QUASILINEAR STATIONARY PROBLEMS WITH A QUADRATIC GRADIENT TERM HAVING SINGULARITIES

DANIELA GIACHETTI AND SERGIO SEGURA DE LEÓN

Abstract. We study the homogeneous Dirichlet problem for some elliptic equations with a first order term $b(u, Du)$ which is quadratic in the gradient variable and singular in the u variable at a positive point. Moreover, the gradient term we consider, changes its sign at the singularity. Dealing with an appropriate concept of solution that gives sense to the equation at the singularity, we prove existence of solutions for every datum belonging to a suitable Lebesgue space. Furthermore, we show that the solution pass through the singularity when data are big enough.

1. Introduction

Since the pioneering works by L. Boccardo, F. Murat and J.P. Puel in the 80's of last century (see, for instance, [12] and [13]), many articles have been published on the Dirichlet problem for elliptic equations having a quadratic gradient term of the form $g(u)|Du|^2$, mainly when g is a continuous real function. The prototype of such a kind of problems is

$$(1.1)\qquad \begin{cases} -\Delta u = g(u)|Du|^2 + f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Recently, attention has been focused on functions g having a singularity, and several authors have studied conditions for existence or non existence of solution when the singularity lies at 0 (see [1], [2], [3] [4], [8] and [17]). The uniqueness of solutions for this type of equations has also been studied (see [5]).

Nevertheless, as far as we know, results concerning equation with a gradient term having a singularity at a positive point (say, 1) can only be found in [11] and [18]. In [11] existence and non existence of solution are considered for $g(s) = -\dfrac{1}{|1-s|^k}$ in (1.1). More precisely, the authors prove that, if $0 < k < 2$, for data $f(x)$ large enough, there is no solution $u \in H_0^1(\Omega)$ satisfying $0 \le u < 1$, while, for $k \ge 2$, there exist always solutions in $H_0^1(\Omega)$, whatever is the size of $f(x)$. In turn, in [18] the authors investigate correlations between problems like (1.1) and some related semilinear problems, via change of unknown.

In the present paper, we deal with the Dirichlet problem for elliptic equations containing a quadratic gradient term with a singularity given by a function of the

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form $g(s) = \pm\dfrac{\operatorname{sign}(s-1)}{|s-1|^\theta}$, where $0 < \theta < 1$. Hence, our model problems will be

$$\left\{\begin{array}{ll} -\Delta u = \pm\dfrac{\operatorname{sign}(u-1)}{|u-1|^\theta}|Du|^2 + f(x)\,, & \text{in } \Omega\,; \\ \\ u = 0\,, & \text{on } \partial\Omega\,. \end{array}\right. \tag{1.2}$$

Actually, instead of the Laplacian, we will consider a general Leray–Lions type operator.

Our aim is to prove that there exists a solution to (1.2) for every datum, whether big or small, which belongs to a suitable Lebesgue space. Moreover, we show that solutions cross the singularity when data are large enough. Note that this is due to the fact that we have gradient terms with singularities. Indeed, the Dirichlet problem for equations with a positive singularity in a zero order term has been studied by Boccardo in [7]. In this case, the singularity makes a bound which cannot be surpassed. Furthermore, in [19] it is shown that this phenomenon persists even if we add, to the singular zero order term, a gradient term which is non singular in the u variable . On the other hand, singularities in the principal term have also been studied (see [6] and, for instance, [16]); in these papers the solutions can achieve the singularity but they cannot pass through it. This is also a feature for the solutions in the problem considered in [11], in this paper is considered problem (1.1) with a function g which could be written as

$$-g(s) = \left\{\begin{array}{ll} \frac{1}{(1-s)^k}\,, & \text{if } 0 \le s < 1\,; \\ +\infty\,, & \text{if } s \ge 1\,. \end{array}\right.$$

In contrast, in our model, the solution can pass through the singularity (see Proposition 2.5 below).

We will obtain our existence results by considering approximating problems, looking for a priori estimates and studying the convergence of the approximate solutions. As usual, when dealing with gradient terms with natural growth, we will get a priori estimates for the solutions to these problems by applying a cancellation result (see [14, 20]). The presence of $\operatorname{sign}(s-1)$ in the gradient term adds some extra snags in getting estimates to this term: we must consider separately what happens on the set $\{u \le 1\}$ and what happens on $\{u \ge 1\}$. Hence, to obtain the desired estimates, we will take test functions whose support lives in each of those sets.

Besides the difficulties that appear when one studies second order equations with gradient terms, there are specific ones arising when we deal with singularities. Among them, we want to highlight two. An obvious, but essential, difficulty is giving a meaning to the gradient term at the singularity. In our cases it becomes in assigning values to $\frac{\operatorname{sign}(u-1)}{|u-1|^\theta}|Du|^2$ in $\{u = 1\}$. We will show that its natural value is 0. This will be justified in the following section (see (2.6) below).

A second hindrance is how we can avoid the singularity in order to obtain convergence of the approximate solutions u_n. Of course, we avoid the singularity by using suitable test functions. But beyond these technical complications, this fact has some important consequences for the convergence of u_n. Indeed, since we are only able to prove the pointwise convergence of the gradients Du_n on the set $\{u \ne 1\}$, it is not easy to deduce the weak convergence of the principal term (when a general

Leray–Lions operator is consider) nor the strong convergence of the gradient term. In both cases, we have to make some cumbersome calculations.

We remark that the kind of lower order terms studied in the present paper, i.e. those terms which include sign $(s-1)$ jointly with a singularity at 1, can be found in the Euler's equation of a problem of minimizing a functional. Consider, for instance, the functional defined by

$$I[u]=\int_\Omega \frac{1+|u-1|^{1-\theta}}{1-\theta}|Du|^2-\int_\Omega fu\,.$$

Even though Euler's equation corresponding to the above I does not satisfy our structural condition (2.2) below, it is not difficult to modify slightly the definition in order to obtain Euler's equation satisfying (2.2). To be more precise, consider a smooth real function T satisfying $T(s)=s$ for all $s\in[0,M]$, with $M>1$, and $T'(s)=0$ for s big enough. Then it is easy to see that Euler's equation corresponding to the functional defined by

$$I[u]=\int_\Omega \frac{1+T(|u-1|^{1-\theta})}{1-\theta}|Du|^2-\int_\Omega fu\,.$$

satisfies all the hypotheses assumed below.

This paper is organized as follows. Next section is devoted to establish our precise assumptions, notation and the statements of the main results; their proof appear in Section 3.

2. Hypotheses and statements of results

Let us state our hypotheses more precisely. Consider a Carathéodory function

$$a(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$$

such that there exist some constants $\alpha>0$ and $\nu>0$ satisfying the following inequalities

$$a(x,s,\xi)\cdot\xi\geq\alpha|\xi|^2, \tag{2.1}$$

$$|a(x,s,\xi)|\leq\nu|\xi|, \tag{2.2}$$

$$(a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta)>0\,; \tag{2.3}$$

for all $\xi,\eta\in\mathbb{R}^N$, with $\xi\neq\eta$, for all $s\in\mathbb{R}$ and for almost all $x\in\Omega$.

We will deal with the following problems

$$\begin{cases}-\mathrm{div}(a(x,u,Du))=\pm\dfrac{\mathrm{sign}\,(u-1)}{|u-1|^\theta}|Du|^2+f(x)\,, & \text{in }\ \Omega\,;\\ u=0\,, & \text{on }\ \partial\Omega\,;\end{cases} \tag{2.4$^\pm$}$$

where Ω is an open bounded set in $\mathbb{R}^N(N>2)$, $\quad 0<\theta<1$ and

$$f(x)\in L^m(\Omega), \tag{2.5}$$

m depending on the chosen sign in $(2.4)^\pm$ (see Theorems 2.3 and 2.4 below). We point out that the equations involve indeterminate quotients on the set $\{u=1\}$, which must be clarified. Indeed, since we are looking for solutions in the energy space $H_0^1(\Omega)$, according to the result of G. Stampacchia in [22], Du vanishes a.e. on $\{u=1\}$, and so the gradient term is indefinite.

Definition 2.1. *If u and $|u-1|^{1-\frac{\theta}{2}}-1$ belong to $H_0^1(\Omega)$, we define*

$$\frac{|Du|^2}{|u-1|^\theta}=\frac{4}{(2-\theta)^2}|D\big(|u-1|^{1-\frac{\theta}{2}}-1\big)|^2\,.$$

Observe that, by definition, $\dfrac{|Du|^2}{|u-1|^\theta}$ always belongs to $L^1(\Omega)$. Moreover, as a consequence of Stampacchia's Theorem, we obtain

$$\frac{|Du|^2}{|u-1|^\theta}=0 \quad \text{a.e. in } \{u=1\}\,. \tag{2.6}$$

In order to check that a function u is actually solution to problem $(2.4)^{\pm}$, we will have to see $|u-1|^{1-\frac{\theta}{2}}-1\in H_0^1(\Omega)$; in this task the following simple claim will be applied.

Lemma 2.2. *Let $u\in H_0^1(\Omega)$. If $\dfrac{|Du|^2}{|u-1|^\theta}$ is integrable on $\{u\neq 1\}$, then*

$$|u-1|^{1-\frac{\theta}{2}}-1\in H_0^1(\Omega)\,,$$

and

$$\int_\Omega \frac{|Du|^2}{|u-1|^\theta}=\int_{\{u\neq 1\}}\frac{|Du|^2}{|u-1|^\theta}\,.$$

To see it, consider $u\in H_0^1(\Omega)$ such that $\dfrac{|Du|^2}{|u-1|^\theta}$ is integrable on $\{u\neq 1\}$. Then

$$\int_\Omega \frac{|Du|^2}{(\frac1n+|u-1|)^\theta}=\int_{\{u\neq 1\}}\frac{|Du|^2}{(\frac1n+|u-1|)^\theta}\le\int_{\{u\neq 1\}}\frac{|Du|^2}{|u-1|^\theta}\,,$$

for all $n\in\mathbb{N}$; in other words

$$\int_\Omega \Big|D\Big(\Big(\frac1n+|u-1|\Big)^{1-\frac{\theta}{2}}-\Big(\frac1n+1\Big)^{1-\frac{\theta}{2}}\Big)\Big|^2\le C\,,\qquad \forall n\in\mathbb{N}\,.$$

Therefore, $(\frac1n+|u-1|)^{1-\frac{\theta}{2}}-(\frac1n+1)^{1-\frac{\theta}{2}}$ is bounded in $H_0^1(\Omega)$. Thus, there exist a subsequence, no relabel, and $v\in H_0^1(\Omega)$ satisfying

$$\Big(\frac1n+|u-1|\Big)^{1-\frac{\theta}{2}}-\Big(\frac1n+1\Big)^{1-\frac{\theta}{2}}\rightharpoonup v\,,\quad \text{weakly in } H_0^1(\Omega)\,.$$

It follows, up to a subsequence, that

$$v=\lim_{n\to\infty}\Big(\frac1n+|u-1|\Big)^{1-\frac{\theta}{2}}-\Big(\frac1n+1\Big)^{1-\frac{\theta}{2}}\,,\quad \text{a.e. in } \Omega\,.$$

Hence, $v=|u-1|^{1-\frac{\theta}{2}}-1$ and so $|u-1|^{1-\frac{\theta}{2}}-1\in H_0^1(\Omega)$. Finally, (2.6) implies the second assertion of our claim.

Next, we introduce some notation. We will denote by $|E|$ the Lebesgue measure of a set $E\subset\Omega$. By C a positive constant depending on parameters of our problem is denoted; its value may vary from line to line. Throughout this paper, $\omega(n)$ will denote a quantity, which only depends on n and on parameters of our problem, and tends to 0 as n goes to ∞; the meaning of $\omega(\varepsilon)$ is similar, but now it tends to 0 as ε goes to 0. We will denote by $T_n(s)$ the usual truncation function at level $\pm n$:

$$T_n(s) = \begin{cases} n & s \geq n \\ s & -n \leq s \leq n \\ -n & s \leq -n. \end{cases} \tag{2.7}$$

Finally, let us introduce the following functions $\gamma(s)$ and $\psi(s)$, $s \in \mathbb{R}$

$$\gamma(s) = \begin{cases} \dfrac{1}{\alpha}\dfrac{1-|s-1|^{1-\theta}}{1-\theta} & \text{if “}-\text{” holds in (2.4)}^{\pm} \\ \dfrac{1}{\alpha}\dfrac{|s-1|^{1-\theta}-1}{1-\theta} & \text{if “}+\text{” holds in (2.4)}^{\pm} \end{cases} \tag{2.8}$$

$$\psi(s) = \int_0^s e^{\gamma(\sigma)}\, d\sigma\,. \tag{2.9}$$

Note that $\gamma(s)$ is a primitive function of the function $\frac{1}{\alpha}\frac{\text{sign}\,(s-1)}{|s-1|^{\theta}}$ if "+" holds in $(2.4)^{\pm}$ and of the function $-\frac{1}{\alpha}\frac{\text{sign}\,(s-1)}{|s-1|^{\theta}} = \frac{1}{\alpha}\frac{\text{sign}\,(1-s)}{|s-1|^{\theta}}$ if "−" holds in $(2.4)^{\pm}$.

The definition of solution u for problem $(2.4)^{+}$ or $(2.4)^{-}$ is precised in the following theorems which are our main results.

Theorem 2.3. *Suppose* (2.1)–(2.3) *hold true. If* $m \geq \frac{2N}{N+2}$, *then there exists at least a function* $u \in H_0^1(\Omega)$ *such that*

$$\psi(u) \in H_0^1(\Omega), \qquad \frac{|Du|^2}{|u-1|^{\theta}} \in L^1(\Omega) \tag{2.10}$$

$$\int_\Omega a(x,u,Du)\cdot D\varphi + \int_\Omega \frac{\text{sign}\,(u-1)}{|u-1|^{\theta}}|Du|^2\varphi = \int_\Omega f\varphi \quad \forall\, \varphi \in H_0^1(\Omega)\cap L^\infty(\Omega). \tag{2.11}$$

Theorem 2.4. *If* (2.1)–(2.3) *hold true and* $m \geq \frac{N}{2}$, *then there exists* $u \in H_0^1(\Omega)$ *satisfying* (2.10) *and*

$$\int_\Omega a(x,u,Du)\cdot D\varphi = \int_\Omega \frac{\text{sign}\,(u-1)}{|u-1|^{\theta}}|Du|^2\varphi + \int_\Omega f\varphi \quad \forall\, \varphi \in H_0^1(\Omega)\cap L^\infty(\Omega). \tag{2.12}$$

One may wonder if the singularity actually occurs, that is if solutions to our problems achieve the singularity. We will show that this is the case when data are large enough, at least in the case of solutions of $(2.4)^{-}$. Indeed, let f be a nonnegative regular enough datum and consider the following problem:

$$\begin{cases} -\Delta\phi_1 = \lambda_1 f\phi_1\,, & \text{in } \Omega\,; \\ \phi_1 = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{2.13}$$

It is well known that there exist $\lambda_1 > 0$ and $\phi_1 \in H_0^1(\Omega)$, with $\phi_1 > 0$, which solve the above eigenvalue problem. Then the following result holds true.

Proposition 2.5. *For every* $\lambda > \lambda_1$, *solutions to*

$$\begin{cases} -\Delta u + \dfrac{\text{sign}\,(u-1)}{|u-1|^{\theta}}|Du|^2 = \lambda f(x)\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,; \end{cases}$$

are not bounded by 1.

REMARK 2.6. It is easy to show that the results of Theorem 2.3 and Theorem 2.4 still hold true if we replace the function $\frac{\mathrm{sign}\,(s-1)}{|1-s|^\theta}$ appearing in $(2.4)^\pm$ with a continuous function $g : \mathbb{R}\backslash\{1\} \to \mathbb{R}$, summable near the singularity $s = 1$, and satisfying $\lim_{|s|\to\infty} g(s) = 0$ and $g(s)\mathrm{sign}\,(s-1) \geq 0$.

3. PROOF OF THE MAIN RESULTS

In order to give the proof of Theorem 2.3 and Theorem 2.4, let us introduce a sequence of approximating problems. Note that the simple choice of replacing the term

$$\frac{\mathrm{sign}\,(u-1)}{|u-1|^\theta}|Du|^2 \qquad \text{with} \qquad \mathrm{sign}\,(u_n-1)T_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 .$$

does not work. Indeed the function $\mathrm{sign}\,(s-1)T_n\big(\frac{1}{|s-1|^\theta}\big)$ is not continuous at point 1, and so known results on existence cannot be applied.

Therefore, we will truncate by means of the real function R_n defined, for $s > 0$, by

$$R_n(s) = \begin{cases} s\,, & \text{if } s \leq n\,; \\ n\left(\dfrac{n}{s}\right)^{1/\theta} & \text{if } s \geq n\,. \end{cases} \tag{3.1}$$

The idea is that the truncate function

$$s \mapsto \mathrm{sign}\,(s-1)R_n\left(\frac{1}{|s-1|^\theta}\right) \tag{3.2}$$

be linear near the singularity, it is written in this way because we want to point out that the function changes its sign at 1 and to highlight the singularity.

Thus, for $n \in \mathbb{N}$, we consider

$$\begin{cases} -\mathrm{div}(a(x,u_n,Du_n)) = \\ \quad = \pm\mathrm{sign}\,(u_n-1)R_n\left(\dfrac{1}{|u_n-1|^\theta}\right)|Du_n|^2 + T_nf(x) & \text{in } \Omega\,; \\ u_n = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.3$_n^\pm$}$$

Note that the existence of such a solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is guaranteed by the results in [20], since the function defined in (3.2) is continuous and

$$\lim_{s\to\infty} R_n\left(\frac{1}{|s-1|^\theta}\right) = 0\,.$$

For every $n \in \mathbb{N}$ and $s \in \mathbb{R}$, we define

$$\gamma_n(s) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^s \mathrm{sign}\,(1-\sigma)\,R_n\left(\frac{1}{|\sigma-1|^\theta}\right)d\sigma & \text{if “$-$” holds in } (3.3)_n^\pm \\ \dfrac{1}{\alpha}\displaystyle\int_1^s \mathrm{sign}\,(\sigma-1)\,R_n\left(\frac{1}{|\sigma-1|^\theta}\right)d\sigma & \text{if “$+$” holds in } (3.3)_n^\pm \end{cases} \tag{3.4}$$

and

$$\psi_n(s) = \int_0^s e^{\gamma_n(\sigma)}\,d\sigma. \tag{3.5}$$

These two functions are approximating versions of those defined in (2.8) and (2.9). The following cancellation result will be used several times in the sequel (see [21] or [14]).

Proposition 3.1. *For every increasing Lipschitz–continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(0) = 0$, it yields that $e^{\gamma_n(u_n)}\phi(u_n)$ can be taken as test function in $(3.3)_n$ and so*

$$\int_\Omega e^{\gamma_n(u_n)}\phi'(u_n)|Du_n|^2 \le \int_\Omega T_n(f)e^{\gamma_n(u_n)}\phi(u_n)\,.$$

Proof of Theorem 2.3: We are dealing with the case of problem $(2.4)^-$ *i.e.*

$$\begin{cases} -\mathrm{div}(a(x,u,Du)) + \dfrac{\mathrm{sign}\,(u-1)}{|u-1|^\theta}|Du|^2 = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As far as the datum is concerned, we have assumed $f \in L^m(\Omega)$, where $m \ge (2^*)'$. Note that, in this case,

$$\lim_{|s|\to+\infty} \gamma(s) = -\infty\,.$$

On the other hand, let us observe that γ_n attains a maximum at 1. So that $e^{\gamma_n(s)} \le e^{\gamma_n(1)}$ for all $s \in \mathbb{R}$. Moreover $\gamma_n(1) \le \gamma(1)$ for all $n \in \mathbb{N}$.

Step 1. Uniform estimates on $(\psi_n(u_n))_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$.

Let us consider $v_n = e^{\gamma_n(u_n)}\psi_n(u_n) \in H_0^1(\Omega)\cap L^\infty(\Omega)$ where $\gamma_n(s)$ and $\psi_n(s)$ are defined in (3.4) and (3.5). Applying Proposition 3.1, we get

$$\int_\Omega |Du_n|^2 e^{2\gamma_n(u_n)} \le \int_\Omega T_n(f)e^{\gamma_n(u_n)}\psi_n(u_n) \le C\int_\Omega |f|\,|\psi_n(u_n)|\,,$$

that is,

$$\int_\Omega |D\psi_n(u_n)|^2 \le C\int_\Omega |f|\,|\psi_n(u_n)|\,.$$

Since $f \in L^m(\Omega)$ with $m \ge (2^*)'$, by the Hölder and Sobolev inequalities, we obtain

$$\int_\Omega |D\psi_n(u_n)|^2 \le C \quad \forall\, n \in \mathbb{N}\,. \tag{3.6}$$

Step 2. Boundedness of the gradient term in $L^1(\Omega)$.

Our aim in this Step is to prove that $\int_\Omega R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \le C$ for all $n \in N$.

Take $e^{\gamma_n(1-(1-u_n)^+)}-1\in H_0^1(\Omega)\cap L^\infty(\Omega)$ as test function in $(3.3)_n^-$. Then

$$\begin{aligned}\frac{1}{\alpha}\int_{\{u_n\le 1\}} & e^{\gamma_n(u_n)}R_n\left(\frac{1}{|u_n-1|^\theta}\right)a(x,u_n,Du_n)\cdot Du_n\\ &-\int_{\{u_n\le 1\}}\Big(e^{\gamma_n(u_n)}-1\Big)R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\\ &+\int_{\{u_n\ge 1\}}\Big(e^{\gamma_n(1)}-1\Big)R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\\ &=\int_{\{u_n\le 1\}}T_n(f)\Big(e^{\gamma_n(u_n)}-1\Big)+\int_{\{u_n\ge 1\}}T_n(f)\Big(e^{\gamma_n(1)}-1\Big).\end{aligned}$$

Using (2.1), cancelling similar terms and disregarding a nonnegative term in the left hand side, we get

$$\int_{\{u_n\le 1\}}R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\le\Big(e^{\gamma_n(1)}-1\Big)\int_\Omega|f|\le c\int_\Omega|f|. \tag{3.7}$$

Taking now $e^{\gamma_n(1)}-e^{\gamma_n((u_n-1)^++1)}$ as test function, we deduce

$$\begin{aligned}\frac{1}{\alpha}\int_{\{u_n\ge 1\}} & \operatorname{sign}(u_n-1)e^{\gamma_n(u_n)}R_n\left(\frac{1}{|u_n-1|^\theta}\right)a(x,u_n,Du_n)\cdot Du_n\\ &+\int_{\{u_n\ge 1\}}\operatorname{sign}(u_n-1)\Big(e^{\gamma_n(1)}-e^{\gamma_n(u_n)}\Big)R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\\ &=\int_\Omega(T_n(f))\Big(e^{\gamma_n(1)}-e^{\gamma_n((u_n-1)^++1)}\Big)\le e^{\gamma_n(1)}\int_\Omega|f|\le c\int_\Omega|f|.\end{aligned}$$

Hence, (2.1) implies

$$\int_{\{u_n\ge 1\}}R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\le c\int_\Omega|f|.$$

Adding (3.7) to this inequality, we prove the desired estimate.

Step 3. Uniform estimates of $(u_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$.

Let us consider u_n as a test function in $(3.3)_n^-$ to get

$$\alpha\int_\Omega|Du_n|^2+\int_\Omega u_n\operatorname{sign}(u_n-1)R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\le\int_\Omega T_n(f)u_n. \tag{3.8}$$

Observe that the second term in the left hand side can be written as

$$\int_\Omega|u_n-1|R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2+\int_\Omega\operatorname{sign}(u_n-1)R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2;$$

so that, dropping a nonnegative term, (3.8) becomes

$$\alpha\int_\Omega|Du_n|^2+\int_\Omega\operatorname{sign}(u_n-1)R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\le\int_\Omega|f|\,|u_n|.$$

Now, Step 2 implies

$$\alpha\int_\Omega|Du_n|^2\le\int_\Omega|f|\,|u_n|+C.$$

On account of $f \in L^m(\Omega)$, with $m \geq (2^*)'$, we may apply first Hölder's inequality and then Sobolev's inequality to obtain

$$\alpha \int_\Omega |Du_n|^2 \leq C\|f\|_m \Big(\int_\Omega |Du_n|^2 \Big)^{1/2} + C\,.$$

It follows now from Young's inequality that

$$\int_\Omega |Du_n|^2 \leq C \quad \forall\, n \in \mathbb{N}\,. \tag{3.9}$$

This estimate implies that there exists $u \in H_0^1(\Omega)$, such that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in} \quad H_0^1(\Omega) \tag{3.10}$$

$$u_n \to u \quad \text{a.e. in } \Omega\,. \tag{3.11}$$

Observe that then (3.6) implies $\psi(u) \in H_0^1(\Omega)$.

Moreover, applying Lemma 1 of [9] (or Theorem 2.1 of [10]) we deduce

$$Du_n \to Du \quad \text{a.e. in } \Omega\,. \tag{3.12}$$

This fact and $Du_n \rightharpoonup Du$ weakly in $L^2(\Omega;\mathbb{R}^N)$ leads to

$$a(x,u_n,Du_n) \rightharpoonup a(x,u,Du) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^N)\,. \tag{3.13}$$

To pass to the limit, we still have to handle the gradient term. We will study it in the following Steps.

Step 4. STRONG CONVERGENCE IN $H_0^1(\Omega)$ OF TRUNCATIONS AWAY FROM THE SINGULARITY.

In this Step, we will prove that the truncations (which do not cross the singularity) of solutions to $(3.3)_n^-$ converge strongly in $H_0^1(\Omega)$. More precisely, we will prove that

$$\begin{gathered} DT_m(G_{1+\varepsilon}(u_n^+)) \overset{n\to+\infty}{\longrightarrow} DT_m(G_{1+\varepsilon}(u^+)) \quad \text{strongly in} \quad L^2(\Omega;\mathbb{R}^N) \\ \forall\, m>1 \quad \forall\, \varepsilon : 0<\varepsilon<m-1 \end{gathered} \tag{3.14}$$

where

$$G_{1+\varepsilon}(s) = \begin{cases} s-1-\varepsilon\,, & \text{if } s \geq 1+\varepsilon\,; \\ 0\,, & \text{if } -1-\varepsilon \leq s \leq 1+\varepsilon\,; \\ s+1+\varepsilon\,, & \text{if } s \leq -1-\varepsilon\,; \end{cases}$$

and

$$\begin{gathered} DT_{1-\varepsilon,m}(u_n) \overset{n\to+\infty}{\longrightarrow} DT_{1-\varepsilon,m}(u) \quad \text{strongly in} \quad L^2(\Omega;\mathbb{R}^N) \\ \forall\, m>0 \quad \forall\, \varepsilon : 0<\varepsilon<1 \end{gathered} \tag{3.15}$$

where

$$T_{1-\varepsilon,m}(s) = \begin{cases} 1-\varepsilon\,, & \text{if } s \geq 1-\varepsilon\,; \\ s\,, & \text{if } -m \leq s \leq 1-\varepsilon\,; \\ -m\,, & \text{if } s \leq -m\,. \end{cases}$$

Proof of (3.14)*:* Let us fix $m > 1$ and $0 < \varepsilon < m - 1$. We have to distinguish between the positive and the negative part of $T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))$. Consider first

$$v_n = \Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+$$

as test function in $(3.3)_n^-$, it follows that

$$\int_\Omega a(x,u_n,Du_n)\cdot Dv_n + \int_\Omega v_n \operatorname{sign}(u_n-1) R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \le \int_\Omega |f|\,|v_n|\,. \tag{3.16}$$

Since

$$\int_\Omega v_n \operatorname{sign}(u_n-1) R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \ge -\int_{\{u_n<1\}} \Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 = 0\,,$$

we obtain that (3.16) becomes

$$I = \int_\Omega a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ \le \int_\Omega |f|\,|T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))| = \omega(n)\,. \tag{3.17}$$

The left hand side of (3.17) may be split into two parts as $I = I_1 + I_2$, where

$$I_1 = \int_{\{1+\varepsilon+m<u_n\}} a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+\,,$$

$$I_2 = \int_{\{1+\varepsilon<u_n\le 1+\varepsilon+m\}} a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+\,,$$

since

$$\int_{\{u_n\le 1+\varepsilon\}} a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ = 0\,.$$

To deal with I_1, first apply (2.1) to get

$$I_1 \ge -\int_{\{1+\varepsilon+m<u_n\}\cap\{u_n^+\ge u^+\}} a(x,u_n,Du_n)\cdot DT_m(G_{1+\varepsilon}(u^+))\,.$$

Observe that the sequence $(|a(x,u_n,Du_n)|)_n$ is bounded in $L^2(\Omega)$ and

$$|DT_m(G_{1+\varepsilon}(u^+))|\chi_{\{m<G_{1+\varepsilon}(u_n^+)\}} \longrightarrow 0$$

strongly in $L^2(\Omega)$, so it implies

$$I_1 \ge -\int_{\{1+\varepsilon+m<u_n\}\cap\{u_n^+\ge u^+\}} a(x,u_n,Du_n)\cdot DT_m(G_{1+\varepsilon}(u^+)) = \omega(n)\,. \tag{3.18}$$

Having in mind that

$$I_2 = \int_\Omega a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+)))\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+\,,$$

it follows from (3.17) and (3.18), that

$$\int_\Omega a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+)))\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^+ \le \omega(n)\,. \tag{3.19}$$

Now is when we will exploit the weak convergence

$$DT_m(G_{1+\varepsilon}u_n^+)\rightharpoonup DT_m(G_{1+\varepsilon}u^+)$$

in $L^2(\Omega;\mathbb{R}^N)$ since this convergence implies

$$\int_\Omega a(x,u_n,DT_m(G_{1+\varepsilon}(u^+)))\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^+ =\omega(n)\,.$$

This fact and (3.19) yield

$$\limsup_{n\to\infty}\int_\Omega\Big(a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+)))-a(x,u_n,DT_m(G_{1+\varepsilon}(u^+)))\Big)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^+\le 0\,. \tag{3.20}$$

A similar inequality can be obtained for $\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-$. To get it, take

$$v_n=-e^{\gamma_n(1+(u_n-1)^+)}\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-$$

as test function in $(3.3)_n^-$ Then, cancelling similar terms and taking into account the sign of the remaining terms,

$$J=-\int_\Omega e^{\gamma_n(1+(u_n-1)^+)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^- \le\int_\Omega |f|\,|v_n|=\omega(n) \tag{3.21}$$

As before, let us split J, but now into three terms as $J=J_1+J_2+J_3$, where

$$J_1=-\int_{\{u_n>1+\varepsilon+m\}} e^{\gamma_n(u_n)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-$$

$$J_2=-\int_{\{1+\varepsilon<u_n\le 1+\varepsilon+m\}} e^{\gamma_n(u_n)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-$$

$$J_3=-\int_{\{u_n\le 1+\varepsilon\}} e^{\gamma_n(1+(u_n-1)^+)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-\,.$$

We have, by (2.1),

$$J_1\ge-\int_{\{u_n>1+\varepsilon+m\}\cap\{u_n^+\le u^+\}} e^{\gamma_n(u_n)}a(x,u_n,Du_n)\cdot DT_m(G_{1+\varepsilon}(u^+))=\omega(n) \tag{3.22}$$

and

$$J_3 \geq -\int_{\{u_n<1+\varepsilon\}\cap\{u_n^+\leq u^+\}} e^{\gamma_n(1+(u_n-1)^+)} a(x,u_n,Du_n)\cdot DT_m(G_{1+\varepsilon}(u^+)) = \omega(n)\,.$$

Therefore, by (3.21), $J_2 \leq \omega(n)$. So that writing J_2 as

$$-\int_{\{1+\varepsilon<u_n\leq 1+\varepsilon+m\}} e^{\gamma_n(u_n)} a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+))) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^-\,,$$

and performing similar computations as before, we obtain

$$\limsup_{n\to\infty}\int_{\{1+\varepsilon<u_n\leq 1+\varepsilon+m\}} -e^{\gamma_n(u_n)} \Big(a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+))) - a(x,u_n,DT_m(G_{1+\varepsilon}(u^+)))\Big) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^- \leq 0\,.$$

Having in mind that the integrand is nonnegative and

$$0 < c = e^{\gamma_n(1+m+\varepsilon)} \leq e^{\gamma_n(u_n)}\chi_{\{1+\varepsilon<u_n\leq 1+\varepsilon+m\}}\,,$$

we deduce

$$\limsup_{n\to\infty}\int_\Omega -\Big(a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+))) - a(x,u_n,DT_m(G_{1+\varepsilon}(u^+)))\Big) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^- \leq 0\,. \tag{3.23}$$

Finally, adding (3.20) and (3.23), we get

$$\limsup_{n\to\infty}\int_\Omega \Big(a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+))) - a(x,u_n,DT_m(G_{1+\varepsilon}(u^+)))\Big) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big) \leq 0\,.$$

from where, applying a well–known lemma by Browder (see [15]), (3.14) follows.

Proof of (3.15)*:* As in the previous proof, we have to distinguish two cases. We begin by taking

$$e^{\gamma_n(1-(1-u_n)^+)}\Big(T_{1-\varepsilon,m}(u_n) - T_{1-\varepsilon,m}(u)\Big)^+$$

as test function in $(3.3)_n^-$ to get

$$\int_\Omega e^{\gamma_n(1-(1-u_n)^+)} a(x,u_n,Du_n)\cdot D\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ \tag{3.24}$$
$$+\int_{\{u_n\le 1\}} \frac{1}{\alpha} R_n\left(\frac{1}{|u_n-1|^\theta}\right) e^{\gamma_n(u_n)} \Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ a(x,u_n,Du_n)\cdot Du_n$$
$$+\int_\Omega \operatorname{sign}(u_n-1) e^{\gamma_n(1-(1-u_n)^+)} \Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2$$
$$= \int_\Omega f e^{\gamma_n(1-(1-u_n)^+)} \Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ .$$

Let us point out that

$$\int_\Omega \operatorname{sign}(u_n-1) e^{\gamma_n(1-(1-u_n)^+)} \Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2$$
$$= \int_{\{u_n\ge 1\}} e^{\gamma_n(1)} \Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2$$
$$- \int_{\{u_n\le 1\}} e^{\gamma_n(u_n)} \Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2$$
$$\ge - \int_{\{u_n\le 1\}} e^{\gamma_n(u_n)} \Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 .$$

Hence, using (2.1), (3.24) becomes

$$\int_\Omega e^{\gamma_n(1-(1-u_n)^+)} a(x,u_n,Du_n)\cdot D\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+$$
$$\le \int_\Omega |f|\, e^{\gamma_n(1-(1-u_n)^+)} |T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)|$$
$$\le e^{\gamma_n(1)} \int_\Omega |f|\,|T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)| = \omega(n) .$$

Splitting suitably the left hand side and taking into account that

$$DT_{1-\varepsilon,m}(u_n) \rightharpoonup DT_{1-\varepsilon,m}(u)$$

weakly in $L^2(\Omega;\mathbb{R}^N)$, we may follow the same procedure as above to obtain

$$\limsup_{n\to\infty} \int_\Omega e^{\gamma_n(T_{1-\varepsilon,m}(u_n))} \Big(a(x,u_n,DT_{1-\varepsilon,m}(u_n)) - a(x,u_n,DT_{1-\varepsilon,m}(u))\Big)$$
$$\cdot D\Big(T_{1-\varepsilon,m}(u_n)) - T_{1-\varepsilon,m}(u))\Big)^+ \le 0 .$$

Since $0 < c \le e^{\gamma_n(-m)} \le e^{\gamma_n(T_{1-\varepsilon,m}(u_n))}$, it follows that

$$\limsup_{n\to\infty} \int_\Omega \Big(a(x,u_n,DT_{1-\varepsilon,m}(u_n)) - a(x,u_n,DT_{1-\varepsilon,m}(u))\Big) \tag{3.25}$$
$$\cdot D\Big(T_{1-\varepsilon,m}(u_n)) - T_{1-\varepsilon,m}(u))\Big)^+ \le 0 .$$

To deal with the negative part, we just take

$$-\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^-$$

as test function in $(3.3)_n^-$. Then we have

$$-\int_\Omega a(x,u_n,Du_n)\cdot D\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^- \leq$$
$$\leq -\int_\Omega f\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^-$$
$$\leq \int_\Omega |f|\,|T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)| = \omega(n)\,.$$

Performing similar computations as above, it implies

$$\limsup_{n\to\infty}\int_\Omega -\Big(a(x,u_n,DT_{1-\varepsilon,m}(u_n))-a(x,u_n,DT_{1-\varepsilon,m}(u))\Big) \cdot D\Big(T_{1-\varepsilon,m}(u_n))-T_{1-\varepsilon,m}(u))\Big)^- \leq 0\,. \tag{3.26}$$

From (3.25) and (3.26), having in mind the Browder lemma, we get (3.15).

Step 5. $\lim_{\varepsilon\to 0}\int_{\{|u_n-1|<\epsilon\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 = 0$ UNIFORMLY IN n.

Fix ε such that $0<\varepsilon<1$, we will see that

$$\int_{\{|u_n-1|<\varepsilon\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 = \omega(\varepsilon)\,.$$

Let us take

$$e^{\left(\gamma_n(1-(1-u_n)^+)-\gamma_n(1-\varepsilon)\right)^+}-1$$

as test function in $(3.3)_n^-$. Observe that the points where this function is different from zero are those belonging to $\{1-\varepsilon<u_n\}$ and its derivative is different from zero in $\{1-\varepsilon<u_n<1\}$. So that

$$\int_{\{1-\varepsilon<u_n<1\}}\frac{1}{\alpha}R_n\left(\frac{1}{|u_n-1|^\theta}\right)e^{\left(\gamma_n(u_n)-\gamma_n(1-\varepsilon)\right)^+}a(x,u_n,Du_n)\cdot Du_n$$
$$-\int_{\{1-\varepsilon<u_n<1\}}R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\left(e^{\left(\gamma_n(1-(1-u_n)^+)-\gamma_n(1-\varepsilon)\right)^+}-1\right)$$
$$+\int_{\{1<u_n\}}R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\left(e^{\left(\gamma_n(1)-\gamma_n(1-\varepsilon)\right)}-1\right)$$
$$\leq\int_\Omega T_n(f)\left(e^{\left(\gamma_n(1-(1-u_n)^+)-\gamma_n(1-\varepsilon)\right)^+}-1\right).$$

Apply (2.1), cancel similar terms and drop a nonnegative one, then

$$\int_{\{1-\varepsilon<u_n<1\}}R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\leq\left(e^{\left(\gamma_n(1)-\gamma_n(1-\varepsilon)\right)}-1\right)\int_\Omega|f|\,.$$

Noting that $\gamma_n(1)-\gamma_n(1-\varepsilon)\leq\gamma(1)-\gamma(1-\varepsilon)=\frac{\varepsilon^{1-\theta}}{\alpha(1-\theta)}$, the previous inequality becomes

$$\int_{\{1-\varepsilon<u_n<1\}}R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\leq\left(e^{\frac{\varepsilon^{1-\theta}}{\alpha(1-\theta)}}-1\right)\int_\Omega|f|=\omega(\varepsilon)\,,$$

so that

$$\int_{\{1-\varepsilon<u_n<1\}}R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2=\omega(\varepsilon)\,,$$

uniformly in n.

Considering now

$$e^{\left(\gamma_n(1)-\gamma_n(1+\varepsilon)\right)} - e^{\left(\gamma_n((u_n-1)^++1)-\gamma_n(1+\varepsilon)\right)^+}$$

as test function in $(3.3)_n^-$ and performing similar manipulations, we get

$$\int_{\{1<u_n<1+\varepsilon\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \le \omega(\varepsilon)\,,$$

uniformly in n. Therefore,

$$\int_{\{|u_n-1|<\varepsilon\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 = \omega(\varepsilon)\,,$$

uniformly in n.

Step 6. Equiintegrability of $\left(R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\right)_{n\in\mathbb{N}}$.

In this Step we are going to prove that for every $\eta>0$ there exists $\delta_\eta>0$ satisfying

$$\forall\, E\subseteq\Omega : |E|<\delta_\eta \quad \sup_n \int_E R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 < \eta\,. \tag{3.27}$$

Indeed, if $m>1$ and $0<\varepsilon<m-1$, then

$$\begin{aligned}\int_E R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 &= \int_{E\cap\{|u_n-1|<\varepsilon\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \\ &+\int_{E\cap\{|u_n|>m\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \\ &+\int_{E\cap\{|u_n|\le m\}\cap\{|u_n-1|\ge\varepsilon\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\,.\end{aligned}\tag{3.28}$$

The first integral at the right hand side of (3.28) goes to zero as $\varepsilon\to0$ uniformly in n, by Step 5.

The second integral at the right hand side of (3.28) can be estimated as

$$\begin{aligned}\int_{E\cap\{|u_n|>m\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 &\le \int_{\{|u_n-1|>m-1\}} \frac{1}{|u_n-1|^\theta}|Du_n|^2 \le \\ &\le \frac{1}{(m-1)^\theta}\int_\Omega |Du_n|^2 \le \omega(m)\,,\end{aligned}$$

due to (3.9). Let $\eta>0$ be given. As a consequence of (3.28) and the previous estimates of its right hand side, we may find $\epsilon\in(0,1)$ and $m\in\mathbb{N}$, with $m\ge2$, satisfying

$$\int_E R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \le \frac{\eta}{2} + \int_{E\cap\{|u_n|\le m\}\cap\{|u_n-1|\ge\varepsilon\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\,.$$

With $\epsilon > 0$ and $m \in \mathbb{N}$ already chosen, we may estimate the last integral in (3.28):

$$\begin{aligned}\int_{E\cap\{|u_n|\le m\}\cap\{|u_n-1|\ge\varepsilon\}} & R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \\ &\le \frac{1}{\varepsilon^\theta}\int_{E\cap\{|u_n|\le m\}\cap\{|u_n-1|\ge\varepsilon\}}|Du_n|^2 \\ &= \frac{1}{\varepsilon^\theta}\int_E |DT_{m-1-\varepsilon}(G_{1+\varepsilon}(u_n^+))|^2 + \frac{1}{\varepsilon^\theta}\int_E |DT_{1-\varepsilon,m}(u_n))|^2 .\end{aligned}$$

By (3.14) and (3.15), the integrals

$$\int_E |DT_{m-1-\varepsilon}(G_{1+\varepsilon}(u_n^+))|^2 \qquad \text{and} \qquad \int_E |DT_{1-\varepsilon,m}(u_n))|^2$$

are uniformly small in n if $|E|$ is small enough. Therefore (3.27) holds true.

As a consequence of this Step 6, (3.11) and (3.12), we obtain that

$$\frac{|Du|^2}{|u-1|^\theta}\chi_{\{u\neq 1\}} \in L^1(\Omega) .$$

Now Lemma 2.2 implies

$$\frac{|Du|^2}{|u-1|^\theta} \in L^1(\Omega) .$$

Step 7. Weak convergence of the gradient term in $L^1(\Omega)$.

As a consequence of (3.11), (3.12) and Step 6, we already know that

$$\operatorname{sign}(u_n-1)\, R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \to \frac{\operatorname{sign}(u-1)}{|u-1|^\theta}|Du|^2 ,$$

strongly in $L^1(\{u\neq 1\})$. However, this Step is not straightforward, since we do not know what happens in $\{u=1\}$.

Fixed $\varphi \in L^\infty(\Omega)$, we will see that

$$\lim_{n\to\infty}\int_\Omega \operatorname{sign}(u_n-1)\, R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\varphi = \int_\Omega \frac{\operatorname{sign}(u-1)}{|u-1|^\theta}|Du|^2\varphi . \tag{3.29}$$

We obviously split the integral

$$\begin{aligned} H_1 &= \int_{\{u\neq 1\}} \operatorname{sign}(u_n-1)\, R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\varphi , \\ H_2 &= \int_{\{u=1\}} \operatorname{sign}(u_n-1)\, R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\varphi .\end{aligned}$$

By (3.11), (3.12), Step 6 and Vitali's theorem, we pass to the limit on n in the term H_1 getting

$$\int_\Omega \frac{\operatorname{sign}(u-1)}{|u-1|^\theta}|Du|^2\chi_{\{u\neq 1\}}\,\varphi .$$

As far as the term H_2 is concerned, we observe that, by Egorov's theorem, for every $\delta > 0$ there exists $\Omega_\delta \subseteq \Omega$ satisfying $|\Omega_\delta| < \delta$ and $u_n \to u$ uniformly in $\Omega\backslash\Omega_\delta$. Then, fixed $\eta > 0$, it yields

$$\int_{\{u=1\}\cap\Omega_\delta} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2\varphi < \frac{\eta}{2} \quad \forall n\in\mathbb{N}$$

for all $\delta \in (0, \delta_{\frac{\eta}{2\|\varphi\|_\infty}})$, where $\delta_{\frac{\eta}{2\|\varphi\|_\infty}}$ is that given by (3.27).

On the other hand, for every $\varepsilon > 0$ we may apply the uniform convergence to find $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|u_n - 1| < \varepsilon$ in the set $\{u = 1\} \cap (\Omega \backslash \Omega_\delta)$. So, it follows from this fact and Step 6 that

$$\int_{\{u=1\}\cap(\Omega\backslash\Omega_\delta)} R_n\left(\frac{1}{|u_n - 1|^\theta}\right)|Du_n|^2\varphi \leq \int_{\{|u_n-1|<\varepsilon\}} R_n\left(\frac{1}{|u_n - 1|^\theta}\right)|Du_n|^2\varphi < \frac{\eta}{2},$$

for n big enough. Thus, $\lim_{n\to+\infty} H_2 = 0$.

Summing up and having in mind Lemma 2.2, we have proved (3.29).

Step 8. Passage to the limit in $(3.3)_n^-$.

Let us fix $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and take it as test function in $(3.3)_n^-$. Then

$$\int_\Omega a(x, u_n, Du_n) \cdot D\varphi + \int_\Omega \operatorname{sign}(u_n - 1)\, R_n\left(\frac{1}{|u_n - 1|^\theta}\right)|Du_n|^2\varphi = \int_\Omega T_n(f)\,\varphi\,. \tag{3.30}$$

To pass to the limit in the principal part, we only have to apply (3.13), while Step 7 is enough to handle the second term. Since obviously $T_n(f) \to f$ in $L^1(\Omega)$, it follows that we pass to the limit in (3.30) obtaining (2.11). □

Remark 3.2. After proving Step 7 in the above proof, we may apply Theorem 3.1 in [10] to improve Step 4. Indeed, we get the strong convergence in $H_0^1(\Omega)$ of every truncation, not only those away from the singularity.

Proof of Theorem 2.4: We pass now to deal with problem $(2.4)^+$ *i.e.*

$$\begin{cases} -\operatorname{div}(a(x, u, Du)) = \dfrac{\operatorname{sign}(u-1)}{|u-1|^\theta}|Du|^2 + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

On the datum, we assume

$$f \in L^m(\Omega)\,, \quad \text{with } m \geq \frac{N}{2}\,. \tag{3.31}$$

Note that in this case, $\lim_{|s|\to+\infty} \gamma(s) = +\infty$ and

$$\lim_{|s|\to+\infty} \frac{e^{\gamma_n(s)}}{\psi_n(s)} = 0 \quad \text{uniformly in } n \tag{3.32}$$

as can easily be proved. Moreover, since each γ_n is bounded from below by $\gamma_n(1)$, and $\gamma_n(1)$ is bounded from below by $\gamma(1)$, there exists a constant C satisfying

$$0 < C < e^{\gamma_n(u_n)}\,, \qquad \text{for all } n \in \mathbb{N}\,. \tag{3.33}$$

We remark that all the test function used in the following proof are admissible since $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

We have to modify Steps 1–5, due to the different behaviour of function $\gamma(s)$ and $\psi(s)$.

Step 1′. Uniform estimates on $(\psi_n(u_n))_{n\in\mathbb{N}}$ and $((u_n))_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$.

We take in $(3.3)_n^+$ the function
$$v_n = e^{\gamma_n(u_n)}\psi_n(u_n) \in H_0^1(\Omega)\cap L^\infty(\Omega)\,.$$
With the same arguments we get the following estimate, for any $\varepsilon > 0$ fixed
$$\int_\Omega |D\psi_n(u_n)|^2 \le \int_\Omega |f|e^{\gamma_n(u_n)}\psi_n(u_n) \le \varepsilon\int_\Omega |f|\psi_n^2(u_n) + C(\varepsilon)\int_\Omega |f|$$
$$\le \varepsilon\|f\|_{L^{\frac{N}{2}}}\cdot\Big(\int_\Omega |\psi_n(u_n)|^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{N}} + C(\varepsilon)\|f\|_{L^1}\,.$$
Therefore, applying Sobolev's inequality and choosing ε sufficiently small, we get the desired estimate:
$$\int_\Omega |D\psi_n(u_n)|^2 \le C \qquad \text{for all } n\in\mathbb{N}\,. \tag{3.34}$$
We point out that as a bonus we have obtained an estimate of $\psi_n(u_n)$ in $L^{2N/(N-2)}(\Omega)$. As a consequence of (3.32), we have also proved the existence of a positive constant C satisfying
$$\int_\Omega \big(e^{\gamma_n(u_n)}\big)^{2N/(N-2)} \le C \qquad \text{for all } n\in\mathbb{N}\,. \tag{3.35}$$
Taking (3.33) into account, the estimate (3.34) also implies an uniform estimate on $(u_n)_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$.

Step 2′. Uniform $L^1(\Omega)$- estimates for $|Du_n|^2R_n(\frac{1}{|u_n-1|^\theta})$.

Define the function
$$\zeta_n(s) = \begin{cases} \gamma_n(s)-\gamma_n(1)\,, & \text{if } s>1\,;\\ 0\,, & \text{if } 0\le s\le 1\,;\\ -\gamma_n(s)\,, & \text{if } s<0\,;\end{cases}$$
Observe that it is a nondecreasing function satisfying $\zeta_n(s)\mathrm{sign}\,(s-1)\ge 0$ and
$$|\zeta_n(s)| \le \gamma_n(s)-\gamma_n(1) \le e^{(\gamma_n(s)-\gamma_n(1))} = e^{-\gamma_n(1)}e^{\gamma_n(s)}\,.$$
We also remark that its derivative is given by
$$\zeta_n'(s) = \frac{1}{\alpha}R_n\Big(\frac{1}{|s-1|^\theta}\Big) \quad \text{if } s\notin[0,1]\,.$$
We now use in $(3.3)_n^+$ the test function $v_n = \zeta_n(u_n)e^{\gamma_n(u_n)}$. Then cancelling similar terms, we get
$$\int_{\{u_n\notin[0,1]\}} e^{\gamma_n(u_n)}R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 \le \int_\Omega T_n(f)e^{\gamma_n(u_n)}\zeta_n(u_n)$$
$$\le e^{-\gamma_n(1)}\int_\Omega |f|e^{2\gamma_n(u_n)}\,.$$
Applying (3.31) and (3.35), we deduce that the last term is bounded. So that, by (3.33),
$$\int_{\{u_n\notin[0,1]\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \le C \quad \forall n\in\mathbb{N}\,. \tag{3.36}$$

On the other hand, consider $0 < k < 1$ and take $T_k(u_n^+)$ as a test function in $(3.3)_n^+$. We obtain

$$\int_{\{0\le u_n\le k\}} a(x,u_n,Du_n)\cdot Du_n = -\int_{\{0\le u_n\le 1\}} T_k(u_n)R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 + k\int_{\{u_n>1\}} R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 + \int_\Omega T_n(f)\,T_k(u_n^+)\,.$$

Since the left hand side is nonnegative, it follows that

$$\int_{\{0\le u_n\le 1\}} T_k(u_n)R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 \le k\int_{\{u_n>1\}} R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 + k\int_\Omega |f|\,.$$

Hence, by (3.36), it yields

$$\int_{\{0\le u_n\le 1\}} \frac{T_k(u_n)}{k}R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 \le \int_{\{u_n>1\}} R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 + \int_\Omega |f| \le C\,.$$

Letting k goes to 0, Fatou's lemma implies

$$\int_{\{0\le u_n\le 1\}} R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 \le C\,,\quad \text{for all } n\in\mathbb{N}\,. \tag{3.37}$$

Adding (3.36) to (3.37), the uniform estimate in $L^1(\Omega)$ of $|Du_n|^2R_n(\frac{1}{|u_n-1|^\theta})$ is proved.

Step 4′. Strong convergence in $H_0^1(\Omega)$ of truncations away from the singularity.

Keeping in mind the notations of Step 4, we have to prove (3.14) and (3.15).
Proof of (3.14). If we take as test function in $(3.3)_n^+$

$$v_n = e^{\gamma_n(1+(u_n-1)^+)}\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+$$

we get, cancelling similar terms and neglecting a negative term of the right hand side,

$$\begin{aligned}\int_\Omega e^{\gamma_n(1+(u_n-1)^+)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ \\ \le \int_{\{u_n\le 1\}} v_n\mathrm{sign}\,(u_n-1)R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 + \int_\Omega |f|\,|v_n| \\ \le \int_\Omega |f|\,e^{\gamma_n(1+(u_n-1)^+)}\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ = \omega(n)\,.\end{aligned} \tag{3.38}$$

The last equality is due to (3.31), (3.35)), and to the fact that the sequence $\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+$ converges strongly to zero in $L^s(\Omega)$ for any

$s \geq 1$. The left hand side of (3.38) may be split into two parts as $I = I_1 + I_2$, where

$$I_1 = \int_{\{1+\varepsilon+m<u_n\}} e^{\gamma_n(u_n)} a(x, u_n, Du_n) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ ,$$

$$I_2 = \int_{\{1+\varepsilon<u_n\leq 1+\varepsilon+m\}} e^{\gamma_n(u_n)} a(x, u_n, Du_n) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ ,$$

since

$$\int_{\{u_n\leq 1+\varepsilon\}} e^{\gamma_n(1+(u_n-1)^+)} a(x, u_n, Du_n) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ = 0 .$$

To deal with I_1, first apply (2.1) to get

$$I_1 \geq -\int_{\{1+\varepsilon+m<u_n\}\cap\{u_n^+\geq u^+\}} e^{\gamma_n(u_n)} a(x, u_n, Du_n) \cdot DT_m(G_{1+\varepsilon}(u^+)) .$$

Observe that the sequence $(e^{\gamma_n(u_n)}|a(x, u_n, Du_n)|)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ due to (2.2) and Step $1'$. Moreover

$$|DT_m(G_{1+\varepsilon}(u^+))|\chi_{\{m<G_{1+\varepsilon}(u_n^+)\}} \longrightarrow 0$$

strongly in $L^2(\Omega)$, so it implies

$$I_1 \geq \omega(n) . \tag{3.39}$$

Having in mind that

$$I_2 = \int_\Omega e^{\gamma_n(u_n)} a(x, u_n, DT_m(G_{1+\varepsilon}(u_n^+))) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ ,$$

it follows from (3.38), (3.39) that

$$\begin{aligned}\int_{\{1+\epsilon\leq u_n\leq m+1+\epsilon\}} & e^{\gamma_n(u_n)} a(x, u_n, DT_m(G_{1+\varepsilon}(u_n^+))) \\ & \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ \leq \omega(n) .\end{aligned} \tag{3.40}$$

Next the weak convergence

$$DT_m(G_{1+\varepsilon}u_n^+) \rightharpoonup DT_m(G_{1+\varepsilon}u^+)$$

in $L^2(\Omega;\mathbb{R}^N)$, (3.40) and (3.33), yield

$$\begin{aligned}\limsup_{n\to\infty} \int_{\{1+\epsilon\leq u_n\leq m+1+\epsilon\}} & e^{\gamma_n(u_n)} \\ & \Big(a(x, u_n, DT_m(G_{1+\varepsilon}(u_n^+))) - a(x, u_n, DT_m(G_{1+\varepsilon}(u^+)))\Big) \\ & \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^+ \leq 0 .\end{aligned} \tag{3.41}$$

A similar inequality can be obtained for $\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^-$. To get it, take

$$v_n = -e^{\gamma_n(1-(1-u_n)^+)}\Big(T_m(G_{1+\varepsilon}(u_n^+)) - T_m(G_{1+\varepsilon}(u^+))\Big)^-$$

as test function in $(3.3)_n^+$. Then, with the usual arguments,

(3.42)
$$J=-\int_\Omega e^{\gamma_n(1-(1-u_n)^+)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^- \leq \int_\Omega |f|\, e^{\gamma_n(1-(1-u_n)^+}\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^- = \omega(n)$$

The last equality follows from inequalities (3.31), (3.35)), and from the convergence of $\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-$ to zero strongly in $L^s(\Omega)$ for any $s\geq 1$. As before, let us split J, but now into three terms as $J=J_1+J_2+J_3$, where

$$J_1=-\int_{\{u_n>1+\varepsilon+m\}} e^{\gamma_n(1)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-$$

$$J_2=-\int_{\{1+\varepsilon<u_n\leq 1+\varepsilon+m\}} e^{\gamma_n(1)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-$$

$$J_3=-\int_{\{u_n\leq 1+\varepsilon\}} e^{\gamma_n(1-(1-u_n)^+)}a(x,u_n,Du_n)\cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^- .$$

We have

$$J_1\geq -\int_{\{u_n>1+\varepsilon+m\}\cap\{u_n^+\leq u^+\}} e^{\gamma_n(1)}a(x,u_n,Du_n)\cdot DT_m(G_{1+\varepsilon}(u^+))=0$$

and

$$J_3\geq -\int_{\{u_n<1+\varepsilon\}\cap\{u_n^+\leq u^+\}} e^{\gamma_n(1-(1-u_n)^+)}a(x,u_n,Du_n)\cdot DT_m(G_{1+\varepsilon}(u^+))=\omega(n)\,.$$

Therefore, by (3.42), $J_2\leq\omega(n)$. So, arguing as before, we easily obtain also

$$\limsup_{n\to\infty}\int_\Omega -\Big(a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+)))-a(x,u_n,DT_m(G_{1+\varepsilon}(u^+)))\Big) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)^-\leq 0\,. \tag{3.43}$$

Finally, adding (3.41) and (3.43), we get

$$\limsup_{n\to\infty}\int_\Omega \Big(a(x,u_n,DT_m(G_{1+\varepsilon}(u_n^+)))-a(x,u_n,DT_m(G_{1+\varepsilon}(u^+)))\Big) \cdot D\Big(T_m(G_{1+\varepsilon}(u_n^+))-T_m(G_{1+\varepsilon}(u^+))\Big)\leq 0\,.$$

from where, applying again Browder's Lemma (see [15]), (3.14) follows.

Proof of (3.15). As in the previous proof, we have to distinguish two cases. We begin by taking

$$v_n=e^{\gamma_n(1+(1-u_n)^-)}\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+$$

as test function in $(3.3)_n^+$ to get, by (3.31) and (3.35), that

$$\int_\Omega e^{\gamma_n(1+(1-u_n)^-)}a(x,u_n,Du_n)\cdot D\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ \\ \le \int_\Omega |f|e^{\gamma_n(1+(1-u_n)^-)}\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ = \omega(n)\,.$$

This estimate implies, splitting as before the first integral and observing that $e^{\gamma_n(1-(1-T_{1-\varepsilon,m}(u_n))^-)} = e^{\gamma(1)}$,

$$\limsup_{n\to\infty}\int_\Omega a(x,u_n,DT_{1-\varepsilon,m}u_n)\cdot D\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^+ \le 0\,.$$

Taking into account that

$$DT_{1-\varepsilon,m}(u_n) \rightharpoonup DT_{1-\varepsilon,m}(u)$$

weakly in $L^2(\Omega;\mathbb{R}^N)$, we have also

$$\limsup_{n\to\infty}\int_\Omega \Big(a(x,u_n,DT_{1-\varepsilon,m}(u_n)) - a(x,u_n,DT_{1-\varepsilon,m}(u))\Big) \\ \cdot D\Big(T_{1-\varepsilon,m}(u_n)) - T_{1-\varepsilon,m}(u))\Big)^+ \le 0\,. \tag{3.44}$$

To deal with the negative part, we just take

$$v_n = -e^{\gamma_n(1-(1-u_n)^+)}\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^-$$

as test function in $(3.3)_n^+$. Then we have

$$-\int_\Omega e^{\gamma_n(1-(1-u_n)^+)}a(x,u_n,Du_n)\cdot D\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^- \le \\ \le \int_\Omega |f|\, e^{(1-(1-u_n)^+)}\Big(T_{1-\varepsilon,m}(u_n)-T_{1-\varepsilon,m}(u)\Big)^- = \omega(n)\,,$$

by (3.31) and (3.35). Performing similar computations as above, it implies

(3.45)

$$\limsup_{n\to\infty}\int_\Omega -e^{\gamma_n(T_{1-\varepsilon,m}(u_n))}\Big(a(x,u_n,DT_{1-\varepsilon,m}(u_n)) - a(x,u_n,DT_{1-\varepsilon,m}(u))\Big) \\ \cdot D\Big(T_{1-\varepsilon,m}(u_n)) - T_{1-\varepsilon,m}(u))\Big)^- \le 0\,.$$

Since $e^{\gamma_n(T_{1-\varepsilon,m}(u_n))} \ge e^{\gamma_n(1)} \ge e^{\gamma(1)}$, from (3.44) and (3.45), on account of the Browder lemma, we get (3.15).

Step 5′. $\lim_{\varepsilon\to 0}\int_{\{|u_n-1|<\varepsilon\}}|Du_n|^2R_n\left(\frac{1}{|u_n-1|^\theta}\right) = 0$ UNIFORMLY IN n.

Consider the function given by

$$\xi_n(s) = \begin{cases} 0\,, & \text{if } s\le 1\,;\\ \gamma_n(s)-\gamma_n(1)\,, & \text{if } 1\le s\le 1+\epsilon\,;\\ \gamma_n(1+\epsilon)-\gamma_n(1)\,, & \text{if } s\ge 1+\epsilon\,. \end{cases}$$

It is a nonnegative and nondecreasing function such that

$$0\le \xi_n(s)\le \gamma_n(1+\epsilon)-\gamma_n(1)\le \gamma(1+\epsilon)-\gamma(1)$$

holds for all $n \in \mathbb{N}$ and all $s \in \mathbb{R}$. We use as test function in $(3.3)_n^+$

$$v_n = \xi_n(u_n)e^{\gamma_n(u_n)} \,.$$

Then we deduce

$$\int_\Omega \xi_n'(u_n)e^{\gamma_n(u_n)}a(x,u_n,Du_n)\cdot Du_n \le \int_\Omega T_n(f)\xi_n(u_n)e^{\gamma_n(u_n)} \,,$$

and so

$$\int_{\{1\le u_n<1+\epsilon\}} e^{\gamma_n(u_n)}R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 \;\le\; (\gamma(1+\epsilon)-\gamma(1))\int_\Omega |f|e^{\gamma_n(u_n)} \,.$$

Finally, it follows from (3.33), (3.31) and (3.35), that

$$\int_{\{1\le u_n<1+\epsilon\}} R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 \le C(\gamma(1+\epsilon)-\gamma(1)) = \omega(\epsilon) \,. \tag{3.46}$$

Consider now the function defined by

$$\bar\xi_n(s) = \begin{cases} 0\,, & \text{if } s\le 1-\epsilon\,;\\ \gamma_n(1-\epsilon)-\gamma_n(s)\,, & \text{if } 1-\epsilon\le s\le 1\,;\\ \gamma_n(1-\epsilon)-\gamma_n(1)\,, & \text{if } s\ge 1\,,\end{cases}$$

which is nonnegative and nondecreasing. It also satisfies

$$0\le \bar\xi_n(s) \le \gamma_n(1-\epsilon)-\gamma_n(1) \le \gamma(1-\epsilon)-\gamma(1)$$

for all $n \in \mathbb{N}$ and all $s \in \mathbb{R}$. Taking

$$v_n = \bar\xi_n(u_n)e^{\gamma_n(1+(u_n-1)^+)}$$

as test function in $(3.3)_n^+$, simplifying and dropping a negative term of the right hand side, it yields

$$\begin{aligned}\int_{\{1-\epsilon<u_n\le 1\}} e^{\gamma_n(1)}R_n\Big(\frac{1}{|u_n-1|^\theta}\Big)|Du_n|^2 &\le \int_\Omega T_n(f)\bar\xi_n(u_n)e^{\gamma_n(1+(u_n-1)^+)}\\ &\le (\gamma(1-\epsilon)-\gamma(1))\int_\Omega |f|e^{\gamma_n(u_n)} \,.\end{aligned}$$

Applying (3.33), (3.31) and (3.35), we obtain

$$\int_{\{1-\epsilon<u_n\le 1\}} R_n\left(\frac{1}{|u_n-1|^\theta}\right)|Du_n|^2 \le \omega(\varepsilon)\,.$$

This estimate and (3.46) prove Step 5′.

The remaining Steps 6, 7, 8 are proved exactly as in the first part. This concludes the proof. □

Proof of Proposition 2.5: Assume, to get a contradiction, that there exists $u \in H_0^1(\Omega)$ which satisfies $u < 1$ and is a solution of

$$\begin{cases} -\Delta u - \frac{|Du|^2}{|u-1|^\theta} = \lambda f\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.47}$$

Take $\left(e^{-\frac{1-(1-u)^{1-\theta}}{1-\theta}} + C\right)\phi_1$ as test function, where C is a positive constant to be determined. Then

$$\int_\Omega \left(e^{-\frac{1-(1-u)^{1-\theta}}{1-\theta}} + C\right) Du\cdot D\phi_1 - \int_\Omega \frac{|Du|^2}{|1-u|^\theta} e^{-\frac{1-(1-u)^{1-\theta}}{1-\theta}}\phi_1$$
$$- \int_\Omega \frac{|Du|^2}{|u-1|^\theta}\left(e^{-\frac{1-(1-u)^{1-\theta}}{1-\theta}} + C\right)\phi_1 = \int_\Omega \lambda f \left(e^{-\frac{1-(1-u)^{1-\theta}}{1-\theta}} + C\right)\phi_1\,.$$

Denoting

$$\Psi(s) = \int_0^s e^{-\frac{1-(1-\sigma)^{1-\theta}}{1-\theta}} + C\, d\sigma\,,$$

the above equality becomes

$$\int_\Omega D\Psi(u)\cdot D\phi_1 - \lambda f\phi_1\left(e^{-\frac{1-(1-u)^{1-\theta}}{1-\theta}} + C\right) = \int_\Omega \left(2e^{-\frac{1-(1-u)^{1-\theta}}{1-\theta}} + C\right)\frac{|Du|^2}{|u-1|^\theta}\phi_1 \ge 0\,. \tag{3.48}$$

On the other hand, we may take $\Psi(u)$ as test function in (2.13) obtaining

$$\int_\Omega D\Psi(u)\cdot D\phi_1 = \int_\Omega \lambda_1 f\phi_1\Psi(u)\,.$$

Substituting this equality in (3.48), we get

$$\int_\Omega f\phi_1\left[\lambda_1\Psi(u) - \lambda\left(e^{-\frac{1-(1-u)^{1-\theta}}{1-\theta}} + C\right)\right] \ge 0\,. \tag{3.49}$$

Consider now the function defined by

$$\Phi(s) = \Psi(s) - \frac{\lambda}{\lambda_1}\left(e^{-\frac{1-(1-s)^{1-\theta}}{1-\theta}} + C\right)\,.$$

Denoting $I = \int_0^1 e^{-\frac{1-(1-\sigma)^{1-\theta}}{1-\theta}}\, d\sigma$, it is straightforward that

$$\Phi'(s) = e^{-\frac{1-(1-s)^{1-\theta}}{1-\theta}} + C + \frac{\lambda}{\lambda_1}\frac{e^{-\frac{1-(1-s)^{1-\theta}}{1-\theta}}}{|1-s|^\theta} > 0$$
$$\Phi(1) = I + C - \frac{\lambda}{\lambda_1}(e^{-\frac{1}{1-\theta}} + C) < 0$$

whenever $C > \dfrac{I - (\lambda/\lambda_1)e^{-\frac{1}{1-\theta}}}{(\lambda/\lambda_1) - 1}$. Hence, $\Phi(s)$ is negative if $s < 1$ and so the integrand in (3.49) is a nonpositive function. Thus, $f(x)\Phi_1(x) = 0$ a.e. in Ω and, by (2.13),

$$0 = \lambda_1\int_\Omega f\phi_1^2 = \int_\Omega |D\phi_1|^2\,.$$

This contradiction proves Proposition 2.5. ☐

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Daniela Giachetti
Dipartimento di Scienze di Base e Applicate per l' Ingegneria,
Università di Roma "La Sapienza",
via Scarpa 16, 00161 Roma, Italy.
E-mail address: daniela.giachetti@sbai.uniroma1.it

Sergio Segura de León
Departament d'Anàlisi Matemàtica, Universitat de València,
Dr. Moliner 50, 46100 Burjassot, València, Spain.
E-mail address: sergio.segura@uv.es