A NON–HOMOGENEOUS ELLIPTIC PROBLEM DEALING WITH THE LEVEL SET FORMULATION OF THE INVERSE MEAN CURVATURE FLOW

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Abstract. In the present paper we study the Dirichlet problem for the equation

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right)+|Du|=f$$

in an unbounded domain $\Omega\subset\mathbb{R}^N$, where the datum f is bounded and nonnegative. We point out that the only hypothesis assumed on $\partial\Omega$ is that of being Lipschitz–continuous. This problem is the non–homogeneous extension of the level set formulation of the inverse mean curvature flow in an Euclidean space. We introduce a suitable concept of weak solution, for which we prove existence, uniqueness and a Comparison Principle.

1. Introduction

The aim of this paper is to study the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right)+|Du|=f\,, & \text{in } \Omega\,;\\ u=0\,, & \text{on } \partial E_0\,;\\ \lim\limits_{|x|\to\infty}u(x)=+\infty\,; & \end{cases} \tag{1.1}$$

where $\Omega=\mathbb{R}^N\backslash\overline{E_0}$, being $N\geq 2$ and E_0 an open bounded set having Lipschitz–continuous boundary, and $0\leq f\in L^\infty(\Omega)$. We introduce a natural concept of weak solution and prove existence, uniqueness and a comparison principle. In bounded domains, the Dirichlet problem for that equation has been considered in [16]. We will use some of the techniques introduced in this paper, but we remark that the proofs of the present paper are more involved due to the unbounded character of the domain. On the other hand, previous results in unbounded domains have dealt with the homogeneous equation (the level set formulation of the inverse mean curvature flow) assuming additional conditions of smoothness on the boundary (see [13, 14, 18, 19]). We improve those papers in the sense that we do not assume the boundary being C^1, only

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Lipschitz–continuous. Nevertheless, when this article was being completed we learned of a preprint by Moser [20] in which this flow is studied from a very general perspective: it is shown that there exists solution under the only assumption on the initial condition E_0 of being open and bounded. Thus, the present paper is actually an extension of the inverse mean curvature flow in an Euclidean space to the inhomogeneous case and using a different concept of solution. In our inhomogeneous case, the datum f plays the role of damping the inverse mean curvature flow.

The inverse mean curvature flow is a one-parameter family of hypersurfaces $\{\Gamma_t\}_{t\geq 0} \subset \mathbb{R}^N$ $(N \geq 2)$ whose normal velocity $V_n(t)$ at each time t equals to the inverse of its mean curvature $H(t)$. If we let $\Gamma_t := F(\Gamma_0, t)$, then the parametric description of the inverse mean curvature flow is to find $F : \Gamma_0 \times [0,T] \to \mathbb{R}^N$ such that

$$\frac{\partial F}{\partial t} = \frac{\nu}{H}, \qquad t \geq 0, \tag{1.2}$$

where ν denotes the unit outward normal to Γ_t.

The inverse mean curvature flow and related geometric evolution problems have been studied by several authors. Among the pioneers works should be quoted [25, 12, 11, 24]. Huisken and Ilmanen in [13] propose a level set formulation of the inverse mean curvature flow (1.2), and define a notion of weak solution using an energy minimizing principle in such a way that the generalized inverse mean curvature flow exists for all time. Using this result they then give a proof of the Penrose Inequality, which says that the total mass of a space–time containing black holes with event horizons of the total area A should be at least $\sqrt{A(16\pi)^{-1}}$, for the particular case of a single black hole.

The level set formulation propose in [13] can be stated as follows. Assume that the flow is given by the level sets of a Lipschitz function $u : \mathbb{R}^N \to \mathbb{R}$ via

$$\Gamma_t = \partial E_t, \quad E_t := \{x \in \mathbb{R}^N \ : \ u(x) < t\}.$$

Wherever u is smooth with $\nabla u \neq 0$, equation (1.2) is equivalent to

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|.$$

Thus, (1.2) give rise to the boundary value problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = |\nabla u|\,, & \text{in } \ \Omega\,; \\[2ex] u = 0\,, & \text{on } \ \partial\Omega\,; \end{cases} \tag{1.3}$$

where $\Omega = \mathbb{R}^N \backslash \overline{E_0}$. Since E_0 is bounded, it follows that Ω is unbounded, and then to get uniqueness Huisken and Ilmanen look for solutions u satisfying

$$\lim_{|x|\to\infty} u(x) = +\infty.$$

To define *weak solution* of problem (1.3), Huisken and Ilmanen in [13] consider, for any compact $K \subset \Omega$ and any function $u \in BV_{loc}(\Omega) \cap C(\Omega)$, the functional

$$J_u^K(v) := \int_K (|\nabla v| + |\nabla u| v)\, dx,$$

and define a *weak solution* of problem (1.3) as a locally Lipschitz function u which satisfies

$$J_u^K(u) \leq J_u^K(v),$$

for every locally Lipschitz function v such that $\{v \neq u\} \subset\subset \Omega$, and any compact K containing $\{v \neq u\}$. They proved the existence of weak solution by elliptic regularity. Later, in [14], Huisken and Ilmanen have proved regularity results of inverse mean curvature flow and as consequence, every weak solution is regular after the first instant where a level set is star shaped.

A different proof for the existence of weak solutions of problem (1.3) is given in [18] by Moser, which relies on the observation that for $p > 1$, a logarithmic change of dependent variable transforms the approximating equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p$ to the homogeneous p-Laplace equation.

Our approach to existence is closer to the one performed by Moser. Indeed, to prove the existence of solution of problem (1.1) we approximate it by the following problems related to the p-Laplacian operator:

$$\begin{cases} -\Delta_p(u) + |\nabla u|^p = f\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial E_0\,; \\ \lim\limits_{|x|\to\infty} u(x) = +\infty\,; & \end{cases} \tag{1.4}$$

where $\Delta_p(u) := \operatorname{div}\,(|\nabla u|^{p-2}\nabla u)$, with $1 < p \leq 2$. We show that problem (1.4) is well–posed. The proof relies on a change of variable, which in the homogeneuos case leads to p-harmonic functions. Nevertheless, in our case, we have not this advantages, so that this result is new, as far as we know, and interesting in itself.

Moreover, our concept of weak solution is different to that used by Huisken and Ilmanen, and Moser, and follows the ideas developed in [2] and [3] (see also [4]) to study the Dirichlet problem associated with the total variation flow. Let us point out that our concept of solution coincides with the alternative formulation proposed by Huisken and Ilmanen in [13, Remark 2, p. 391], but we do not impose Lipschitz continuity to the solutions. Since this definition is not based on a functional depending on the solution being searched, it seems more natural.

Let us briefly summarize the contents of this paper. In Section 2 we fix the notation and give some preliminaries results that we will need. Section 3 is devoted to the study of the approximating problems, of the p–Laplacian type, that we used to prove the existence of solution. In Section 4 we introduce our notion of solution to problem (1.1) and derive some consequences of the definition. In Section 5 we prove the existence of solution and in Section 6 the uniqueness of solution and a Comparison Principle. In Section 7 we show some explicit examples. Finally, in Section 8 we make some remarks concerning the particular case of the level set formulation of the inverse mean curvature flow.

2. Preliminary results

In this section we introduce some notation and some preliminary results that we need. Throughout this paper $N \geq 2$, $\mathcal{H}^{N-1}$ will denote the $(N-1)$–dimensional Hausdorff measure and $\mathcal{L}^N$ the Lebesgue measure. If μ is a measure on Ω and $q \geq 1$, the symbol $L^q(\Omega, \mu)$ will denote the usual Lebesgue space of q–summable functions from Ω to $\mathbb{R}$. The measure will not be written when referring to the Lebesgue measure. Given a nonnegative $f \in L^1_{loc}(\Omega)$, the Lebesgue space in which the measure be the Lebesgue measure with weight f, will be denoted by $L^q(\Omega, f\mathcal{L}^N)$. Moreover, $L^q(\Omega; \mathbb{R}^N)$ will denote the space corresponding to $\mathbb{R}^N$–valued functions with the Lebesgue measure. The symbol $W^{1,q}(\Omega)$ will denote the Sobolev space of functions with distributional derivatives in $L^q(\Omega)$ and $W^{1,q}_0(\Omega)$ the subspace of functions in $W^{1,q}(\Omega)$ having zero traces on $\partial\Omega$.

2.1. **Functions of bounded variations and some generalizations.** The natural energy space to study the problems we are interested in is the space of functions of bounded variation. Recall that if Ω is an open subset of $\mathbb{R}^N$, a function $u \in L^1(\Omega)$ whose distributional gradient Du is a vector valued Radon measure with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. We denote by $BV_{loc}(\Omega)$ the space of functions $u \in L^1_{loc}(\Omega)$, such that $u \in BV(\omega)$ for all $\omega \subset\subset \Omega$. For every $u \in BV(\Omega)$, the Radon measure Du is decomposed into its absolutely continuous and singular parts with respect to the Lebesgue measure: $Du = D^a u + D^s u$. So $D^a u = \nabla u \; \mathcal{L}^N$, where ∇u is the Radon–Nikodým derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$.

We denote by S_u the set of all $x \in \Omega$ such that u does not have an approximate limit at x. We say that $x \in \Omega$ is an *approximate jump point of* u if there exist $u^+(x) > u^-(x) \in \mathbb{R}$ and $\nu_u(x) \in S^{N-1}$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{\mathcal{L}^N(B^+_\rho(x, \nu_u(x)))} \int_{B^+_\rho(x,\nu_u(x))} |u(y) - u^+(x)| \, dy = 0$$

$$\lim_{\rho \downarrow 0} \frac{1}{\mathcal{L}^N(B^-_\rho(x, \nu_u(x)))} \int_{B^-_\rho(x,\nu_u(x))} |u(y) - u^-(x)| \, dy = 0,$$

where

$$B^+_\rho(x, \nu_u(x)) = \{ y \in B_\rho(x) \;:\; \langle y - x, \nu_u(x) \rangle > 0 \}$$

and

$$B_\rho^-(x,\nu_u(x)) = \{y \in B_\rho(x) \ : \ \langle y-x, \nu_u(x)\rangle < 0\}.$$

We recall that for a Radon measure μ in Ω and a Borel set $A \subseteq \Omega$ the measure $\mu \llcorner A$ is defined by $(\mu \llcorner A)(B) = \mu(A \cap B)$ for any Borel set $B \subseteq \Omega$. If a measure μ is such that $\mu = \mu \llcorner A$ for a certain Borel set A, the measure μ is said to be concentrated on A.

We denote by J_u the set of approximate jump points of u. By the Federer-Vol'pert Theorem [1, Theorem 3.78], it is known that the set S_u is countably $\mathcal{H}^{N-1}$–rectifiable and $\mathcal{H}^{N-1}(S_u \backslash J_u) = 0$. Moreover, $Du \llcorner J_u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u$. Using S_u and J_u, we may split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$ defined by

$$D^j u = D^s u \llcorner J_u \quad \text{and} \quad D^c u = D^s u \llcorner (\Omega \backslash S_u).$$

Then, we have

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

Moreover, if $x \in J_u$, then $\nu_u(x) = \frac{Du}{|Du|}(x)$, $\frac{Du}{|Du|}$ being the Radon–Nikodým derivative of Du with respect to its total variation $|Du|$.

If x is a Lebesgue point of u, then $u^+(x) = u^-(x)$ for any choice of the normal vector and we say that x is an approximate continuity point of u. We define the approximate limit of u by $\tilde{u}(x) = u^+(x) = u^-(x)$. The *precise representative* $u^* : \Omega \backslash (S_u \backslash J_u) \to \mathbb{R}$ of u is defined as equal to $\tilde{u}$ on $\Omega \backslash S_u$ and equal to $\frac{u^+ + u^-}{2}$ on J_u. It is well–know (see for instance [1, Corollary 3.80]) that if ρ is a symmetric mollifier, then the mollified functions $u \star \rho_\epsilon$ pointwise converges to u^* in its domain.

Recall that a $\mathcal{L}^N$–measurable set $E \subset \Omega$ is said to have *finite perimeter* in Ω if $\chi_E \in BV(\Omega)$, and then the perimeter of E in Ω is defined as

$$Per(E,\Omega) := \int_\Omega |D\chi_E|.$$

We also denote

$$Per(E) := Per(E,\mathbb{R}^N).$$

When E has finite perimeter in Ω, the *reduced boundary* $\partial^* E$ is the set of all points $x \in \operatorname{supp}(|D\chi_E|)$ such that the limit

$$\nu_E(x) := \lim_{\rho \downarrow 0} \frac{D\chi_E(B_\rho(x))}{|D\chi_E(B_\rho(x))|}$$

exists in $\mathbb{R}^N$ and satisfies $|\nu_E(x)| = 1$. The function $\nu_E : \partial^* E \to \mathbb{S}^{N-1}$ is called the *generalized inner normal* to E.

Two well–known facts concerning the reduced boundary can be found in [1, Theorem 3.59]:

$$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E\,, \tag{2.1}$$

and in [1, Theorem 3.61]:

$$\lim_{\rho\downarrow 0}\frac{\mathcal{L}^N(B_\rho(x)\cap E)}{\mathcal{L}^N(B_\rho(x))}=\frac{1}{2}\qquad \mathcal{H}^{N-1}\text{–a.e. } x\in\partial^* E\cap\Omega\,. \tag{2.2}$$

For further information concerning functions of bounded variation we refer to [1], [10] or [27].

2.2. **A generalized Green's formula.** We shall need several results from [6] (see also [4]) in order to give sense to the dot product of bounded vector fields whose divergence is a measure and the gradient of a BV function. This theory was also studied in [8] from a different point of view.

Assume that Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz–continuous boundary. Set

$$\mathcal{DM}_\infty(\Omega):=\left\{\mathbf{z}\in L^\infty(\Omega,\mathbb{R}^N)\ :\ \mathrm{div}\,(\mathbf{z})\text{ is a bounded Radon measure in }\Omega\right\}.$$

If $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ and $w\in BV(\Omega)\cap C(\Omega)\cap L^\infty(\Omega)$, we define the functional $(\mathbf{z},Dw):C_0^\infty(\Omega)\to\mathbb{R}$ by the formula

$$\langle(\mathbf{z},Dw),\varphi\rangle:=-\int_\Omega w\,\varphi\,d(\mathrm{div}\,(\mathbf{z}))-\int_\Omega w\,\mathbf{z}\cdot\nabla\varphi\,dx. \tag{2.3}$$

In [6] (see also [4, Corollary C.7, C.16]) the following result is proved.

Proposition 2.1. *The distribution $(\mathbf{z},Dw)$ is actually a Radon measure with finite total variation.*

The measures $(\mathbf{z},Dw)$, $|(\mathbf{z},Dw)|$ are absolutely continuous with respect to the measure $|Dw|$ and

$$\left|\int_B(\mathbf{z},Dw)\right|\le\int_B|(\mathbf{z},Dw)|\le\|\mathbf{z}\|_{L^\infty(U)}\int_B|Dw|$$

for all Borel sets B and for all open sets U such that $B\subset U\subset\Omega$.

Moreover, if $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous increasing function, then

$$\theta(\mathbf{z},D(f\circ w),x)=\theta(\mathbf{z},Dw,x),\qquad |Dw|-\text{a.e. in }\ \Omega, \tag{2.4}$$

where $\theta(\mathbf{z},Dw,\cdot)$ is the Radon–Nikodým derivative of $(\mathbf{z},Dw)$ with respect to $|Dw|$.

We denote by ν the outward unit normal to $\partial\Omega$. In [6], a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ is defined. More precisely, it is proved that there exists a linear operator $\gamma:\mathcal{DM}_\infty(\Omega)\to L^\infty(\partial\Omega)$ such that

$$\|\gamma(\mathbf{z})\|_\infty\le\|\mathbf{z}\|_\infty$$

$$\gamma(\mathbf{z})(x)=\mathbf{z}(x)\cdot\nu(x)\quad\text{ for all } x\in\partial\Omega\ \text{ if }\ \mathbf{z}\in C^1(\overline{\Omega},\mathbb{R}^N).$$

We shall denote $\gamma(\mathbf{z})(x)$ by $[\mathbf{z},\nu](x)$. Moreover, the following *Green's formula*, relating the function $[\mathbf{z},\nu]$ and the measure $(\mathbf{z},Dw)$, for $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ and $w\in BV(\Omega)\cap C(\Omega)\cap L^\infty(\Omega)$, is established

$$\int_\Omega w\,d(\operatorname{div}(\mathbf{z}))+\int_\Omega(\mathbf{z},Dw)=\int_{\partial\Omega}[\mathbf{z},\nu]\,w\;d\mathcal{H}^{N-1}. \tag{2.5}$$

Applying a Meyers–Serrin type Theorem, it was observed in [17] (see also [7]) that it is possible to get a Green's formula like (2.5) for $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ and $w\in BV(\Omega)\cap L^\infty(\Omega)$, that is, without assuming the continuity of w. To do that, for $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ and $w\in BV(\Omega)\cap L^\infty(\Omega)$ is defined the functional $(\mathbf{z},Dw):C_0^\infty(\Omega)\to\mathbb{R}$ by the formula

$$\langle(\mathbf{z},Dw),\varphi\rangle:=-\int_\Omega w^*\,\varphi\,d(\operatorname{div}(\mathbf{z}))-\int_\Omega w\,\mathbf{z}\cdot\nabla\varphi\,dx, \tag{2.6}$$

which is well defined since $|\operatorname{div}(\mathbf{z})|$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$ (see [8, Proposition 3.1]). With this definition of $(\mathbf{z},Dw)$, in [17] it is proved that $(\mathbf{z},Dw)$ is a Radon measure such that

$$\left|\int_B(\mathbf{z},Dw)\right|\le\|\mathbf{z}\|_{L^\infty(U)}|Dw|(B) \tag{2.7}$$

for every Borel set B and for every open set U such that $B\subset U\subset\Omega$, and verifies the Green formula

$$\int_\Omega w^*\,d(\operatorname{div}(\mathbf{z}))+\int_\Omega(\mathbf{z},Dw)=\int_{\partial\Omega}[\mathbf{z},\nu]w\;d\mathcal{H}^{N-1}. \tag{2.8}$$

Observe that for $\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$ and $w\in BV(\Omega)\cap L^\infty(\Omega)$, we have the following equality as Radon measures

$$\operatorname{div}(w\mathbf{z})=(\mathbf{z},Dw)+w^*\operatorname{div}(\mathbf{z})\,, \tag{2.9}$$

so that $w\mathbf{z}\in\mathcal{DM}_\infty(\Omega)$.

Remark 2.2. When Ω is unbounded, we will say that a Radon measure is locally bounded in Ω if its total variation is finite in each open bounded $\omega\subset\Omega$. We then denote

$$\mathcal{DM}_\infty^{loc}(\Omega):=\left\{\mathbf{z}\in L^\infty(\Omega,\mathbb{R}^N)\ :\ \operatorname{div}(\mathbf{z})\text{ is a locally bounded Radon measure in }\Omega\right\}.$$

It is easy to see that if $w\in BV_{loc}(\Omega)\cap L^\infty_{loc}(\Omega)$ and $\mathbf{z}\in\mathcal{DM}^{loc}_\infty(\Omega)$, then the functional $(\mathbf{z},Dw):C_0^\infty(\Omega)\to\mathbb{R}$ defined by the formula

$$\langle(\mathbf{z},Dw),\varphi\rangle:=-\int_\Omega w^*\,\varphi\,d(\operatorname{div}(\mathbf{z}))-\int_\Omega w\,\mathbf{z}\cdot\nabla\varphi\,dx \tag{2.10}$$

is a Radon measure on Ω that also satisfy

$$\left|\int_B(\mathbf{z},Dw)\right|\le\|\mathbf{z}\|_{L^\infty(U)}|Dw|(B) \tag{2.11}$$

for every Borel set B and for every open set U such that $B\subset U\subset\subset\Omega$.

Nevertheless, Green's formula does not hold in general. Hence, to apply Green's formula, we will have to restrict to an open bounded $\omega \subset \Omega$.

In principle it is not clear that (2.4) holds in the case that $\mathbf{z} \in \mathcal{DM}_\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. However, in [16, Proposition 2.2] we have showed that (2.4) holds if we assume that the jump part is $\mathcal{H}^{N-1}$–null, that is $D^j u = 0$. With a similar proof, we can establish the following result.

Proposition 2.3. *Let $\mathbf{z} \in \mathcal{DM}^{loc}_\infty(\Omega)$ and consider $u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ with $D^j u = 0$. If $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous increasing function, then*

$$\theta(\mathbf{z}, D(f \circ u), x) = \theta(\mathbf{z}, Du, x), \qquad |Du| - \text{a.e. in } \Omega \tag{2.12}$$

We also have the following result with the same proof of [16, Proposition 2.3].

Proposition 2.4. *If $\mathbf{z} \in \mathcal{DM}^{loc}_\infty(\Omega)$ and $u, w \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ with $D^j u = D^j w = 0$, then*

$$(w\mathbf{z}, Du) = w^*(\mathbf{z}, Du) \quad \text{as Radon measures.} \tag{2.13}$$

Finally, let us remark that with a similar proof to [7, Lemma 5.6] (see also [5, Proposition 1]), we can obtain the following result.

Proposition 2.5. *If $\mathbf{z} \in \mathcal{DM}^{loc}_\infty(\Omega)$ and $u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$, then*

$$[u\mathbf{z}, \nu] = u[\mathbf{z}, \nu] \quad \mathcal{H}^{N-1} - a.e. \text{ in } \partial\Omega.$$

3. Approximating problems: Existence and uniqueness

From now on we will assume that $E_0 \subset \mathbb{R}^N$ is an open bounded set with Lipschitz continuous boundary and $\Omega = \mathbb{R}^N \setminus \overline{E_0}$.

To prove the existence of solution of problem (1.1) we approximate it by the following problems related to the p–Laplacian operator:

$$\begin{cases} -\Delta_p(u) + |\nabla u|^p = f\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial E_0\,; \\ \lim\limits_{|x|\to\infty} u(x) = +\infty\,. \end{cases} \tag{3.1}$$

Our aim in this Section is to prove the following results.

Theorem 3.1. *For each $1 < p < 2$ and each nonnegative $f \in L^\infty(\Omega)$, there exists a unique solution of problem (3.1) in the sense of Definition 3.4. Moreover, if $x_0 \in E_0$ and $s > 0$ satisfy $E_0 \subset B_s(x_0)$, then*

$$u(x) \geq (N-p) \log\left(\frac{|x - x_0|}{s}\right), \qquad x \in \Omega\,. \tag{3.2}$$

Theorem 3.2. *Fix $1 < p < 2$. For $i \in \{1,2\}$, let E_0^i a bounded and open set and let u_i be the solution to problem* (3.1)*, in the sense of Definition 3.4, in the domain $\Omega_i = \mathbb{R}^N \backslash \overline{E_0^i}$ with datum $f_i \in L^\infty(\Omega_i)$, $i = 1, 2$. If $E_0^2 \subset E_0^1$ and $f_1 \le f_2$ in Ω_1, then $u_1 \le u_2$ in Ω_1.*

Remark 3.3. Taking $E_0 = B_s(x_0)$ and $f \equiv 0$, it is easy to check that

$$u(x) = (N-p)\log\left(\frac{|x-x_0|}{s}\right), \qquad x \in \Omega\,,$$

is a solution to problem (3.1). Thus, the bound in (3.2) is achieved. Furthermore, observe that then

$$u(x) = t \Longleftrightarrow |x-x_0| = s e^{t/(N-p)}\,, \qquad \text{for } |x-x_0| \ge s \text{ and } t \ge 0\,,$$

defines a p–approximation to the inverse mean curvature flow starting from $B_s(x_0)$.

We will prove Theorem 3.1 as a consequence of Proposition 3.9 and Theorem 3.12 below, while Theorem 3.2 relies on Proposition 3.9 and Corollary 3.11.

We will begin by defining what is meant by a solution to this problem.

Definition 3.4. *We say that a nonnegative function u is a solution to* (3.1) *if*

(1) $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$
(2) its trace on ∂E_0 *is* 0
(3) $\lim\limits_{|x|\to\infty} u(x) = +\infty$
(4) $e^{-u/(p-1)} \in L^p(\Omega, f\mathcal{L}^N)$
(5) $e^{-u/(p-1)} \in W^{1,p}(\Omega)$

and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx + \int_\Omega |\nabla u|^p \varphi\,dx = \int_\Omega f\varphi\,dx\,, \tag{3.3}$$

holds for every φ satisfying $\varphi|_\omega \in W_0^{1,p}(\omega) \cap L^\infty(\omega)$ and $\varphi \equiv 0$ in $\Omega\backslash\omega$, where $\omega \subset \Omega$ is open and bounded.

Remark 3.5. Since E_0 is bounded and $u \in W^{1,p}_{loc}(\Omega)$, it follows that the trace of u on the boundary of E_0 is well–defined.

Remark 3.6. For a solution u to (3.1), consider the function $v = e^{-u/(p-1)}$. It follows that $v \in L^p(\Omega, f\mathcal{L}^N) \cap W^{1,p}(\Omega)$ and its trace on ∂E_0 is 1. Taking $v(x) = 1$ for all $x \in E_0$, we may assume that v is defined on $\mathbb{R}^N$ and so $v \in W^{1,p}(\mathbb{R}^N)$. Thus, Sobolev's inequality implies that $v \in L^{p^*}(\mathbb{R}^N)$, where $p^* = \dfrac{Np}{N-p}$. This type of arguments will also be applied to every function belonging to $W^{1,p}(\Omega)$ whose trace is constant on E_0 (as those test functions in Proposition 3.9 below), without further comments.

Observe that, formally, under the change of unknown

$$v = (p-1)e^{-\frac{u}{p-1}} \qquad u = -(p-1)\log\left(\frac{v}{p-1}\right) \tag{3.4}$$

problem (3.1) becomes equivalent to

$$\begin{cases} -\Delta_p(v) + f\left(\dfrac{v}{p-1}\right)^{p-1} = 0\,, & \text{in } \ \Omega\,;\\ v > 0\,, & \text{in } \ \Omega\,;\\ v = p-1\,, & \text{on } \ \partial E_0\,;\\ \lim\limits_{|x|\to\infty} v(x) = 0\,. \end{cases} \tag{3.5}$$

Actually, for any $\alpha > 0$ given, we will consider a slightly more general problem

$$\begin{cases} -\Delta_p(v) + f v^{p-1} = 0\,, & \text{in } \ \Omega\,;\\ v > 0\,, & \text{in } \ \Omega\,;\\ v = \alpha\,, & \text{on } \ \partial E_0\,;\\ \lim\limits_{|x|\to\infty} v(x) = 0\,. \end{cases} \tag{3.6}$$

Definition 3.7. *We say that v is a solution to* (3.6) *if $v \in W^{1,p}(\Omega) \cap L^p(\Omega, f\mathcal{L}^N)$ satisfies that $0 < v \le \alpha$ in Ω, its trace on ∂E_0 is α, $\lim\limits_{|x|\to\infty} v(x) = 0$ and*

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi\, dx + \int_\Omega f v^{p-1}\varphi\, dx = 0\,, \tag{3.7}$$

holds for every $\varphi \in W_0^{1,p}(\Omega) \cap L^p(\Omega, f\mathcal{L}^N)$.

Moreover, for each open bounded subset $\omega \subset \Omega$ there exists a constant $C_\omega > 0$ such that

$$v \ge C_\omega\,, \quad \text{in } \omega\,. \tag{3.8}$$

Remark 3.8. As in Remark 3.5, we may guarantee that the trace of v on the boundary of E_0 is well–defined and, as in Remark 3.6, $v \in L^{p^*}(\mathbb{R}^N)$.

We next prove that the change of unknown (3.4) transforms a solution to (3.1) in a solution to (3.5) and reciprocally.

Proposition 3.9. *The function u is a solution to* (3.1) *in the sense of Definition 3.4 if, and only if, $v = (p-1)e^{-\frac{u}{p-1}}$ is a solution to* (3.5) *in the sense of Definition 3.7.*

Proof. Assume first that u is a solution to (3.1) and set $v = (p-1)e^{-\frac{u}{p-1}}$. It is straightforward that $v \in W^{1,p}(\Omega) \cap L^p(\Omega, f\mathcal{L}^N)$ satisfies that its trace on ∂E_0 is $p-1$.

Consider an open bounded set $\omega \subset \Omega$. Observe that $u \in L^\infty(\omega)$ implies that there exists a constant $C_\omega > 0$ such that $e^{-u} \geq C_\omega$ on ω. Thus, $v \geq \big((p-1)C_\omega\big)^{p-1} > 0$ on ω.

Consider again $\omega \subset \Omega$ and the constant $C_\omega > 0$ such that $e^{-u} \geq C_\omega$ on ω. Let φ satisfy $\varphi|_\omega \in W_0^{1,p}(\omega) \cap L^\infty(\omega)$ and $\varphi \equiv 0$ in $\Omega \backslash \omega$. Since the function $g(s) = s^{p-1}$ is always Lipschitz–continuous away from 0, it follows from $e^{-\frac{u}{p-1}} \in W^{1,p}(\omega)$ and Stampacchia's Theorem that $e^{-u} \in W^{1,p}(\omega)$. Hence, $e^{-u}\varphi \in W_0^{1,p}(\omega) \cap L^\infty(\omega)$ and so it can be chosen as test function in (3.3). Then, taking it and simplifying, we obtain

$$\int_\Omega e^{-u}|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx = \int_\Omega f e^{-u}\varphi\, dx\,.$$

Performing easy computations, it yields

$$-\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi\, dx = \int_\Omega f\left(\frac{v}{p-1}\right)^{p-1}\varphi\, dx\,.$$

It remains to extend the class of admissible test functions. To this end, consider $\varphi \in W_0^{1,p}(\Omega) \cap L^p(\Omega, f\mathcal{L}^N)$. Let ζ be a smooth function satisfying $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on the unit ball $B_1(0)$ and $\operatorname{supp}(\zeta) \subset B_2(0)$. Consider the sequence $\{\zeta_k\}$ given by $\zeta_k(x) = \zeta\big(\frac{x}{k}\big)$ and define $\varphi_k = \zeta_k\varphi$ for all $k \in \mathbb{N}$.

We claim that

$$\varphi_k \to \varphi\,, \quad \text{in } \; L^p(\Omega, f\mathcal{L}^N) \tag{3.9}$$

$$\nabla\varphi_k \to \nabla\varphi\,, \quad \text{in } \; L^p(\Omega;\mathbb{R}^N)\,. \tag{3.10}$$

To see (3.9), observe that $\varphi_k \equiv \varphi$ in $B_k(0)$ and $|\varphi_k| \leq |\varphi|$. Thus,

$$\Big(\int_\Omega f|\varphi_k-\varphi|^p\, dx\Big)^{1/p} = \Big(\int_{\{|x|>k\}} f|\varphi_k-\varphi|^p\, dx\Big)^{1/p} \leq 2\Big(\int_{\{|x|>k\}} f|\varphi|^p\, dx\Big)^{1/p},$$

and the latter integral goes to 0 as $k \to \infty$ since $\varphi \in L^p(\Omega, f\mathcal{L}^N)$.

We now show (3.10). As above, we have $\nabla\varphi_k \equiv \nabla\varphi$ in $B_k(0)$. Hence, by Hölder's inequality and the identity $\nabla\varphi_k = \zeta_k\nabla\varphi + \varphi\nabla\zeta_k$,

$$\begin{aligned}\Big(\int_\Omega |\nabla\varphi_k-\nabla\varphi|^p\, dx\Big)^{1/p} &= \Big(\int_{\{|x|>k\}} |\nabla\varphi_k-\nabla\varphi|^p\, dx\Big)^{1/p} \\ &\leq \Big(\int_{\{|x|>k\}} |\zeta_k\nabla\varphi-\nabla\varphi|^p\, dx\Big)^{1/p} + \Big(\int_{\{|x|>k\}} |\varphi|^p|\nabla\zeta_k|^p\, dx\Big)^{1/p} \\ &\leq 2\Big(\int_{\{|x|>k\}} |\nabla\varphi|^p\, dx\Big)^{1/p} + \Big(\int_{\{|x|>k\}} |\varphi|^{p^*}\, dx\Big)^{1/p^*}\Big(\int_\Omega |\nabla\zeta|^N\, dx\Big)^{1/N}.\end{aligned} \tag{3.11}$$

Since $\nabla\varphi \in L^p(\Omega;\mathbb{R}^N)$ and, by Sobolev's inequality, $\varphi \in L^{p^*}(\Omega)$, it follows that

$$\lim_{k\to\infty}\int_{\{|x|>k\}} |\nabla\varphi|^p\, dx = 0 \qquad \lim_{k\to\infty}\int_{\{|x|>k\}} |\varphi|^{p^*}\, dx = 0\,.$$

Thus, (3.11) yields (3.10).

Finally, note that $\varphi_k \in W_0^{1,p}(\Omega \cap B_{2k}(0)) \cap L^\infty(\Omega \cap B_{2k}(0))$, and so we may apply (3.7) to each φ_k. Then (3.9) and (3.10) allow us to let k go to ∞ and conclude that (3.7) holds true in general.

Reciprocally, assume that v is a solution to (3.5) and set $u = -(p-1)\log\left(\frac{v}{p-1}\right)$. It is easy to check that $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$, $e^{-u/(p-1)} \in L^p(\Omega, f\mathcal{L}^N) \cap W^{1,p}(\Omega)$ and its trace on ∂E_0 is 0.

Consider an open bounded set $\omega \subset \Omega$ and let $C_\omega > 0$ be a constant such that $v \geq C_\omega$ on ω. Since $g(s) = s^{-(p-1)}$ is a Lipschitz–continuous function away from 0, it follows from Stampacchia's Theorem that $\left(\frac{p-1}{v}\right)^{p-1} \in W^{1,p}(\omega)$. So $\left(\frac{p-1}{v}\right)^{p-1}\varphi \in W_0^{1,p}(\omega) \cap L^\infty(\omega)$. Taking it as test function in (3.7) when f is replaced with $f/(p-1)^{p-1}$, we have

$$\int_\Omega \left(\frac{p-1}{v}\right)^{p-1} |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi\, dx - \int_\Omega \left(\frac{p-1}{v}\right)^{p} \varphi |\nabla v|^p\, dx + \int_\Omega f\varphi\, dx = 0\,.$$

By simple manipulations this equality becomes

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx - \int_\Omega \varphi|\nabla u|^p\, dx + \int_\Omega f\varphi\, dx = 0\,,$$

and so the proof is complete. □

The previous result implies that Theorem 3.1 is a consequence of the existence and uniqueness for problem (3.6). To begin our study of problem (3.6), we will see that it is subject to a comparison principle.

By a supersolution (respectively, subsolution) to problem (3.6) we mean a positive function v satisfying $v \in W^{1,p}(\Omega) \cap L^p(\mathbb{R}^N, f_1\mathcal{L}^N)$, with $f_1 \leq$(resp.,$\geq$) f, its trace on ∂E_0 is greater (resp., less) than α and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi\, dx + \int_\Omega f_1 v^{p-1}\varphi\, dx \geq (\text{resp.}, \leq) 0\,, \tag{3.12}$$

holds for every nonnegative $\varphi \in W_0^{1,p}(\Omega) \cap L^p(\Omega, f_1\mathcal{L}^N)$.

Proposition 3.10. *Let v be a solution to problem* (3.6) *in the sense of Definition 3.7.*

(1) If v_1 is a supersolution to problem (3.6)*, then $v \leq v_1$.*
(2) If v_2 is a subsolution to problem (3.6)*, then $v_2 \leq v$.*

Proof. We just have to prove the first assertion, since the second is analogously proved.

Observe that, since the trace of v_1 on ∂E_0 is greater than that of v, we deduce that the trace of $\varphi = (v - v_1)^+$ on ∂E_0 is equal to 0. Taking this φ in (3.7) and in the formulation (3.12) corresponding to v_1, and subtracting them, we obtain

$$\int_\Omega (|\nabla v|^{p-2}\nabla v - |\nabla v_1|^{p-2}\nabla v_1)\cdot\nabla(v-v_1)^+\,dx \\ + \int_\Omega f_1(v^{p-1}-v_1^{p-1})(v-v_1)^+\,dx + \int_\Omega (f-f_1)v^{p-1}(v-v_1)^+\,dx \le 0\,.$$

Since the integrands are nonnegative, we deduce that they vanish. So

$$(|\nabla v|^{p-2}\nabla v - |\nabla v_1|^{p-2}\nabla v_1)\cdot\nabla(v-v_1)^+ = 0 \quad \text{in } \Omega\,,$$

and it implies $\nabla(v-v_1)^+=0$ in Ω. Since Sobolev's inequality leads to $(v-v_1)^+=0$ in Ω, we conclude that $v\le v_1$. □

Corollary 3.11. *Fix $1<p<2$. Let v_i be the solution to problem* (3.6)*, in the sense of Definition 3.7, in the domain $\Omega_i=\mathbb{R}^N\backslash\overline{E_0^i}$ with datum $f_i\in L^\infty(\Omega_i)$, $i=1,2$. If $E_0^2\subset E_0^1$ and $f_1\le f_2$ in Ω_1, then $v_2\le v_1$ in Ω_1.*

Proof. We only have to see that v_2 is a subsolution to problem (3.6) in the domain Ω_1 with datum f_1. This is indeed the case, since $f_2\ge f_1$ and $v_2|_{\partial E_0^1}\le\alpha=v_1|_{\partial E_0^1}$. □

Once Theorem 3.2 is proved (as a consequence of Corollary 3.11 and Proposition 3.9), it only remains to see existence and uniqueness for problem (3.6).

Theorem 3.12. *Let $\alpha>0$ and let $f\in L^\infty(\Omega)$ be nonnegative. Then, for each $1<p<2$, there exists one and only one solution v to problem* (3.6) *in the sense of Definition 3.7. Moreover, if $x_0\in E_0$ and $s>0$ satisfy $E_0\subset B_s(x_0)$, then*

$$v(x)\le\alpha\left(\frac{|x-x_0|}{s}\right)^{-\frac{N-p}{p-1}}\,,\qquad x\in\Omega\,. \tag{3.13}$$

Proof. Existence

When $f\equiv0$, we may apply the same approach used in [18]. Thus, we will assume that f is positive on a set of positive $\mathcal{L}^N$–measure. Now we extend f to be 0 in E_0 and consider $\{v\in W^{1,p}(\mathbb{R}^N)\cap L^p(\mathbb{R}^N,f\mathcal{L}^N)\,:\,v=\alpha \text{ on } E_0\}$. Then we define on this affine space the functional given by

$$I[v]=\int_{\mathbb{R}^N}|\nabla v|^p\,dx+\int_{\mathbb{R}^N}f|v|^p\,dx\,. \tag{3.14}$$

This functional is convex and coercive, since $I[v]^{1/p}$ defines a norm in $W^{1,p}(\mathbb{R}^N)$ that is equivalent to its usual norm. By well–known results, there exists a function v which minimizes I and so it satisfies (3.7) for all $\varphi\in W_0^{1,p}(\Omega)\cap L^p(\Omega,f\mathcal{L}^N)$.

The solution v turns to be nonnegative. Indeed, we may take $\varphi=v^-$ in (3.7) which becomes

$$-\int_{\{v\le0\}}|\nabla v|^p\,dx-\int_{\{v\le0\}}f|v|^p\,dx=0\,,$$

so that we deduce that v must vanish on the set $\{v \le 0\}$.

On the other hand, if $k \ge \alpha$ then $T_k(v) \in W^{1,p}(\mathbb{R}^N) \cap L^p(\Omega, f\mathcal{L}^N)$ is such that $v = \alpha$ on E_0, where $T_k(s) = \sup(-k, \inf(s,k))$. Hence denoting $G_k(s) := s - T_k(s)$ and taking $\varphi = G_k(v)$ as test function in (3.7), we get

$$\int_{\mathbb{R}^N} |\nabla G_k(v)|^p dx + \int_{\mathbb{R}^N} f v^{p-1} G_k(v) dx = 0\,,$$

and so $G_k(v) = 0$. Therefore, we obtain that

$$0 \le v \le \alpha\,. \tag{3.15}$$

Uniqueness

It is a straightforward consequence of Proposition 3.10.

Positivity of v

Set $\lambda = \|f\|_\infty$ and consider the problem

$$\begin{cases} -\Delta_p(w) + \lambda w^{p-1} = 0\,, & \text{in } \mathbb{R}^N \backslash \overline{E_0}\,; \\ w = \dfrac{\alpha}{2}\,, & \text{on } \partial E_0\,; \end{cases} \tag{3.16}$$

From the Steps 1–4, already proved, we deduce that there is a nonnegative weak solution $w \in W^{1,p}(\mathbb{R}^N \backslash E_0)$ to the problem (3.16). This solution turns to be regular enough (see [9, 23]) to apply the Strong Maximum Principle given in [21, 26]. Hence, for every open bounded $\omega \subset \Omega$ we may find a constant $C_\omega > 0$ satisfying

$$w > C_\omega \quad \text{on} \quad \omega\,.$$

On the other hand, applying Proposition 3.10, we obtain that $w \le v$. Therefore,

$$v > C_\omega \quad \text{on} \quad \omega\,.$$

Compatibility of v with (3.13)

Given $x_0 \in E_0$ and $s > 0$ satisfying $E_0 \subset B_s(x_0)$, consider the function defined by $w(x) = \alpha\Big(\frac{|x-x_0|}{s}\Big)^{-\frac{N-p}{p-1}}$. Since $w \ge \alpha = v$ on ∂E_0 and $-\Delta_p(w) = 0 \ge -\Delta_p(v)$ in Ω, it follows from Proposition 3.10 that $w \ge v$ in Ω, as desired. □

4. Definition of solutions

We introduce the following concept of solution to problem (1.1).

Definition 4.1. *We say that a nonnegative function $u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ is a weak solution to* (1.1) *if $D^j u = 0$,*

$$\lim_{|x| \to \infty} u(x) = +\infty\,, \tag{4.1}$$

$$u|_{\partial E_0} = 0 \quad \mathcal{H}^{N-1} - a.e., \tag{4.2}$$

and there exists a vector field $\mathbf{z} \in \mathcal{DM}^{loc}_\infty(\Omega; \mathbb{R}^N)$*, with* $\|\mathbf{z}\|_\infty \leq 1$*, satisfying*

$$-\mathrm{div}\,(\mathbf{z}) + |Du| = f \quad \text{in } \mathcal{D}'(\Omega) \tag{4.3}$$

and

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures in } \Omega\,. \tag{4.4}$$

Remark 4.2. Let us point out that in Remark 8.1 we shall see that in the case $f \equiv 0$, for locally Lipschitz functions, the above concept of solution coincides with the one introduced by Huisken and Ilmanen.

Remark 4.3. Having in mind (4.2), we can extend every solution u to be 0 in $\overline{E_0}$ and consider $u \in BV_{loc}(\mathbb{R}^N)$. We shall use this extension without further comments. We may also extend f to vanish in $\overline{E_0}$. We explicitly point out that the associated vector field $\mathbf{z}$ will not be extended.

Remark 4.4. The condition $D^j u = 0$ does not imply that u is a continuous function. Nevertheless, then $\mathcal{H}^{N-1}(S_u) = 0$ and so the points of discontinuity of its precise representative u^* make up a $\mathcal{H}^{N-1}$–null set. Having a negligible jump part implies important consequences. Among them, the chain rule is as simple as the one for Sobolev spaces, namely:
If $u \in BV(\Omega) \cap L^\infty(\Omega)$ satisfies $D^j u = 0$, and f is a Lipschitz–continuous function, then $v = f \circ u$ belongs to $BV(\Omega)$ and $Dv = f'(u^*)(D^a u + D^c u)$.

It follows from this result that

$$Du \llcorner \{u^* = t\} = 0 \qquad \text{for all } t \geq 0. \tag{4.5}$$

Indeed, $Du = D(u-t) = D(u-t)^+ - D(u-t)^-$ and, on the other hand,

$$D(u-t)^+ = \begin{cases} Du\,, & \text{if } u > t\,; \\ 0\,, & \text{if } u \leq t\,; \end{cases} \quad \text{and} \quad D(u-t)^- = \begin{cases} 0\,, & \text{if } u \geq t\,; \\ -Du\,, & \text{if } u < t\,; \end{cases}$$

which imply $D(u-t)^+ \llcorner \{u^* = t\} = 0$ and $D(u-t)^- \llcorner \{u^* = t\} = 0$. Hence,

$$Du \llcorner \{u^* = t\} = D(u-t)^+ \llcorner \{u^* = t\} - D(u-t)^- \llcorner \{u^* = t\} = 0\,.$$

Proposition 4.5. *Let u be a weak solution to problem* (1.1) *with associated vector field* $\mathbf{z}$*. Then it satisfies*

$$-\mathrm{div}(e^{-u}\mathbf{z}) = e^{-u} f \qquad \text{in } \mathcal{D}'(\Omega) \tag{4.6}$$

$$(\mathbf{z}, D(1-e^{-u})) = |D(1-e^{-u})| \quad \text{as Radon measures in } \Omega. \tag{4.7}$$

Proof. In fact, by (4.4), it yields

$$\theta(\mathbf{z}, Du, x) = 1 \quad |Du| - \text{a.e. in } \Omega.$$

Then, applying Proposition 2.3 we deduce (4.7). From here, applying (2.9), (4.7), (4.3) and the chain rule (on account of $D^j u = 0$), we have

$$\begin{aligned} -\operatorname{div}(e^{-u}\mathbf{z}) &= \operatorname{div}\big((1-e^{-u})\,\mathbf{z}\big) - \operatorname{div}(\mathbf{z}) \\ &= (\mathbf{z}, D(1-e^{-u})) + (1-e^{-u})^* \operatorname{div}(\mathbf{z}) - \operatorname{div}(\mathbf{z}) \\ &= |D(1-e^{-u})| - (e^{-u})^* \operatorname{div}(\mathbf{z}) \\ &= (e^{-u})^*|Du| - (e^{-u})^*(-f+|Du|) = e^{-u} f, \end{aligned}$$

and the proof concludes. □

Since our problem describes a geometric flow, all level sets $\{u \le t\}$ should have finite perimeter. Belonging u to the space $BV_{loc}(\Omega)$ and having $\lim_{|x|\to\infty} u(x) = +\infty$, by the coarea formula, we are sure that this fact is true for almost all $t \ge 0$. The next result shows that it actually holds for all $t \ge 0$. In the spirit of Remark 4.3, in the next result we consider u and f extended to $\mathbb{R}^N$.

Theorem 4.6. *The following conditions hold for a solution u to problem* (1.1) *with associated vector field* $\mathbf{z}$.

(1) For every $t \ge 0$, the set $\{u \le t\}$ has finite perimeter.
(2) For every $t > 0$, the set $\{u < t\}$ has finite perimeter.
(3) For every $t > 0$,

$$|D\chi_{\{u\le t\}}| = -(\mathbf{z}, D\chi_{\{u\le t\}})\,, \quad \textit{as measures on } \Omega\,. \tag{4.8}$$

Moreover,

$$|D\chi_{\{u=0\}}| = -(\mathbf{z}, D\chi_{\{u=0\}})\,, \quad \textit{as measures on } \Omega\,.$$

(4) For every $t > 0$,

$$|D\chi_{\{u<t\}}| = -(\mathbf{z}, D\chi_{\{u<t\}})\,, \quad \textit{as measures on } \Omega\,.$$

(5) The function

$$t \mapsto Per(\{u \le t\}) \tag{4.9}$$

is right–continuous in $[0,+\infty[$.
(6) The function

$$t \mapsto Per(\{u < t\}) \tag{4.10}$$

is left–continuous in $]0,+\infty[$.
(7) The following identity connecting both functions holds true:

$$Per(\{u \le t\}) = Per(\{u < t\}) - \int_{\{u=t\}} f(x)\,dx\,, \quad \textit{for all } t > 0\,; \tag{4.11}$$

Proof. We will consider several auxiliary real functions:

(i) $g(t) = \displaystyle\int_{\mathbb{R}^N} |D\chi_{\{u\le t\}}| = \operatorname{Per}(\{u \le t\}), \qquad t \ge 0.$

(ii) $h(t) = \displaystyle\int_{\mathbb{R}^N} |D\chi_{\{u<t\}}| = \mathrm{Per}(\{u<t\}), \qquad t>0.$

(iii) $G(t) = \displaystyle\int_{\{u\le t\}} |Du|, \qquad t\ge 0.$

(iv) $F(t) = \displaystyle\int_{\{u\le t\}} f(x)\,dx, \qquad t\ge 0.$

Observe that all functions are nonnegative and that $G(0)=0$ due to (4.5).

We will obtain Theorem 4.6 as a consequence of studying the connection among the above functions.

Step 1: $g,h\in L^1_{loc}(0,+\infty)$ and G is absolutely continuous on each bounded interval of $[0,+\infty)$ and nondecreasing.

Let $t\ge 0$. Since $\lim_{|x|\to\infty} u(x)=+\infty$, given t, we may find $r(t)>0$ satisfying

$$\{u\le t\}\subset B_{r(t)}(0)\,. \tag{4.12}$$

Hence, it follows from $u\in BV_{loc}(\Omega)$ that $u\in BV(B_{r(t)}(0))$ and so the coarea formula implies $g,h\in L^1(0,t)$. We conclude that $g,h\in L^1_{loc}(0,+\infty)$.

Considering the truncation of u at level t, which is a function of bounded variation in $\mathbb{R}^N$, the coarea formula also leads to

$$G(t)=\int_0^t\int_{\mathbb{R}^N}|D\chi_{\{u>s\}}|\,ds=\int_0^t\int_{\mathbb{R}^N}|D\chi_{\{u\le s\}}|\,ds=\int_0^t g(s)\,ds\,. \tag{4.13}$$

Thus, G is absolutely continuous on each bounded interval. The nonnegativeness of g yields that G is nondecreasing.

Step 2: Function F is nondecreasing and right–continuous.

Since $f\ge 0$, we have that function F is nondecreasing, so that the set of its discontinuities is at most countable. It is right–continuous due to

$$\lim_{h\to 0^+}(F(t+h)-F(t))=\lim_{h\to 0^+}\int_{\{t<u\le t+h\}} f(x)\,dx=0\,.$$

It is not left–continuous in those t satisfying $\displaystyle\int_{\{u=t\}} f(x)\,dx>0$ since

$$\lim_{h\to 0^-}(F(t)-F(t+h))=\lim_{h\to 0^-}\int_{\{t+h<u\le t\}} f(x)\,dx=\int_{\{u=t\}} f(x)\,dx\,. \tag{4.14}$$

Step 3: If there exists the right derivative $G'_+(t)$, then $g(t)\le G'_+(t)$ for all $t\ge 0$.

For $t \geq 0$ and $h > 0$ fixed, consider the real function defined by

$$\eta(s) = \begin{cases} 1\,, & \text{if } s \leq t\,; \\ \dfrac{t-s}{h} + 1\,, & \text{if } t \leq s \leq t+h\,; \\ 0\,, & \text{if } s \geq t+h\,. \end{cases} \tag{4.15}$$

It is straightforward that η is Lipschitz–continuous and its derivative is given by $\eta'(s) = -\frac{1}{h}\chi_{]t,t+h[}(s)$ for $s \neq t, t+h$. Take now $\phi \in C_0^1(\mathbb{R}^N, \mathbb{R}^N)$ such that $\|\phi\|_\infty \leq 1$. Then, having in mind (4.5) and Remark 4.3, we have

$$\begin{aligned} -\int_{\mathbb{R}^N} \eta(u(x)) \operatorname{div} \phi(x)\, dx &= \int_{\mathbb{R}^N} D\eta(u) \cdot \phi \leq \int_{\mathbb{R}^N} |D\eta(u)| \\ &= \frac{1}{h} \int_{\{t<u<t+h\}} |Du| = \frac{1}{h} \Big[\int_{\{u \leq t+h\}} |Du| - \int_{\{u \leq t\}} |Du| \Big] = \frac{G(t+h) - G(t)}{h}\,. \end{aligned}$$

Letting $h \to 0^+$, we deduce that

$$-\int_{\{u \leq t\}} \operatorname{div} \phi(x)\, dx \leq G'_+(t)\,.$$

Taking now the supremum over all such ϕ, it yields

$$\int_{\mathbb{R}^N} |D\chi_{\{u \leq t\}}| \leq G'_+(t)\,.$$

Step 4: If there exists the left derivative $G'_-(t)$, then $h(t) \leq G'_-(t)$ for all $t > 0$.

Just follow the same argument as in the previous step now considering $t > 0$, $h < 0$ and the function given by

$$\eta(s) = \begin{cases} 1\,, & \text{if } s \leq t+h\,; \\ \dfrac{s-t}{h}\,, & \text{if } t+h \leq s \leq t\,; \\ 0\,, & \text{if } s \geq t\,. \end{cases} \tag{4.16}$$

Step 5: There exists a constant A such that $g(t) = A + G(t) - F(t)$ holds for almost all $t > 0$.

First, we claim that

$$(\mathbf{z}, D\chi_{\{u>t\}}) = |D\chi_{\{u>t\}}| \quad \text{as measures for almost all } t > 0. \tag{4.17}$$

In fact, having in mind Proposition 2.3 and the proof of [16, Proposition 2.2], we have

$$\langle (\mathbf{z}, Du), \varphi \rangle = \int_{-\infty}^{+\infty} \langle (\mathbf{z}, D\chi_{\{u>t\}}), \varphi \rangle\, dt \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

Then, since $(\mathbf{z}, Du) = |Du|$ as measures on Ω, by the coarea formula we get (4.17). Now, by (4.17), we have

$$-(\mathbf{z}, D\chi_{\{u\leq t\}}) = |D\chi_{\{u\leq t\}}| \quad \text{as measures on } \Omega \text{ for almost all } t>0.$$

Fix $t_2 > t_1 > 0$ satisfying $-(\mathbf{z}, D\chi_{\{u\leq t_i\}}) = |D\chi_{\{u\leq t_i\}}|$ as measures on Ω and being Lebesgue points of g for $i = 1, 2$. Then there exists $G'(t_i) = g(t_i)$ and, by Step 3, the set $\{u \leq t_i\}$ has a finite perimeter, $i = 1, 2$. Noting that $\{t_1 < u \leq t_2\} \subset \Omega$ and applying Green's formula in $B_{r(t_2)+1}(0)\cap\Omega$, on account of (4.12), we obtain

$$\begin{aligned}\int_\Omega |D\chi_{\{u\leq t_2\}}| &= -\int_\Omega (\mathbf{z}, D\chi_{\{u\leq t_2\}}) = -\int_\Omega (\mathbf{z}, D\chi_{\{t_1<u\leq t_2\}}) - \int_\Omega (\mathbf{z}, D\chi_{\{u\leq t_1\}}) \\ &= -\int_{B_{r(t_2)+1}(0)\cap\Omega} (\mathbf{z}, D\chi_{\{t_1<u\leq t_2\}}) - \int_\Omega (\mathbf{z}, D\chi_{\{u\leq t_1\}}) \\ &= \int_{B_{r(t_2)+1}(0)\cap\Omega} \chi^*_{\{t_1<u\leq t_2\}}\, d(\operatorname{div}\mathbf{z}) - \int_\Omega (\mathbf{z}, D\chi_{\{u\leq t_1\}}) \\ &= \int_{\{t_1<u\leq t_2\}} d(\operatorname{div}\mathbf{z}) + \int_\Omega |D\chi_{\{u\leq t_1\}}|\,,\end{aligned}$$

and consequently

$$\int_{\{t_1<u\leq t_2\}} d(\operatorname{div}\mathbf{z}) = \int_\Omega |D\chi_{\{u\leq t_2\}}| - \int_\Omega |D\chi_{\{u\leq t_1\}}|. \tag{4.18}$$

Therefore, applying (4.18), (4.3) and (4.13), we get

$$\begin{aligned}G(t_2) - G(t_1) &= \int_{\{t_1<u\leq t_2\}} |Du| = \int_{\{t_1<u\leq t_2\}} d(\operatorname{div}\mathbf{z}) + \int_{\{t_1<u\leq t_2\}} f(x)\,dx \\ &= \int_\Omega |D\chi_{\{u\leq t_2\}}| - \int_\Omega |D\chi_{\{u\leq t_1\}}| + F(t_2) - F(t_1) \\ &= \int_{\mathbb{R}^N} |D\chi_{\{u\leq t_2\}}| - \int_{\mathbb{R}^N} |D\chi_{\{u\leq t_1\}}| + F(t_2) - F(t_1)\,.\end{aligned}$$

It yields that $G(t_2)-G(t_1) = g(t_2)-g(t_1)+F(t_2)-F(t_1)$, so that $-G(t_i)+g(t_i)+F(t_i)$ is constant. Since this argument is true for each pair of points, up to a null set, we conclude that $-G(t) + g(t) + F(t)$ is a constant for almost all $t > 0$.

Step 6: For every $t \geq 0$ there exists $G'_+(t)$ and $G'_+(t) = A + G(t) - F(t)$ holds.

Let $h > 0$. By (4.13) and the previous step, we have

$$G(t+h) = \int_0^{t+h} g(s)\,ds = \int_0^{t+h} A\,ds + \int_0^{t+h} G(s)\,ds - \int_0^{t+h} F(s)\,ds\,,$$

and similarly

$$G(t) = \int_0^t A\,ds + \int_0^t G(s)\,ds - \int_0^t F(s)\,ds\,.$$

Thus,

$$\frac{G(t+h)-G(t)}{h} = A + \frac{1}{h}\Big[\int_t^{t+h} G(s)\,ds - \int_t^{t+h} F(s)\,ds\Big]$$

and letting $h \to 0^+$, it yields that there exists $G'_+(t)$ and $G'_+(t) = A + G(t) - F(t)$ holds.

As a consequence of Step 3 and Step 6, we deduce that the set $\{u \le t\}$ has a finite perimeter for all $t \ge 0$. Thus (1) is proved.

Step 7: For every $t > 0$ there exists $G'_-(t)$ and $G'_-(t) = A + G(t) - F(t^-)$ holds (where $F(t^-)$ denotes the limit from the left).

It is a straightforward consequence of having

$$\frac{G(t+h)-G(t)}{h} = A + \frac{1}{h}\Big[\int_{t+h}^{t} G(s)\,ds - \int_{t+h}^{t} F(s)\,ds\Big]$$

for $h < 0$.

It follows from Step 4 and Step 7, that the set $\{u < t\}$ has a finite perimeter for all $t > 0$ and so (2) is proved.

Step 8: The identity $g(t) = G'_+(t)$ holds and $-(\mathbf{z}, D\chi_{\{u\le t\}}) = |D\chi_{\{u\le t\}}|$ for every $t \ge 0$.

Fix any $t \ge 0$, and consider the same function η defined in (4.15). Applying Green's formula in $B_{r(t)+1}(0) \cap \Omega$ as in Step 5, Proposition 2.3, Proposition 2.5 and the chain rule, we may perform the following manipulations

$$\begin{aligned}\int_\Omega \eta(u(x))\, d(\operatorname{div}\mathbf{z}) &= -\int_\Omega (\mathbf{z}, D\eta(u)) + \int_{\partial\Omega} \eta(u)[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} \\ &= \int_\Omega (\mathbf{z}, D(-\eta(u))) + \int_{\partial\Omega} [\mathbf{z},\nu]\, d\mathcal{H}^{N-1} = \int_\Omega |D(-\eta(u))| + \int_{\partial\Omega} [\mathbf{z},\nu]\, d\mathcal{H}^{N-1} \\ &= -\int_\Omega \eta'(u^*)|Du| + \int_{\partial\Omega} [\mathbf{z},\nu]\, d\mathcal{H}^{N-1} = \frac{1}{h}\int_{\{t<u<t+h\}} |Du| + \int_{\partial\Omega} [\mathbf{z},\nu]\, d\mathcal{H}^{N-1} \\ &= \frac{G(t+h)-G(t)}{h} + \int_{\partial\Omega} [\mathbf{z},\nu]\, d\mathcal{H}^{N-1}\,.\end{aligned}$$

Letting $h \to 0^+$, it follows that

$$\int_\Omega \chi_{\{u\le t\}} d(\operatorname{div}\mathbf{z}) = G'_+(t) + \int_{\partial\Omega} [\mathbf{z},\nu]\, d\mathcal{H}^{N-1}\,.$$

Next, since we know that $\chi_{\{u\le t\}} \in BV(\Omega)$, apply Green's formula in $B_{r(t)+1}(0) \cap \Omega$ again to deduce

$$-\int_\Omega (\mathbf{z}, D\chi_{\{u\le t\}}) + \int_{\partial\Omega} [\mathbf{z},\nu]\, d\mathcal{H}^{N-1} = \int_\Omega \chi_{\{u\le t\}} d(\operatorname{div}\mathbf{z}) = G'_+(t) + \int_{\partial\Omega} [\mathbf{z},\nu]\, d\mathcal{H}^{N-1}\,.$$

In other words:

$$G'_+(t) = -\int_\Omega (\mathbf{z}, D\chi_{\{u\le t\}}) \le \int_\Omega |D\chi_{\{u\le t\}}| = g(t)\,. \tag{4.19}$$

Finally, it follows from Step 3 that $G'_+(t) = g(t)$ and so the inequality in (4.19) becomes an equality. Thus, $-\int_\Omega (\mathbf{z}, D\chi_{\{u\le t\}}) = \int_\Omega |D\chi_{\{u\le t\}}|$. Now it is easy to deduce that

$$-(\mathbf{z}, D\chi_{\{u\le t\}}) = |D\chi_{\{u\le t\}}| \quad \text{as measures on } \Omega\,.$$

Hence, (3) is proved.

Step 9: The identity $h(t) = G'_-(t)$ holds and $-(\mathbf{z}, D\chi_{\{u<t\}}) = |D\chi_{\{u<t\}}|$ for every $t > 0$.

It is enough to argue as in the previous step, but using the function η defined in (4.16). So (4) is proved.

Step 10: The identity $g(t) = A + G(t) - F(t)$ holds for every $t \ge 0$.

It follows from Step 6 and Step 8; as a consequence, (5) is proved.

Step 11: The identity $g(t^-) = A + G(t) - F(t^-) = G'_-(t) = h(t)$ holds for every $t > 0$.

It is straightforward that Step 11 implies $g(t^-) = A + G(t) - F(t^-)$ for all $t > 0$. By Step 7 and Step 9, we are done. It follows that (6) is proved.

Step 12: The identity (4.11) *holds.*

Taking into account the previous steps, we have

$$\begin{aligned}\mathrm{Per}(\{u \le t\}) - \mathrm{Per}(\{u < t\}) &= g(t) - h(t) = g(t) - g(t^-)\\ &= F(t^-) - F(t) = -\int_{\{u=t\}} f(x)\,dx\,,\end{aligned}$$

for every $t \ge 0$. ☐

Remark 4.7. As a consequence of (4.8), we have

$$\theta(\mathbf{z}, D\chi_{\{u\le t\}}, \cdot) = -1 \qquad |D\chi_{\{u\le t\}}| - \text{a.e. in } \Omega.$$

Now, assuming $\mathbf{z}$ is continuous, by [6, Proposition 3.2] (see also [4, Theorem C.14]),

$$\theta(\mathbf{z}, D\chi_{\{u\le t\}}, \cdot) = \mathbf{z}\cdot \frac{D\chi_{\{u\le t\}}}{|D\chi_{\{u\le t\}}|} \qquad |D\chi_{\{u\le t\}}| - \text{a.e. in } \Omega.$$

Therefore,

$$\mathbf{z}\cdot \nu_{\{u\le t\}} = -1 \qquad \mathcal{H}^{N-1} - \text{a.e. in } \partial^*\{u \le t\},$$

which means that the vector field $\mathbf{z}$ has the direction of the generalized outward unit normal to $\partial^*\{u \le t\}$.

Corollary 4.8. *If u is a solution to problem* (1.1) *with associated vector field* $\mathbf{z}$, *then*

$$Per(\{u \le t\}) = e^t Per(\{u=0\}) - \int_{\{0<u\le t\}} e^{t-u(x)} f(x)\,dx\,, \tag{4.20}$$

for all $t>0$.

Proof. Given $t>0$ take $r(t)$ satisfying (4.12) as in Theorem 4.6. Starting with Proposition 4.5, applying Green's formula in $B_{r(t)+1}(0)$ and having in mind Proposition 2.4, Proposition 2.5 and Theorem 4.6, we obtain for $0<s<t$

$$\begin{aligned} &\int_{\{s<u\le t\}} e^{-u(x)} f(x)\,dx \\ &\qquad = -\int_{\{s<u\le t\}} \operatorname{div}\left(e^{-u}\mathbf{z}\right) = \int_\Omega (e^{-u}\mathbf{z}, D\chi_{\{s<u\le t\}}) \\ &= \int_\Omega [e^{-u}]^* (\mathbf{z}, D\chi_{\{s<u\le t\}}) = \int_\Omega [e^{-u}]^*(\mathbf{z}, D\chi_{\{u\le t\}}) - \int_\Omega [e^{-u}]^*(\mathbf{z}, D\chi_{\{u\le s\}}) \\ &\qquad\qquad = -\int_\Omega [e^{-u}]^* |D\chi_{\{u\le t\}}| + \int_\Omega [e^{-u}]^* |D\chi_{\{u\le s\}}|\,. \end{aligned} \tag{4.21}$$

We next analyze the term $\int_\Omega [e^{-u}]^*|D\chi_{\{u\le t\}}|$. On the one hand, we claim that

$$u^*(x) = t \quad \text{for } \mathcal{H}^{N-1}\text{–almost all } x\in \partial^*\{u\le t\}\,. \tag{4.22}$$

In fact, by (2.2) and having in mind that $D^j u = 0$, it is enough to prove (4.22) for the points of $\partial^*\{u\le t\}$ that are Lebesgue points of u of density $\frac{1}{2}$. Let x be one of such points, it follows that

$$\lim_{\rho\downarrow 0} \frac{\mathcal{L}^N(B_\rho(x)\cap\{u\le t\})}{\mathcal{L}^N(B_\rho(x))} = \frac{1}{2} \tag{4.23}$$

and

$$\lim_{\rho\downarrow 0} \frac{\mathcal{L}^N(B_\rho(x)\cap\{u> t\})}{\mathcal{L}^N(B_\rho(x))} = \frac{1}{2}\,. \tag{4.24}$$

Then, from (4.23), taking into account that x is a Lebesgue point of u we get

$$\begin{aligned} &\lim_{\rho\downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho(x)\cap\{u\le t\})} \int_{B_\rho(x)\cap\{u\le t\}} |u(y)-u^*(x)|\,dy \\ &\quad = \lim_{\rho\downarrow 0} \frac{\mathcal{L}^N(B_\rho(x))}{\mathcal{L}^N(B_\rho(x)\cap\{u\le t\})}\ \frac{1}{\mathcal{L}^N(B_\rho(x))} \int_{B_\rho(x)\cap\{u\le t\}} |u(y)-u^*(x)|\,dy \\ &\qquad\qquad \le 2\lim_{\rho\downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho(x))}\int_{B_\rho(x)} |u(y)-u^*(x)|\,dy = 0\,. \end{aligned}$$

Thus,

$$u^*(x) = \lim_{\rho\downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho(x)\cap\{u\le t\})}\int_{B_\rho(x)\cap\{u\le t\}} u(y)\,dy \le t\,.$$

Similarly, from (4.24), we obtain

$$u^*(x) = \lim_{\rho\downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho(x)\cap\{u> t\})}\int_{B_\rho(x)\cap\{u> t\}} u(y)\,dy \ge t,$$

and therefore (4.22) holds. On the other hand, we apply that, by (2.1), the Radon measure $|D\chi_{\{u\le t\}}|$ is supported on $\partial^*\{u\le t\}$. Hence, it yields that

$$\int_{\mathbb{R}^N}[e^{-u}]^*|D\chi_{\{u\le t\}}| = e^{-t}\int_{\mathbb{R}^N}|D\chi_{\{u\le t\}}|\,.$$

Analogously, we have $\int_{\mathbb{R}^N}[e^{-u}]^*|D\chi_{\{u\le s\}}| = e^{-s}\int_{\mathbb{R}^N}|D\chi_{\{u\le s\}}|$. Once these equalities are obtained, it follows from (4.21) that

$$\begin{aligned}\int_{\{s<u\le t\}} e^{-u(x)}f(x)\,dx &= -e^{-t}\int_{\mathbb{R}^N}|D\chi_{\{u\le t\}}| + e^{-s}\int_{\mathbb{R}^N}|D\chi_{\{u\le s\}}|\\ &= -e^{-t}\mathrm{Per}(\{u\le t\}) + e^{-s}\mathrm{Per}(\{u\le s\})\,.\end{aligned}$$

Remembering (4.9), let $s\to 0^+$, we then deduce that

$$\int_{\{0<u\le t\}} e^{-u(x)}f(x)\,dx = -e^{-t}\mathrm{Per}(\{u\le t\}) + \mathrm{Per}(\{u=0\})\,,$$

from where (4.20) follows. □

5. Existence of solutions

The aim of this section is to prove the following existence result.

Theorem 5.1. *For each nonnegative $f\in L^\infty(\Omega)$, there exists a solution to problem* (1.1)*. Moreover, if $x_0\in E_0$ and $s>0$ satisfy $E_0\subset B_s(x_0)$, then*

$$u(x)\ge (N-1)\log\left(\frac{|x-x_0|}{s}\right),\qquad x\in\Omega\,. \tag{5.1}$$

Proof. Throughout this proof, we will assume that $1<p<\frac{3}{2}$. By Theorem 3.1, we know that for each of these p there exists a unique solution u_p of problem (3.1), that is, $0\le u_p\in W^{1,p}_{loc}(\Omega)\cap L^\infty_{loc}(\Omega)$, $e^{-u_p/(p-1)}\in L^p(\Omega, f\mathcal{L}^N)$, $\nabla\big(e^{-u_p/(p-1)}\big)\in L^p(\Omega;\mathbb{R}^N)$, $\lim_{|x|\to\infty} u_p(x) = +\infty$, its trace on ∂E_0 is 0 and

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx + \int_\Omega |\nabla u_p|^p\varphi\,dx = \int_\Omega f\varphi\,dx\,, \tag{5.2}$$

holds for every $\varphi \in W^{1,p}_{loc}(\Omega) \cap L^\infty(\Omega) \cap L^1(\Omega, f\mathcal{L}^N)$ such that its distributional gradient $\nabla\varphi$ belongs to $L^p(\Omega;\mathbb{R}^N)$ and its trace on ∂E_0 is 0. Moreover, if $x_0 \in E_0$ and $s > 0$ satisfy $E_0 \subset B_s(x_0)$, then

$$u_p(x) \geq (N-p)\log\left(\frac{|x-x_0|}{s}\right), \qquad x \in \Omega\,. \tag{5.3}$$

Our aim is to see that some subsequence of $\{u_p\}_{p>1}$ tends to the solution of problem (1.1) as $p \to 1^+$. We proceed by dividing the proof into several steps.

Step 1: Local BV–estimate

Given a bounded open subset $\omega \subset \Omega$, let $\varphi \in \mathcal{D}(\Omega)$ such that $0 \leq \varphi \leq 1$, with $\varphi \equiv 1$ in ω. Then, taking φ^p as test function in (5.2) and applying Young's inequality, we have

$$\begin{aligned}\int_\Omega |\nabla u_p|^p\varphi^p\,dx &= -p\int_\Omega \varphi^{p-1}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx + \int_\Omega f\varphi^p\,dx\\ &\leq \frac{p}{p'}\int_\Omega |\nabla u_p|^p\varphi^p\,dx + \int_\Omega |\nabla\varphi|^p\,dx + \int_{\mathrm{supp}(\varphi)} f\,dx\,.\end{aligned}$$

Hence, recalling that $1 < p < \frac{3}{2}$, we have

$$\begin{aligned}\int_\omega |\nabla u_p|^p\,dx &\leq \frac{1}{2-p}\int_\Omega |\nabla\varphi|^p\,dx + \frac{1}{2-p}\int_{\mathrm{supp}(\varphi)} f\,dx\\ &\leq \frac{1}{2-p}\int_\Omega (1+|\nabla\varphi|^{\frac{3}{2}})\,dx + \frac{1}{2-p}\int_{\mathrm{supp}(\varphi)} f\,dx\\ &\leq 2\int_\Omega (1+|\nabla\varphi|^{\frac{3}{2}})\,dx + 2\int_{\mathrm{supp}(\varphi)} f\,dx\,,\end{aligned}$$

that is:

$$\int_\omega |\nabla u_p|^p\,dx \leq C = C(\omega, f) \quad \forall 1 < p < \frac{3}{2}. \tag{5.4}$$

From here, applying again Hölder's and Young's inequality, we get

$$\int_\omega |\nabla u_p|\,dx \leq C = C(\omega, f) \quad \forall 1 < p < \frac{3}{2}. \tag{5.5}$$

By a diagonal argument we deduce there exists $u \in BV_{loc}(\Omega)$ such that, up to subsequences, we have

$$\lim_{p\downarrow 1} u_p = u \quad \text{in } L^q_{loc}(\Omega)\,,\ 1 \leq q < \frac{N}{N-1}\,, \quad \text{and a.e. in } \Omega, \tag{5.6}$$

$$\nabla u_p \llcorner \omega \xrightharpoonup{p\downarrow 1} Du\llcorner\omega \quad \text{weakly-* as measures for each open bounded } \omega \subset \Omega\,. \tag{5.7}$$

Step 2: u is nonnegative and u satisfies (5.1)*, so that* $\lim_{|x|\to+\infty} u(x) = +\infty$

Since each u_p is nonnegative, it follows from the poinwise convergence that so is u.
On the other hand, letting $p \to 1$ in (5.3), the poinwise convergence (5.6) also implies the inequality (5.1). The limit as $|x|$ goes to ∞ is then a straightforward consequence.

Step 3: Local L^∞–estimate

In this step, using the De Giorgi-Stampacchia methods, we are going to prove that $u \in L^\infty_{loc}(\Omega)$. We start by proving the following local Caccioppoli's inequality. For any $x_0 \in \Omega$ and $R > 0$ small enough, there exists a constant $C > 0$, which does not depend on p, such that

$$\int_{B_\rho(x_0)} |DG_k(u)\,dx| \le \frac{C}{R-\rho}\int_{B_R(x_0)} G_k(u)\,dx \quad \text{for } 0<\rho<R,\ k>0\,; \tag{5.8}$$

here G_k denotes the real function defined by $G_k(s) = (|s|-k)^+\mathrm{sign}\,(s)$.
To this end, we consider $0 < \epsilon < \min\{\frac{1}{2}, \frac{1}{8\|f\|_\infty}\}$, and we fix $R_0 > 0$ satisfying $\mathcal{L}^N(B_{R_0}(0))^{\frac{1}{N}} < \epsilon$. Then, for every $0 < R \le R_0$, we have $\mathcal{L}^N(B_R(0))^{\frac{1}{N}} < \epsilon$ and so

$$\mathcal{L}^N(B_R(x_0))^{\frac{(N+1)p}{N}-1} < \epsilon^p \quad \text{for } p > 1. \tag{5.9}$$

Next fix $0 < R \le R_0$ and $0 < \rho < R$, and let $\eta \in C_0^\infty(B_R(x_0))$ be such that $0 \le \eta \le 1$, with $\eta \equiv 1$ in $B_\rho(x_0)$ and $|\nabla\eta| \le \frac{2}{R-\rho}$. Taking $\eta^p G_k(u_p)$ as test function in (5.2), and neglecting some positive terms, we get

$$\int_\Omega |\nabla G_k(u_p)|^p \eta^p\,dx + p\int_\Omega \eta^{p-1} G_k(u_p)|\nabla G_k(u_p)|^{p-2}\nabla G_k(u_p)\cdot\nabla\eta\,dx \le \int_\Omega f\eta^p G_k(u_p)\,dx.$$

Hence, applying Young's inequality, we have

$$\int_\Omega |\nabla G_k(u_p)|^p\eta^p\,dx$$
$$\le (p-1)\int_\Omega \eta^p|\nabla G_k(u_p)|^p\,dx + \int_\Omega G_k(u_p)^p|\nabla\eta|^p\,dx + \|f\|_\infty\int_\Omega \eta\,G_k(u_p)\,dx.$$

Thus,

$$\int_{B_R(x_0)} |\nabla G_k(u_p)|^p\eta^p \tag{5.10}$$
$$\le \frac{1}{2-p}\,dx\int_{B_R(x_0)} G_k(u_p)^p|\nabla\eta|^p\,dx + \frac{\|f\|_\infty}{2-p}\int_{B_R(x_0)} \eta\,G_k(u_p)\,dx\,.$$

Now, the second term of the right hand side of (5.10) can be estimated applying the Hölder, Young and Sobolev inequalities:

$$\begin{aligned}\int_{B_R(x_0)} \eta\, G_k(u_p)\, dx &\le \mathcal{L}^N(B_R(x_0))^{\frac{Np-N+p}{Np}} \left(\int_{B_R(x_0)} \eta^{p^*} |G_k(u_p)|^{p^*}\, dx\right)^{\frac{1}{p^*}} \\ &\le \frac{1}{p'\epsilon^{p'}} \mathcal{L}^N(B_R(x_0))^{\frac{Np-N+p}{N(p-1)}} + \frac{\epsilon^p}{p} \left(\int_{B_R(x_0)} \eta^{p^*} |G_k(u_p)|^{p^*}\, dx\right)^{\frac{p}{p^*}} \\ &\le \frac{p-1}{p} \left(\frac{\mathcal{L}^N(B_R(x_0))^{\frac{Np-N+p}{N}}}{\epsilon^p}\right)^{\frac{1}{p-1}} + \frac{\epsilon^p(N-1)}{N-p} \int_{B_R(x_0)} |\nabla(\eta\, G_k(u_p)|^p\, dx\,,\end{aligned}$$

where we have estimated the Sobolev constant by $\frac{(N-1)p}{N-p}$. Consequently, having in mind (5.9), we arrive to

$$\begin{aligned}&\int_{B_R(x_0)} \eta\, G_k(u_p)\, dx \\ &\le \frac{p-1}{p} + \frac{\epsilon^p(N-1)}{N-p} 2^{p-1} \left(\int_{B_R(x_0)} \eta^p |\nabla(G_k(u_p)|^p\, dx + \int_{B_R(x_0)} G_k(u_p)^p |\nabla\eta|^p\, dx\right).\end{aligned}$$

Therefore, (5.10) becomes

$$\begin{aligned}\int_{B_R(x_0)} |\nabla G_k(u_p)|^p \eta^p\, dx &\le 2\int_{B_R(x_0)} G_k(u_p)^p |\nabla\eta|^p\, dx + 2\|f\|_\infty \int_{B_R(x_0)} \eta\, G_k(u_p)\, dx \\ &\le 2\int_{B_R(x_0)} G_k(u_p)^p |\nabla\eta|^p\, dx + 2\|f\|_\infty \frac{p-1}{p} \\ &\quad + \|f\|_\infty \frac{\epsilon^p(N-1)}{N-p} 2^p \left(\int_{B_R(x_0)} \eta^p |\nabla(G_k(u_p)|^p\, dx + \int_{B_R(x_0)} G_k(u_p)^p |\nabla\eta|^p\, dx\right).\end{aligned}$$

Now, by our choice of ϵ, we have $\|f\|_\infty \frac{\epsilon^p(N-1)}{N-p} 2^p \le \|f\|_\infty \frac{\epsilon(N-1)}{N-p} 2 \le \frac{1}{2}$ for $1 < p < \frac{3}{2}$, and consequently we obtain that

$$\begin{aligned}&\frac{1}{2}\int_{B_R(x_0)} |\nabla G_k(u_p)|^p \eta^p\, dx \\ &\qquad \le 2\|f\|_\infty \frac{p-1}{p} + \left(2 + \|f\|_\infty \frac{\epsilon^p(N-1)}{N-p} 2^p\right) \int_{B_R(x_0)} G_k(u_p)^p |\nabla\eta|^p\, dx\,,\end{aligned}$$

from where it follows that

$$\begin{aligned}\int_{B_R(x_0)} |\nabla G_k(u_p)|^p \eta^p\, dx &\le 4\|f\|_\infty \frac{p-1}{p} + 5\int_{B_R(x_0)} G_k(u_p)^p |\nabla\eta|^p\, dx \\ &\le 4\|f\|_\infty \frac{p-1}{p} + \frac{10}{(R-\rho)^p} \int_{B_R(x_0)} G_k(u_p)^p\, dx\,.\end{aligned}$$

Then, applying Young's inequality, we get

$$
\begin{aligned}
\int_{B_\rho(x_0)} |\nabla G_k(u_p)|\,dx &\le \int_{B_R(x_0)} |\nabla G_k(u_p)|\eta\,dx \\
&\le \frac{1}{p}\int_{B_R(x_0)} |\nabla G_k(u_p)|^p\eta^p\,dx + \frac{p-1}{p}\mathcal{L}^N(B_R(x_0)) \\
&\le \frac{10}{p(R-\rho)^p}\int_{B_R(x_0)} G_k(u_p)^p\,dx + (p-1)\left(\frac{4}{p^2}\|f\|_\infty + \frac{1}{p}\mathcal{L}^N(B_R(x_0))\right).
\end{aligned} \tag{5.11}
$$

Since, for $1 < p \le q < \frac{N}{N-1}$,

$$G_k(u_p)^p \le 1 + G_k(u_p)^q$$

and

$$\lim_{p\downarrow 1} G_k(u_p) = G_k(u) \quad \text{in} \ \ L^q(B_R(x_0)),\ 1 \le q < \frac{N}{N-1},$$

we may pass to the limit on the right hand side. Having in mind the lower semicontinuity of the total variation on the left hand side, and letting $p \to 1^+$ in (5.11), we obtain that

$$\int_{B_\rho(x_0)} |DG_k(u)| \le \frac{C}{R-\rho}\int_{B_R(x_0)} G_k(u)\,dx$$

and the proof of (5.8) is finished.

We claim now that there exists a constant $C>0$, not depending on p, such that

$$\int_{B_\rho(x_0)} G_k(u)\,dx \le \frac{C}{(R-\rho)(k-h)^{\frac{1}{N}}}\left(\int_{B_R(x_0)} G_h(u)\,dx\right)^{1+\frac{1}{N}} \quad \text{for } 0<\rho<R,\ k>h. \tag{5.12}$$

In fact, for $k>0$, we denote

$$A_k := \{x \in B_R(x_0) \ : \ u(x) \ge k\}.$$

Fixed $0<\rho<R$, let $\eta \in C_0^\infty(B_{\frac{R+\rho}{2}}(x_0))$ such that $0 \le \eta \le 1$, with $\eta \equiv 1$ in $B_\rho(x_0)$ and $|\nabla \eta| \le \frac{4}{R-\rho}$. If we take $\psi := \eta\, G_k(u) \in BV(B_R(x_0))$, applying the Hölder and Sobolev

inequalities, we obtain the following estimate

$$\int_{B_\rho(x_0)} G_k(u)\,dx \le \int_{B_{\frac{R+\rho}{2}}(x_0)} \psi\,dx \le \left(\int_{B_{\frac{R+\rho}{2}}(x_0)} \psi^{\frac{N}{N-1}}\,dx\right)^{\frac{N-1}{N}} \mathcal{L}^N(A_k)^{\frac{1}{N}}$$
$$\le \mathcal{L}^N(A_k)^{\frac{1}{N}} \int_{B_{\frac{R+\rho}{2}}(x_0)} |D\psi|\,dx$$
$$\le \mathcal{L}^N(A_k)^{\frac{1}{N}} \left(\int_{B_{\frac{R+\rho}{2}}(x_0)} \eta|DG_k(u)|\,dx + \int_{B_{\frac{R+\rho}{2}}(x_0)} G_k(u)|\nabla\eta|\,dx\right)$$
$$\le \mathcal{L}^N(A_k)^{\frac{1}{N}} \left(\int_{B_{\frac{R+\rho}{2}}(x_0)} |DG_k(u)|\,dx + \frac{4}{R-\rho}\int_{B_{\frac{R+\rho}{2}}(x_0)} G_k(u)\,dx\right).$$

Then, by (5.8) there exists a constat $C>0$, not depending on p, such that

$$\int_{B_\rho(x_0)} G_k(u)\,dx \le \frac{C}{R-\rho}\mathcal{L}^N(A_k)^{\frac{1}{N}} \int_{B_R(x_0)} G_k(u)\,dx\,.$$

Now, if $0<h<k$, we have

$$\int_{B_R(x_0)} G_k(u)\,dx \le \int_{B_R(x_0)} G_h(u)\,dx$$

and

$$\mathcal{L}^N(A_k)^{\frac{1}{N}} \le \frac{1}{(k-h)^{\frac{1}{N}}}\left(\int_{B_R(x_0)} G_h(u)\,dx\right)^{\frac{1}{N}}.$$

Therefore,

$$\int_{B_\rho(x_0)} G_k(u)\,dx \le \frac{C}{(R-\rho)(k-h)^{\frac{1}{N}}}\left(\int_{B_R(x_0)} G_h(u)\,dx\right)^{1+\frac{1}{N}}$$

and (5.12) holds.

Finally, from (5.12) and applying [22, Lemma 5.1.] we have there exists $d>0$ such that

$$\int_{B_{\frac{R}{2}}(x_0)} G_d(u)\,dx = 0,$$

and thus

$$\operatorname*{ess\,sup}_{x\in B_{\frac{R}{2}}(x_0)} u(x) \le d.$$

Hence, $u\in L^\infty_{loc}(\Omega)$.

Step 4: Existence of a vector field $\mathbf{z}\in L^\infty(\Omega;\mathbb{R}^N)$ *such that* $\|\mathbf{z}\|_\infty \le 1$

First, fixed $\omega \subset\subset \Omega$, we will see that $\{|\nabla u_p|^{p-2}\nabla u_p\}_{p>1}$ is weakly relatively compact in $L^1(\omega;\mathbb{R}^N)$. Indeed, applying (5.4) and Hölder's inequality, it yields

$$\int_\omega |\nabla u_p|^{p-1}\,dx \le \Big(\int_\omega |\nabla u_p|^p\,dx\Big)^{(p-1)/p}|\omega|^{1/p} \le C(\omega)\,.$$

On the other hand, the sequence is equi–integrable since, for each measurable set $E \subset \omega$, we obtain

$$\int_E |\nabla u_p|^{p-1}\,dx \le \Big(\int_\omega |\nabla u_p|^p\,dx\Big)^{(p-1)/p}|E|^{1/p} \le C(\omega)|E|^{1/p}\,.$$

Therefore, we get a subsequence (without relabeling) and a vector field $\mathbf{z}_\omega \in L^1(\omega;\mathbb{R}^N)$ satisfying

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}_\omega\,, \qquad \text{weakly in } L^1(\omega;\mathbb{R}^N)\,.$$

Furthermore, arguing as in [2, Lemma 1], it follows from (5.4) that $\mathbf{z}_\omega \in L^\infty(\omega;\mathbb{R}^N)$ and $\|\mathbf{z}_\omega\|_\infty \le 1$ holds.

Next we consider an increasing sequence $\{\omega_n\}_n$ such that $\omega_n \subset\subset \Omega$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty \omega_n = \Omega$. A diagonal argument shows that there exist a subsequence (no relabel) and a vector field $\mathbf{z} : \Omega \to \mathbb{R}^N$ such that its restriction to each $\omega \subset\subset \Omega$ is equal to $\mathbf{z}_\omega$ and

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}\,, \qquad \text{weakly in } L^1(\omega;\mathbb{R}^N)\,.$$

Hence, $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\mathbf{z}\|_\infty \le 1$ and

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}\,, \qquad \text{weakly in } L^1_{loc}(\Omega;\mathbb{R}^N)\,. \tag{5.13}$$

Step 5: $\operatorname{div}(\mathbf{z})$ *is a Radon measure having locally bounded total variation*

We will apply (5.4) and (5.13) to see that $\operatorname{div}(\mathbf{z})$ is a Radon measure. Observe that (5.2) and (5.4) imply that the sequence $\operatorname{div}\Big(|\nabla u_p|^{p-2}\nabla u_p\Big)$ is locally bounded in $L^1(\Omega;\mathbb{R}^N)$. Hence, up to subsequences, it converges weakly–* in the sense of measures to a Radon measure having locally bounded total variation. It follows from (5.13) that the limit must be $\operatorname{div}(\mathbf{z})$.

Step 6: The equation $-\operatorname{div}(\mathbf{z}) + |Du| = f$ *holds in* $\mathcal{D}'(\Omega)$

Take $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$ as test function in (5.2) to get

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx + \int_\Omega |\nabla u_p|^p\varphi\,dx = \int_\Omega f\varphi\,dx\,.$$

Now Young's inequality implies

$$\begin{aligned}\int_\Omega |\nabla u_p|\varphi\,dx &\le \frac{1}{p}\int_\Omega |\nabla u_p|^p\varphi\,dx + \frac{p-1}{p}\int_\Omega\varphi\,dx\\ &= \frac{1}{p}\int_\Omega f\varphi\,dx - \frac{1}{p}\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx + \frac{p-1}{p}\int_\Omega\varphi\,dx\,.\end{aligned}$$

Thus, we may let p go to 1, applying the lower semicontinuity on the left hand side and (5.13) on the right. It yields

$$\int_\Omega \varphi |Du| \leq \int_\Omega f\varphi\, dx - \int_\Omega \mathbf{z}\cdot\nabla\varphi\, dx\,,$$

in other words, the inequality

$$-\mathrm{div}\,(\mathbf{z}) + |Du| \leq f\,, \qquad \text{holds in } \mathcal{D}'(\Omega)\,. \tag{5.14}$$

The reverse inequality is not straightforward, we first need to establish a related equality. Given $\varphi \in C_0^\infty(\Omega)$, take φe^{-u_p} as test function in (5.2); simplifying we get

$$\int_\Omega e^{-u_p}|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi = \int_\Omega f e^{-u_p}\varphi\,.$$

On account of (5.13), (5.6) and $e^{-u_p}\leq 1$ for all $p>1$, we may let p go to 1 and obtain

$$\int_\Omega e^{-u}\mathbf{z}\cdot\nabla\varphi = \int_\Omega f e^{-u}\varphi\,.$$

Thus,

$$-\mathrm{div}\left(e^{-u}\mathbf{z}\right) = f e^{-u}\,, \qquad \text{holds in } \mathcal{D}'(\Omega)\,. \tag{5.15}$$

Next we will define a function in Ω, up to a $\mathcal{H}^{N-1}$–null set,

$$(e^{-u})^\sharp = \begin{cases} \dfrac{e^{-u^-} - e^{-u^+}}{u^+ - u^-}\,, & \text{in } J_u\,; \\[2mm] e^{-\tilde u}\,, & \text{in } \Omega\backslash S_u\,. \end{cases}$$

This is a representative of the function e^{-u}, which is different of the precise representative (when J_u is not a negligible set). With this notation, the chain rule becomes

$$|D(e^{-u})| = (e^{-u})^\sharp |Du|\,. \tag{5.16}$$

In [16], we defined a Radon measure by

$$\left(\mathbf{z}, D(e^{-u})^\sharp\right) = -(e^{-u})^\sharp \mathrm{div}\,(\mathbf{z}) + \mathrm{div}\left(e^{-u}\,\mathbf{z}\right)$$

and we proved (see [16, (2.28)]) that

$$|\left(\mathbf{z}, D(e^{-u})^\sharp\right)| \leq \|\mathbf{z}\|_\infty |De^{-u}| \tag{5.17}$$

as measures. Then, it follows from (5.14), (5.15), (5.16) and (5.17) that

$$\begin{aligned} -\,\mathrm{div}\,(e^{-u}\,\mathbf{z}) &= -(e^{-u})^\sharp\,\mathrm{div}\,(\mathbf{z}) - (\mathbf{z}, D(e^{-u})^\sharp) \\ &\leq e^{-u}\,f - (e^{-u})^\sharp\,|Du| - (\mathbf{z}, D(e^{-u})^\sharp) \\ &\leq e^{-u}\,f - (e^{-u})^\sharp\,|Du| + |D(e^{-u})| = e^{-u}\,f = -\mathrm{div}\,(e^{-u}\,\mathbf{z})\,. \end{aligned}$$

Hence, all the inequalities become equalities and so

$$(e^{-u})^\sharp\,(-\mathrm{div}\,(\mathbf{z}) + |Du|) = e^{-u} f, \quad \text{as Radon measures on } \Omega,$$

from where it follows that

$$-\mathrm{div}\,(\mathbf{z}) + |Du| = f\,, \quad \text{holds in } \mathcal{D}'(\Omega)\,. \tag{5.18}$$

Step 7: $(\mathbf{z}, Du) = |Du|$ *as measures*

This Step is proven exactly as in [16, proof of Theorem 3.5], having in mind that in that proof Green's formula is only applied on open bounded sets.

Step 8: $D^j u = 0$

It also follows the argument of [16, proof of Theorem 3.5].

Step 9: $u = 0$ *on* ∂E_0

Let $R > 0$ be large enough to have $E_0 \subset B_R(0)$ and consider a cut–off function $\zeta \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \le \zeta \le 1$ and

$$\zeta(x) = \begin{cases} 1\,, & \text{if } |x| \le R\,; \\ 0\,, & \text{if } |x| \ge 2R\,. \end{cases}$$

Taking $\zeta\, u_p$ as test function in (5.2), it yields

$$\int_\Omega \zeta |\nabla u_p|^p\, dx + \int_\Omega u_p |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \zeta\, dx + \int_\Omega \zeta\, u_p |\nabla u_p|^p\, dx = \int_\Omega f \zeta\, u_p\, dx\,,$$

and applying Young's inequality we obtain

$$\begin{aligned} &\int_\Omega \zeta |\nabla u_p|\, dx + \int_\Omega \zeta\, u_p |\nabla u_p|\, dx \\ &\quad \le \frac{1}{p} \int_\Omega \zeta |\nabla u_p|^p\, dx + \frac{1}{p} \int_\Omega \zeta\, u_p |\nabla u_p|^p\, dx + \frac{p-1}{p} \int_\Omega \zeta (1 + u_p)\, dx \\ &\quad = \frac{1}{p} \int_\Omega f \zeta\, u_p\, dx - \frac{1}{p} \int_\Omega u_p |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \zeta\, dx + \frac{p-1}{p} \int_\Omega \zeta (1 + u_p)\, dx\,. \end{aligned} \tag{5.19}$$

In order to let p go to 1, we have to use, on the left hand side, the lower semi-continuity of the functionals given by

$$\mathcal{F}_1(u) = \int_\Omega \zeta |Du| + \int_{\partial\Omega} \zeta |u|\, d\mathcal{H}^{N-1}$$

and

$$\mathcal{F}_2(u) = \int_\Omega \zeta\, u^* |Du| + \frac{1}{2} \int_{\partial\Omega} \zeta\, u^2\, d\mathcal{H}^{N-1}\,.$$

On the right hand side of (5.19), we apply Step 3, (5.13) and the fact that ζ has compact support. Thus, it follows from (5.19) that

$$\int_\Omega \zeta|Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \int_\Omega \zeta\, u^*|Du| + \frac{1}{2}\int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1} \leq \int_\Omega f\zeta\, u\, dx - \int_\Omega u\mathbf{z}\cdot\nabla\zeta\, dx\,. \tag{5.20}$$

Observe that Steps 6–7 imply, by Green's formula on $\tilde{\Omega} = B_{2R}(0)\backslash\overline{E_0}$, that

$$\int_{\tilde{\Omega}} f\zeta\, u\, dx = \int_{\tilde{\Omega}} (\mathbf{z}, D(\zeta\, u)) - \int_{\partial\tilde{\Omega}} \zeta\, u[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\tilde{\Omega}} \zeta\, u^*|Du|\,. \tag{5.21}$$

Taking into account that

$$\langle(\mathbf{z}, Du), (\zeta\varphi)\rangle = \langle(\mathbf{z}, D(\zeta\, u)), \varphi\rangle - \int_\Omega u\varphi\mathbf{z}\cdot\nabla\zeta\, dx$$

holds for every $\varphi \in C_0^\infty(\Omega)$, we have

$$(\mathbf{z}, D(\zeta\, u)) = \zeta(\mathbf{z}, Du) + u\mathbf{z}\cdot\nabla\zeta\mathcal{L}^N \llcorner \Omega \quad \text{as measures.}$$

Then, since $(\mathbf{z}, Du) = |Du|$, we deduce from (5.21) that

$$\int_{\tilde{\Omega}} f\zeta\, u\, dx - \int_{\tilde{\Omega}} u\mathbf{z}\cdot\nabla\zeta\, dx = \int_{\tilde{\Omega}} \zeta|Du| - \int_{\partial\tilde{\Omega}} \zeta u[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_{\tilde{\Omega}} \zeta\, u^*|Du|\,,$$

and having in mind that every integrand vanishes outside $\tilde{\Omega}$,

$$\int_\Omega f\zeta\, u\, dx - \int_\Omega u\mathbf{z}\cdot\nabla\zeta\, dx = \int_\Omega \zeta|Du| - \int_{\partial\Omega} u[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \int_\Omega \zeta\, u^*|Du|\,.$$

Substituting it in (5.20) and simplifying, we get

$$\int_{\partial\Omega} (|u| + u[\mathbf{z},\nu])\, d\mathcal{H}^{N-1} + \frac{1}{2}\int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1} \leq 0\,.$$

Since both terms are nonnegative, it yields that the trace of u vanishes on $\partial\Omega = \partial E_0$, as desired. ☐

6. Uniqueness and Comparison Principle

This Section is devoted to prove uniqueness of solutions to problem (1.1) and deduce a Comparison Principle.

Theorem 6.1. *For each nonnegative $f \in L^\infty(\Omega)$, the weak solution to problem* (1.1) *is unique.*

Proof. Let u_i, $i = 1, 2$, be two solutions to problem (1.1). Then, $u_i \in BV_{loc}(\Omega)\cap L^\infty_{loc}(\Omega)$ with $u_i \geq 0$, $\lim_{|x|\to\infty} u_i(x) = +\infty$, $u_i|_{\partial\Omega} = 0$ $\mathcal{H}^{N-1}$-a.e. and $D^j u_i = 0$, and there exist vector fields $\mathbf{z}_i \in L^\infty(\Omega;\mathbb{R}^N)$ $(i = 1, 2)$ satisfying $\|\mathbf{z}_i\|_\infty \leq 1$,

$$\operatorname{div}(\mathbf{z}_i) \quad \text{is a Radon measure having locally bounded total variation,} \tag{6.1}$$

$$-\mathrm{div}\,(\mathbf{z}_i) + |Du_i| = f \quad \text{in } \mathcal{D}'(\Omega), \tag{6.2}$$

and

$$(\mathbf{z}_i, Du_i) = |Du_i| \quad \text{as measures in } \Omega\,. \tag{6.3}$$

Moreover, due to Proposition 4.5, we also have that

$$-\mathrm{div}(e^{-u_i}\mathbf{z}_i) = e^{-u_i} f \qquad \text{in } \mathcal{D}'(\Omega). \tag{6.4}$$

and

$$(\mathbf{z}_i, D(1-e^{-u_i})) = |D(1-e^{-u_i})|. \tag{6.5}$$

In the following two steps, we will fix $k > 0$ and denote $T_k(s) = \sup(-k, \inf(s,k))$ and $G_k(s) = s - T_k(s)$, as in the proof of Theorem 3.12.

Step 1: The Radon measure $(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1)-u_2)^+)$ is positive

Having in mind (6.3) and the fact $(\mathbf{z}_1, Du_2) \le |Du_2|$ and $(\mathbf{z}_2, DT_k(u_1)) \le |DT_k(u_1)|$, it follows that

$$\int_\Omega \phi(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1)-u_2)^+)$$
$$= \int_{\{T_k(u_1)\ge u_2\}} \phi\,[|DT_k(u_1)| + |Du_2| - (\mathbf{z}_1, Du_2) - (\mathbf{z}_2, DT_k(u_1))] \ge 0,$$

for any nonnegative $\phi \in C_c(\Omega)$, so that Step 1 is proved.

Step 2: The Radon measure $(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1)-u_2)^+)$ vanishes

Since $\lim_{|x|\to\infty} u_2(x) = +\infty$, we may find $R_k > 0$ such that $|x| \ge R_k$ implies $|u_2(x)| > k$. It follows that $[(e^{-u_2} - e^{-T_k(u_1)})^+]^*$ vanishes on $\{|x| \ge R_k\}$. Now multiplying (6.2) by $[(e^{-u_2} - e^{-T_k(u_1)})^+]^*$ and applying Green's formula (2.8) in $B_{R_k}(0)\cap\Omega$, we obtain

$$\begin{aligned}
&\int_\Omega f(x)(e^{-u_2} - e^{-T_k(u_1)})^+(x)\,dx \\
&= \int_{B_{R_k}(0)\cap\Omega} f(x)(e^{-u_2} - e^{-T_k(u_1)})^+(x)\,dx \\
&= \int_{B_{R_k}(0)\cap\Omega} (\mathbf{z}_i, D((e^{-u_2} - e^{-T_k(u_1)})^+) + \int_{B_{R_k}(0)\cap\Omega} [(e^{-u_2} - e^{-T_k(u_1)})^+]^*|Du_i| \\
&= \int_\Omega (\mathbf{z}_i, D((e^{-u_2} - e^{-T_k(u_1)})^+) + \int_\Omega [(e^{-u_2} - e^{-T_k(u_1)})^+]^*|Du_i|,
\end{aligned} \tag{6.6}$$

$i = 1, 2$.

We begin by analyzing (6.6) for $i = 1$. We first remark that it follows from (6.3), Proposition 2.3 and the chain rule that

$$(\mathbf{z}_1, D(1-e^{-T_k(u_1)})) = |D(1-e^{-T_k(u_1)})| = [e^{-T_k(u_1)}]^*|DT_k(u_1)|\,.$$

Applying it, we deduce from (6.6) that

$$\begin{aligned}
&\int_\Omega f(x)(e^{-u_2}-e^{-T_k(u_1)})^+(x)\,dx\\
&\qquad=-\int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_1,D(1-e^{-u_2}))+\int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_1,D(1-e^{-T_k(u_1)}))\\
&\qquad\quad+\int_{\{T_k(u_1)>u_2\}}[e^{-u_2}]^*|Du_1|-\int_{\{T_k(u_1)>u_2\}}[e^{-T_k(u_1)}]^*|Du_1|\\
&\qquad=-\int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_1,D(1-e^{-u_2}))+\int_{\{T_k(u_1)>u_2\}}[e^{-u_2}]^*|Du_1|\\
&\qquad\qquad\qquad-\int_{\{k>u_2\}}[e^{-T_k(u_1)}]^*|DG_k(u_1)|
\end{aligned}$$

Similarly, it follows from (6.5), the chain rule and (6.6) for $i=2$ that

$$\begin{aligned}
&\int_\Omega f(x)(e^{-u_2}-e^{-T_k(u_1)})^+(x)\,dx\\
&\qquad=-\int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_2,D(1-e^{-u_2}))+\int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_2,D(1-e^{-T_k(u_1)}))\\
&\qquad\quad+\int_{\{T_k(u_1)>u_2\}}[e^{-u_2}]^*|Du_2|-\int_{\{T_k(u_1)>u_2\}}[e^{-T_k(u_1)}]^*|Du_2|\\
&\qquad=\int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_2,D(1-e^{-T_k(u_1)}))-\int_{\{T_k(u_1)>u_2\}}[e^{-T_k(u_1)}]^*|Du_2|.
\end{aligned}$$

As a consequence of the previous equations, we obtain that

$$\begin{aligned}
&-\int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_1,D(1-e^{-u_2}))+\int_{\{T_k(u_1)>u_2\}}[e^{-u_2}]^*|Du_1|-e^{-k}\int_{\{k>u_2\}}|DG_k(u_1)|\\
&\qquad=\int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_2,D(1-e^{-T_k(u_1)}))-\int_{\{T_k(u_1)>u_2\}}[e^{-T_k(u_1)}]^*|Du_2|\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\{T_k(u_1)>u_2\}}[e^{-u_2}]^*|Du_1| + \int_{\{T_k(u_1)>u_2\}}[e^{-T_k(u_1)}]^*|Du_2| \\
&= \int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_1, D(1-e^{-u_2})) + \int_{\{T_k(u_1)>u_2\}}(\mathbf{z}_2, D(1-e^{-T_k(u_1)})) + e^{-k}\int_{\{k>u_2\}}|DG_k(u_1)| \\
&\le \int_{\{T_k(u_1)>u_2\}}|D(1-e^{-T_k(u_1)})| + \int_{\{T_k(u_1)>u_2\}}|D(1-e^{-u_2})| + e^{-k}\int_{\{k>u_2\}}|DG_k(u_1)| \\
&\qquad = \int_{\{T_k(u_1)>u_2\}}[e^{-T_k(u_1)}]^*|Du_1| + \int_{\{T_k(u_1)>u_2\}}[e^{-u_2}]^*|Du_2|,
\end{aligned}$$

and consequently,

$$\int_{\{T_k(u_1)>u_2\}}\left([e^{-T_k(u_1)}]^* - [e^{-u_2}]^*\right)(|Du_1| - |Du_2|) \ge 0. \tag{6.7}$$

On the other hand, it follows from (6.4) (for $i = 1$), (2.9), Proposition 2.3 and the chain rule that

$$\begin{aligned}
&-\operatorname{div}\left(e^{-T_k(u_1)}\mathbf{z}_1\right) = -\operatorname{div}\left(e^{G_k(u_1)}(e^{-u_1}\mathbf{z}_1)\right) \\
&\quad = e^{G_k(u_1)}(e^{-u_1}f) - [e^{-u_1}]^*\big(\mathbf{z}_1, D\big(e^{G_k(u_1)}\big)\big) = e^{-T_k(u_1)}f - [e^{-u_1}]^*\big|D\big(e^{G_k(u_1)}\big)\big| \\
&\qquad = e^{-T_k(u_1)}f - [e^{-T_k(u_1)}]^*|DG_k(u_1)| = e^{-T_k(u_1)}f - e^{-k}|DG_k(u_1)|,
\end{aligned}$$

and so

$$-\operatorname{div}\left(e^{-T_k(u_1)}\mathbf{z}_1\right) + e^{-k}|DG_k(u_1)| = e^{-T_k(u_1)}f\,. \tag{6.8}$$

Observing that $[(T_k(u_1)-u_2)^+]^*$ vanishes on $\{|x| \ge R_k\}$, multiplying (6.8) by $[(T_k(u_1)-u_2)^+]^*$, applying Green's formula (2.8) in $B_{R_k}(0)\cap\Omega$ and dropping a nonnegative term, we obtain

$$\int_\Omega (e^{-T_k(u_1)}\mathbf{z}_1, D(T_k(u_1) - u_2)^+) \le \int_\Omega e^{-T_k(u_1)}f(x)(T_k(u_1) - u_2)^+\,dx\,. \tag{6.9}$$

Next, multiplying (6.4) (for $i = 2$) by $[(T_k(u_1) - u_2)^+]^*$ and applying Green's formula (2.8) in $B_{R_k}(0)\cap\Omega$, it yields

$$\int_\Omega (e^{-u_2}\mathbf{z}_2, D(T_k(u_1) - u_2)^+) = \int_\Omega e^{-u_2}f(x)(T_k(u_1) - u_2)^+\,dx\,. \tag{6.10}$$

Hence, subtracting (6.10) from (6.9),

$$\begin{aligned}
&\int_\Omega (e^{-T_k(u_1)}\mathbf{z}_1, D(T_k(u_1) - u_2)^+) - \int_\Omega (e^{-u_2}\mathbf{z}_2, D(T_k(u_1) - u_2)^+) \\
&\qquad \le \int_\Omega \left(e^{-T_k(u_1)} - e^{-u_2}\right) f(x)(T_k(u_1) - u_2)^+\,dx \le 0\,.
\end{aligned}$$

Thus,

$$\int_\Omega (e^{-T_k(u_1)}\mathbf{z}_1 - e^{-u_2}\mathbf{z}_2, D(T_k(u_1)-u_2)^+) \le 0\,,$$

and so

$$\begin{aligned} 0 &\ge \int_{\{T_k(u_1)>u_2\}} (e^{-T_k(u_1)}\mathbf{z}_1 - e^{-u_2}\mathbf{z}_2, D(T_k(u_1)-u_2)) \\ &= \int_{\{T_k(u_1)>u_2\}} ((e^{-T_k(u_1)} - e^{-u_2})\mathbf{z}_2, D(T_k(u_1)-u_2)) \\ &+ \int_{\{T_k(u_1)>u_2\}} (e^{-T_k(u_1)}(\mathbf{z}_1 - \mathbf{z}_2), D(T_k(u_1)-u_2)). \end{aligned} \tag{6.11}$$

The first term on the right hand side is analyzed having in mind (6.3), (6.7), Proposition 2.4 and the inequality $|(\mathbf{z}_2, DT_k(u_1))| \le |DT_k(u_1)|$, and performing some easy calculations:

$$\begin{aligned} &\int_{\{T_k(u_1)>u_2\}} ((e^{-T_k(u_1)} - e^{-u_2})\mathbf{z}_2, D(T_k(u_1)-u_2)) \\ &= \int_{\{T_k(u_1)>u_2\}} (e^{-T_k(u_1)} - e^{-u_2})^*(\mathbf{z}_2, D(T_k(u_1)-u_2)) \\ &= \int_{\{T_k(u_1)>u_2\}} (e^{-T_k(u_1)} - e^{-u_2})^*(\mathbf{z}_2, DT_k(u_1)) - \int_{\{T_k(u_1)>u_2\}} (e^{-T_k(u_1)} - e^{-u_2})^*|Du_2| \\ &\ge \int_{\{T_k(u_1)>u_2\}} (e^{-T_k(u_1)} - e^{-u_2})^*|DT_k(u_1)| - \int_{\{T_k(u_1)>u_2\}} (e^{-T_k(u_1)} - e^{-u_2})^*|Du_2| \\ &= \int_{\{T_k(u_1)>u_2\}} \left([e^{-T_k(u_1)}]^* - [e^{-u_2}]^*\right)(|DT_k(u_1)| - |Du_2|) \\ &= \int_{\{T_k(u_1)>u_2\}} \left([e^{-T_k(u_1)}]^* - [e^{-u_2}]^*\right)(|Du_1| - |Du_2|) \\ &\quad - \int_{\{T_k(u_1)>u_2\}} \left(e^{-k} - [e^{-u_2}]^*\right)|DG_k(u_1)| \ge 0\,. \end{aligned}$$

Hence, dropping this nonnegative term in (6.11) and applying Proposition 2.4 again, we obtain

$$\int_{\{T_k(u_1)>u_2\}} [e^{-T_k(u_1)}]^*(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1)-u_2)) \le 0\,,$$

and so

$$e^{-k}\int_\Omega (\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1)-u_2)^+) \le 0\,.$$

Finally, since the Radon measure $(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - u_2)^+)$ is nonnegative, we deduce that

$$(\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - u_2)^+) = 0\,.$$

Step 3: The Radon measure $(\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2))$ *vanishes*

Given $R > 0$, consider $k > \|u_1\|_{L^\infty(B_R(0)\cap\Omega)}$. Then

$$(\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2)^+))\llcorner B_R(0) \cap \Omega = (\mathbf{z}_1 - \mathbf{z}_2, D(T_k(u_1) - u_2)^+))\llcorner B_R(0) \cap \Omega = 0\,.$$

Since this fact holds for every $R > 0$, it follows that

$$(\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2)^+)) = 0\,.$$

Then

$$(\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2)^-) = (\mathbf{z}_2 - \mathbf{z}_1, D(u_2 - u_1)^+) = 0\,.$$

Therefore,

$$(\mathbf{z}_1 - \mathbf{z}_2, D(u_1 - u_2)) = 0.$$

Step 4: It holds $u_1 = u_2$ in Ω.

Following the arguments of the proof of [16, Theorem 3.8] in each $\Omega \cap B_R(0)$ and letting $R \to +\infty$, we get $Du_1 = Du_2$ as measures in Ω. Consequently, $u_1 - u_2$ is a constant in each connected component of Ω. Since $u_1 - u_2 = 0$ on $\partial\Omega$, the conclusion follows. □

As a consequence of uniqueness and Theorem 3.2, we deduce the following Comparison Principle for problem (1.1).

Theorem 6.2. *Let u_i be the solution to problem* (1.1) *in the domain $\Omega_i = \mathbb{R}^N \backslash \overline{E_0^i}$ with datum $f_i \in L^\infty(\Omega_i)$, $i = 1, 2$. If $E_0^2 \subset E_0^1$ and $f_1 \le f_2$ in Ω_1, then $u_1 \le u_2$ in Ω_1.*

Proof. Applying Theorems 5.1 and 6.1, we know that each u_i is the pointwise limit of the sequence of approximate solutions $u_{i,p}$ to problems (3.1) in the domain Ω_i with datum f_i. Since Theorem 3.2, $E_0^2 \subset E_0^1$ and $f_1 \le f_2$ in Ω_1 imply that $u_{1,p} \le u_{2,p}$ in Ω_1, it follows that $u_1 \le u_2$ in Ω_1. □

7. Examples

We are going to find explicit radial solutions to problem (1.1).

Example 7.1. Let $r > 0$. Consider $E_0 := B_r(0)$, take $\Omega = \mathbb{R}^N \setminus \overline{E_0}$ and $f(x) := \tilde f(|x|)$, with $\tilde f :]r, +\infty[\to [0, +\infty[$. We are looking for a radial solution $u(x) = g(|x|)$, where $g : [r, +\infty[\to [0, +\infty[$ is continuous and such that $g(s) \ge 0$ for $s \in [r, +\infty[$ and $g(r) = 0$. Moreover, we assume that g is nondecreasing.

Assuming that $g'(|x|) > 0$, we have $Du(x) = g'(|x|)\frac{x}{|x|}$ and consequently $\mathbf{z}(x) = \frac{x}{|x|}$. Obviously, this latest identification need not be occur where g' vanishes. We next see that g is actually increasing. Assume, on the contrary, that there exist $r \le s_1 < s_2$

such that $g(s_1) = g(s_2) = t$. Let s_1 be the smallest number and let s_2 be the greatest one satisfying that equality. Then $\{u < t\} = B_{s_1}(0)$ and $\{u \le t\} = \overline{B_{s_2}(0)}$, so that

$$\mathrm{Per}(\{u < t\}) < \mathrm{Per}(\{u \le t\}) .$$

On the other hand, by (4.11),

$$\mathrm{Per}(\{u \le t\}) = \mathrm{Per}(\{u < t\}) - \int_{\{u=t\}} f(x)\,dx \le \mathrm{Per}(\{u < t\}) ,$$

which is a contradiction.

Once we have seen that g is increasing, we may deduce that $\mathbf{z}(x) = \frac{x}{|x|}$ for almost all $|x| > r$. Actually, $\mathbf{z}(x) = \frac{x}{|x|}$ for all $|x| > r$; otherwise, the vector field $\mathbf{z}$ would jump and g would too. Thus, $\operatorname{div}\mathbf{z}(x) = \dfrac{N-1}{|x|}$ for all $|x| > r$. Then

$$-\mathrm{div}\ \mathbf{z}(x) + |Du(x)| = f(x),$$

leads to the equation

$$g'(s) = \tilde{f}(s) + \frac{N-1}{s}. \tag{7.1}$$

Hence,

$$g(s) = \int_r^s \tilde{f}(\sigma)\,d\sigma + (N-1)\log\left(\frac{s}{r}\right) .$$

Therefore the solution to problem (1.1) is given by

$$u(x) = \int_r^{|x|} \tilde{f}(s)\,ds + (N-1)\log\left(\frac{|x|}{r}\right) . \tag{7.2}$$

It is now straightforward to find the solution to problem (1.1) starting from $E_0 = B_r(a)$. We next exemplify some particular cases.

(1) If

$$\tilde{f}(s) = ps^{p-1} - \frac{N-1}{s}, \quad p > 0 ,$$

then we have that the solution is given by

$$u(x) = |x-a|^p - r^p .$$

(2) If $\tilde{f} \equiv \lambda > 0$, then

$$u(x) = \lambda(|x-a| - r) + (N-1)\log\left(\frac{|x-a|}{r}\right) . \tag{7.3}$$

(3) If $\tilde{f} \equiv 0$, that is, for the inverse mean curvature flow, we obtain that the solution is

$$u(x) = (N-1)\log\left(\frac{|x-a|}{r}\right) .$$

(4) If we take $\tilde{f}(s) = \left(1 - \frac{N-1}{s}\right)\chi_{]r_0,r_1[}(s)$, with $r < r_0 < r_1$ and $r_0 > N-1$, then the solution is given by

$$u(x) = \begin{cases} (N-1)\log\left(\frac{|x-a|}{r}\right), & \text{if } r < |x-a| \le r_0\,; \\ |x-a| - r_0 + (N-1)\log\left(\frac{r_0}{r}\right), & \text{if } r_0 < |x-a| \le r_1\,; \\ (r_1 - r_0) + (N-1)\log\left(\frac{|x-a|\,r_0}{r\,r_1}\right), & \text{if } |x-a| > r_1\,. \end{cases}$$

Observe that the level sets of the above solutions are expanding spheres but with different speed.

We point out that, having in mind Theorem 6.2, the radial solution (7.3) implies estimates on the solution to (1.1).

Proposition 7.2. *Assume that $x_0 \in \mathbb{R}^N$ and $r > 0$ satisfy $B_r(x_0) \subset E_0$. Then*

$$u(x) \le \|f\|_\infty(|x-x_0| - r) + (N-1)\log\left(\frac{|x-x_0|}{r}\right), \qquad x \in \Omega.$$

Example 7.3. Let $0 < r_1 < r_2$. Consider $E_0 := B_{r_2}(0)\backslash\overline{B_{r_1}(0)}$, take $\Omega = \mathbb{R}^N \setminus \overline{E_0}$ and $f(x) := \tilde{f}(|x|)$, with $\tilde{f} : [0, r_1[\cap]r_2, +\infty[\to [0,+\infty[$. We are looking for a radial solution $u(x) = g(|x|)$, where $g : [0, r_1] \cap [r_2, +\infty[\to [0, +\infty[$ is continuous and such that $g(s) \ge 0$ for $s \in [0, r_1] \cap [r_2, +\infty[$ and $g(r_i) = 0$, $i = 1, 2$. Furthermore, we will assume that g is nonincreasing on $[0, r_1]$ and nondecreasing on $[r_2, +\infty[$.

It is straightforward that u is a solution in $\mathbb{R}^N \setminus \overline{B_{r_2}(0)}$ and in $B_{r_1}(0)$. Hence, by the arguments of Example 7.1, we deduce that

$$u(x) = \int_{r_2}^{|x|} \tilde{f}(s)\,ds + (N-1)\log\left(\frac{|x|}{r_2}\right), \quad |x| > r_2\,;$$

while the arguments of [16, Section 4] provide us the solution in $B_{r_1}(0)$. We point out that in this last zone, the solution is bounded, so that we get a level set where this hole disappears (this level may be 0 when the datum is small enough, for instance, for $\tilde{f}(s) = \lambda$, $0 \le s < r_1$, with $0 \le \lambda \le N$). There is another feature that distinguish the solution in both zones, namely there are no flat zones in $\mathbb{R}^N \setminus \overline{B_{r_2}(0)}$, as we have seen, while there can be in $B_{r_1}(0)$ (see [16, Example 4.1]).

8. The level set formulation of the inverse mean curvature flow

As we have mentioned in the introduction, the level set formulation of the inverse mean curvature flow, corresponds to the homogeneous case $f \equiv 0$, that is to the problem

$$
(8.1) \qquad \begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = 0\,, & \text{in } \ \Omega\,; \\[2ex] u = 0\,, & \text{on } \ \partial E_0\,; \\[2ex] \displaystyle\lim_{|x|\to\infty} u(x) = +\infty\,; & \end{cases}
$$

where $\Omega = \mathbb{R}^N \backslash \overline{E_0}$. The existence and uniqueness of weak solutions was proved in [13] and [18], under the assumption that the boundary of E_0 is of class C^1; they also require a uniform one–sided bound for the mean curvature of ∂E_0 since in the proof this hypothesis is needed. One improvement of this assumption was done by Moser in [19], where he only imposes that ∂E_0 is continuously differentiable. Let us point out that here we also improve this result since we only assume that ∂E_0 is Lipschitz–continuous.

Remark 8.1. Let us see that for problem (8.1), assuming that u is locally Lipschitz–continuous, our concept of solution coincides with the one given by Huisken and Ilmanen. Due to uniqueness it is enough to show that if u is a solution to problem (8.1) in the sense of Definition 4.1 and u is locally Lipschitz, then it is also a solution in the sense of Huisken and Ilmanen. To check it, we need to show that u satisfies

$$
J_u^K(u) \le J_u^K(v), \quad J_u^K(v) := \int_K (|\nabla v| + |\nabla u| v)\, dx
$$

for every locally Lipschitz function v such that $\{v \ne u\} \subset\subset \Omega$ and every compact K containing $\{v \ne u\}$. In fact, since u is a solution to problem (8.1), there exists a vector field $\mathbf{z} \in \mathcal{DM}_\infty^{loc}(\Omega; \mathbb{R}^N)$ satisfying $\|\mathbf{z}\|_\infty \le 1$, (4.3) and (4.4). Fix one of those v and a compact K containing $\{v \ne u\}$. Consider ω an open bounded set with Lipschitz–continuous boundary satisfying $K \subset \omega \subset \Omega$. Multiplying (4.3) by v and applying Green's formula in ω, we get

$$
(8.2) \qquad \int_{\partial\omega} [\mathbf{z}, \nu] v \, d\mathcal{H}^{N-1} = \int_\omega (\mathbf{z}, Dv) + v|Du|.
$$

Since $\|\mathbf{z}\|_\infty \le 1$, it follows that $(\mathbf{z}, Dv) \le |Dv|$ and so we deduce that

$$
\int_{\partial\omega} [\mathbf{z}, \nu] v \, d\mathcal{H}^{N-1} \le \int_\omega |Dv| + v|Du|\,.
$$

On the other hand, taking $v = u$ in (8.2) and having in mind (4.4), we get

$$
\int_{\partial\omega} [\mathbf{z}, \nu] u \, d\mathcal{H}^{N-1} = \int_\omega |Du| + u|Du|.
$$

Then $u = v$ on $\partial\omega$, implies $\int_\omega |Du| + u|Du| \le \int_\omega |Dv| + v|Du|$. Since it holds for every open bounded ω with Lipschitz continuous boundary containing K, it follows that $J_u^K(u) \leq J_u^K(v)$. Therefore, among all admissible functions, the minimum of J_u^K is attained at u.

Example 8.2. Consider the initial datum $E_0 = B_{r_1}(a)\backslash\overline{B_{r_2}(a)} \subset \mathbb{R}^N$, with $r_1 > r_2$. Then the solution of problem (8.1) is given by

$$u(x) := \begin{cases} (N-1)\log\left(\dfrac{|x-a|}{r_1}\right), & \text{if } |x-a| > r_1\,; \\ 0\,, & \text{if } |x-a| < r_2\,; \end{cases}$$

producing a sudden phenomenon of fattening. The corresponding vector field $\mathbf{z}$ is given by

$$\mathbf{z}(x) := \frac{x-a}{|x-a|}\,, \quad \text{if } |x-a| > r_1\,,$$

and any vector field $\mathbf{z}$ satisfying $\|\mathbf{z}\|_\infty \leq 1$ and $\operatorname{div}\mathbf{z} = 0$ in $B_{r_2}(a)$.

To check that u must vanish in $B_{r_2}(a)$, recall that $-\operatorname{div}\mathbf{z} + |Du| = 0$ holds in $B_{r_2}(a)$ in the sense of distributions. Applying Green's formula, it yields

$$\int_{B_{r_2}(a)} (\mathbf{z}, Du) + \int_{B_{r_2}(a)} u|Du| = \int_{\partial B_{r_2}(a)} u[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} = 0\,,$$

due to $u\big|_{\partial B_{r_2}(a)} = 0$. Since $(\mathbf{z}, Du) = |Du|$, it follows that

$$\int_{B_{r_2}(a)} (u+1)|Du| = 0\,.$$

We then deduce that the Radon measure Du vanishes on $B_{r_2}(a)$, so that $u \equiv 0$, as desired.

This phenomenon of sudden jump of the evolving surface is also possible at some instants $t > 0$, and at these instants, its perimeter is preserved (see [13, Example 1.5]). In [13] this property is explained using the notion of strictly minimizing hull. In our formulation of the level set approach, this fact can be deduced from Theorem 4.6.

To see it, we consider the level corresponding to one of these instants t. We have to prove

$$Per(\{u < t\}) = Per(\{u \leq t\}),$$

but it is just (4.11) with $f \equiv 0$.

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