THE 1–LAPLACIAN ELLIPTIC EQUATION WITH INHOMOGENEOUS ROBIN BOUNDARY CONDITIONS

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Abstract. In this paper we study existence of solutions to the 1–Laplacian elliptic equation with inhomogeneous Robin boundary conditions. It is also analyzed from the point of view of the Euler–Lagrange equation of a lower semicontinuous functional. We see the equivalence between the solutions of the elliptic problem and the minimizers of the functional.

1. Introduction

In this paper, we are concerned with the following Robin problem:

$$\begin{cases} -\mathrm{div}\Big(\dfrac{Du}{|Du|}\Big) = 0, & \text{in } \Omega, \\ \lambda u + \Big[\dfrac{Du}{|Du|}, \nu\Big] = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$. Here and in what follows, Ω will denote an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. So there exists ν, a unit outward normal vector field defined at $\mathcal{H}^{N-1}$–almost every point of $\partial\Omega$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure. As far as the datum g is concerned, it belongs to $L^2(\partial\Omega)$.

Boundary value problems for this type of elliptic equations has been studied by several authors. The homogeneous Neumann problem was tackled in [3] (where the correct concept of solution is introduced giving sense to $\frac{Du}{|Du|}$) and also in [2], where the authors deal with a nonlinear boundary condition: $-\frac{Du}{|Du|} \cdot \nu \in \beta(u)$. For inhomogeneous Neumann boundary conditions we refer to [12]. On the other hand, several papers have dealt with Dirichlet boundary conditions either with a nontrivial right hand side (see [4], [8], [9], [13], [14] and the book [5]), or studying its connection with functions of least gradient [11]. However, up to our knowledge, this is the first time that nonhomogeneous Robin boundary conditions for the 1–Laplacian are studied. When one compares this problem with the previous ones there are important differences.

In this kind of problems, a mayor difficulty appears to make sense the boundary condition, which need not be achieved (see Example 2.5). From a variational approach we would like to minimize the functional defined by

$$I[u] = \int_\Omega |Du| + \int_{\partial\Omega} \Big(\frac{\lambda}{2}u^2 - g\Big)\, d\mathcal{H}^{N-1}\,.$$

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Keeping in mind that this functional is not lower semi–continuous with respect to the $L^1(\Omega)$–convergence, we have to consider its semicontinuous envelope as given by (2.12) below.

From the viewpoint of the equation, we need to truncate the expression appearing in the boundary condition due to the condition $\|[\frac{Du}{|Du|},\nu]\|_\infty \leq 1$. Indeed, we replace $\lambda u - g$ with $T_1(\lambda u - g)$, where the truncature operator is defined by

$$T_k(r) := [k - (k - |r|)^+]\,\mathrm{sign}\,(r)\,, \qquad r \in \mathbb{R},\ k > 0\,.$$

Our strategy to obtain existence of a solution to (1.1) is to take the limit as $p \searrow 1$ of solutions to the following problems

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0, & \text{in } \Omega, \\ -|\nabla u|^{p-2}\big[\nabla u, \nu\big] = T_1(\lambda u - g), & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

Our main result states that the functions u_p converge pointwise to a measurable function u that is a solution to our problem.

Theorem 1.1. *For every $g \in L^2(\partial\Omega)$ there exists a weak solution (in the sense of Definition 2.3, see Section 2) to* (1.1) *that can be obtained taking the limit as $p \searrow 1$ of a sequence of solutions u_p to* (1.2).

In Section 2, we prove this main Theorem and then we relate the solutions to problem (1.1) with the minimizers of the lower semicontinuous functional defined by (2.12). The last section is concerned with the corresponding results that holds true in the limiting problem.

2. An elliptic problem with Robin boundary conditions

This Section is devoted to the study of problem (1.1). We study approximating problems involving the p–Laplacian, introduce our concept of solution, prove the existence result and relate the solutions with the minimizers of the corresponding functional.

2.1. Approximating problems.

As was mentioned in the introduction, to approximate the problem (1.1) we consider the following ones:

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0, & \text{in } \Omega\,, \\ -|\nabla u|^{p-2}\big[\nabla u, \nu\big] = T_1(\lambda u - g), & \text{on } \partial\Omega\,, \end{cases} \tag{2.3}$$

with $1 < p < 2$.

Definition 2.1. *We say that $u \in W^{1,p}(\Omega)$ is a weak solution to* (2.3) *if it holds*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx + \int_{\partial\Omega} T_1(\lambda u - g)\varphi\, d\mathcal{H}^{N-1} = 0 \tag{2.4}$$

for every $\varphi \in W^{1,p}(\Omega)$.

Given $g \in L^2(\partial\Omega)$, there exists a Borel function $\Gamma_g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ such that for all $x \in \partial\Omega$, the function $r \mapsto \Gamma_g(x,r)$ is convex and Lipschitz continuous with Lipschitz constant less or equal to 1 and satisfying

$$\partial_r \Gamma_g(x,r) = T_1(\lambda r - g(x)). \tag{2.5}$$

Note that the function $\Gamma_g(x,r)$ can be defined explicitly as

$$\Gamma_g(x,r) = \frac{1}{\lambda}\left(\phi(\lambda r - g(x)) - \frac{|g(x)|^2}{2}\right),$$

where

$$\phi(t) := \begin{cases} \dfrac{|t|^2}{2} & \text{if } |t| \le 1 \\ |t| - \frac{1}{2} & \text{if } |t| > 1. \end{cases} \tag{2.6}$$

Therefore

$$\Gamma_g(x,r) = \begin{cases} \dfrac{\lambda r^2}{2} - rg & \text{if } |\lambda r - g| \le 1 \\ \left|r - \dfrac{g}{\lambda}\right| - \dfrac{1}{2\lambda} - \dfrac{|g|^2}{2\lambda} & \text{if } |\lambda r - g| > 1. \end{cases}$$

Theorem 2.2. *For every $g \in L^2(\partial\Omega)$ there exists a unique weak solution to* (2.3).

Proof. Consider in $W^{1,p}(\Omega)$ the functional defined by

$$I_p(u) := \frac{1}{p}\int_\Omega |\nabla u(x)|^p\,dx + \int_{\partial\Omega} \Gamma_g(x,u(x))\,d\mathcal{H}^{N-1}.$$

It is straightforward that this functional is strictly convex. In regards to lower semicontinuity with respect to the weak convergence, it is a consequence of the compactness of the trace embedding (see [16, Theorem 6.2]).

As far as coerciveness is concerned, observe that

$$\phi(t) \ge |t| - \frac{1}{2}, \qquad \forall t \in \mathbb{R},$$

and so

$$\Gamma_g(x,u(x)) \ge \left|u(x) - \frac{g(x)}{\lambda}\right| - \frac{1}{2\lambda} - \frac{g(x)^2}{2\lambda} \ge |u(x)| - \frac{1}{\lambda}|g(x)| - \frac{1}{2\lambda} - \frac{g(x)^2}{2\lambda}.$$

Therefore, we have that I_p is bigger (up to a constant) than the functional given by

$$J[u] = \frac{1}{p}\int_\Omega |\nabla u|^p + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1}.$$

Now, by the generalized Poincaré inequality (see [19]) and the Trace Theorem (see [16]), we have that $\left(\int_\Omega |\nabla u|^p\right)^{1/p} + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1}$ defines a norm on $W^{1,p}(\Omega)$, equivalent to the usual one. Thus, it follows that J is coercive, and consequently I_p is so. Hence (by standard arguments) it has a unique minimum in $W^{1,p}(\Omega)$, which is the unique weak solution to (2.3). ■

2.2. **Definition of solution.** Let us state precisely what is our definition of a solution to (1.1). This concept of solution was introduced in [3] (see also the book [5]). Here we only adapt this definition to the appearance of Robin boundary conditions.

Since we study equations in which the 1–Laplacian occurs, our natural energy space is the space of bounded variation functions in Ω, that is functions $u \in L^1(\Omega)$ such that its distributional gradient Du is a bounded Radon measure with finite total variation $|Du|$. We will denote it by $BV(\Omega)$. For background in functions of bounded variation, we refer to [1]. We also need to recall some definitions introduced by Anzellotti in [6] (see also [5]) and the Green formula he derives there. Let

$$X_N(\Omega) = \left\{ \mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) \ : \ \mathrm{div}\,(\mathbf{z}) \in L^N(\Omega) \right\}.$$

If $\mathbf{z} \in X_N(\Omega)$ and $w \in BV(\Omega)$, we define $(\mathbf{z}, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Dw), \varphi \rangle := -\int_\Omega w\,\varphi\,\mathrm{div}\,(\mathbf{z})\,dx - \int_\Omega w\,\mathbf{z}\cdot\nabla\varphi\,dx.$$

The distribution $(\mathbf{z}, Dw)$ is actually a Radon measure with finite total variation. The measures $(\mathbf{z}, Dw)$, $|(\mathbf{z}, Dw)|$ are absolutely continuous with respect to the measure $|Dw|$ and

$$\left| \int_B (\mathbf{z}, Dw) \right| \le \int_B |(\mathbf{z}, Dw)| \le \|\mathbf{z}\|_{L^\infty(U)} \int_B |Dw|$$

holds for all Borel sets B and for all open sets U such that $B \subset U \subset \Omega$.

In [6], a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z} \in X_N(\Omega)$ is defined. More precisely, it is proved that there exists a linear operator $\gamma : X_N(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(\mathbf{z})\|_\infty \le \|\mathbf{z}\|_\infty$$

and

$$\gamma(\mathbf{z})(x) = \mathbf{z}(x)\cdot\nu(x) \qquad \text{for all } x \in \partial\Omega \ \text{ if } \ \mathbf{z} \in C^1(\overline{\Omega}, \mathbb{R}^N).$$

We shall denote $\gamma(\mathbf{z})(x)$ by $[\mathbf{z}, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[\mathbf{z}, \nu]$ and the measure $(\mathbf{z}, Dw)$, for $\mathbf{z} \in X_N(\Omega)$ and $w \in BV(\Omega)$, is established

$$\int_\Omega w \ \mathrm{div}\,(\mathbf{z}) \ dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w \ d\mathcal{H}^{m-1}. \tag{2.7}$$

We are now ready to introduce our concept of solution.

Definition 2.3. *Given $g \in L^2(\partial\Omega)$, we say that $u \in BV(\Omega) \cap L^2(\partial\Omega)$ is a weak solution to* (1.1) *if there exists a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ satisfying*

$$\|\mathbf{z}\|_\infty \le 1\,, \tag{2.8}$$

$$\mathrm{div}\,(\mathbf{z}) = 0\,, \qquad \textit{in } \mathcal{D}'(\Omega)\,, \tag{2.9}$$

$$(\mathbf{z}, Du) = |Du|\,, \tag{2.10}$$

$$-[\mathbf{z}, \nu] = T_1(\lambda u - g)\,, \qquad \mathcal{H}^{N-1}\textit{-a.e. on } \partial\Omega\,. \tag{2.11}$$

Remark 2.4. The truncation of the boundary condition (2.11) might seem an arbitrary condition and give the feeling that does not define “the right solution” to the Robin problem. Evidences that our definition provides the correct solution are:

(1) The Robin problem for our equation is the Euler–Lagrange equation associated to the minimization of the functional

$$I[u] = \int_\Omega |Du| + \int_{\partial\Omega} \left(\frac{\lambda}{2} u^2 - g \right) d\mathcal{H}^{N-1}\,.$$

This functional is not lower semicontinuous with respect to the $L^1(\Omega)$–convergence; its lower semicontinuous envelope is the functional given by

$$I_1(u) := \int_\Omega |Du| + \int_{\partial\Omega} \Gamma_g(x, u(x))\, d\mathcal{H}^{N-1}\,. \tag{2.12}$$

Since $\partial_r \Gamma_g(x,r) = T_1(\lambda r - g(x))$, hence the corresponding Euler–Lagrange equation should involve the boundary condition $-[\mathbf{z},\nu] = T_1(\lambda u - g)$.

(2) The boundary condition $\lambda u + [\mathbf{z},\nu] = \lambda h$ becomes $u = h$ as λ tends to ∞. Likewise, $-[\mathbf{z},\nu] = T_1(\lambda(u-h))$ becomes $-[\mathbf{z},\nu] = \operatorname{sign}(u-h)$ when λ goes to ∞. This condition $-[\mathbf{z},\nu] = \operatorname{sign}(u-h)$ is the one used to study the Dirichlet problem for the 1-Laplacian.

Another evidence of the need of truncate the boundary condition can be found in Remark 2.7 and Proposition 2.13.

We see in the next example that the boundary condition need not be achieved.

Example 2.5. Let Ω be a halfmoon–shaped set, whose boundary consists of a concave zone and a convex one. In order to be more concrete, take $\Omega = (\mathbb{R}^2 \backslash B_1(0,0)) \cap B_1(0,1)$. Then, $\partial\Omega = A \cup B$, where

$$A := \partial\Omega \cap \partial B_1(0,0) = \left\{ (x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1, \ -\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2} \right\}$$

and

$$B := \partial\Omega \cap \partial B_1(0,1) = \left\{ (x,y) \in \mathbb{R}^2 \ : \ x^2 + (y-1)^2 = 1, \ y \ge \frac{1}{2} \right\}.$$

Consider the vector field $\mathbf{z}(x,y) := -\dfrac{(x,y)}{x^2+y^2}$. Observe that $[\mathbf{z},\nu] = 1$ on A and, owing to $y = \frac{x^2+y^2}{2}$ on B, $[\mathbf{z},\nu] = -\dfrac{x^2+y^2-y}{x^2+y^2} = -\dfrac{1}{2}$ on B. Fix $\lambda > 1$ and take the boundary datum g defined by

$$g(x,y) := \begin{cases} \lambda & \text{if } (x,y) \in A \\ [\mathbf{z},\nu] & \text{if } (x,y) \in B. \end{cases}$$

Then, it is easy to check that $u \equiv 0$ is a weak solution of problem (1.1) according to Definition 2.3. Nevertheless, we have

$$\lambda u + [\mathbf{z},\nu] = 1 \ne \lambda = g \qquad \text{on } A\,.$$

2.3. **Proof of the existence result.** In this subsection we prove our main result for the elliptic problem (1.1).

Proof of Theorem 1.1. Consider u_p the unique solution to (2.3); taking λu_p as test function in (2.4), we get

$$0 = \lambda \int_\Omega |\nabla u_p|^p + \int_{\partial\Omega} T_1(\lambda u_p - g)(\lambda u_p - g)\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} T_1(\lambda u_p - g) g\, d\mathcal{H}^{N-1}$$

and this implies

$$\lambda \int_\Omega |\nabla u_p|^p + \int_{\partial\Omega} T_1(\lambda u_p - g)(\lambda u_p - g)\, d\mathcal{H}^{N-1} \le \int_{\partial\Omega} |g|\, d\mathcal{H}^{N-1}\,. \tag{2.13}$$

Since

$$|\lambda u_p - g| \le T_1(\lambda u_p - g)(\lambda u_p - g) + 1\,,$$

we deduce

$$\begin{aligned}\int_{\partial\Omega} |\lambda u_p|\, d\mathcal{H}^{N-1} &\le \int_{\partial\Omega} |g|\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} |\lambda u_p - g|\, d\mathcal{H}^{N-1} \\ &\le \int_{\partial\Omega} |g|\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} T_1(\lambda u_p - g)(\lambda u_p - g)\, d\mathcal{H}^{N-1} + \mathcal{H}^{N-1}(\partial\Omega)\,.\end{aligned}$$

Going back to (2.13), it yields

$$\int_\Omega |\nabla u_p|^p + \int_{\partial\Omega} |u_p|\, d\mathcal{H}^{N-1} \le C\Big(1 + \int_{\partial\Omega} |g|\, d\mathcal{H}^{N-1}\Big)\,.$$

Thus, $(u_p)_p$ is bounded in $BV(\Omega)$ and so we may find $u \in BV(\Omega)$ satisfying, up to subsequences,

$$\nabla u_p \rightharpoonup Du \quad \text{*–weakly in the sense of measures;} \tag{2.14}$$

$$u_p \to u \quad \text{a.e. in } \Omega\,; \tag{2.15}$$

$$u_p \to u \quad \text{strongly in } L^r(\Omega) \quad \text{for } 1 \le r < \frac{N}{N-1}\,; \tag{2.16}$$

Moreover, working as in [3] (see also the proof of [4, Proposition 3] and of [13, Proposition 4.1]), we can prove that there exists $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \le 1$ and

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}\,, \quad \text{weakly in } L^q(\Omega) \quad \forall q < \infty\,. \tag{2.17}$$

Now take $\varphi \in C_0^\infty(\Omega)$ as test function in (2.4) to get

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi = 0\,.$$

Letting p go to 1, we obtain (2.9).

To prove (2.10), consider $\varphi \in C_0^\infty(\Omega)$, with $\varphi \ge 0$, and take φu_p as test function in (2.4). Then

$$\int_\Omega u_p |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi + \int_\Omega |\nabla u_p|^p \varphi = 0\,,$$

To deal with the second term, we apply Young's inequality and the lower semi–continuity of the total variation, while the limit in the first one follows from using (2.16) and (2.17); so that, we deduce

$$\int_\Omega u\mathbf{z} \cdot \nabla\varphi + \int_\Omega |Du|\varphi \le 0\,. \tag{2.18}$$

On the other hand, $\operatorname{div}\mathbf{z} = 0$ implies that

$$\int_\Omega u\mathbf{z} \cdot \nabla\varphi + \int_\Omega (\mathbf{z}, Du)\varphi = 0\,,$$

from here and (2.18) we obtain that

$$\int_\Omega |Du|\varphi \le \int_\Omega (\mathbf{z}, Du)\varphi$$

holds for all $\varphi \in C_0^\infty(\Omega)$, with $\varphi \ge 0$. Hence, $|Du| \le (\mathbf{z}, Du)$ as measures. The reverse inequality follows from being $\mathbf{z}$ a vector field with $\|\mathbf{z}\|_\infty \le 1$.

To finish the existence part of the proof, we only have to see (2.11). For $w \in W^{1,\infty}(\Omega)$, taking $w - u_p$ as test function in (2.4), we have

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla(w - u_p) + \int_{\partial\Omega} T_1(\lambda u_p - g)(w - u_p) = 0.$$

Having in mind (2.5), we arrive to

$$\int_\Omega |\nabla u_p|^p + \int_{\partial\Omega} \Gamma_g(x, u_p(x))\, d\mathcal{H}^{N-1}$$
$$\leq \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla w + \int_{\partial\Omega} \Gamma_g(x, w(x))\, d\mathcal{H}^{N-1}.$$

Then, by Young's inequality we obtain that

$$p\int_\Omega |\nabla u_p| + \int_{\partial\Omega} \Gamma_g(x, u_p(x))\, d\mathcal{H}^{N-1}$$
$$\leq (p-1)|\Omega| + \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla w + \int_{\partial\Omega} \Gamma_g(x, w(x))\, d\mathcal{H}^{N-1}.$$

Now, by a result of Modica [15, Proposition 1.2] we know that the functional

$$I_1(u) := \int_\Omega |Du| + \int_{\partial\Omega} \Gamma_g(x, u(x))\, d\mathcal{H}^{N-1}$$

is lower semi–continuous with respect to the $L^1(\Omega)$–convergence, hence having in mind (2.16) and (2.17), we may let p go to 1 and obtain

$$\int_\Omega |Du| + \int_{\partial\Omega} \Gamma_g(x, u(x))\, d\mathcal{H}^{N-1} \leq \int_\Omega \mathbf{z} \cdot \nabla w + \int_{\partial\Omega} \Gamma_g(x, w(x))\, d\mathcal{H}^{N-1}\,. \tag{2.19}$$

Let $w \in BV(\Omega)$, applying results from [6] and [7], we know that there exists a sequence $(w_n) \subset W^{1,\infty}(\Omega)$ such that

$$w_n \to w \quad \text{in} \quad L^1(\Omega),$$
$$\int_\Omega |\nabla w_n(x)|\, dx \to \int_\Omega |Dw|,$$
$$\int_\Omega \mathbf{z} \cdot \nabla w_n\, dx \to \int_\Omega (\mathbf{z}, Dw).$$

In particular we have that w_n strictly converges to w in $BV(\Omega)$. Then, we have $w_n \to w$ in $L^1(\partial\Omega)$ (see [1]) and therefore, from the continuity of the function $r \mapsto \Gamma_g(x, r)$ and the inequality $\Gamma_g(x, r) \leq |r - \frac{g(x)}{\lambda}|$, we also obtain

$$\lim_{n\to\infty} \int_{\partial\Omega} \Gamma_g(x, w_n(x))\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} \Gamma_g(x, w(x))\, d\mathcal{H}^{N-1}.$$

Then, taking w_n as test functions in (2.19) and letting $n \to \infty$ we get

$$\int_\Omega |Du| + \int_{\partial\Omega} \Gamma_g(x, u(x))\, d\mathcal{H}^{N-1} \leq \int_\Omega (\mathbf{z}, Dw) + \int_{\partial\Omega} \Gamma_g(x, w(x))\, d\mathcal{H}^{N-1} \tag{2.20}$$

for every $w \in BV(\Omega)$.

From the above inequality, having in mind (2.9) and (2.10), and applying Green's formula, for every $w \in BV(\Omega)$, we have

$$-\int_{\partial\Omega}[\mathbf{z},\nu](w-u)\,d\mathcal{H}^{N-1} \leq \int_{\partial\Omega}\Gamma_g(x,w(x))\,d\mathcal{H}^{N-1} - \int_{\partial\Omega}\Gamma_g(x,u(x))\,d\mathcal{H}^{N-1}. \tag{2.21}$$

Due to the trace embedding, (2.21) holds for every $w \in L^1(\partial\Omega)$. Given $w \in L^1(\partial\Omega)$ and $0 \leq \varphi \in L^\infty(\partial\Omega)$, taking $u + \frac{\varphi}{\varphi+1}(w-u)$ as test function in (2.21) we get

$$-\int_{\partial\Omega}[\mathbf{z},\nu]\frac{\varphi}{\varphi+1}(w-u)\,d\mathcal{H}^{N-1}$$
$$\leq \int_{\partial\Omega}\Gamma_g(x,u+\frac{\varphi}{\varphi+1}(w-u))\,d\mathcal{H}^{N-1} - \int_{\partial\Omega}\Gamma_g(x,u(x))\,d\mathcal{H}^{N-1}$$

and, by the convexity of the function $r \mapsto \Gamma_g(x,r)$,

$$-\int_{\partial\Omega}[\mathbf{z},\nu]\frac{\varphi}{\varphi+1}(w-u)\,d\mathcal{H}^{N-1} \leq \int_{\partial\Omega}\frac{\varphi}{\varphi+1}[\Gamma_g(x,w(x))-\Gamma_g(x,u(x))]\,d\mathcal{H}^{N-1},$$

which implies

$$\int_{\partial\Omega}\varphi\left([\Gamma_g(x,w(x))-\Gamma_g(x,u(x))]+[\mathbf{z},\nu](w-u)\right)\,d\mathcal{H}^{N-1} \geq 0$$

for every $0 \leq \varphi \in L^\infty(\partial\Omega)$, from where we finally obtain

$$-[\mathbf{z},\nu] \in \partial_r\Gamma_g(x,u(x)) = T_1(\lambda u(x)-g(x)) \quad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega$$

and so (2.11) holds true. ■

Proposition 2.6. *Let $g \in L^2(\partial\Omega)$ and $u \in BV(\Omega)$.*
u is a solution to (1.1) *if and only if u minimizes the functional given by*

$$I_1(u) := \int_\Omega |Du| + \int_{\partial\Omega}\Gamma_g(x,u(x))\,d\mathcal{H}^{N-1}.$$

Proof. Assume, first that u is a solution to (1.1). Then there exists a vector field $\mathbf{z}$ satisfying (2.8–2.11). Since $\operatorname{div}\mathbf{z} = 0$, for every $v \in BV(\Omega)$, it yields

$$\int_\Omega(\mathbf{z},D(u-v)) = \int_{\partial\Omega}(u-v)[\mathbf{z},\nu]\,d\mathcal{H}^{N-1}.$$

Applying (2.10) on the left hand side and (2.11) on the right hand side, it leads to

$$\int_\Omega |Du| - (\mathbf{z},Dv) = -\int_{\partial\Omega}(u-v)T_1(\lambda u-g)\,d\mathcal{H}^{N-1}.$$

Taking into account that $T_1(\lambda u - g)$ belongs to the subdifferential $\partial_r\Gamma_g(x,u(x))$, we deduce that

$$T_1(\lambda u(x)-g(x))(v(x)-u(x)) \leq \Gamma_g(x,v(x)) - \Gamma_g(x,u(x)).$$

So

$$\int_\Omega |Du| - (\mathbf{z},Dv) \leq \int_{\partial\Omega}\Gamma_g(x,v(x)) - \Gamma_g(x,u(x))\,d\mathcal{H}^{N-1}.$$

from where it is straightforward to derive $I_1(u) \leq I_1(v)$.

To see the converse, assume now that u minimizes I_1. If v denotes the solution to (1.1) we have found in the proof of Theorem 1.1, then there exists a vector field $\mathbf{z}$ satisfying (2.8–2.11). Arguing as above, we obtain

$$\begin{aligned} I_1(v) = \int_\Omega |Dv| + \int_{\partial\Omega} \Gamma_g(x, v(x))\, d\mathcal{H}^{N-1} \\ \le \int_\Omega (\mathbf{z}, Du) + \int_{\partial\Omega} \Gamma_g(x, u(x))\, d\mathcal{H}^{N-1} \\ \le \int_\Omega |Du| + \int_{\partial\Omega} \Gamma_g(x, u(x))\, d\mathcal{H}^{N-1} = I_1(u)\,. \end{aligned}$$

Since $I_1(v) \ge I_1(u)$ holds, it follows that the above inequalities becomes equalities. Hence, $\int_\Omega (\mathbf{z}, Du) = \int_\Omega |Du|$ and, by $\|\mathbf{z}\|_\infty \le 1$, we conclude that $(\mathbf{z}, Du) = |Du|$ as measures.

On the other hand, thanks to Green's formula, it also follows that

$$\begin{aligned} -\int_{\partial\Omega} (v-u)[\mathbf{z}, \nu]\, d\mathcal{H}^{N-1} = \int_\Omega (\mathbf{z}, D(u-v)) \\ = \int_\Omega |Du| - \int_\Omega |Dv| = \int_{\partial\Omega} \Gamma_g(x, v(x)) - \Gamma_g(x, u(x))\, d\mathcal{H}^{N-1}\,. \end{aligned} \tag{2.22}$$

Having in mind that v is the solution found in the proof of Theorem 1.1, we use (2.21) to get

$$-\int_{\partial\Omega} [\mathbf{z}, \nu](w-v)\, d\mathcal{H}^{N-1} \le \int_{\partial\Omega} \Gamma_g(x, w(x)) - \Gamma_g(x, v(x))\, d\mathcal{H}^{N-1}\,, \tag{2.23}$$

for every $w \in BV(\Omega)$. Adding (2.22) and (2.23), it yields

$$-\int_{\partial\Omega} [\mathbf{z}, \nu](w-u)\, d\mathcal{H}^{N-1} \le \int_{\partial\Omega} \Gamma_g(x, w(x)) - \Gamma_g(x, u(x))\, d\mathcal{H}^{N-1}\,,$$

for every $w \in BV(\Omega)$. Following next the argument of the proof of Theorem 1.1 after (2.21), we conclude that $-[\mathbf{z}, \nu] = T_1(\lambda u - g)$ $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$.

Therefore, u is a solution to (1.1) in the sense of Definition 2.3. ■

Remark 2.7. It is worth trying to pass to the limit in the approximating problems

$$\begin{cases} -\operatorname{div}\big(|\nabla u_p|^{p-2}\nabla u_p\big) = 0\,, & \text{in } \Omega\,; \\ -|\nabla u_p|^{p-2}\big[\nabla u_p, \nu\big] = \lambda u_p - g\,, & \text{on } \partial\Omega\,; \end{cases} \tag{2.24}$$

when $p \to 1$. Taking u_p as test function, it yields

$$\int_\Omega |\nabla u_p|^p + \int_{\partial\Omega} \lambda u_p^2\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} g u_p\, d\mathcal{H}^{N-1}\,, \tag{2.25}$$

which implies

$$\lambda \int_{\partial\Omega} u_p^2\, d\mathcal{H}^{N-1} \le \left(\int_{\partial\Omega} g^2\, d\mathcal{H}^{N-1}\right)^{1/2} \left(\int_{\partial\Omega} u_p^2\, d\mathcal{H}^{N-1}\right)^{1/2}.$$

It leads to an $L^2(\partial\Omega)$–estimate that, jointly with the $BV(\Omega)$ obtained as in the proof of Theorem 1.1, provide $h \in L^2(\partial\Omega)$, $u \in BV(\Omega)$ and $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ satisfying

$$u_p \to h \quad \text{weakly in } L^2(\partial\Omega)\,;$$
$$\nabla u_p \rightharpoonup Du \quad *\text{–weakly in the sense of measures;}$$
$$u_p \to u \quad \text{a.e. in } \Omega\,;$$
$$u_p \to u \quad \text{strongly in } L^r(\Omega) \quad \text{for } 1 \le r < \frac{N}{N-1}\,;$$
$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z} \quad \text{strongly in } L^q(\Omega;\mathbb{R}^N) \quad \text{for } 1 \le q < \infty\,.$$

Now, the lower semicontinuity of the Total Variation and the L^2–norm turn (2.25) in

$$\int_\Omega |Du| + \int_{\partial\Omega} \lambda h^2 \, d\mathcal{H}^{N-1} \le \int_{\partial\Omega} gh \, d\mathcal{H}^{N-1}\,. \tag{2.26}$$

One may further continue as in the proof of Theorem 1.1, and so seeing that $\operatorname{div}\mathbf{z} = 0$ and $(\mathbf{z}, Du) = |Du|$ as measures. Moreover, if we take $v \in W^{1,1}(\Omega) \cap L^2(\partial\Omega)$ as test function in (2.24) and then let p go to 1, we obtain

$$\int_\Omega \mathbf{z}\cdot\nabla v = -\int_{\partial\Omega} (\lambda h - g) v \, d\mathcal{H}^{N-1}\,.$$

Hence, Green's formula implies

$$\int_{\partial\Omega} [\mathbf{z},\nu] v + (\lambda h - g) v \, d\mathcal{H}^{N-1} = 0$$

for all $v \in L^2(\partial\Omega)$. It follows the identity $-[\mathbf{z},\nu] = \lambda h - g$ $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$.

Therefore, the function $\lambda h - g$ here plays the same role that $T_1(\lambda u - g)$ in Definition 2.3. We will next see that these functions coincide. The proof will be split into two steps.

Step 1: We will see that

$$\int_{\partial\Omega} u[\mathbf{z},\nu] \, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \Gamma_g(u) \, d\mathcal{H}^{N-1} \le \int_{\partial\Omega} w[\mathbf{z},\nu] \, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \Big(\frac{\lambda}{2} w^2 - gw\Big) \, d\mathcal{H}^{N-1}\,, \tag{2.27}$$

holds for every $w \in W^{1,2}(\Omega)$. To this end, fix $w \in W^{1,2}(\Omega)$ and take $u_p - w$ as test function in (2.24). Then Young's inequality yields

$$\begin{aligned}\int_\Omega |\nabla u_p|^p + \int_{\partial\Omega} (\lambda u_p^2 - g u_p) \, d\mathcal{H}^{N-1} &= \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla w + \int_{\partial\Omega} w(\lambda u_p - g) \, d\mathcal{H}^{N-1} \\ &\le \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla w + \frac{\lambda}{2}\int_{\partial\Omega} u_p^2 \, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\Big(\frac{\lambda}{2}w^2 - gw\Big) \, d\mathcal{H}^{N-1}\,.\end{aligned}$$

Simplifying, we obtain

$$\int_\Omega |\nabla u_p|^p + \int_{\partial\Omega} \Big(\frac{\lambda}{2} u_p^2 - g u_p\Big) \, d\mathcal{H}^{N-1} \le \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla w + \int_{\partial\Omega}\Big(\frac{\lambda}{2}w^2 - gw\Big) \, d\mathcal{H}^{N-1}\,.$$

It follows that

$$\begin{aligned}
&\int_\Omega |\nabla u_p|^p + \int_{\partial\Omega} \Gamma_g(u_p)\, d\mathcal{H}^{N-1} \\
&\qquad\le \frac{1}{p}\int_\Omega |\nabla u_p|^p + \frac{p-1}{p}|\Omega| + \int_{\partial\Omega}\Big(\frac{\lambda}{2}u_p^2 - g u_p\Big)\, d\mathcal{H}^{N-1} \\
&\qquad\le \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla w + \frac{p-1}{p}|\Omega| + \int_{\partial\Omega}\Big(\frac{\lambda}{2}w^2 - g w\Big)\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

Applying the lower–semicontinuity on the left hand side, we deduce

$$\int_\Omega |Du| + \int_{\partial\Omega}\Gamma_g(u)\, d\mathcal{H}^{N-1} \le \int_\Omega \mathbf{z}\cdot\nabla w + \int_{\partial\Omega}\Big(\frac{\lambda}{2}w^2 - g w\Big)\, d\mathcal{H}^{N-1}\,,$$

and, due to the identity $(\mathbf{z}, Du) = |Du|$, Green's formula implies (2.27).

Step 2: Consider now a sequence $(w_n)_n$ in $W^{1,2}(\Omega)$ such that $w_n\big|_{\partial\Omega} \to h$ strongly in $L^2(\partial\Omega)$ and apply Step 1. It follows from

$$\begin{aligned}
&\int_{\partial\Omega} u[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\Gamma_g(u)\, d\mathcal{H}^{N-1} \\
&\qquad\qquad\le \int_{\partial\Omega} w[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\Big(\frac{\lambda}{2}w_n^2 - g w_n\Big)\, d\mathcal{H}^{N-1}\,,
\end{aligned}$$

for all $n\in\mathbb{N}$, that

$$\begin{aligned}
&\int_{\partial\Omega} u[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\Gamma_g(u)\, d\mathcal{H}^{N-1} \\
&\qquad\qquad\le \int_{\partial\Omega} h[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\Big(\frac{\lambda}{2}h^2 - g h\Big)\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

Taking into account (2.6), it becomes

$$\begin{aligned}
&\int_{\partial\Omega} u[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \frac{1}{\lambda}\int_{\partial\Omega}\phi(\lambda u - g)\, d\mathcal{H}^{N-1} - \frac{1}{\lambda}\int_{\partial\Omega}\frac{g^2}{2}\, d\mathcal{H}^{N-1} \\
&\qquad\qquad\le \int_{\partial\Omega} h[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\Big(\frac{\lambda}{2}h^2 - g h\Big)\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

Performing easy manipulations, it yields

$$\begin{aligned}
&\int_{\partial\Omega}(\lambda u - g)[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\phi(\lambda u - g)\, d\mathcal{H}^{N-1} - \int_{\partial\Omega}\frac{g^2}{2}\, d\mathcal{H}^{N-1} \\
&\qquad\qquad\le \int_{\partial\Omega}(\lambda h - g)[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\frac{\lambda^2h^2 - 2\lambda g h}{2}\, d\mathcal{H}^{N-1}\,.
\end{aligned}$$

Having in mind $-[\mathbf{z},\nu] = \lambda h - g$, it follows that

$$\begin{aligned}
&\int_{\partial\Omega}(\lambda u - g)[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\phi(\lambda u - g)\, d\mathcal{H}^{N-1} \\
&\qquad\le -\int_{\partial\Omega}(\lambda h - g)^2\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}\frac{\lambda^2h^2 - 2\lambda g h + g^2}{2}\, d\mathcal{H}^{N-1} \\
&\qquad\qquad = -\int_{\partial\Omega}\frac{(\lambda h - g)^2}{2}\, d\mathcal{H}^{N-1} = -\int_{\partial\Omega}\frac{[\mathbf{z},\nu]^2}{2}\, d\mathcal{H}^{N-1}
\end{aligned}$$

and so

$$\int_{\partial\Omega}\Big((\lambda u-g)[\mathbf{z},\nu]+\phi(\lambda u-g)+\frac{[\mathbf{z},\nu]^2}{2}\Big)\,d\mathcal{H}^{N-1}\le 0\,. \tag{2.28}$$

To see that the integrand in (2.28) is nonnegative, we will distinguish two cases.

1) Case $|\lambda u-g|\le 1$: Then

$$(\lambda u-g)[\mathbf{z},\nu]+\frac{(\lambda u-g)^2}{2}+\frac{[\mathbf{z},\nu]^2}{2}=\frac{(\lambda u-g+[\mathbf{z},\nu])^2}{2}\ge 0\,.$$

2) Case $|\lambda u-g|>1$: In this case, we have

$$\begin{aligned}(\lambda u-g)[\mathbf{z},\nu]+|\lambda u-g|-\frac12+\frac{[\mathbf{z},\nu]^2}{2}\\ &=\frac{(\lambda u-g+[\mathbf{z},\nu])^2}{2}-\frac{(|\lambda u-g|-1)^2}{2}\\ &=\frac{(|\lambda u-g|+\operatorname{sign}(\lambda u-g)[\mathbf{z},\nu])^2}{2}-\frac{(|\lambda u-g|-1)^2}{2}\ge 0\,,\end{aligned} \tag{2.29}$$

due to the inequality $\operatorname{sign}(\lambda u-g)[\mathbf{z},\nu]\ge -1$.

Hence, we deduce from (2.28) that its integrand vanishes, that is,

$$(\lambda u-g)[\mathbf{z},\nu]+\phi(\lambda u-g)+\frac{[\mathbf{z},\nu]^2}{2}=0\,,\qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega\,.$$

It is straightforward that if $|\lambda u-g|\le 1$, then

$$\lambda u-g=-[\mathbf{z},\nu]=\lambda h-g\,.$$

On the other hand, if $|\lambda u-g|>1$, then (2.29) implies $\operatorname{sign}(\lambda u-g)[\mathbf{z},\nu]=-1$, so that $\operatorname{sign}(\lambda u-g)=-[\mathbf{z},\nu]=\lambda h-g$.

Therefore, the equality $T_1(\lambda u-g)=\lambda h-g$ is proved.

2.4. **Remarks on uniqueness.** It is easy to see that, in general, uniqueness for the Robin problem does not hold.

Example 2.8. Consider $\Omega:=]0,1[$, $\lambda=1$ and g such that $g(0)=-1$ and $g(1)=2$. Then, any increasing function u in $]0,1[$ such that $u(0)=0$ and $u(1)=1$ is a weak solution with associated vector field $\mathbf{z}\equiv 1$.

Remark 2.9. We explicitly point out that, even though the solution need not be unique, the same vector field $\mathbf{z}$ can be used for all possible solution and, moreover, the weak trace $[\mathbf{z},\nu]$ is univocally determined on the boundary. As a consequence, given a solution u, the function $T_1(\lambda u-g)$ is univocally determined on the boundary.

In fact, let u_1,u_2 be two weak solutions. Then there exist two bounded vector fields $\mathbf{z}_1,\mathbf{z}_2$ satisfying (2.8–2.11). Thus, multiplying the equation (2.9) for $\mathbf{z}_1$ by (u_1-u_2) and applying Green's formula, we have,

$$\int_\Omega(\mathbf{z}_1,D(u_1-u_2))+\int_{\partial\Omega}T_1(\lambda u_1-g)(u_1-u_2)\,d\mathcal{H}^{N-1}=0\,.$$

Similarly, we obtain

$$\int_\Omega(\mathbf{z}_2,D(u_2-u_1))+\int_{\partial\Omega}T_1(\lambda u_2-g)(u_2-u_1)\,d\mathcal{H}^{N-1}=0\,.$$

Therefore, adding the two equalities, we get

$$\int_\Omega (\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2)) + \int_{\partial\Omega} \big[T_1(\lambda u_1 - g) - T_1(\lambda u_2 - g)\big](u_1 - u_2)\, d\mathcal{H}^{N-1} = 0\,. \tag{2.30}$$

Now,

$$(\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2)) = |Du_1| + |Du_2| - (\mathbf{z}_1, Du_2) - (\mathbf{z}_2, Du_1) \geq 0\,, \tag{2.31}$$

since $(\mathbf{z}_1, Du_2) \leq |Du_2|$ and $(\mathbf{z}_2, Du_1) \leq |Du_1|$. On the other hand, the function given by $s \mapsto T_1(\lambda s - g(x))$ is nondecreasing, so that

$$\big[T_1(\lambda u_1 - g) - T_1(\lambda u_2 - g)\big](u_1 - u_2) \geq 0\,. \tag{2.32}$$

Hence, the two terms in (2.30) have to vanish. Consequently, it follows from (2.31) that $(\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2)) = 0$ and so $(\mathbf{z}_1, Du_2) = |Du_2|$ and $(\mathbf{z}_2, Du_1) = |Du_1|$, as measures. Furthermore, it follows from (2.32) that $\big[T_1(\lambda u_1 - g) - T_1(\lambda u_2 - g)\big](u_1 - u_2) = 0$ and then $T_1(\lambda u_1 - g) = T_1(\lambda u_2 - g)$ $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$. So we deduce that $T_1(g - \lambda u_1) = [\mathbf{z}_2, \nu]$ and $T_1(g - \lambda u_2) = [\mathbf{z}_1, \nu]$ $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$.

In summary, we may check that u_1 is a solution using the vector field $\mathbf{z}_2$ and, reciprocally, that u_2 is a solution using the vector field $\mathbf{z}_1$, and we have seen that $[\mathbf{z}_1, \nu] = [\mathbf{z}_2, \nu] = T_1(\lambda u_1 - g) = T_1(\lambda u_2 - g)$ as well.

Example 2.8 strongly relies on the 1–dimensional setting. Actually, it is a consequence of the non-uniqueness of the Dirichlet problem for 1–harmonic functions. This phenomenon also explain the nonuniqueness of the Robin problem in higher dimensions. We next turn to show the connections between the Robin and the Dirichlet problems for the 1–Laplacian.

The Dirichlet problem for the 1–Laplacian is

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = 0\,, & \text{in } \Omega\,; \\ u = h\,, & \text{on } \partial\Omega\,; \end{cases} \tag{2.33}$$

where $h \in L^1(\partial\Omega)$ and the concept of solution is the following:

Definition 2.10. *We will say that $u \in BV(\Omega)$ is a solution to problem* (2.33) *if there exists a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$, with $\|\mathbf{z}\|_\infty \leq 1$, satisfying*

$$-\mathrm{div}\,(\mathbf{z}) = 0\,, \qquad \textit{in } \mathcal{D}'(\Omega)\,; \tag{2.34}$$

$$(\mathbf{z}, Du) = |Du|\,, \tag{2.35}$$

$$[\mathbf{z}, \nu] \in \mathrm{sign}(h - u)\,, \qquad \mathcal{H}^{N-1}\textit{–a.e. on } \partial\Omega\,. \tag{2.36}$$

In [11] we study existence and uniqueness for problem (2.33). We now see the 1–dimensional setting in the spirit of Example 2.8.

Remark 2.11. Given an open interval $]a, b[$ and $\alpha, \beta \in \mathbb{R}$, if $\alpha < \beta$, then every nondecreasing function $u :]a, b[\to \mathbb{R}$ satisfying $u(a) = \alpha$ and $u(b) = \beta$ is a solution to problem

$$\begin{cases} -\left(\dfrac{u'}{|u'|}\right)' = 0\,, & \text{in }]a, b[\,, \\ u(a) = \alpha\,, \quad u(b) = \beta\,. \end{cases} \tag{2.37}$$

To check it, just consider as associated vector field $\mathbf{z} \equiv 1$. Analogously, if $\alpha > \beta$, then every nonincreasing function $u \ :]a, b[\to \mathbb{R}$ satisfying $u(a) = \alpha$ and $u(b) = \beta$ is a solution to problem (2.37), taking $\mathbf{z} \equiv -1$ as associated vector field. On the other hand, uniqueness holds when $\alpha = \beta$ since the unique solution to (2.37) is the constant function: just integrate by parts to get $\int_a^b |u'| = \int_a^b \mathbf{z}u' = 0$.

One might think that the above examples are specific of monotone functions and they do not occur in higher dimensions. Nevertheless, this is not so as we next see. To do that we first study the relation between the Robin and the Dirichlet problems. We need to show that, as we have seen in Remark 2.9 for the Robin problem, the same vector field $\mathbf{z}$ can be used for all possible solutions of the Dirichlet problem (2.33).

Remark 2.12. Assume that u_i, $i = 1, 2$, are solutions to problem (2.33) with associated vector fields $\mathbf{z}_i$. Then arguing as in Remark 2.9, we obtain

$$\int_\Omega (\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2)) - \int_{\partial\Omega} (u_1 - u_2)[\mathbf{z}_1 - \mathbf{z}_2, \nu]\, d\mathcal{H}^{N-1} = 0 \tag{2.38}$$

The first term is nonnegative, as we already check. With respect to the second one, observe that

$$\begin{aligned} &-(u_1 - u_2)[\mathbf{z}_1 - \mathbf{z}_2, \nu] \\ &\quad = (h - u_1)[\mathbf{z}_1, \nu] + (h - u_2)[\mathbf{z}_2, \nu] - (h - u_2)[\mathbf{z}_1, \nu] - (h - u_1)[\mathbf{z}_2, \nu] \\ &\qquad = |h - u_1| + |h - u_2| - (h - u_2)[\mathbf{z}_1, \nu] - (h - u_1)[\mathbf{z}_2, \nu] \geq 0\,. \end{aligned}$$

Hence, the two terms in (2.38) vanish. The first term implies $(\mathbf{z}_i, Du_j) = |Du_j|$, while the second yields $|h - u_j| = (h - u_j)[\mathbf{z}_i, \nu]$. The desired fact follows.

Proposition 2.13. *Let $g, h \in L^2(\partial\Omega)$ and $u \in BV(\Omega)$.*

(i) *If u is a solution to the Robin problem* (1.1)*, then u is a solution to the Dirichlet problem*

$$\begin{cases} -\mathrm{div}\Big(\dfrac{Du}{|Du|}\Big) = 0, & \text{in } \Omega, \\ u = \dfrac{1}{\lambda}(g - [\mathbf{z}, \nu]), & \text{on } \partial\Omega, \end{cases} \tag{2.39}$$

where $\mathbf{z}$ is any vector field associated with the solution u.

(ii) *If u is a solution to the Dirichlet problem* (2.33) *and $\mathbf{z}$ is a vector field associated with this solution, then u is a solution of the Robin problem* (1.1) *for $g = \lambda h - [\mathbf{z}, \nu]$.*

Proof. (i): Let u be a solution to (1.1). Then, by Remark 2.9, the weak trace $[\mathbf{z}, \nu]$ is independent of the vector field $\mathbf{z}$. Thus, g is univocally determined, so that we may define $h = \dfrac{1}{\lambda}\big[T_1(\lambda u - g) + g\big]$ on $\partial\Omega$. It follows from (2.11) that

$$-[\mathbf{z}, \nu] = T_1(\lambda u - g) = \lambda h - g\,, \quad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega\,.$$

We must consider three cases:

(1) $|\lambda u - g| \leq 1$: Here we deduce $u = h$
(2) $\lambda u - g > 1$: We have that $\lambda h - g = 1 < \lambda u - g$, so that $u > h$ and $[\mathbf{z}, \nu] = -1$. Thus, $[\mathbf{z}, \nu] = \mathrm{sign}\,(h - u)$.

(3) $\lambda u - g < -1$: We now have $u < h$ and $[\mathbf{z}, \nu] = 1$, and so $[\mathbf{z}, \nu] = \operatorname{sign}(h-u)$ as well.

In any case, we have proved that $[\mathbf{z}, \nu] \in \operatorname{sign}(h-u)$, wherewith u is a solution to (2.39).

(ii): Let u be a solution to the Dirichlet problem (2.33) and $\mathbf{z}$ a vector field associated with the solution u. Take $g := \lambda h + [\mathbf{z}, \nu]$. It follows from the definition of solution that

$$[\mathbf{z}, \nu] \in \operatorname{sign}(h-u)\,.$$

Three cases must be considered:

(1) $u > h$: Then $[\mathbf{z}, \nu] = -1$ and so $1 = \lambda h - g < \lambda u - g$. Hence, $-[\mathbf{z}, \nu] = T_1(\lambda u - g)$.
(2) $u < h$: Here $[\mathbf{z}, \nu] = 1$, so that we also deduce $-[\mathbf{z}, \nu] = T_1(\lambda u - g)$.
(3) $u = h$: Since $\lambda u - g = \lambda h - g = -[\mathbf{z}, \nu]$, it follows that $-[\mathbf{z}, \nu] = T_1(\lambda u - g)$.

We have obtained, in any case, that $-[\mathbf{z}, \nu] = T_1(\lambda u - g)$ and so u is a solution to (1.1). ■

An example of non–uniqueness of the Dirichlet problem in higher dimensions is considered in [11]. We now modify it to show that uniqueness also fails for the Robin problem (1.1). For the sake of simplicity, we will choose $\lambda = 1$.

Example 2.14. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and consider the boundary datum $g : \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \to \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} x^2 - y^2 + 1 + x\,, & \text{if } x > \frac{\sqrt{2}}{2}\,; \\ x^2 - y^2 + 1 - x\,, & \text{if } x < -\frac{\sqrt{2}}{2}\,; \\ x^2 - y^2 - 1 - y\,, & \text{if } y > \frac{\sqrt{2}}{2}\,; \\ x^2 - y^2 - 1 + y\,, & \text{if } y < -\frac{\sqrt{2}}{2}\,. \end{cases} \tag{2.40}$$

At the points satisfying $|x| = |y| = \frac{\sqrt{2}}{2}$, function g is not defined, it can take any value.

Now define another function $h : \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \to \mathbb{R}$ by

$$h(x,y) = \begin{cases} x^2 - y^2 + 1\,, & \text{if } |x| > \frac{\sqrt{2}}{2}\,; \\ x^2 - y^2 - 1\,, & \text{if } |y| > \frac{\sqrt{2}}{2}\,. \end{cases}$$

In [11] it has been proved that those functions given by

$$u_\lambda(x,y) = \begin{cases} 2x^2\,, & \text{if } |x| > \frac{\sqrt{2}}{2}\,,\ |y| < \frac{\sqrt{2}}{2}\,; \\ \lambda\,, & \text{if } |x| < \frac{\sqrt{2}}{2}\,,\ |y| < \frac{\sqrt{2}}{2}\,; \\ -2y^2\,, & \text{if } |x| < \frac{\sqrt{2}}{2}\,,\ |y| > \frac{\sqrt{2}}{2}\,; \end{cases}$$

with $-1 \le \lambda \le 1$, are solutions to the Dirichlet problem (2.39). Actually, there exists a unique vector field that satisfies all the requirements. This vector field is such that

$$[\mathbf{z}, \nu](x,y) = \begin{cases} x\,, & \text{if } x > \frac{\sqrt{2}}{2}\,; \\ -x\,, & \text{if } x < -\frac{\sqrt{2}}{2}\,; \\ -y\,, & \text{if } y > \frac{\sqrt{2}}{2}\,; \\ y\,, & \text{if } y < -\frac{\sqrt{2}}{2}\,. \end{cases}$$

Since $\lambda h - g = -[\mathbf{z}, \nu]$ holds on $\partial\Omega$, we may invoke Proposition 2.13, to conclude that functions u_λ are solutions to the Robin problem with datum (2.40).

In [17] and [18], functions of least gradient are studied. Under some smoothing assumptions on the boundary, a general theory of existence and uniqueness is proved for continuous data. This theory was completed in [11] by showing that functions of least gradient are the solutions to the Dirichlet problem for the 1–Laplacian. As a consequence of these results and Proposition 2.13 we may state a uniqueness result for our Robin problem.

Corollary 2.15. *Let Ω be a bounded Lipschitz domain satisfying a uniform exterior ball condition. Assume also that for every $x \in \partial\Omega$ there exists $\epsilon > 0$ such that for every set of finite perimeter $A \subset\subset B_\epsilon(x)$,*

$$\mathcal{H}^{N-1}(\partial\Omega) \leq \mathcal{H}^{N-1}(\partial(\Omega \cup A)).$$

Take $g \in L^2(\partial\Omega)$. Let $u \in BV(\Omega)$ be a solution to the Robin problem (1.1) *with associated vector field $\mathbf{z}$. If $h := \frac{1}{\lambda}(g - [\mathbf{z}, \nu])$ is continuous on $\partial\Omega$, then u is the unique solution to the Robin problem* (1.1).

3. The limiting case: the Neumann problem

The inhomogeneous Neumann problem for the 1–Laplacian is problem (1.1) with $\lambda = 0$, that is:

$$\begin{cases} -\operatorname{div}\Big(\dfrac{Du}{|Du|}\Big) = 0, & \text{in } \Omega, \\ \Big[\dfrac{Du}{|Du|}, \nu\Big] = g, & \text{on } \partial\Omega, \end{cases} \tag{3.41}$$

This problem was studied in [12] where it is shown that the approximate solutions involving the p–Laplacian converge to a solution only when the datum is small enough. On the contrary, we have seen here that there is no need of assumptions on the size of the datum to obtain a solution in the case of the Robin problem. As always, the presence of the term λu on the boundary plays a regularizing role.

In this Section we study both the Neumann problem for the 1–Laplacian and the corresponding minimization problem. We explicitly prove that both problems are solvable only for small data. To specify how small must be the datum, we start by introducing a norm in $L^\infty(\partial\Omega)$. This norm, which turns to be equivalent to the usual one $\|\cdot\|_{L^\infty(\partial\Omega)}$, is defined by

$$\|g\|_* = \sup\left\{ \frac{\displaystyle\int_{\partial\Omega} gw\, d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |\nabla w|} \ : \ w \in W^{1,1}(\Omega)\backslash\{0\}\,, \quad \int_{\partial\Omega} w\, d\mathcal{H}^{N-1} = 0 \right\}.$$

Proposition 3.1. *For each $g \in L^\infty(\partial\Omega)$ the following conditions are equivalent.*

(1) There exists a minimum of the functional defined in $BV(\Omega)$ by

$$I_0(u) = \int_\Omega |Du| - \int_{\partial\Omega} ug\,.$$

(2) $\|g\|_* \leq 1$.

Proof. Assume first that $\|g\|_* \leq 1$. Then, by [12, Theorem 4.4], there exists u which is a solution to problem (3.41). Thus, there is a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \leq 1$, $\operatorname{div} \mathbf{z} = 0$, $(\mathbf{z}, Du) = |Du|$ as measures and $[\mathbf{z}, \nu] = g$ on $\partial\Omega$.

For every $v \in BV(\Omega)$, we obtain

$$\int_\Omega (\mathbf{z}, D(u-v)) = \int_{\partial\Omega} (u-v)[\mathbf{z}, \nu]\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} (u-v) g\, d\mathcal{H}^{N-1}\,.$$

Thus,

$$\int_\Omega |Du| - \int_\Omega (\mathbf{z}, Dv) = \int_{\partial\Omega} (u-v) g\, d\mathcal{H}^{N-1}\,,$$

and so $I_0(u) \leq I_0(v)$.

Assume now that $\|g\|_* > 1$, say $\|g\|_* > 1+\epsilon$ for certain $\epsilon > 0$. By the definition of the norm $\|\cdot\|_*$, there exists $w \in W^{1,1}(\Omega)\backslash\{0\}$ such that $\int_{\partial\Omega} w\, d\mathcal{H}^{N-1} = 0$ and

$$\int_{\partial\Omega} g w\, d\mathcal{H}^{N-1} > (1+\epsilon) \int_\Omega |\nabla w|\, dx\,.$$

Then $I_0(w) < -\epsilon \int_\Omega |\nabla w|\, dx$. Taking $u = Mw$ with $M > 0$, it yields

$$I_0(u) < -M\epsilon \int_\Omega |\nabla w|\, dx\,,$$

wherewith $\inf I_0(u) = -\infty$. ■

Proposition 3.2. *For each $g \in L^\infty(\partial\Omega)$ the following conditions are equivalent.*

(1) There exists a solution to problem (3.41)*.*
(2) $\|g\|_* \leq 1$*.*

Proof. If $\|g\|_* \leq 1$ it is already proved in [12, Theorem 4.4] that there exists a solution to problem (3.41).

Conversely, if there exist $u \in BV(\Omega)$ and $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ satisfying $\|\mathbf{z}\|_\infty \leq 1$, $\operatorname{div} \mathbf{z} = 0$, $(\mathbf{z}, Du) = |Du|$ as measures and $[\mathbf{z}, \nu] = g$ on $\partial\Omega$. Then, for every $w \in W^{1,1}(\Omega)\backslash\{0\}$ such that $\int_{\partial\Omega} w\, d\mathcal{H}^{N-1} = 0$, we have

$$\int_{\partial\Omega} g w\, d\mathcal{H}^{N-1} = \int_\Omega \mathbf{z} \cdot \nabla w \leq \int_\Omega |\nabla w|\,.$$

Hence, $\|g\|_* \leq 1$. ■

Proposition 3.3. *Let $g \in L^\infty(\partial\Omega)$ satisfy $\|g\|_* \leq 1$ and let $u \in BV(\Omega)$. Then the following assertions are equivalent.*

(1) u minimizes functional I_0.
(2) u is solution to problem (3.41)*.*

Proof. The implication (1) $\Rightarrow$ (2) can be seen simplifying the argument of the second part of Proposition 2.6.

On the other hand, (2) $\Rightarrow$ (1) is included in Proposition 3.1. ■

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