$ and $\Psi(s) = \frac{1}{2}|s|s$, we may consider $f(x) = \frac{\lambda_1 v(x)}{E\left(\Psi^{-1}(v(x))\right)}$ and $u(x) = \Psi^{-1}(v(x))$.

We will see that $u \notin H^1_{loc}(\Omega)$. Writing η as the unit normal outward vector to Ω_+ on the set $\Omega \cap \partial\Omega_+ = \{(x', 0) \,:\, |x'| < 1\}$ and applying Hopf's

Lemma, we obtain $\dfrac{\partial w}{\partial \eta} < 0$. Hence,

$$\lim_{h\to 0^+} \frac{w(x',h)}{h} = \frac{\partial w}{\partial x_N}(x',0) = -\frac{\partial w}{\partial \eta}(x',0) > 0\,,$$

and there exists $\delta > 0$ such that $0 < h < \delta$ implies $w(x',h) < 2\frac{\partial w}{\partial x_N}(x',0)h$. As a consequence,

$$\frac{\left|\frac{\partial v}{\partial x_N}(x',0)\right|}{|v(x',h)|} > \frac{1}{2|h|}$$

for all $0 < |h| < \delta$. Taking into account the continuity of $\frac{\partial v}{\partial x_N}$ and

$$|\nabla u|^2 = \frac{|\nabla v|^2}{E(u)^2} = \frac{|\nabla v|^2}{2|v|} \geq \frac{|\frac{\partial v}{\partial x_N}|^2}{2|v|}\,,$$

it follows that there exists a positive constant C satisfying

$$|\nabla u(x',x_N)|^2 \geq \frac{C}{|x_N|}$$

for x_N small enough. Therefore, $|\nabla u|^2$ is not summable on any measurable set containing $\Omega \cap \partial\Omega_+$.

While $u \notin H^1_{loc}(\Omega)$, we have that $\Psi(u) \in H^1_0(\Omega)$ and u is a solution to

$$\begin{cases} -\Delta u = \frac{|\nabla u|^2}{u} + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,; \end{cases}$$

by means of the Cole–Hopf change of unknown. Moreover, we may perform the same procedure as in Example 4.1 by introducing the approximate auxiliary functions b_n, E_n and Ψ_n, and by considering the problems

$$\begin{cases} -\Delta u_n = b_n(u_n)|\nabla u_n|^2 + f_n(x)\,, & \text{in } \Omega\,; \\ u_n = 0\,, & \text{in } \partial\Omega\,; \end{cases}$$

where $f_n(x) = \frac{\lambda_1 v(x)}{E_n\left(\Psi_n^{-1}(v(x))\right)}$. Therefore, u is a solution obtained as a limit of regular solutions.

Finally a remark is in order. It follows from $\dfrac{\partial w}{\partial \eta} < 0$ on $\Omega \cap \partial\Omega_+$ that $E(u)\dfrac{\partial u}{\partial \eta} < 0$ and so $\dfrac{\partial u}{\partial \eta} = -\infty$ on $\Omega \cap \partial\Omega_+$. As a consequence, we have that $|\nabla u| = +\infty$ when crossing the singular set $\{u = 0\}$.

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