THE DIRICHLET PROBLEM FOR A SINGULAR ELLIPTIC EQUATION ARISING IN THE LEVEL SET FORMULATION OF THE INVERSE MEAN CURVATURE FLOW

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ABSTRACT. In the present paper we consider the Dirichlet problem associated with a nonlinear singular elliptic equation, whose differential operator arises in the level set formulation of the inverse mean curvature flow; namely, we study

$$-\mathrm{div}\left(\frac{\mathrm{Du}}{|\mathrm{Du}|}\right) + |\mathrm{Du}| = \mathrm{f}\,.$$

We introduce a suitable concept of weak solution, for which we prove existence and uniqueness of the homogeneous Dirichlet problem in a bounded open set of $\mathbb{R}^N$ for data f belonging to suitable Lebesgue spaces. Moreover, examples of explicit solutions are shown.

1. INTRODUCTION

Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial\Omega$. Let us consider the problem

$$(1.1)\qquad \begin{cases} -\mathrm{div}\left(\frac{\mathrm{Du}}{|\mathrm{Du}|}\right) + |\mathrm{Du}| = \mathrm{f} & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

being $0 \leq f \in L^q(\Omega)$, $q > N$. This differential operator appears in the level set formulation of the inverse mean curvature flow ([12], see also [11], [17] and [18]).

The *inverse mean curvature flow* is a one-parameter family of hypersurfaces $\{\Gamma_t\}_{t\geq 0} \subset \mathbb{R}^N$ ($N \geq 2$) whose normal velocity $V_n(t)$ at each time t equals the inverse of its mean curvature $H(t)$. If we let $\Gamma_t := F(\Gamma_0, t)$, then the parametric description of the inverse mean curvature flow is to find $F : \Gamma_0 \times [0,T] \to \mathbb{R}^N$ such that

$$(1.2)\qquad \frac{\partial F}{\partial t} = \frac{\nu}{H}, \qquad t \geq 0,$$

where ν denotes the unit outward normal to Γ_t.

The inverse mean curvature flow was originally introduced as a mathematical method for proving well-known conjectures from the black hole theory such as *Penrose Inequality*

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(which says that the total mass of a spacetime containing black holes with event horizons of the total area A should be at least $\sqrt{A(16\pi)^{-1}}$). Huisken and Ilmanen in [12] propose a level set formulation for the inverse mean curvature flow (1.2), and define a weak notion of solution using an energy minimizing principle in such a way that the generalized inverse mean curvature flow exists for all time. Using this result they then give a proof of the Penrose Inequality for the particular case of a single black hole.

The level set formulation proposed in [12] is the following. Assume that the flow is given by the level sets of a function $u : \mathbb{R}^N \to \mathbb{R}$ via

$$\Gamma_t = \partial E_t, \quad E_t := \{x \in \mathbb{R}^N \ : \ u(x) < t\}.$$

Wherever u is smooth with $\nabla u \neq 0$, equation (1.2) is equivalent to

$$\operatorname{div}\left(\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|}\right) = |\nabla \mathrm{u}|.$$

Thus, (1.2) give rise to the boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|}\right) = |\nabla \mathrm{u}| & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega = \mathbb{R}^N \backslash E_0$.

Huisken and Ilmanen [12] define a *weak solution* of problem (1.3) as a locally Lipschitz function u which minimizes

$$J_u(v) := \int_\Omega (|\nabla v| + |\nabla u| v)\, dx$$

for every locally Lipschitz function v such that $\{v \neq u\} \subset\subset \Omega$. They proved the existence of weak solution by elliptic regularity. Afterwards, Huisken and Ilmanen in [13] have proved regularity results for the inverse mean curvature flow and as consequence that every weak solution is regular after the first instant where a level set is star shaped. A different proof for the existence of weak solution of problem (1.3) is given in [17], which is based on the observation that for $p > 1$, a logarithmic change of dependent variable transforms the approximating equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p$ to the homogeneous p-Laplace equation.

Our aim is to prove existence and uniqueness of solutions for problem (1.1). Note that when $f \equiv 0$ problem (1.1) coincides with (1.3). Now, even if $f \equiv 0$, there are important differences between the two problems. One is that Huisken and Ilmanen study the problem on other manifolds than $\mathbb{R}^N$. On the other hand, we assume that Ω is an open bounded set, while Huisken, Ilmanen and Moser assume that it is an unbounded one, since E_0 is compact. Therefore we only consider the homogeneous

Dirichlet condition $u\big|_{\partial\Omega} \equiv 0$ and they, besides it, need another condition at infinity. As expected, this condition states that u must satisfy

$$\lim_{|x|\to\infty} u(x) = +\infty.$$

Our approach to the existence is closer to the one followed by Moser, but our concept of weak solution is different and follows the ideas developed in [3] (see also [4]) to study the Dirichlet problem associated with the total variation flow. Indeed, our definition of solution relies on the existence of a bounded vector field $\mathbf{z}$ which plays the role of $\frac{Du}{|Du|}$, even if Du may vanish (see Definition 3.1 below). We prove that, given any nonnegative $f \in L^q(\Omega)$, with $q > N$, there exists a unique nonnegative bounded solution to (1.1). This solution belongs to $BV(\Omega)$, but it has a negligible jump part. We explicitly point out that it is the presence of the gradient term in (1.1) which implies uniqueness and leads to certain regularizing effects (see Remarks 3.9 and 3.10).

Let us briefly summarize the contents of this paper. In Section 2 we fix the notation and give some preliminaries results that we need. The next section is devoted to establish the existence and uniqueness results. Finally, in the last section we give some explicit solutions of our problem.

2. Preliminary results

In this section we introduce some notation and some preliminary results that we need. Throughout this paper $\mathcal{H}^{N-1}$ will denote the $(N-1)$–dimensional Hausdorff measure and $\mathcal{L}^N$ the Lebesgue measure.

2.1. **Functions of bounded variations and some generalizations.** The natural energy space to study the problems we are interested in is the space of functions of bounded variation. Recall that if Ω is an open subset of $\mathbb{R}^N$, a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. For every $u \in BV(\Omega)$, the Radon measure Du is decomposed into its absolutely continuous and singular parts with respect to the Lebesgue measure: $Du = D^a u + D^s u$. So $D^a u = \nabla u \, \mathcal{L}^N$, where ∇u is the Radon–Nikodým derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$.

We denote by S_u the set of all $x \in \Omega$ such that x is not a Lebesgue point of u, that is, $x \in \Omega \backslash S_u$ if there exists $\tilde{u}(x)$ such that

$$\lim_{\rho\downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho(x))} \int_{B_\rho(x)} |u(y) - \tilde{u}(x)| \, dy = 0.$$

We say that $x \in \Omega$ is an *approximate jump point of* u if there exist $u_+(x) > u_-(x) \in \mathbb{R}$ and $\nu_u(x) \in S^{N-1}$ such that

$$\lim_{\rho\downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho^+(x, \nu_u(x)))} \int_{B_\rho^+(x,\nu_u(x))} |u(y) - u_+(x)| \, dy = 0$$

$$\lim_{\rho\downarrow 0}\frac{1}{\mathcal{L}^N(B_\rho^-(x,\nu_u(x)))}\int_{B_\rho^-(x,\nu_u(x))}|u(y)-u_-(x)|\,dy=0,$$

where

$$B_\rho^+(x,\nu_u(x))=\{y\in B_\rho(x)\ :\ \langle y-x,\nu_u(x)\rangle>0\}$$

and

$$B_\rho^-(x,\nu_u(x))=\{y\in B_\rho(x)\ :\ \langle y-x,\nu_u(x)\rangle<0\}.$$

We recall that for a Radon measure μ in Ω and a Borel set $A\subseteq\Omega$ the measure $\mu\llcorner A$ is defined by $(\mu\llcorner A)(B)=\mu(A\cap B)$ for any Borel set $B\subseteq\Omega$. If a measure μ is such that $\mu=\mu\llcorner A$ for a certain Borel set A, the measure μ is said to be concentrated on A.

We denote by J_u the set of approximate jump points of u. By the Federer-Vol'pert Theorem [2, Theorem 3.78], we know that S_u is countably $\mathcal{H}^{N-1}$–rectifiable and $\mathcal{H}^{N-1}(S_u\backslash J_u)=0$. Moreover, $Du\llcorner J_u=(u^+-u^-)\nu_u\mathcal{H}^{N-1}\llcorner J_u$. Using S_u and J_u, we may split D^su in two parts: the *jump* part D^ju and the *Cantor* part D^cu defined by

$$D^ju=D^su\llcorner J_u\quad\text{and}\quad D^cu=D^su\llcorner(\Omega\backslash S_u).$$

Then, we have

$$D^ju=(u^+-u^-)\nu_u\mathcal{H}^{N-1}\llcorner J_u.$$

Moreover, if $x\in J_u$, then $\nu_u(x)=\frac{Du}{|Du|}(x)$, $\frac{Du}{|Du|}$ being the Radon–Nikodým derivative of Du with respect to its total variation $|Du|$.

The *precise representative* $u^*:\Omega\backslash(S_u\backslash J_u)\to\mathbb{R}$ of u is defined as equal to $\tilde u$ on $\Omega\backslash S_u$ and equal to $\frac{u_-+u_+}{2}$ on J_u. It is well know (see for instance [2, Corollary 3.80]) that if ρ is a symmetric mollifier, then the mollified functions $u\star\rho_\epsilon$ pointwise converges to u^* in its domain.

For further information concerning functions of bounded variation we refer to [2], [10] or [20].

2.2. **A generalized Green's formula.** We shall need several results from [6] (see also [4]) in order to give sense to the integrals of bounded vector fields whose divergence is a measure integrated with respect to the gradient of a BV function. This theory was also studied in [8] from a different point of view.

Assume that Ω is an open bounded set of $\mathbb{R}^N$ with Lipschitz continuous boundary. Let $p\geq 1$ and $p'\geq 1$ be such that $\frac{1}{p}+\frac{1}{p'}=1$. We will consider the spaces

$$\left\{\mathbf{z}\in L^\infty(\Omega,\mathbb{R}^N)\ :\ \operatorname{div}(\mathbf{z})\in \mathrm{L}^{\mathrm{p}}(\Omega)\right\}$$

and

$$\left\{\mathbf{z}\in L^\infty(\Omega,\mathbb{R}^N)\ :\ \operatorname{div}(\mathbf{z})\text{ is a bounded measure in }\Omega\right\}.$$

The first space is denoted as $X_p(\Omega)$ in [6] while the second (the space of divergence–measure vector fields) is denoted as $X_\mu(\Omega)$ in [6] and as $\mathcal{DM}_\infty(\Omega)$ in [8]. From now on we shall use the notation $\mathcal{DM}_\infty(\Omega)$.

If $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$ or $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $w \in BV(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$, we define the functional $(\mathbf{z}, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Dw), \varphi \rangle := -\int_\Omega w\,\varphi\,\mathrm{div}\,(\mathbf{z})\,\mathrm{dx} - \int_\Omega \mathrm{w}\,\mathbf{z} \cdot \nabla\varphi\,\mathrm{dx}. \tag{2.1}$$

In [6] (see also [4, Corollary C.7, C.16]) it is proved the following result.

Proposition 2.1. *The distribution* $(\mathbf{z}, Dw)$ *is actually a Radon measure with finite total variation.*

The measures $(\mathbf{z}, Dw)$, $|(\mathbf{z}, Dw)|$ *are absolutely continuous with respect to the measure* $|Dw|$ *and*

$$\left| \int_B (\mathbf{z}, Dw) \right| \le \int_B |(\mathbf{z}, Dw)| \le \|\mathbf{z}\|_{L^\infty(U)} \int_B |Dw|$$

for all Borel sets B *and for all open sets* U *such that* $B \subset U \subset \Omega$.

Denoting by

$$\theta(\mathbf{z}, Dw, \cdot) : \Omega \to \mathbb{R}$$

the Radon–Nikodým derivative of $(\mathbf{z}, Dw)$ *with respect to* $|Dw|$, *it follows that*

$$\int_B (\mathbf{z}, Dw) = \int_B \theta(\mathbf{z}, Dw, x)\,|Dw| \qquad \text{for all Borel sets} \quad B \subset \Omega$$

and

$$\|\theta(z, Dw, \cdot)\|_{L^\infty(\Omega, |Dw|)} \le \|\mathbf{z}\|_\infty.$$

Moreover, if $f : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz continuous increasing function, then*

$$\theta(\mathbf{z}, D(f \circ w), x) = \theta(\mathbf{z}, Dw, x), \qquad |Dw| - \text{a.e. in} \quad \Omega \tag{2.2}$$

In [6], a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ is defined. More precisely, it is proved that there exists a linear operator $\gamma : \mathcal{DM}_\infty(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(\mathbf{z})\|_\infty \le \|\mathbf{z}\|_\infty$$

$$\gamma(\mathbf{z})(x) = \mathbf{z}(x) \cdot \nu(x) \qquad \text{for all } x \in \partial\Omega \;\text{ if }\; \mathbf{z} \in C^1(\overline{\Omega}, \mathbb{R}^N).$$

We shall denote $\gamma(\mathbf{z})(x)$ by $[\mathbf{z}, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[\mathbf{z}, \nu]$ and the measure $(\mathbf{z}, Dw)$, for $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$ or $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $w \in BV(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$, is established

$$\int_\Omega w\;\mathrm{div}\,(\mathbf{z})\;\mathrm{dx} + \int_\Omega (\mathbf{z}, \mathrm{Dw}) = \int_{\partial\Omega} [\mathbf{z}, \nu]\mathrm{w}\;\mathrm{d}\mathcal{H}^{\mathrm{N}-1}. \tag{2.3}$$

Applying a Meyers–Serrin type Theorem, it was observed in [16] that it is possible to get a Green's formula like (2.3) for $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, that is, without assuming the continuity of w. To do that, for $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ the functional $(\mathbf{z}, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ is defined by the formula

$$\langle (\mathbf{z}, Dw), \varphi \rangle := -\int_\Omega w^*\,\varphi\,d\mu - \int_\Omega w\,\mathbf{z} \cdot \nabla\varphi\,dx \tag{2.4}$$

where $\mu := \operatorname{div}(\mathbf{z})$; which is well defined since $|\mu|$ is absolutely continuous respect to $\mathcal{H}^{N-1}$ (see [8, Proposition 3.1]). We explicitly remark that this definition depends on the precise representative of u; if we change the representative, in general, we will get a different definition (see another definition in (2.27) below).

With the above definition of $(\mathbf{z}, Dw)$, in [16] it is proved that $(\mathbf{z}, Dw)$ is a Radon measure such that

$$\left| \int_B (\mathbf{z}, Dw) \right| \le \|\mathbf{z}\|_{L^\infty(U)} |Dw|(B) \tag{2.5}$$

for every Borel set B and for every open set U such that $B \subset U \subset \Omega$, and verifies the Green formula

$$\int_\Omega w^* \, d\mu + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z}, \nu] w \, d\mathcal{H}^{N-1}. \tag{2.6}$$

Observe that for $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, we have the following equality as Radon measures

$$\operatorname{div}(\mathrm{w}\mathbf{z}) = (\mathbf{z}, \mathrm{Dw}) + \mathrm{w}^* \operatorname{div}(\mathbf{z}), \tag{2.7}$$

so that $w\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$.

In principle it is not clear that (2.2) holds in the case that $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. However, let us see that (2.2) holds if we assume the jump part $D^j u$ vanishes.

Proposition 2.2. *Let $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and consider $u \in BV(\Omega) \cap L^\infty(\Omega)$ with $D^j u = 0$. If $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous increasing function, then*

$$\theta(\mathbf{z}, D(f \circ u), x) = \theta(\mathbf{z}, Du, x), \qquad |Du| - \text{a.e. in } \Omega \tag{2.8}$$

Proof. By the proof of (2.2) in [6, Proposition 2.8], it is enough to prove that

$$\theta(\mathbf{z}, Du, x) = \theta(\mathbf{z}, D\chi_{E_{u,t}}, x) \;\; |D\chi_{E_{u,t}}| - \text{a.e. in } \Omega \text{ for } \mathcal{L}^1 - \text{almost all } t \in \mathbb{R}, \tag{2.9}$$

where $E_{u,t} := \{x \in \Omega \; : \; u(x) > t\}$. We remark that the main ingredient to prove the above formula is a "slicing" result that links the measure $(\mathbf{z}, Du)$ with the measures $(\mathbf{z}, D\chi_{E_{u,t}})$. This result can be stated as: for all $\varphi \in C_0^\infty(\Omega)$, the function $t \mapsto \langle (\mathbf{z}, D\chi_{E_{u,t}}), \varphi \rangle$ is $\mathcal{L}^1$- measurable and

$$\langle (\mathbf{z}, Du), \varphi \rangle = \int_{-\infty}^{+\infty} \langle (\mathbf{z}, D\chi_{E_{u,t}}), \varphi \rangle \, dt. \tag{2.10}$$

Given $\mathbf{z}$, by results in [8], there exists a sequence $\mathbf{z}_n \in C^\infty(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ such that

$$\mathbf{z}_n \rightharpoonup \mathbf{z} \quad \text{in} \;\; L^\infty(\Omega; \mathbb{R}^N) - \text{weak}^*$$

and

$$\operatorname{div}(\mathbf{z}_\mathrm{n}) \rightharpoonup \operatorname{div}(\mathbf{z}) \;\; \text{and} \;\; |\operatorname{div}(\mathbf{z}_\mathrm{n})| \rightharpoonup |\operatorname{div}(\mathbf{z})| \quad \text{weakly-* as measures.}$$

Fix $\varphi \in C_0^\infty(\Omega)$. By the coarea formula we have

$$\langle (\mathbf{z}_n, Du), \varphi \rangle = \int_\Omega \mathbf{z}_n(x) \cdot \frac{Du}{|Du|}(x)\varphi(x)\, |Du|$$

$$(2.11) \qquad = \int_{-\infty}^{+\infty} \left(\int_\Omega \mathbf{z}_n(x) \cdot \frac{D\chi_{E_{u,t}}}{|D\chi_{E_{u,t}}|}(x)\varphi(x)\, |D\chi_{E_{u,t}}| \right) dt$$

$$= \int_{-\infty}^{+\infty} \langle (\mathbf{z}_n, D\chi_{E_{u,t}}), \varphi \rangle \, dt.$$

On the other hand,

$$(2.12) \qquad \langle (\mathbf{z}_n, Du), \varphi \rangle = -\int_\Omega u\varphi \operatorname{div}(\mathbf{z}_n)\, dx - \int_\Omega u\mathbf{z}_n \cdot \nabla\varphi \, dx.$$

Now, by [8, Proposition 3.1], $|\mathrm{div}\,(\mathbf{z})| \ll \mathcal{H}^{\mathrm{N}-1}$, and since $D^j u = 0$, we have $u\varphi$ is a bounded Borel function with compact support such that the set of its discontinuity points is $|\mathrm{div}\,(\mathbf{z})|$-negligible. Thus, from [2, Proposition 1.62], we can pass to the limit in the first term of the right hand of (2.12). Since obviously we can pass to the limit in the second term, taking limit in (2.12), we get

$$(2.13) \qquad \langle (\mathbf{z}_n, Du), \varphi \rangle \to \langle (\mathbf{z}, Du), \varphi \rangle.$$

Next we will pass to the limit in the right hand side of (2.11). Observe that, since $D^j u = 0$, we have $\mathcal{H}^{N-1}\left(\partial E_{u,t} \cap \partial E_{u,s}\right) = 0$ if $s \neq t$. Then, applying again $|\mathrm{div}\,(\mathbf{z})| \ll \mathcal{H}^{\mathrm{N}-1}$, we have

$$|\mathrm{div}\,(\mathbf{z})|\,(\partial \mathrm{E_{u,t}} \cap \partial \mathrm{E_{u,s}}) = 0 \quad \text{ if } \ \mathrm{s} \neq \mathrm{t}.$$

Therefore, there exists $A \subset \mathbb{R}$ numerable such that

$$|\mathrm{div}\,(\mathbf{z})|\,(\partial \mathrm{E_{u,t}}) = 0 \quad \text{if } \ \mathrm{t} \in \mathbb{R}\backslash \mathrm{A}.$$

Then we may apply the same argument as above to let n goes to $+\infty$ in

$$\langle (\mathbf{z}_n, D\chi_{E_{u,t}}), \varphi \rangle = -\int_{E_{u,t}} \varphi \operatorname{div}(\mathbf{z}_n)\, dx - \int_{E_{u,t}} \mathbf{z}_n \cdot \nabla\varphi \, dx$$

and deduce that

$$(2.14) \qquad \langle (\mathbf{z}_n, D\chi_{E_{u,t}}), \varphi \rangle \to \langle (\mathbf{z}, Du\chi_{E_{u,t}}), \varphi \rangle \qquad \text{for all } t \in \mathbb{R}\backslash A\,.$$

Moreover, it is straightforward that

$$|\langle (\mathbf{z}_n, D\chi_{E_{u,t}}), \varphi \rangle| \le \|\mathbf{z}_n\|_\infty \|\varphi\|_\infty \int_\Omega |D\chi_{E_{u,t}}| \le C \int_\Omega |D\chi_{E_{u,t}}|,$$

for all $n \in \mathbb{N}$. By the Dominated Convergence Theorem, we obtain that

$$(2.15) \qquad \int_{-\infty}^{+\infty} \langle (\mathbf{z}_n, D\chi_{E_{u,t}}), \varphi \rangle \, dt \to \int_{-\infty}^{+\infty} \langle (\mathbf{z}, D\chi_{E_{u,t}}), \varphi \rangle \, dt.$$

Finally, from (2.13) and (2.15), we get (2.10). □

Proposition 2.3. *If* $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ *and* $u, w \in BV(\Omega) \cap L^\infty(\Omega)$ *with* $D^j u = D^j w = 0$, *then*

$$(wz, Du) = w^*(\mathbf{z}, Du) \quad \textit{as Radon measures.} \tag{2.16}$$

Proof. By (2.7), we have $w\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and also,

$$\begin{aligned} \mathrm{div}\,(\mathrm{uw}\mathbf{z}) - \mathrm{u}^*\mathrm{w}^*\mathrm{div}\,(\mathbf{z}) &= (\mathrm{w}\mathbf{z}, \mathrm{Du}) + \mathrm{u}^*\,\mathrm{div}\,(\mathrm{w}\mathbf{z}) - \mathrm{u}^*\mathrm{w}^*\mathrm{div}\,(\mathbf{z}) \\ = (w\mathbf{z}, Du) + u^*w^*\mathrm{div}\,(\mathbf{z}) + \mathrm{u}^*(\mathbf{z}, \mathrm{Dw}) &- \mathrm{u}^*\mathrm{w}^*\mathrm{div}\,(\mathbf{z}) = (\mathrm{w}\mathbf{z}, \mathrm{Du}) + \mathrm{u}^*(\mathbf{z}, \mathrm{Dw}). \end{aligned}$$

Interchanging the roles of u and w, we get

$$\mathrm{div}\,(\mathrm{uw}\mathbf{z}) - \mathrm{u}^*\mathrm{w}^*\mathrm{div}\,(\mathbf{z}) = (\mathrm{u}\mathbf{z}, \mathrm{Dw}) + \mathrm{w}^*(\mathbf{z}, \mathrm{Du}).$$

Hence, we obtain that

$$(w\mathbf{z}, Du) + u^*(\mathbf{z}, Dw) = (u\mathbf{z}, Dw) + w^*(\mathbf{z}, Du). \tag{2.17}$$

We claim now that

$$u^*(\mathbf{z}, Du) = (u\mathbf{z}, Du) \quad \text{as Radon measures.} \tag{2.18}$$

Assume first that $u \geq 0$. Since, by Proposition 2.2,

$$\theta(\mathbf{z}, Du^2, x) = \theta(\mathbf{z}, Du, x), \qquad |Du| - \text{a.e. in } \Omega, \tag{2.19}$$

we have

$$\begin{aligned} (\mathbf{z}, Du^2) = \theta(\mathbf{z}, Du^2, x)|Du^2| &= \theta(\mathbf{z}, Du^2, x)2u^*|Du| \\ &= 2u^*\theta(\mathbf{z}, Du, x)|Du| = 2u^*(\mathbf{z}, Du). \end{aligned}$$

On the other hand, by (2.7)

$$\mathrm{div}\,(\mathrm{u}^2\mathbf{z}) = (\mathbf{z}, \mathrm{Du}^2) + (\mathrm{u}^2)^*\,\mathrm{div}\,(\mathbf{z}).$$

Now, since $D^j u = 0$, we have $(u^2)^* = (u^*)^2$ up to a $\mathcal{H}^{N-1}$–null set, therefore

$$\begin{aligned} 2u^*(\mathbf{z}, Du) = (\mathbf{z}, Du^2) &= \mathrm{div}\,(\mathrm{u}^2\mathbf{z}) - (\mathrm{u}^2)^*\,\mathrm{div}\,(\mathbf{z}) \\ = \mathrm{div}\,(\mathrm{u}^2\mathbf{z}) - (\mathrm{u}^*)^2\,\mathrm{div}\,(\mathbf{z}) &= \mathrm{u}^*(\mathbf{z}, \mathrm{Du}) + (\mathrm{u}\mathbf{z}, \mathrm{Du}), \end{aligned}$$

and the claim (2.18) holds when u is nonnegative. When the sign of u can change, we apply the claim to $u + \|u\|_\infty$. It follows that

$$\begin{aligned} (u\mathbf{z}, Du) + \|u\|_\infty(\mathbf{z}, Du) &= ((u + \|u\|_\infty)\mathbf{z}, D(u + \|u\|_\infty)) \\ = (u + \|u\|_\infty)^*(\mathbf{z}, D(u + \|u\|_\infty)) &= u^*(\mathbf{z}, Du) + \|u\|_\infty(\mathbf{z}, Du)\,. \end{aligned}$$

Simplifying, we get (2.18) in general.

Applying (2.18) to the function $u + w$, we obtain that

$$(u + w)^*(\mathbf{z}, D(u + w)) = ((u + w)\mathbf{z}, D(u + w)),$$

from where it follows that

$$u^*(\mathbf{z}, Dw) + w^*(\mathbf{z}, Du) = (u\mathbf{z}, Dw) + (w\mathbf{z}, Du) \quad \text{as Radon measures.} \tag{2.20}$$

Finally, from (2.17) and (2.20), we obtain (2.16). □

As noted after (2.4), our definition of $(\mathbf{z}, w)$ changes by changing the representative of w. Since another definition will be useful in the sequel, we next introduce it. To this end, we will need the following approximation of a BV–function by smooth functions. Before giving the approximation result let us introduce the following notation. For each $w \in BV(\Omega)$, we consider the representatives $\tilde{w}_+, \tilde{w}_- : \Omega\backslash(S_w\backslash J_w) \to \mathbb{R}$ of w defined as equal to $\tilde{w}$ on $\Omega\backslash S_w$ and equal to w_+, respectively w_- on J_u.

Proposition 2.4. *For each $w \in BV(\Omega) \cap L^\infty(\Omega)$ there exists a sequence $\{w_n\}_n$ in $BV(\Omega) \cap C^\infty(\Omega)$ satisfying*

$$w_n \to \tilde{w}_+ \,, \quad \text{pointwise } \mathcal{H}^{N-1}\text{–a.e.} \tag{2.21}$$

$$\|w_n\|_\infty \le \|w\|_\infty \,, \quad \text{for all } n \in \mathbb{N} \,. \tag{2.22}$$

$$\sup_{n\in\mathbb{N}} \int_\Omega |\nabla w_n| < \infty \,. \tag{2.23}$$

A similar approximation holds for $\tilde{w}_-$.

Proof. For each $n \in \mathbb{N}$, set $F_n = \Omega\backslash\{x - t\,\nu_w(x) \; : \; x \in J_w \,, \quad 0 < t < \frac{1}{n}\}$. Observe that $\Omega = \bigcup_{n=1}^\infty F_n$. Let ρ be a symmetric mollifier supported in $B_1(0)$ and consider the sequence defined by $\rho_n(x) = n^N \rho(nx)$. We define

$$w_n = \rho_n \star (w\chi_{F_n}) \,.$$

We will prove that $w_n(x) \to \tilde{w}_+(x)$ for all $x \in (\Omega\backslash S_w) \cup J_w$; since $\mathcal{H}^{N-1}(S_w\backslash J_w) = 0$, we will deduce (2.21).

If $x \in \Omega\backslash S_w$, then

$$\begin{aligned} |w_n(x) - \tilde{w}_+(x)| &\le \int_{\mathbb{R}^N} |(w\chi_{F_n})\left(x - \frac{1}{n}z\right) - \tilde{w}(x)|\rho(z)\,dz \\ &\le \frac{\|\rho\|_\infty}{(1/n)^N} \int_{B_{1/n}(x)\cap F_n} |w(y) - \tilde{w}(x)|\,dy \le \frac{\|\rho\|_\infty}{(1/n)^N} \int_{B_{1/n}(x)} |w(y) - \tilde{w}(x)|\,dy \end{aligned}$$

and the last term tends to 0 as n goes to ∞. On the other hand, if $x \in J_w$, note that $B_{1/n}(x) \cap F_n = B^+_{1/n}(x, \nu_w(x))$, at least for n large enough. Thus, an analogous argument yields

$$|w_n(x) - \tilde{w}_+(x)| \le \frac{\|\rho\|_\infty}{(1/n)^N} \int_{B^+_{1/n}(x,\nu_w(x))} |w(y) - w_+(x)|\,dy \,,$$

which tends to 0 as n goes to ∞.

Condition (2.22) follows easily from the definition of w_n. To see condition (2.23), it is enough to perform the following computations

$$\limsup_{n\to\infty}\int_\Omega |\nabla w_n| \leq \limsup_{n\to\infty}\int_\Omega |D(w\chi_{F_n})|$$
$$= \limsup_{n\to\infty}\int_{F_n} |Dw| + \|w\|_\infty \limsup_{n\to\infty}\int_\Omega |D\chi_{F_n}| \leq \int_\Omega |Dw| + 2\|w\|_\infty \mathcal{H}^{N-1}(J_w)\,.$$

□

Now, given a C^1–real function g and $a \leq b$, we write

$$⨍_a^b g(s)\,ds = \begin{cases} \dfrac{1}{b-a}\displaystyle\int_a^b g(s)\,ds\,, & \text{if } a<b\,;\\ g(a)\,, & \text{if } a=b\,.\end{cases}$$

For $w \in BV(\Omega)\cap L^\infty(\Omega)$, we denote

$$g(w)^\sharp = ⨍_{\tilde w_-}^{\tilde w_+} g(s)\,ds.$$

Note that $g(w)^\sharp$ is a particular representative of $g(w) \in BV(\Omega)\cap L^\infty(\Omega)$. It is equal to the precise representative $g(w)^*$ of $g(w)$ if g is an affine function but, in general, $g(w)^* \neq g(w)^\sharp$. We point out that, by the chain rule BV (Theorem 3.96 in [2]), we have

$$Dg(w) = g'(w)^\sharp Dw. \tag{2.24}$$

In the next result we will see that $g(w)^\sharp$ is $\mathcal{H}^{N-1}$–a.e. equal to a Borel measurable function.

Lemma 2.5. *For every $w \in BV(\Omega)\cap L^\infty(\Omega)$, there exists a sequence $\{v_n\}_n$ in $W^{1,1}(\Omega)\cap L^\infty(\Omega)$ such that*

$$\begin{cases} v_n \to g(w)^\sharp\,, & \textit{pointwise } \mathcal{H}^{N-1}\textit{–a.e.}\\ v_n \rightharpoonup g(w)^\sharp\,, & \textit{weakly-* in } BV(\Omega)\,.\end{cases} \tag{2.25}$$

Proof. Applying Proposition 2.4, we find two sequences $\{w_n^1\}_n$ and $\{w_n^2\}_n$ in $BV(\Omega)\cap C^\infty(\Omega)$ satisfying

$$w_n^1 \to \tilde w_+ \quad\text{and}\quad w_n^2 \to \tilde w_-\,, \quad \text{pointwise } \mathcal{H}^{N-1}\text{–a.e.}$$
$$\|w_n^i\|_\infty \leq \|w\|_\infty\,, \quad \text{for } i=1,2 \text{ and for all } n\in\mathbb{N}\,.$$
$$\sup_{n\in\mathbb{N}}\int_\Omega |\nabla w_n^i| < \infty\,, \quad \text{for } i=1,2\,.$$

We may assume that $w_n^2 \le w_n^1$ without loss of generality; if necessary, we will take $w_n^1 \vee w_n^2$ instead of w_n^1 and $w_n^1 \wedge w_n^2$ instead of w_n^2. Defining

$$v_n = \fint_{w_n^2}^{w_n^1} g(s)\,ds\,,$$

it is straightforward that

$$v_n \to g(w)^\sharp\,, \quad \text{pointwise } \mathcal{H}^{N-1}\text{–a.e.}$$

and

$$\|v_n\|_\infty \le \sup_{|s|\le\|w\|_\infty} |g(s)|\,, \quad \text{for all } n\in\mathbb{N}\,.$$

Thus, as a consequence of Lebesgue's Theorem, we obtain

$$v_n \to g(w)^\sharp\,, \quad \text{strongly in } L^1(\Omega)\,. \tag{2.26}$$

On the other hand, performing easy computations, it yields

$$\nabla v_n = \Big[\frac{g(w_n^1)-v_n}{w_n^1-w_n^2}\chi_{\{w_n^2<w_n^1\}} + \frac{1}{2}g'(w_n^1)\chi_{\{w_n^2=w_n^1\}}\Big]\nabla w_n^1 + \Big[\frac{v_n-g(w_n^2)}{w_n^1-w_n^2}\chi_{\{w_n^2<w_n^1\}} + \frac{1}{2}g'(w_n^2)\chi_{\{w_n^2=w_n^1\}}\Big]\nabla w_n^2\,.$$

Using Taylor's Theorem, we also obtain that

$$\Big|\frac{v_n-g(w_n^1)}{w_n^1-w_n^2}\Big| = \frac{1}{2}|g'(z_1)| \quad\text{and}\quad \Big|\frac{v_n-g(w_n^2)}{w_n^1-w_n^2}\Big| = \frac{1}{2}|g'(z_2)|$$

for some $z_i \in [w_n^2, w_n^1]$, $i=1,2$. Therefore,

$$|\nabla v_n| \le \frac{1}{2}\sup_{|s|\le\|w\|_\infty}|g'(s)|\big[|\nabla w_n^1| + |\nabla w_n^2|\big]\,, \quad \text{for all } n\in\mathbb{N}\,,$$

and so

$$\sup_{n\in\mathbb{N}}\int_\Omega |\nabla v_n| < \infty\,.$$

This estimate and (2.26) imply $v_n \rightharpoonup g(w)^\sharp$ weakly-* in $BV(\Omega)$; hence, (2.25) holds. □

Now we are ready to provide another definition of a pairing between $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ and $g(w)\in BV(\Omega)\cap L^\infty(\Omega)$, where g is a C^1–real function. In fact, we introduce the functional $(\mathbf{z}, Dg(w)^\sharp) : C_0^\infty(\Omega)\to\mathbb{R}$ by the formula

$$\langle(\mathbf{z}, Dg(w)^\sharp), \varphi\rangle := -\int_\Omega g(w)^\sharp\,\varphi\,d\mu - \int_\Omega g(w)\,\mathbf{z}\cdot\nabla\varphi\,dx \tag{2.27}$$

where $\mu := \operatorname{div}(\mathbf{z})$. Having in mind (2.25) and $|\mu| \ll \mathcal{H}^{N-1}$, we deduce that $g(w)^\sharp$ is μ–measurable and so $(\mathbf{z}, Dg(w)^\sharp)$ is well–defined.

Moreover, by using again (2.25) and the arguments of [6], it follows that $(\mathbf{z}, Dg(w)^\sharp)$ is actually a Radon measure such that

$$\left| \int_B (\mathbf{z}, Dg(w)^\sharp) \right| \leq \|\mathbf{z}\|_{L^\infty(U)} |Dg(w)|(B) \tag{2.28}$$

for every Borel set B and for every open set U such that $B \subset U \subset \Omega$.

2.3. **Functionals defined on** BV**.** Let Ω be an open subset of $\mathbb{R}^N$. Let $l : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$ be a Borel function such that

$$C(x)|\xi| - D(x) \leq l(x, z, \xi) \leq M'(x) + M|\xi| \tag{2.29}$$

for any $(x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, $|z| \leq R$, where M is a positive constant and $C, D, M' \geq 0$ are bounded Borel functions which may depend on R. Assume that $C, D, M' \in L^1(\Omega)$.

Following Dal Maso [9], we consider the following functional for $u \in BV(\Omega) \cap L^\infty(\Omega)$:

$$\begin{aligned} \mathcal{R}_l(u) := &\int_\Omega l(x, u(x), \nabla u(x))\, dx + \int_\Omega l^0 \left(x, \tilde{u}(x), \frac{Du}{|Du|}(x) \right) |D^c u| \\ &+ \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} l^0(x, s, \nu_u(x))\, ds \right) d\mathcal{H}^{N-1}(x), \end{aligned} \tag{2.30}$$

where the recession function l^0 of l is defined by

$$l^0(x, z, \xi) = \lim_{t \to 0^+} t l \left(x, z, \frac{\xi}{t} \right), \tag{2.31}$$

it is convex and homogeneous of degree 1 with respect to ξ.

Assume that $l : \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$ is a continuous function convex in its last variable such that

$$0 \leq l(z, \xi) \leq M(1 + |\xi|) \qquad \forall (z, \xi) \in \mathbb{R} \times \mathbb{R}^N \tag{2.32}$$

for some constant $M \geq 0$ which may depend on R. Given a function $u \in BV(\Omega) \cap L^\infty(\Omega)$, we define the Radon measure $l(u, Du)$ in Ω by

$$\langle l(u, Du), \phi \rangle := \mathcal{R}_{\phi l}(u) \qquad \phi \in C_c(\Omega)^+. \tag{2.33}$$

If $\phi \in C_c(\Omega)$, we write $\phi = \phi^+ - \phi^-$ with $\phi^+ = \max(\phi, 0)$, $\phi^- = -\min(\phi, 0)$, and we define $\langle l(u, Du), \phi \rangle := \mathcal{R}_{\phi^+ l}(u) - \mathcal{R}_{\phi^- l}(u)$.

Let us observe that in the special case

$$l^0(z, \xi) = \varphi(z) \psi^0(\xi), \tag{2.34}$$

where φ is Lipschitz continuous and ψ^0 is an homogeneous function of degree 1, by applying the chain rule for BV–functions (see [2]), we have

$$\mathcal{R}_{\phi l}(u) = \int_\Omega \phi(x) l(u, \nabla u) dx + \int_\Omega \phi(x) \psi^0 \left(\frac{Du}{|Du|} \right) |D^s J_\varphi(u)|, \tag{2.35}$$

where $J_\varphi(r)$ denote the primitive of φ given by

$$J_\varphi(r) = \int_0^r \varphi(s)\, ds.$$

In this case,

$$l(u, Du)^s = \psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(u)|. \tag{2.36}$$

In [5] we have established the following result.

Theorem 2.6. *Let l verifying (2.32) and (2.34), $g \in L^\infty(\partial\Omega)$ and $\phi \in C(\overline{\Omega})^+$ be given. Then, the functional $\mathcal{F}^g_{\phi l} : BV(\Omega) \longrightarrow \mathbb{R}$ defined by*

$$\mathcal{F}^g_{\phi l}(u) := \mathcal{R}_{\phi l}(u) + \int_{\partial\Omega} \phi\, |J_\varphi(g) - J_\varphi(u)|\psi^0(\nu_u)\; d\mathcal{H}^{N-1}$$

is lower semi-continuous with respect to the L^1-convergence.

For the particular case $l(z,\xi) := |\xi|$, and $\phi(x) = 1$ for all $x \in \overline{\Omega}$, by Theorem 2.6, we have that the functional $\mathcal{F}^g : BV(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{F}^g(u) := \int_\Omega |Du| + \int_{\partial\Omega} |g(x) - u(x)|\, d\mathcal{H}^{N-1}(x) \tag{2.37}$$

is lower semi-continuous with respect to the L^1-convergence, which is a well known result

For the particular case $l(z,\xi) := e^{-z}|\xi|$, $g = 0$, by Theorem 2.6, we have that for any $\phi \in C(\overline{\Omega})^+$, the functional $\mathcal{F}_\phi : BV(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{F}_\phi(u) &:= \int_\Omega \phi(x)e^{-u(x)}|\nabla u(x)|\, dx + \int_\Omega \phi(x)e^{-\tilde{u}(x)}\, |D^c u| \\ &+ \int_{J_u} \phi(x)\left(\int_{u^-(x)}^{u^+(x)} e^{-s}\, ds\right) d\mathcal{H}^{N-1}(x) + \int_{\partial\Omega} \phi(x)\, |e^{-u(x)} - 1|\; d\mathcal{H}^{N-1}(x)\end{aligned} \tag{2.38}$$

is lower semi-continuous with respect to the L^1-convergence.

For $u \in BV(\Omega)$, we define the Radon measure $\mathbb{E}(u)$ in Ω by

$$\langle \mathbb{E}(u), \phi\rangle := \mathcal{F}_\phi(u) \qquad \forall\, \phi \in C_c(\Omega). \tag{2.39}$$

Note that for $\phi \in C_c(\Omega)$, we have

$$\langle \mathbb{E}(u), \phi\rangle = \int_\Omega \phi(x)e^{-u(x)}|\nabla u(x)|\, dx + \int_\Omega \phi(x)e^{-\tilde{u}(x)}\, |D^c u|$$

$$+\int_{J_u} \phi(x) \left(\int_{u_-(x)}^{u_+(x)} e^{-s}\, ds \right) d\mathcal{H}^{N-1}(x).$$

Now, applying the chain rule for BV–functions (see [2]), we have

$$|D(1-e^{-u})| = e^{-u}|\nabla u|\mathcal{L}^N + e^{-\tilde{u}}|D^c u| + |e^{-u_+} - e^{-u_-}|\mathcal{H}^{N-1} \llcorner J_u.$$

Therefore, we have obtained that

$$\mathbb{E}(u) = |D(1-e^{-u})| \quad \text{as Radon measures on } \ \Omega. \tag{2.40}$$

Similarly, taking $l(z,\xi) := z|\xi|$ and $g = 0$, by Theorem 2.6, we have that for any $\phi \in C(\overline{\Omega})^+$, the functional $\mathcal{F}_\phi : BV(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{F}_\phi(u) &:= \int_\Omega \phi(x)u(x)|\nabla u(x)|\, dx + \int_\Omega \tilde{u}(x)\, |D^c u| \\ &+ \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} s\, ds \right) d\mathcal{H}^{N-1}(x) + \int_{\partial\Omega} \phi(x)\, \frac{1}{2} u(x)^2 \ d\mathcal{H}^{N-1}(x) \end{aligned} \tag{2.41}$$

is lower semi-continuous with respect to the L^1-convergence.

For $u \in BV(\Omega)$, we define the Radon measure $\mathbb{I}(u)$ in Ω by

$$\langle \mathbb{I}(u), \phi \rangle := \mathcal{F}_\phi(u) \qquad \forall\, \phi \in C_c(\Omega). \tag{2.42}$$

Now, observe that for $\phi \in C_c(\Omega)$, we have

$$\langle \mathbb{I}(u), \phi \rangle = \int_\Omega \phi(x)u(x)|\nabla u(x)|\, dx + \int_\Omega \tilde{u}(x)\, |D^c u| + \int_{J_u} \frac{1}{2} \left(u_+(x)^2 - u_-(x)^2\right) \, d\mathcal{H}^{N-1}(x)$$

$$= \int_\Omega \phi(x)u^*(x)|\nabla u(x)|\, dx + \int_\Omega u^*(x)\, |D^c u| + \int_{J_u} u^*(x)\left(u_+(x) - u_-(x)\right)\, d\mathcal{H}^{N-1}(x)$$

and consequently we have that

$$\mathbb{I}(u) = u^*|Du| \quad \text{as Radon measures on } \ \Omega. \tag{2.43}$$

3. Existence and uniqueness of solutions

We introduce the following concept of solution to problem (1.1).

Definition 3.1. Let f be a non-negative function in $L^N(\Omega)$. We say that u is a *weak solution* of problem (1.1), if $0 \le u \in BV(\Omega) \cap L^\infty(\Omega)$ with $D^j u = 0$ and there exists a vector field $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$, with $\|\mathbf{z}\|_\infty \le 1$, satisfying

$$-\mathrm{div}\,(\mathbf{z}) + |\mathrm{Du}| = \mathrm{f} \quad \text{in} \ \ \mathcal{D}'(\Omega)\,, \tag{3.1}$$

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures in} \ \ \Omega\,, \tag{3.2}$$

and

$$u|_{\partial\Omega} = 0 \quad \mathcal{H}^{N-1} - a.e. \tag{3.3}$$

Remark 3.2. It is worth noting that if u is a solution to problem (1.1) in the sense of the above definition, then it is also a solution in the sense of Huisken and Ilmanen: namely, it minimizes a suitable functional. To see it, assume that u is a solution to problem (1.1) in the sense of the above definition. Then there exists a vector field $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ satisfying $\|\mathbf{z}\|_\infty \le 1$, (3.1) and (3.2). Define

$$I_u(v) = \int_\Omega (|Dv| + v^*|Du| - fv)\,.$$

We will see that u minimizes this functional among all $v \in BV(\Omega) \cap L^\infty(\Omega)$ satisfying $D^j v = 0$ and $v|_{\partial\Omega} = 0$. Indeed, fix one of those v, multiply (3.1) by v and apply Green's formula to obtain

$$0 = \int_\Omega (\mathbf{z}, Dv) + v^*|Du| - fv\,. \tag{3.4}$$

Since $\|\mathbf{z}\|_\infty \le 1$, it follows that $(\mathbf{z}, Dv) \le |Dv|$ and so we deduce that

$$0 \le I_u(v)\,.$$

On the other hand, taking $v = u$ in (3.4) and having in mind (3.2), we get that $I_u(u) = 0$. Therefore, among all admissible functions, the minimum of I_u is attained at u.

Remark 3.3. The condition $D^j u = 0$ does not imply that u is a continuous function. Nevertheless, then $\mathcal{H}^{N-1}(S_u) = 0$ and so the points of discontinuity of its precise representative u^* make up a $\mathcal{H}^{N-1}$–null set.

Remark 3.4. Let us see that if u is a weak solution of problem (1.1) and $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$, with $\|\mathbf{z}\|_\infty \le 1$, satisfying (3.1) and (3.2), then

$$-\mathrm{div}(e^{-u}\mathbf{z}) = e^{-u} f \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.5}$$

In fact, by (3.2)

$$\theta(\mathbf{z}, Du, x) = 1 \quad |Du| - \text{a.e. in } \Omega.$$

Then, by Proposition 2.2 we have

$$\theta(\mathbf{z}, D(1 - e^{-u}), x) = 1 \quad |D(1 - e^{-u})| - \text{a.e. in } \Omega,$$

and consequently, for all Borel sets $B \subset \Omega$,

$$\int_B (\mathbf{z}, D(1 - e^{-u})) = \int_B \theta(\mathbf{z}, D(1 - e^{-u}), x)\,|D(1 - e^{-u})| = \int_B |D(1 - e^{-u})|.$$

Therefore

$$(\mathbf{z}, D(1 - e^{-u})) = |D(1 - e^{-u})| \quad \text{as Radon measures in } \Omega. \tag{3.6}$$

On the other hand, by (2.7), (3.6), (3.1) and (2.40), we have

$$-\mathrm{div}\,(\mathrm{e}^{-\mathrm{u}}\mathbf{z}) = \mathrm{div}\,\big((1-\mathrm{e}^{-\mathrm{u}})\,\mathbf{z}\big) - \mathrm{div}\,(\mathbf{z}) = (\mathbf{z}, \mathrm{D}(1-\mathrm{e}^{-\mathrm{u}})) + (1-\mathrm{e}^{-\mathrm{u}})^*\,\mathrm{div}\,(\mathbf{z}) - \mathrm{div}\,(\mathbf{z})$$

$$= |D(1-e^{-u})| - (e^{-u})^*\,\mathrm{div}\,(\mathbf{z}) = (\mathrm{e}^{-\mathrm{u}})^*|\mathrm{Du}| - (\mathrm{e}^{-\mathrm{u}})^*(-\mathrm{f} + |\mathrm{Du}|) = \mathrm{e}^{-\mathrm{u}}\mathrm{f},$$

and (3.5) hold.

We have the following existence result.

Theorem 3.5. *Given $0 \le f \in L^q(\Omega)$, with $q > N$, there exists, at least, one weak solution u of problem (1.1).*

Proof. To prove the existence of solution of problem (1.1) we approximate it by the following problems related with the p-Laplacian.

$$(3.7)\qquad \begin{cases} -\Delta_p(u) + |\nabla u|^p = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where, for $1 < p \le N$, the p-Laplacian operator $\Delta_p(u) := \mathrm{div}\ (|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u})$.

Now we proceed by dividing the proof into several steps.

Step 1. *Existence of non–negative approximate solutions*

It is well known (see for instance [7]) that for any $0 \le f \in L^q(\Omega)$, $q > N$, there exists a weak solution of the problem (3.7), that is, a function $u_p \in W_0^{1,p}(\Omega)$ satisfying

$$(3.8)\qquad \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla w + \int_\Omega |\nabla u_p|^p w = \int_\Omega f w \quad \forall\, w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Moreover, this function is bounded, since $q > \frac{N}{p}$ and it is unique (it is enough to apply the same argument used in [1]).

Taking in (3.8) $w = -e^{-u_p}u_p^-$, we have

$$\int_\Omega e^{-u_p}|\nabla u_p^-|^p = -\int_\Omega f e^{-u_p} u_p^- \le 0.$$

Hence, $|\nabla u_p^-| = 0$ in Ω, and consequently, $u_p \ge 0$.

Step 2. *BV–estimate*

For $k > 0$, we consider the truncatures $T_k(s) = \sup(-k, \inf(s,k))$. Taking $w = \frac{T_k(u_p)}{k}$ in (3.8), we get

$$\int_\Omega \frac{T_k(u_p)}{k}|\nabla u_p|^p \le \int_\Omega f\frac{T_k(u_p)}{k} \le \int_\Omega f,$$

and, by Fatou's Lemma, we may let $k \to 0^+$ and obtain

$$(3.9)\qquad \int_\Omega |\nabla u_p|^p \le \int_\Omega f\,.$$

Applying Young's inequality we get

$$\int_\Omega |\nabla u_p| \le C \qquad \forall\, p > 1\,, \tag{3.10}$$

where C does not depend on p. Thus, $\{u_p\}_{p>1}$ is bounded in $W^{1,1}(\Omega)$ and we may extract a subsequence such that u_p converges in $L^1(\Omega)$ and almost everywhere to some $u \in BV(\Omega)$ as $p \to 1^+$. Moreover, $u \ge 0$.

Step 3. L^∞*–estimate*

Now, using the Stampacchia methods, we are going to prove that $u \in L^\infty(\Omega)$. Since $q > N$, then $\frac{N}{q'(N-1)} > 1$. Fix p_0, such that $1 < p_0 < \frac{N}{q'(N-1)}$, and take p such that $1 < p \le p_0$. For any $k > 0$ consider $G_k(s) := s - T_k(s)$, $s \in \mathbb{R}$. Taking $w = G_k(u_p)$ as test function in (3.8) and using Hölder's inequality, we get

$$\int_\Omega |\nabla G_k(u_p)|^p \le \int_\Omega f G_k(u_p) \le \|f\|_q \left(\int_\Omega |G_k(u_p)|^{q'}\right)^{\frac{1}{q'}}$$
$$\le \|f\|_q \left(\int_\Omega |G_k(u_p)|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} |A_k^p|^{\frac{1}{q'}-\frac{N-1}{N}},$$

where

$$A_k^p := \{x \in \Omega \ : \ u_p(x) > k\}.$$

Hence, by Sobolev's and Hölder's inequalities, we have

$$\int_\Omega |\nabla G_k(u_p)|^p \le \|f\|_q S \left(\int_\Omega |\nabla G_k(u_p)|\right) |A_k^p|^{\frac{1}{q'}-\frac{N-1}{N}}$$
$$\le \|f\|_q S |\Omega|^{\frac{p-1}{p}} \left(\int_\Omega |\nabla G_k(u_p)|^p\right)^{\frac{1}{p}} |A_k^p|^{\frac{1}{q'}-\frac{N-1}{N}},$$

and consequently

$$\left(\int_\Omega |\nabla G_k(u_p)|^p\right)^{\frac{p-1}{p}} \le \|f\|_q S |\Omega|^{\frac{p-1}{p}} |A_k^p|^{\frac{1}{q'}-\frac{N-1}{N}}.$$

On the other hand, by (3.10) and Sobolev's inequality, we have $\|u_p\|_{\frac{N}{N-1}} \le C$ for all $p > 1$, hence

$$|A_k^p| \le C k^{-\frac{N}{N-1}} \quad \text{for all } p > 1,$$

and consequently we get $\lim_{k\to\infty} |A_k^p| = 0$ uniformly in $p > 1$. Then, if we fix a k_0 sufficiently large such that

$$\|f\|_q S |A_{k_0}^p|^{\frac{1}{q'}-\frac{N-1}{N}} < 1 \quad \forall\, p \in]1, p_0], \tag{3.11}$$

then

$$\left(\int_\Omega |\nabla G_k(u_p)|^p\right)^{\frac{1}{p}} \le \left(\|f\|_q S |A_k^p|^{\frac{1}{q'}-\frac{N-1}{N}}\right)^{\frac{1}{p-1}} |\Omega|^{\frac{1}{p}} \le \left(\|f\|_q S |A_k^p|^{\frac{1}{q'}-\frac{N-1}{N}}\right)^{\frac{1}{p_0-1}} |\Omega|^{\frac{1}{p}}.$$

Therefore,

$$\int_\Omega |\nabla G_k(u_p)| \le \left(\int_\Omega |\nabla G_k(u_p)|^p\right)^{\frac{1}{p}} |\Omega|^{\frac{p-1}{p}}$$
$$\le \left(\|f\|_q S |A_k^p|^{\frac{1}{q'}-\frac{N-1}{N}}\right)^{\frac{1}{p_0-1}} |\Omega| = C|A_k^p|^{\frac{1}{p_0-1}\left(\frac{1}{q'}-\frac{N-1}{N}\right)},$$

where C does not depend of p. Then, applying again Sobolev's inequality, we obtain that

$$\left(\int_\Omega |G_k(u_p)|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} \le S\int_\Omega |\nabla G_k(u_p)| \le SC|A_k^p|^{\frac{1}{p_0-1}\left(\frac{1}{q'}-\frac{N-1}{N}\right)}.$$

Hence

$$\int_\Omega |G_k(u_p)|^{\frac{N}{N-1}} \le \tilde{C}|A_k^p|^{\frac{1}{p_0-1}\left(\frac{N}{q'(N-1)}-1\right)}.$$

Then, if $h > k \ge k_0$, we have

$$(h-k)^{\frac{N}{N-1}}|A_h^p| \le \int_{A_h^p} |G_k(u_p)|^{\frac{N}{N-1}} \le \tilde{C}|A_k^p|^{\frac{1}{p_0-1}\left(\frac{N}{q'(N-1)}-1\right)},$$

from where it follows that

$$|A_h^p| \le \frac{\tilde{C}}{(h-k)^{\frac{N}{N-1}}}|A_k^p|^{\frac{1}{p_0-1}\left(\frac{N}{q'(N-1)}-1\right)}.$$

Then, since

$$\beta := \frac{1}{p_0-1}\left(\frac{N}{q'(N-1)}-1\right) > 1,$$

by Stampacchia's Lemma (see [19, Lemme 5.1] or [14, Lemma B.1]), there is d_p such that

$$|A_{k_0+d_p}^p| = 0 \quad \text{for all } 1 < p \le p_0,$$

being

$$d_p = 2^{\frac{\beta}{\beta-1}} C^{\frac{N-1}{N}} |A_{k_0}^p|^{\frac{(\beta-1)(N-1)}{N}}.$$

Now, by (3.11), there exists a constant Q depending on $\|f\|_q$, but independent on p, for $1 < p \le p_0$, such that

$$|A_{k_0}^p| \le Q.$$

Therefore, there exists a constant $\tilde{Q}$ depending on $\|f\|_q$ and N, but independent on p, for $1 < p \le p_0$, such that $d_p \le \tilde{Q}$, and consequently

$$|A_{k_0+\tilde{Q}}^p| = 0 \quad \text{for all } 1 < p \le p_0,$$

from where it follows that

$$\|u_p\|_\infty \le k_0 + \tilde{Q} \quad \text{for all } 1 < p \le p_0, \tag{3.12}$$

and so

$$\|u\|_\infty \le k_0 + \tilde{Q}. \tag{3.13}$$

Step 4. *Existence of the vector field* $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ *with* $\|\mathbf{z}\|_\infty \le 1$

Let us prove that $\{|\nabla u_p|^{p-2}\nabla u_p\}_{p>1}$ is weakly relatively compact in $L^1(\Omega, \mathbb{R}^N)$. For that, using (3.9) we observe that

$$\int_\Omega |\nabla u_p|^{p-1} \le \left(\int_\Omega |\nabla u_p|^p\right)^{\frac{p-1}{p}} |\Omega|^{\frac{1}{p}} \le C,$$

where C does not depend on p. On the other hand, for any measurable subset $E \subseteq \Omega$,

$$\Big|\int_E |\nabla u_p|^{p-2}\nabla u_p\Big| \le \int_E |\nabla u_p|^{p-1} \le M_1^{\frac{p-1}{p}} |E|^{\frac{1}{p}}.$$

Thus, $\{|\nabla u_p|^{p-2}\nabla u_p\}_{p>1}$, being bounded and equi–integrable in $L^1(\Omega, \mathbb{R}^N)$, is weakly relatively compact in $L^1(\Omega, \mathbb{R}^N)$. There is not loss of generality in assuming that the whole "sequence" converges. Therefore, there exists $\mathbf{z} \in L^1(\Omega, \mathbb{R}^N)$ such that

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z} \quad \text{as } p \to 1, \text{ weakly in } L^1(\Omega, \mathbb{R}^N). \tag{3.14}$$

Given $0 \le \varphi \in C_0^\infty(\Omega)$, taking $w = \varphi$ in (3.8), we have

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi + \int_\Omega |\nabla u_p|^p \varphi = \int_\Omega f\varphi. \tag{3.15}$$

Now, by the lower semi-continuity of the total variation, and applying Young's inequality, we have

$$\int_\Omega \varphi|Du| \le \liminf_{p\downarrow 1} \int_\Omega \varphi|\nabla u_p|\,dx \le \liminf_{p\downarrow 1} \left(\frac{1}{p}\int_\Omega \varphi|\nabla u_p|^p\,dx + \frac{p-1}{p}\int_\Omega \varphi\,dx\right)$$
$$= \liminf_{p\downarrow 1} \int_\Omega \varphi|\nabla u_p|^p\,dx.$$

Then, having in mind (3.14), if we take limit in (3.15), we get

$$\int_\Omega \mathbf{z}\cdot\nabla\varphi + \int_\Omega |Du|\varphi \le \int_\Omega f\varphi. \tag{3.16}$$

Thus,

$$-\mathrm{div}\,(\mathbf{z}) + |\mathrm{Du}| \le \mathrm{f} \qquad \text{in } \mathcal{D}'(\Omega), \tag{3.17}$$

and consequently, $\mathrm{div}\,(\mathbf{z})$ is a Radon measure in Ω.

The next step is to see that

$$\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N) \quad \text{with} \quad \|\mathbf{z}\|_\infty \le 1. \tag{3.18}$$

On account of (3.10), this fact can be proved exactly as in [3].

To finish the proof of this step, we will see that the Radon measure $\operatorname{div}(\mathbf{z})$ has bounded total variation. To this end, it is enough to show that the Radon measure $\nu = f + \operatorname{div}(\mathbf{z})$ is finite (observe that, by (3.17), ν is actually positive). First note that ν defines a bounded linear map on $W_0^{1,1}(\Omega)$, a consequence of the Hölder and Sobolev inequalities. Indeed, if $\phi \in C_0^\infty(\Omega)$, then

$$\Big|\int_\Omega \phi\, d\nu\Big| \le \Big|\int_\Omega f\phi - \int_\Omega \mathbf{z}\cdot\nabla\phi\Big| \le \Big|\|f\|_N \|\phi\|_{\frac{N}{N-1}} + \|\mathbf{z}\|_\infty \int_\Omega |\nabla\phi|\Big|$$
$$\le \Big|S\|f\|_N \int_\Omega |\nabla\phi| + \|\mathbf{z}\|_\infty \int_\Omega |\nabla\phi|\Big| = C\int_\Omega |\nabla\phi|\,,$$

and this estimate holds for all $\phi \in W_0^{1,1}(\Omega)$ by density. Now consider

$$w_n := T_1(n\,\operatorname{dist}(x,\partial\Omega))$$

which defines an increasing sequence satisfying

$$\int_\Omega |\nabla w_n| = \int_{\{x:\ \operatorname{dist}(x,\partial\Omega)<1/n\}} |\nabla w_n|$$

and $\sup w_n = \chi_\Omega$. Thus,

$$\nu(\Omega) = \lim_{n\to\infty} \int_\Omega w_n\, d\nu \le C\, \liminf_{n\to\infty} \int_\Omega |\nabla w_n| = C\,\mathcal{H}^{N-1}(\partial\Omega)\,.$$

Hence, since ν is finite, $\operatorname{div}(\mathbf{z})$ has bounded total variation, and consequently $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$.

Step 5. *The equation* $-\operatorname{div}\mathbf{z} + |Du| = f$ *holds in* $\mathcal{D}'(\Omega)$

Given $\phi \in C_0^\infty(\Omega)$, taking $w = e^{-u_p}\phi$ in (3.8), we have

$$\int_\Omega e^{-u_p}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\phi = \int_\Omega f e^{-u_p}\phi.$$

Then, letting $p \to 1$, we obtain that

$$\int_\Omega e^{-u}\mathbf{z}\cdot\nabla\phi = \int_\Omega f e^{-u}\phi, \tag{3.19}$$

consequently

$$-\operatorname{div}(e^{-u}\mathbf{z}) = e^{-u} f \qquad \text{in } \mathcal{D}'(\Omega), \tag{3.20}$$

and $e^{-u}\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$.

On the other hand, by (2.27),

$$(\mathbf{z}, D(e^{-u})^\sharp) = \operatorname{div}(\mathrm{e}^{-\mathrm{u}}\mathbf{z}) - (\mathrm{e}^{-\mathrm{u}})^\sharp \operatorname{div}(\mathbf{z}) \quad \text{as Radon measures on } \Omega.$$

Then, having in mind (3.17), (2.28), (3.18), (2.24) and (3.20), we obtain

$$
\begin{aligned}
-\operatorname{div}(\mathrm{e}^{-\mathrm{u}}\,\mathbf{z}) &= -(\mathrm{e}^{-\mathrm{u}})^\sharp \operatorname{div}(\mathbf{z}) - (\mathbf{z}, \mathrm{D}(\mathrm{e}^{-\mathrm{u}})^\sharp) \\
&\le e^{-u} f - (e^{-u})^\sharp\, |Du| - (\mathbf{z}, D(e^{-u})^\sharp) \\
&\le e^{-u} f - (e^{-u})^\sharp\, |Du| + |D(e^{-u})| = e^{-u} f = -\operatorname{div}(\mathrm{e}^{-\mathrm{u}}\,\mathbf{z})
\end{aligned}
$$

Therefore, the above inequalities become equalities and so

$$
(e^{-u})^\sharp\, (-\operatorname{div}(\mathbf{z}) + |\mathrm{Du}|) = e^{-u} f, \quad \text{as Radon measures on } \Omega,
$$

from where it follows that

$$
-\operatorname{div}(\mathbf{z}) + |\mathrm{Du}| = \mathrm{f} \quad \text{as Radon measures on } \Omega. \tag{3.21}
$$

Step 6. $|Du| = (\mathbf{z}, Du)$ *as measures*

Given $0 \le \phi \in C_0^\infty(\Omega)$, taking $w = u_p \phi$ in (3.8), we have

$$
\int_\Omega \phi |\nabla u_p|^p + \int_\Omega u_p \phi |\nabla u_p|^p + \int_\Omega u_p |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \phi = \int_\Omega f u_p \phi. \tag{3.22}
$$

By the lower semi-continuity of the total variation and the operator $\mathcal{F}_\phi$ defined by (2.41), and Young's inequality, taking limit in (3.22) when $p \downarrow 1$ we have

$$
\int_\Omega \phi |Du| + \int_\Omega \phi u^* |Du| + \int_\Omega u \mathbf{z} \cdot \nabla \phi \, dx \le \int_\Omega \phi f u \, dx.
$$

Then, applying (3.21), we get

$$
\int_\Omega \phi |Du| + \int_\Omega \phi u^* |Du| + \int_\Omega u \mathbf{z} \cdot \nabla \phi \, dx \le \int_\Omega \phi u^* |Du| - \int_\Omega \phi u^* \, d(\operatorname{div}(\mathbf{z})),
$$

from where it follows that

$$
\int_\Omega \phi |Du| \le \langle (\mathbf{z}, Du), \phi \rangle,
$$

and consequently

$$
|Du| \le (\mathbf{z}, Du) \quad \text{as Radon measures on } \Omega.
$$

Therefore, since $\|\mathbf{z}\|_\infty \le 1$, we obtain that

$$
|Du| = (\mathbf{z}, Du). \tag{3.23}
$$

Step 7. $D^j u = 0$.

Since S_u is countably $\mathcal{H}^{N-1}$–rectifiable and $\mathcal{H}^{N-1}(S_u \backslash J_u) = 0$, there exist countably many Lipschitz hypersurfaces Γ_k de class C^1 such that

$$\mathcal{H}^{N-1}\left(S_u \backslash \bigcup_{k=1}^{\infty} \Gamma_k\right) = 0$$

(see [10, Theorem 12, Sec. 5.9] or [2, Theorem 3.78 and Chapter 2.9]). Therefore, it is enough to show that

$$|D^j u|(\Gamma_k) = 0 \quad \forall\, k \in \mathbb{N}\,.$$

Fixed k denote by ν the unit normal to Γ_k, chosen in such a way that $\nu(x) = \nu_u(x)$ for $\mathcal{H}^{N-1}$–almost all $x \in \Gamma_k \cap J_u$. Thereby the traces of u on $\Gamma_k \cap J_u$ are given by $(u^+_{\Gamma_k}, u^+_{\Gamma_k}) = (u_+, u_-)$.

The proof of Step 7 relies on the following claim:

(3.24) *For any $x_0 \in \Gamma_k$ there exists an open set U, with $x_0 \in U$ and $|D^j u|(U \cap \Gamma_k) = 0\,$.*

In fact, take U to be a smooth open neighbourhood of x_0 satisfying $\overline{U \cap \Gamma_k} \subset \Omega$, set $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < d(\overline{U \cap \Gamma_k}, \partial\Omega)$ and consider

$$U_n = \left\{ x + t\nu(x) \ : \ x \in U \cap \Gamma_k\,, \ |t| < \frac{1}{n} \right\} \subset \Omega \ \text{ for all } n \geq n_0.$$

Then U_n is an open generalized cylinder with Lipschitz boundary and

$$\bigcap_{n \geq n_0} U_n = U \cap \Gamma_k\,.$$

Since $-\mathrm{div}\,(\mathbf{z}) + |Du| = f$ in the sense of distributions, we deduce

$$-\int_{U_n} u^* \mathrm{div}\,(\mathbf{z}) + \int_{U_n} u^* |Du| = \int_{U_n} fu\,.$$

By applying Green's formula and Step 6, it yields

$$\int_{U_n} |Du| - \int_{\partial U_n} u[\mathbf{z}, \eta]\, d\mathcal{H}^{N-1} + \int_{U_n} u^* |Du| = \int_{U_n} fu\,, \tag{3.25}$$

η denoting the unit outward normal to ∂U_n.

Now, we are going to analyze each term in the previous equation. It is straightforward that

$$\lim_{n\to\infty} \int_{U_n} (1 + u^*)|Du| = \int_{U \cap \Gamma_k} (1 + u^*)|Du|. \tag{3.26}$$

Also, since $\mathcal{L}^N(\Gamma_k) = 0$, we have

$$\lim_{n\to\infty} \int_{U_n} fu = \int_{U \cap \Gamma_k} fu = 0. \tag{3.27}$$

To study the remainder term, we split the boundary ∂U_n into three parts:

$$\partial U_n = E_n^+ \cup E_n^- \cup E_n^0 \,,$$

where

$$E_n^+ \subset \left\{ x + \frac{1}{n}\nu(x) \ : \ x \in U \cap \Gamma_k \right\} \,,$$
$$E_n^- \subset \left\{ x - \frac{1}{n}\nu(x) \ : \ x \in U \cap \Gamma_k \right\}$$

and E_n^0 denotes the lateral surface of the generalized cylinder U_n, namely that obtained from $\partial(U \cap \Gamma_k)$. It satisfies

$$\bigcap_{n \geq n_0} E_n^0 = \partial(U \cap \Gamma_k).$$

Thus,

$$\lim_{n\to\infty} \int_{E_n^+} u[\mathbf{z}, \eta] \, d\mathcal{H}^{N-1} = \int_{U\cap\Gamma_k} u_{\Gamma_k}^+[\mathbf{z}, \nu] \, d\mathcal{H}^{N-1} \,,$$
$$\lim_{n\to\infty} \int_{E_n^-} u[\mathbf{z}, \eta] \, d\mathcal{H}^{N-1} = \int_{U\cap\Gamma_k} u_{\Gamma_k}^-[\mathbf{z}, -\nu] \, d\mathcal{H}^{N-1} \,,$$
$$\lim_{n\to\infty} \int_{E_n^0} u[\mathbf{z}, \eta] \, d\mathcal{H}^{N-1} = 0 \,,$$

this last equality holds since u and $[\mathbf{z}, \eta]$ are bounded, and $\mathcal{H}^{N-1}(\partial(U \cap \Gamma_k)) = 0$. It follows that

$$\lim_{n\to\infty} \int_{\partial U_n} u[\mathbf{z}, \eta] \, d\mathcal{H}^{N-1} = \int_{U\cap\Gamma_k} (u_{\Gamma_k}^+ - u_{\Gamma_k}^-)[\mathbf{z}, \nu] \, d\mathcal{H}^{N-1} \,. \tag{3.28}$$

Taking (3.26), (3.27) and (3.28) into account, if we let n goes to $+\infty$ in (3.25), then we get

$$\int_{U\cap\Gamma_k} (1 + u^*)|Du| = \int_{U\cap\Gamma_k} (u_{\Gamma_k}^+ - u_{\Gamma_k}^-)[\mathbf{z}, \nu] \, d\mathcal{H}^{N-1} \,. \tag{3.29}$$

Now, by [2, Proposition 3.92], both the absolutely continuous part $D^a u$ and the Cantor part $D^c u$ of the gradient vanish on sets which are σ–finite with respect to $\mathcal{H}^{N-1}$. So we know that

$$|Du| \llcorner \Gamma_k = |D^j u| \llcorner \Gamma_k \,.$$

Hence,

$$\left| \int_{U\cap\Gamma_k} (u_{\Gamma_k}^+ - u_{\Gamma_k}^-)[\mathbf{z}, \nu] \, d\mathcal{H}^{N-1} \right| \leq \int_{U\cap\Gamma_k} |u_{\Gamma_k}^+ - u_{\Gamma_k}^-| \, d\mathcal{H}^{N-1} = \int_{U\cap\Gamma_k} |D^j u| \,,$$

and it follows from (3.29) that

$$\int_{U\cap\Gamma_k} (1 + u^*)|D^j u| \leq \int_{U\cap\Gamma_k} |D^j u|$$

and consequently

$$\int_{U\cap\Gamma_k} u^*|D^j u| \le 0\,.$$

Since the measure appearing in the integrand is nonnegative, we actually have that it vanishes:

$$u^*|D^j u| \llcorner (U\cap\Gamma_k) = 0\,. \tag{3.30}$$

Having in mind $u^*|D^j u| \llcorner (U\cap\Gamma_k) = \frac{1}{2}(u_+ + u_-)(u_+ - u_-)\mathcal{H}^{N-1}\llcorner(U\cap\Gamma_k\cap J_u)$, we deduce $u_+ = u_-$ on $U\cap\Gamma_k\cap J_u$, up to a $\mathcal{H}^{N-1}$–null set, from where (3.24) follows.

Finally, as consequence of (3.24), we obtain that any compact subset of Γ_k is $|D^j u|$–null, and we get $|D^j u|(\Gamma_k) = 0$.

Step 8. $u = 0$ *on* $\partial\Omega$

Finally let us prove the boundary condition. Taking $w = u_p$ in (3.8), we have

$$\int_\Omega |\nabla u_p|^p + \int_\Omega u_p|\nabla u_p|^p = \int_\Omega f u_p. \tag{3.31}$$

By the lower semi-continuity of the operators $\mathcal{F}^0$ defined by (2.37) and $\mathcal{F}_\phi$ defined by (2.41) with $\phi\equiv 1$, taking limit in (3.31) when $p\downarrow 1$, we get

$$\int_\Omega |Du| + \int_{\partial\Omega}|u(x)|\,d\mathcal{H}^{N-1}(x) + \int_\Omega u^*|Du| + \frac{1}{2}\int_{\partial\Omega} u(x)^2\,d\mathcal{H}^{N-1}(x) \le \int_\Omega f(x)u(x)\,dx.$$

Hence, having in mind (3.21), (3.23) and using Green's formula (2.6), we obtain that

$$\int_{\partial\Omega}\left(|u(x)| + [\mathbf{z},\nu](x)u(x)\right)\,d\mathcal{H}^{N-1}(x) + \frac{1}{2}\int_{\partial\Omega} u(x)^2\,d\mathcal{H}^{N-1}(x) \le 0.$$

Then, since both integrand are non-negative, we deduce that $u|_{\partial\Omega} = 0$ $\mathcal{H}^{N-1}$-a.e. □

To prove the uniqueness, we need the following results which are of interest in themselves. Every $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ is determined up to sets of zero Lebesgue measure but, as pointed out in [8], a precise representative may be defined. This precise representative is given by

$$\mathbf{z}(x) = \begin{cases} \lim\limits_{\rho\to 0^+}\dfrac{1}{\mathcal{L}^N(B_\rho(x))}\displaystyle\int_{B_\rho(x)}\mathbf{z}(y)\,dy\,, & \text{if } x\notin S(\mathbf{z})\,;\\ 0\,, & \text{if } x\in S(\mathbf{z})\,; \end{cases} \tag{3.32}$$

where $S(\mathbf{z})$ denote the approximate discontinuity set of $\mathbf{z}$. If we deal with precise representatives, every $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ is pointwise determined. In the following result is stated that the precise representatives of two vector fields differ in a $|Du|$–null subset of the set of approximate continuity of both.

Proposition 3.6. *Let* $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{DM}_\infty(\Omega)$ *satisfying* $\|\mathbf{z}_i\|_\infty \le 1$ *and denote by* $S(\mathbf{z}_i)$ *the approximate discontinuity set of* $\mathbf{z}_i$*, for* $i = 1, 2$*. Let* $u \in BV(\Omega) \cap L^\infty(\Omega)$ *be such that* $D^j u = 0$*. If* $(\mathbf{z}_i, Du) = |Du|$*, for* $i = 1, 2$*, then*

$$\mathbf{z}_1 = \mathbf{z}_2 \quad \textit{holds} \ \ |Du| - \textit{a.e. in} \ \ \Omega \backslash (S(\mathbf{z}_1) \cup S(\mathbf{z}_2)).$$

Proof. Given $u \in BV(\Omega) \cap L^\infty(\Omega)$, we define

$$\mathcal{Z}(u) = \{\mathbf{z} \in \mathcal{DM}_\infty(\Omega) \ : \ \|\mathbf{z}\|_\infty \le 1\,, \ (\mathbf{z}, Du) = |Du| \text{ as measures } \} \tag{3.33}$$

An easy computation shows that

$$\mathcal{Z}(u) \quad \text{is a convex set.} \tag{3.34}$$

We claim that, for every $\mathbf{z} \in \mathcal{Z}(u)$, up to a $|Du|$–null set,

$$|\mathbf{z}(x)| = 1 \quad \text{for all } x \in \Omega \backslash S(\mathbf{z}). \tag{3.35}$$

Let $\mathbf{z} \in \mathcal{Z}(u)$ be fixed, actually we will consider its precise representative. Since $(\mathbf{z}, Du) = |Du|$ as measures, it follows that $\theta(\mathbf{z}, Du, x) = 1$ holds for $|Du|$–almost all $x \in \Omega$. On the other hand, for every $\rho > 0$, if we set

$$\mathbf{z}_\rho(x) = \frac{1}{\mathcal{L}^N(B_\rho(x))} \int_{B_\rho(x)} \mathbf{z}(y)\, dy,$$

by [6, Remark 2.5], we have

$$\mathbf{z}_\rho \cdot \frac{Du}{|Du|} \rightharpoonup \theta(\mathbf{z}, Du, \cdot)\,, \quad \text{in } L^\infty_{loc}(\Omega, |Du|)\text{–weak}^*$$

where $\frac{Du}{|Du|}$ denotes the Radon–Nikodým derivative of Du with respect to the measure $|Du|$. Thus, up to a $|Du|$–null set, we have

$$1 = \theta(\mathbf{z}, Du, x) \le \limsup_{\rho \to 0^+} \mathbf{z}_\rho(x) \cdot \frac{Du}{|Du|}(x) \le \limsup_{\rho \to 0^+} |\mathbf{z}_\rho(x)|\,. \tag{3.36}$$

Disregarding the $|Du|$–null set and having in mind that $\lim_{\rho\to 0^+} |\mathbf{z}_\rho(x)| = |\mathbf{z}(x)|$ holds for every $x \notin S(\mathbf{z})$ and that $\|\mathbf{z}\|_\infty \le 1$, we deduce from (3.36) that $|\mathbf{z}(x)| = 1$ for every $x \notin S(\mathbf{z})$, hence our claim is proved.

Now we are ready to see that $\mathcal{Z}(u)$ has, at most, a vector field. Assume that $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{Z}(u)$. By (3.34), we know that $\frac{1}{2}(\mathbf{z}_1 + \mathbf{z}_2) \in \mathcal{Z}(u)$. By the parallelogram identity,

$$\left|\frac{1}{2}(\mathbf{z}_1 - \mathbf{z}_2)\right|^2 = \frac{1}{2}|\mathbf{z}_1|^2 + \frac{1}{2}|\mathbf{z}_1|^2 - \left|\frac{1}{2}(\mathbf{z}_1 + \mathbf{z}_2)\right|^2 .$$

Then, having in mind (3.35), we have that the right hand side vanishes $|Du|$–a.e. in $\Omega \backslash (S(\mathbf{z}_1) \cup S(\mathbf{z}_2))$, and consequently the left hand side so does. Therefore, $\mathbf{z}_1 = \mathbf{z}_2$ $|Du|$–a.e. in $\Omega \backslash (S(\mathbf{z}_1) \cup S(\mathbf{z}_2))$. □

Proposition 3.7. *Let* $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ *be such that* $\|\mathbf{z}\|_\infty \le 1$*. If* $u_1, u_2 \in BV(\Omega) \cap L^\infty(\Omega)$ *satisfy* $(\mathbf{z}, Du_1) = |Du_1| = |Du_2| = (\mathbf{z}, Du_2)$*, then* $Du_1 = Du_2$*.*

Proof. First notice that

$$\left(\mathbf{z}, D\left(\frac{u_1+u_2}{2}\right)\right) = \frac{1}{2}(\mathbf{z}, Du_1) + \frac{1}{2}(\mathbf{z}, Du_2) = \frac{1}{2}|Du_1| + \frac{1}{2}|Du_2| = |Du_1|\,.$$

Thus, performing easy manipulations, we obtain

$$\begin{aligned} |Du_1| &= \left(\mathbf{z}, D\left(\frac{u_1+u_2}{2}\right)\right) \le \left|D\left(\frac{u_1+u_2}{2}\right)\right| \\ &= \frac{1}{2}|Du_1+Du_2| \le \frac{1}{2}|Du_1| + \frac{1}{2}|Du_2| = |Du_1|\,, \end{aligned}$$

and so all the inequalities become equalities. Therefore,

$$\left|D\left(\frac{u_1+u_2}{2}\right)\right| = |Du_1| = |Du_2|\,.$$

Applying the Radon–Nikodým Theorem we get two vector functions f_1, f_2, with $|f_1| = 1$ $|Du_1|$-a.e., $|f_2| = 1$ $|Du_1|$-a.e., and satisfying

$$Du_1 = f_1|Du_1|\,, \quad Du_2 = f_2|Du_2| = f_2|Du_1|\,.$$

Hence,

$$D\left(\frac{u_1+u_2}{2}\right) = \left(\frac{f_1+f_2}{2}\right)|Du_1|.$$

Since $\left|D\left(\frac{u_1+u_2}{2}\right)\right| = |Du_1|$, it follows that $\left|\frac{f_1+f_2}{2}\right| = 1$ $|Du_1|$-a.e. Thus, by the parallelogram identity,

$$\left|\frac{f_1-f_2}{2}\right|^2 = \frac{1}{2}\left(|f_1|^2+|f_2|^2\right) - \left|\frac{f_1+f_2}{2}\right|^2 = 0 \qquad |Du_1|\text{–a.e.}$$

Then $f_1 = f_2$ $|Du_1|$–a.e. and so $Du_1 = Du_2$. □

Theorem 3.8. *Given $0 \le f \in L^q(\Omega)$, with $q > N$, there exists, at most, one weak solution of problem* (1.1)

Proof. Let u_i, $i = 1, 2$, two solutions of problem (1.1). Then, $0 \le u_i \in BV(\Omega) \cap L^\infty(\Omega)$ with $u_i|_{\partial\Omega} = 0$ $\mathcal{H}^{N-1}$-a.e. and $D^j u_i = 0$, and there exists a vector field $\mathbf{z}_i \in \mathcal{DM}_\infty(\Omega)$, with $\|\mathbf{z}_i\|_\infty \le 1$, satisfying

$$-\mathrm{div}\,(\mathbf{z}_\mathrm{i}) + |\mathrm{Du}_\mathrm{i}| = \mathrm{f} \quad \text{in} \;\; \mathcal{D}'(\Omega),\; \mathrm{i} = 1, 2, \tag{3.37}$$

and

$$(\mathbf{z}_i, Du_i) = |Du_i| \quad \text{as measures in} \;\; \Omega,\; i = 1, 2. \tag{3.38}$$

Having in mind (3.38), since $|(\mathbf{z}_i, Du_j)| \le |Du_j|$, for any $\phi \in C_c(\Omega)^+$, we get

$$\int_\Omega \phi(\mathbf{z}_1 - \mathbf{z}_2, D(u_1-u_2)^+) = \int_{\{u_1 \ge u_2\}} \phi\left[|Du_1| + |Du_2| - (\mathbf{z}_1, Du_2) - (\mathbf{z}_2, Du_1)\right] \ge 0,$$

and consequently the Radon measure $(\mathbf{z}_1 - \mathbf{z}_2, D(u_1-u_2)^+)$ is positive.

On the other hand, multiplying (3.37) by $[(e^{-u_2} - e^{-u_1})^+]^*$ and applying Green's formula (2.6), we obtain

$$\int_\Omega (\mathbf{z}_i, D((e^{-u_2} - e^{-u_1})^+) + \int_\Omega [(e^{-u_2} - e^{-u_1})^+]^* |Du_i| = \int_\Omega f(x)[(e^{-u_2} - e^{-u_1})^+]^*(x)\, dx,$$

$i = 1, 2$. From here, having in mind (3.6), we can write,

$$\int_\Omega f(x)[(e^{-u_2} - e^{-u_1})^+]^*(x)\, dx = -\int_{\{u_1 > u_2\}} (\mathbf{z}_1, D(1 - e^{-u_2}))$$

$$+ \int_{\{u_1 > u_2\}} (\mathbf{z}_1, D(1 - e^{-u_1})) + \int_{\{u_1 > u_2\}} [e^{-u_2}]^* |Du_1| - \int_{\{u_1 > u_2\}} [e^{-u_1}]^* |Du_1|$$

$$= -\int_{\{u_1 > u_2\}} (\mathbf{z}_1, D(1 - e^{-u_2})) + \int_{\{u_1 > u_2\}} [e^{-u_2}]^* |Du_1|$$

and

$$\int_\Omega f(x)[(e^{-u_2} - e^{-u_1})^+]^*(x)\, dx = -\int_{\{u_1 > u_2\}} (\mathbf{z}_2, D(1 - e^{-u_2}))$$

$$+ \int_{\{u_1 > u_2\}} (\mathbf{z}_2, D(1 - e^{-u_1})) + \int_{\{u_1 > u_2\}} [e^{-u_2}]^* |Du_2| - \int_{\{u_1 > u_2\}} [e^{-u_1}]^* |Du_2|$$

$$= -\int_{\{u_1 > u_2\}} (\mathbf{z}_2, D(1 - e^{-u_1})) - \int_{\{u_1 > u_2\}} [e^{-u_1}]^* |Du_2|.$$

Therefore, we obtain that

$$\int_{\{u_1 > u_2\}} [e^{-u_2}]^* |Du_1| + \int_{\{u_1 > u_2\}} [e^{-u_1}]^* |Du_2|$$

$$= \int_{\{u_1 > u_2\}} (\mathbf{z}_1, D(1 - e^{-u_2})) - \int_{\{u_1 > u_2\}} (\mathbf{z}_2, D(1 - e^{-u_1}))$$

$$\leq \int_{\{u_1 > u_2\}} |D(1 - e^{-u_1})| + \int_{\{u_1 > u_2\}} |D(1 - e^{-u_2})|$$

$$= \int_{\{u_1 > u_2\}} [e^{-u_1}]^* |Du_1| + \int_{\{u_1 > u_2\}} [e^{-u_2}]^* |Du_2|,$$

and consequently,

$$\int_{\{u_1 > u_2\}} \left([e^{-u_2}]^* - [e^{-u_1}]^*\right) (|Du_2| - |Du_1|) \geq 0. \tag{3.39}$$

On the other hand, by (3.5), we have

$$-\mathrm{div}(e^{-u_i}\mathbf{z}_i) = e^{-u_i} f \qquad \text{in } \mathcal{D}'(\Omega), \quad i = 1, 2. \tag{3.40}$$

Multiplying (3.40) by $[(u_1 - u_2)^+]^*$ and applying Green's formula (2.6), we obtain

$$\int_\Omega (e^{-u_i}\mathbf{z}_i, D(u_1-u_2)^+) = \int_\Omega e^{-u_i} f(x)[(u_1-u_2)^+]^*\, dx, \quad i=1,2.$$

Hence,

$$\int_\Omega (e^{-u_1}\mathbf{z}_1, D(u_1-u_2)^+) - \int_\Omega (e^{-u_2}\mathbf{z}_2, D(u_1-u_2)^+)$$
$$= \int_\Omega \left(e^{-u_1} - e^{-u_2}\right) f(x)[(u_1-u_2)^+]^*\, dx \le 0.$$

Therefore,

$$\begin{aligned} 0 &\ge \int_{\{u_1>u_2\}} (e^{-u_2}\mathbf{z}_2 - e^{-u_1}\mathbf{z}_1, D(u_2-u_1)) \\ &= \int_{\{u_1>u_2\}} ((e^{-u_2}-e^{-u_1})\mathbf{z}_2, D(u_2-u_1)) \\ &\quad + \int_{\{u_1>u_2\}} (e^{-u_1}(\mathbf{z}_2-\mathbf{z}_1), D(u_2-u_1)). \end{aligned} \tag{3.41}$$

Having in mind (3.38), (3.39) and Proposition 2.3, and since $|(\mathbf{z}_2, Du_1)| \le |Du_1|$, we have

$$\int_{\{u_1>u_2\}} ((e^{-u_2}-e^{-u_1})\mathbf{z}_2, D(u_2-u_1)) = \int_{\{u_1>u_2\}} (e^{-u_2}-e^{-u_1})^*(\mathbf{z}_2, D(u_2-u_1))$$
$$= \int_{\{u_1>u_2\}} (e^{-u_2}-e^{-u_1})^*|Du_2| - \int_{\{u_1>u_2\}} (e^{-u_2}-e^{-u_1})^*(\mathbf{z}_2, Du_1)$$
$$\ge \int_{\{u_1>u_2\}} \left([e^{-u_2}]^* - [e^{-u_1}]^*\right)\left(|Du_2| - |Du_1|\right) \ge 0.$$

Hence, from (3.41) and applying again Proposition 2.3, we get

$$\int_\Omega [e^{-u_1}]^*(\mathbf{z}_1-\mathbf{z}_2, D(u_1-u_2)^+) = \int_{\{u_1>u_2\}} [e^{-u_1}]^*((\mathbf{z}_2-\mathbf{z}_1), D(u_2-u_1)) \le 0.$$

Then, since the Radon measure $(\mathbf{z}_1-\mathbf{z}_2, D(u_1-u_2)^+)$ is positive, we obtain that

$$(\mathbf{z}_1-\mathbf{z}_2, D(u_1-u_2)^+) = 0.$$

Similarly, we get

$$(\mathbf{z}_1-\mathbf{z}_2, D(u_1-u_2)^-) = (\mathbf{z}_2-\mathbf{z}_1, D(u_2-u_1)^+) = 0.$$

Therefore,

$$(\mathbf{z}_1-\mathbf{z}_2, D(u_1-u_2)) = 0. \tag{3.42}$$

Now, from (3.42), and (3.38), since $|(\mathbf{z}_i, Du_j)| \le |Du_j|$, we get

$$|Du_1| + |Du_2| = (\mathbf{z}_1, Du_1) + (\mathbf{z}_2, Du_2) = (\mathbf{z}_1, Du_2) + (\mathbf{z}_2, Du_1) \le |Du_1| + |Du_1|,$$

from where it follows that

$$(\mathbf{z}_1, Du_2) = |Du_2|$$

and

$$(\mathbf{z}_2, Du_1) = |Du_1|.$$

These equalities and (3.38) imply, applying Proposition 3.6 (and considering the precise representatives of $\mathbf{z}_1$ and $\mathbf{z}_2$, as defined in (3.32)), that $\mathbf{z}_1 \neq \mathbf{z}_2$ just holds in a null set with respect to both $|Du_1|$ and $|Du_2|$ in $\Omega \backslash (S_1(\mathbf{z}_1) \cup S_2(\mathbf{z}_2))$. We next prove that

$$\operatorname{div}(\mathbf{z}_1) = \operatorname{div}(\mathbf{z}_2) \qquad \text{as measures in } \Omega\,. \tag{3.43}$$

Denote $\Lambda = \{\mathbf{z}_1 \neq \mathbf{z}_2\} \cap (\Omega \backslash (S_1(\mathbf{z}_1) \cup S_2(\mathbf{z}_2)))$. If $A \subset \Lambda$ is a Borel set, then we have that $|Du_i|(A) = 0$, for $i = 1, 2$, and (3.37) hold, so that

$$\operatorname{div}(\mathbf{z}_1)(A) = |Du_1|(A) - \int_A f = -\int_A f = |Du_2|(A) - \int_A f = \operatorname{div}(\mathbf{z}_2)(A)\,.$$

Hence, $\operatorname{div}(\mathbf{z}_1) = \operatorname{div}(\mathbf{z}_2)$ as measures in Λ.

On the other hand, given $\epsilon > 0$, we may find a regular open set U satisfying $\Lambda \subset U$ and $|\operatorname{div} \mathbf{z}_i|(U \backslash \Lambda) < \epsilon$ for $i = 1, 2$. We point out that $\mathbf{z}_1(x) = \mathbf{z}_2(x)$ for all $x \in \Omega \backslash \overline{U}$. Indeed, we already know that the points of $\Omega \backslash \overline{U}$ where those vector fields may be different make up a subset of $S_1(\mathbf{z}_1) \cup S_2(\mathbf{z}_2)$. As a consequence, $\mathbf{z}_1 = \mathbf{z}_2$ holds $\mathcal{L}^N$–a.e. in the open set $\Omega \backslash \overline{U}$ and so

$$S_1(\mathbf{z}_1) \cap \big(\Omega \backslash \overline{U}\big) = S_2(\mathbf{z}_2) \cap \big(\Omega \backslash \overline{U}\big)\,.$$

On account of being both vector fields precise representatives, it follows that $\mathbf{z}_1(x) = \mathbf{z}_2(x)$ for all $x \in \Omega \backslash \overline{U}$. As a straightforward consequence, we obtain that $\operatorname{div}(\mathbf{z}_1) = \operatorname{div}(\mathbf{z}_2)$ as measures in $\Omega \backslash \overline{U}$. Another consequence is

$$[\mathbf{z}_1 - \mathbf{z}_2, \nu] = 0 \qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial(\Omega \backslash \overline{U})\,, \tag{3.44}$$

where ν denotes the unit outward normal to $\partial(\Omega \backslash \overline{U})$. To prove it, we consider $h \in L^\infty\big(\partial(\Omega \backslash \overline{U})\big)$ and take $v \in W^{1,1}(\Omega \backslash \overline{U})$ such that $h = v|_{\partial(\Omega \backslash \overline{U})}$. Applying Green's formula, it yields

$$\int_{\partial(\Omega \backslash \overline{U})} h[\mathbf{z}_1 - \mathbf{z}_2, \nu]\, d\mathcal{H}^{N-1} = \int_{\Omega \backslash \overline{U}} v \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) + \int_{\Omega \backslash \overline{U}} (\mathbf{z}_1 - \mathbf{z}_2) \cdot \nabla v = 0\,.$$

Since

$$\int_{\partial(\Omega \backslash \overline{U})} h[\mathbf{z}_1 - \mathbf{z}_2, \nu]\, d\mathcal{H}^{N-1} = 0 \quad \text{ for every } h \in L^\infty\left(\partial(\Omega \backslash \overline{U})\right),$$

it follows that (3.44) holds true.

To go on with the proof of (3.43), fix $\varphi \in C_0^\infty(\Omega)$. On account of $\Omega \backslash \overline{U} \subset \{\mathbf{z}_1 = \mathbf{z}_2\}$, we have

$$\int_\Omega (\mathbf{z}_1 - \mathbf{z}_2) \cdot \nabla\varphi = \int_U (\mathbf{z}_1 - \mathbf{z}_2) \cdot \nabla\varphi + \int_{\Omega\backslash\overline{U}} (\mathbf{z}_1 - \mathbf{z}_2) \cdot \nabla\varphi = \int_U (\mathbf{z}_1 - \mathbf{z}_2) \cdot \nabla\varphi \,.$$

Applying Green's formula in both sides, it yields

$$-\int_\Omega \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) = -\int_U \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) + \int_{\partial U} \varphi[\mathbf{z}_1 - \mathbf{z}_2, \nu_0] \, d\mathcal{H}^{N-1} \,, \tag{3.45}$$

with ν_0 denoting the unit outward normal to ∂U. The last term can be split as

$$\int_{\partial U} \varphi[\mathbf{z}_1 - \mathbf{z}_2, \nu_0] \, d\mathcal{H}^{N-1} = \int_{\partial U \cap \partial\Omega} \varphi[\mathbf{z}_1 - \mathbf{z}_2, \nu_0] \, d\mathcal{H}^{N-1} + \int_{\partial U \backslash \partial\Omega} \varphi[\mathbf{z}_1 - \mathbf{z}_2, \nu_0] \, d\mathcal{H}^{N-1} \,.$$

It is straightforward that $\int_{\partial U \cap \partial\Omega} \varphi[\mathbf{z}_1 - \mathbf{z}_2, \nu_0] \, d\mathcal{H}^{N-1} = 0$ since $\varphi \in C_0^\infty(\Omega)$. To analyze the other term, observe that

$$\int_{\partial U \backslash \partial\Omega} \varphi[\mathbf{z}_1 - \mathbf{z}_2, \nu_0] \, d\mathcal{H}^{N-1} = -\int_{\partial(\Omega\backslash\overline{U})\backslash\partial\Omega} \varphi[\mathbf{z}_1 - \mathbf{z}_2, \nu] \, d\mathcal{H}^{N-1} = 0 \,,$$

due to (3.44). Hence, $\int_{\partial U} \varphi[\mathbf{z}_1 - \mathbf{z}_2, \nu_0] \, d\mathcal{H}^{N-1} = 0$ and (3.45) becomes

$$-\int_\Omega \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) = -\int_U \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) \,.$$

So that

$$\int_\Omega \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) = \int_{U\backslash\Lambda} \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) \,,$$

since $\operatorname{div}(\mathbf{z}_1) = \operatorname{div}(\mathbf{z}_2)$ as measures in Λ. It follows from the way U has been chosen, that

$$\left| \int_\Omega \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) \right| = \left| \int_{U\backslash\Lambda} \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) \right| < 2\epsilon \|\varphi\|_\infty \,.$$

Thus, we deduce that

$$\int_\Omega \varphi \operatorname{div}(\mathbf{z}_1 - \mathbf{z}_2) = 0 \,.$$

Since it holds for every $\varphi \in C_0^\infty(\Omega)$, we have finish the proof of (3.43).

Finally, it follows from (3.37) and (3.43) that $|Du_1| = |Du_2|$ as measures in Ω, and by Proposition 3.7 we obtain $Du_1 = Du_2$. Since $u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0$ hold $\mathcal{H}^{N-1}$-a.e., we conclude that $u_1 = u_2$, as desired. □

We finish this section pointing out several differences between the problem we have studied and the Dirichlet problem associated with the total variation flow.

Remark 3.9. In [3] (see also [4]) we give the following definition of solution of problem

$$
\begin{cases} -\mathrm{div}\left(\dfrac{\mathrm{Du}}{|\mathrm{Du}|}\right) = \mathrm{f}\,, & \text{in } \Omega\,; \\ \\ u = 0\,, & \text{on } \partial\Omega\,; \end{cases} \tag{3.46}
$$

with $f \in L^N(\Omega)$. A function $u \in BV(\Omega)$ is a solution of (3.46) if there exists a vector field $\mathbf{z} \in X_N(\Omega)$, with $\|\mathbf{z}\|_\infty \le 1$, satisfying

$$
-\mathrm{div}\,(\mathbf{z}) = \mathrm{f} \quad \text{in } \mathcal{D}'(\Omega)\,, \tag{3.47}
$$

$$
(\mathbf{z}, Du) = |Du| \quad \text{as measures in } \Omega\,, \tag{3.48}
$$

and

$$
[\mathbf{z}, \nu] \in \mathrm{sign}(-u)\,, \qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega\,, \tag{3.49}
$$

where ν denotes the unit outward vector. This condition (3.49) is a weak form of the boundary condition; it is introduced since the trace on the boundary of solutions to (3.46) does not always vanish.

It is known (see [16] and references therein) that there exists a solution to problem (3.46) only when $\|f\|_N$ is small enough. On the contrary, if $\|f\|_N$ is larger, then problem (3.46) has no solution (or alternatively, modifying a little bit the concept of solution, solutions of (3.46) take the value ∞ in a set of positive measure).

Furthermore, as a consequence of (2.2), it is easy to see that if u is a solution of (3.46) then, for every $\psi : \mathbb{R} \to \mathbb{R}$ positive, increasing and Lipschitz continuous function, $\psi(u)$ is also a solution of the same problem.

Therefore, the term $|Du|$ produces a regularizing effect and without its presence, solutions have no regularity (even may take the value ∞), the boundary condition only holds in a weak sense and there is not uniqueness at all.

Remark 3.10. In [3] we have proved existence and uniqueness of solution for every datum (regardless of its size) if we deal with the problem

$$
\begin{cases} u - \mathrm{div}\left(\dfrac{\mathrm{Du}}{|\mathrm{Du}|}\right) = \mathrm{f}\,, & \text{in } \Omega\,; \\ \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.50}
$$

Thus the presence of the zero order term u has a certain regularizing effect. Nevertheless, solutions to (3.50) may jump and the boundary condition only holds in the sense of (3.49). Let us show these features with some simple examples.

Let $\Omega = B_R(0) \subset \mathbb{R}^N$, with $N < R$. Consider $f = \chi_{B_r(0)}$, with $N < r < R$; it is easy to see that the function $u = \left(1 - \frac{N}{r}\right)\chi_{B_r(0)}$ is the unique solution of (3.50) with vector field given by

$$\mathbf{z}(x) := \begin{cases} -\frac{x}{r} & \text{if } \ x \in B_r(0) \\ \\ -r^{N-1}\frac{x}{|x|^N} & \text{if } \ x \in \Omega \backslash B_r(0). \end{cases}$$

Observe that the above solution u satisfies $J_u = \{x \in \mathbb{R}^N \ : \ |x| = r\}$.

On the other hand, if $f \equiv 1$, then the constant function $u \equiv 1 - \frac{N}{R}$ is the unique solution of (3.50) with vector field given by $\mathbf{z}(x) := -\frac{x}{R}$. We remark that this solution does not vanish on the boundary, but satisfies

$$[\mathbf{z}, \nu] = -\frac{x}{R} \cdot \frac{x}{R} = -1 \in \operatorname{sign}(-u)\,.$$

4. Explicit radial solutions

This section is devoted to find explicit radial solutions to the problem (1.1). Consider $\Omega = B_R(0) \subset \mathbb{R}^N$, and $f(x) := \tilde{f}(|x|)$, with $\tilde{f} : [0,R] \to [0,+\infty[$. We are looking for a radial solution $u(x) = g(|x|)$ such that $g(r) \geq 0$ for $r \in [0,R]$ and $g(R) = 0$. We will not find an explicit expression of g.

Assuming that $g' < 0$, we have $Du(x) = -|g'(|x|)|\frac{x}{|x|}$ and consequently $\mathbf{z}(x) = -\frac{x}{|x|}$. Then

$$-\operatorname{div}(\mathbf{z}) = \frac{\mathrm{N}-1}{|\mathrm{x}|},$$

and we arrive to the equation

$$0 < -g'(r) = \tilde{f}(r) - \frac{N-1}{r}. \tag{4.1}$$

Hence,

$$\frac{N-1}{r} < \tilde{f}(r). \tag{4.2}$$

In summary, if $\frac{N-1}{r} \geq \tilde{f}(r)$, then $g'(r) = 0$ and so $Du(x) = 0$. Now an expression of $\mathbf{z}$ must be searched, taking into account that $-\operatorname{div}(\mathbf{z}) = \mathrm{f}$ have to be satisfied. On the contrary, if $\frac{N-1}{r} < \tilde{f}(r)$, then perhaps $g'(r) < 0$ holds. In this case we may apply (4.1), to obtain

$$g(r) = \int_r^R \tilde{f}(s) - \frac{N-1}{s}\,ds\,,$$

and deduce $u(x) = g(|x|)$ and $\mathbf{z}(x) = -\frac{x}{|x|}$.

Nevertheless, the condition $\frac{N-1}{r} < \tilde{f}(r)$ is necessary but not sufficient to obtain $g'(r) < 0$. We point out that $\frac{N-1}{r} < \tilde{f}(r)$ and $g'(r) = 0$ can occur. So that, the set of all points for which $\frac{N-1}{r} < \tilde{f}(r)$ holds, must be split into two parts: the part where $g'(r) < 0$ and the part $g'(r) = 0$.

In the following explicit examples, the main criterion that allow us to decide how to split the zone where $\frac{N-1}{r} < \tilde{f}(r)$ holds, is to get a continuous vector field $\mathbf{z}$. For the sake of simplicity, we will take $R = 1$.

Example 4.1. (1) Take $f(x) := \lambda$. If $0 \le \lambda \le N$, we have that $u \equiv 0$ is solution with

$$\mathbf{z}(x) = -\frac{\lambda x}{N}.$$

Assume now that $\lambda > N$. Then, for $\frac{N}{\lambda} < r < 1$, (4.2) holds and we have

$$g(r) = \int_r^1 \left(\tilde{f}(s) - \frac{N-1}{s}\right) ds = \int_r^1 \lambda - \frac{N-1}{s}\, ds = \lambda(1-r) + \log{(r)}^{N-1}.$$

Then, it is easy to see that in this case the solution is given by

$$u(x) := \begin{cases} \lambda(1-\frac{N}{\lambda}) + \log\left(\frac{N}{\lambda}\right)^{N-1} & \text{if} \quad |x| \le \frac{N}{\lambda} \\ \lambda(1-|x|) + \log{(|x|)}^{N-1} & \text{if} \quad \frac{N}{\lambda} < |x| < 1, \end{cases}$$

with

$$\mathbf{z}(x) = -\frac{\lambda x}{N}\chi_{B_{\frac{N}{\lambda}}(0)}(x) - \frac{x}{|x|}\chi_{B_1(0)\setminus\overline{B_{\frac{N}{\lambda}}(0)}}(x).$$

(2) Take $f(x) := \frac{\lambda}{|x|^q}$ with $0 \le \lambda$ and $0 < q < 1$. Then, $f \in L^m(\Omega)$, for $m < \frac{N}{q}$. If $0 \le \lambda \le N-q$, we have $u \equiv 0$ is solution with

$$\mathbf{z}(x) = -\frac{\lambda x}{(N-q)|x|^q}.$$

Assume now that $\lambda > N-q$. Then, for $\left(\frac{N-q}{\lambda}\right)^{\frac{1}{1-q}} < r < 1$, (4.2) holds and we have

$$g(r) = \int_r^1 \left(\tilde{f}(s) - \frac{N-1}{s}\right) ds = \int_r^1 \frac{\lambda}{s^q} - \frac{N-1}{s}\, ds$$
$$= \frac{\lambda}{1-q}(1-r^{1-q}) + \log{(r)}^{N-1}.$$

Then, it is easy to see that in this case the solution is given by

$$u(x) := \frac{\lambda}{1-q}\left(1-|x|^{1-q}\right) + (N-1)\log|x|,$$

with

$$\mathbf{z}(x) = \frac{-x}{|x|}.$$

In the case $r \le \left(\frac{N-q}{\lambda}\right)^{\frac{1}{1-q}}$, we have $g'(r) = 0$ and

$$u(x) = \frac{1}{1-q}(\lambda - N + q) + \frac{N-1}{1-q}\log\left(\frac{N-q}{\lambda}\right),$$

with

$$\mathbf{z}(x) = -\frac{\lambda}{N-q}\frac{x}{|x|^q}.$$

(3) Take $f(x) := \lambda\chi_{B_{\frac{1}{2}}(0)}$. If $0 \le \lambda \le 2N$, we have $u \equiv 0$ is solution with

$$\mathbf{z}(x) = \begin{cases} -\frac{\lambda x}{N} & \text{if} \quad |x| \le \frac{1}{2} \\ \\ -\frac{\lambda x}{2^N N |x|^N} & \text{if} \quad \frac{1}{2} < |x| < 1. \end{cases}$$

Assume now that $\lambda > 2N$. Then, for $\frac{N}{\lambda} < r < \frac{1}{2}$, (4.2) holds and we have

$$g(r) = \int_r^{\frac{1}{2}} \left(\tilde{f}(s) - \frac{N-1}{s} \right) ds = \int_r^{\frac{1}{2}} \lambda - \frac{N-1}{s}\, ds = \lambda\left(\frac{1}{2} - r\right) + \log{(r)}^{N-1}.$$

Then, it is easy to see that in this case the solution is given by

$$u(x) := \begin{cases} \lambda(\frac{1}{2} - \frac{N}{\lambda}) + \log\left(\frac{2N}{\lambda}\right)^{N-1} & \text{if} \quad |x| \le \frac{N}{\lambda} \\ \\ \lambda(\frac{1}{2} - |x|) + \log{(2|x|)}^{N-1} & \text{if} \quad \frac{N}{\lambda} < |x| \le \frac{1}{2} \\ \\ 0 & \text{if} \quad \frac{1}{2} < |x| < 1, \end{cases}$$

with

$$\mathbf{z}(x) = -\frac{\lambda x}{N}\chi_{B_{\frac{N}{\lambda}}(0)}(x) - \frac{x}{|x|}\chi_{B_{\frac{1}{2}}(0)\setminus\overline{B_{\frac{N}{\lambda}}(0)}}(x) - \frac{\lambda x}{2^N N|x|^N}\chi_{B_1(0)\setminus\overline{B_{\frac{1}{2}}(0)}}(x).$$

Remark 4.2. We explicitly remark that, in the previous examples, the solution is trivial (identically 0) when the considered datum f is small enough. This situation occurs until the vector field $\mathbf{z}$ satisfies $\|\mathbf{z}\|_\infty = 1$. When the norm of the datum increases, the gradient term $|Du|$ absorbs the excess and so a finite solution can always be obtained. This fact contrasts with what happens if the equation has no gradient term. Then, once $\|\mathbf{z}\|_\infty = 1$ is attained, solutions blow up (see [16]). However, this phenomenon of absorbing the excess is not new, since it was already remarked in the case of an anisotropic equation (see [15]).

In the above examples, we have consider data belonging to $L^m(B_1(0))$, with $m > N$, and we have found bounded continuous solutions. When the condition $m > N$ does not hold, no bounded continuous solution must be expected, as shown in the following example.

Example 4.3. Take $f(x) := \dfrac{\lambda}{|x|}$ and observe that $f \in L^m(B_1(0))$ for all $m < N$. If $\lambda > N-1$, we have $\tilde{f}(r) = \frac{\lambda}{r} > \frac{N-1}{r}$ and (4.2) holds. In this case, we have

$$g(r) = \int_r^1 \left(\tilde{f}(s) - \frac{N-1}{s} \right) ds = \int_r^1 \frac{\lambda - N + 1}{s}\, ds = \log\left(\frac{1}{r}\right)^{\lambda - N + 1},$$

and the solution is

$$u(x) = -(\lambda - N + 1) \log |x|,$$

with

$$\mathbf{z}(x) = -\frac{x}{|x|}.$$

Now in the case $\lambda \leq N - 1$, we have that the solution is $u \equiv 0$, with

$$\mathbf{z}(x) = -\frac{\lambda x}{(N-1)|x|}.$$

We have prove that there exists a unique solution in the class of bounded $BV-$ functions whose jump part is empty. Outside that class is possible to find other distributional solutions. Indeed, we already know that $u \equiv 0$ is the weak solution to problem (1.1) with datum $f \equiv 0$. We now include an example of a non trivial solution.

Example 4.4. Let $N = 2$. Defining $u(x, y) = \frac{1}{2} \log(x^2 + y^2)$, we get

$$Du(x,y) = \frac{(x,y)}{x^2+y^2}, \qquad |Du(x,y)| = \frac{1}{\sqrt{x^2+y^2}} \quad \text{and} \quad \mathbf{z}(x,y) = \frac{(x,y)}{\sqrt{x^2+y^2}}.$$

Then

$$-\operatorname{div} \mathbf{z}(\mathrm{x}, \mathrm{y}) + |\mathrm{Du}(\mathrm{x}, \mathrm{y})| = \frac{-1}{\sqrt{\mathrm{x}^2+\mathrm{y}^2}} + \frac{1}{\sqrt{\mathrm{x}^2+\mathrm{y}^2}} = 0$$

and

$$(\mathbf{z}(x,y), Du(x,y)) = \frac{1}{\sqrt{x^2+y^2}} = |Du(x,y)|.$$

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