ON THE BEHAVIOUR OF THE SOLUTIONS TO P-LAPLACIAN EQUATIONS AS P GOES TO 1

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Abstract. In the present paper we study the behaviour as p goes to 1 of the weak solutions to the problems

$$\begin{cases} -\mathrm{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = f & \text{in } \Omega \\ u_p = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary and $p > 1$. As far as the datum f is concerned, we analyze several cases: the most general one is $f \in W^{-1,\infty}(\Omega)$. We also illustrate our results by means of remarks and examples.

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1. Introduction

In the present paper we study the behaviour, when p goes to 1, of the solutions $u_p \in W_0^{1,p}(\Omega)$ to the problems

$$\begin{cases} -\mathrm{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = f & \text{in } \Omega \\ u_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$ and Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary. We analyze the case where Ω is a ball and the datum f is a non–negative radially decreasing function belonging to the Lorentz space $L^{N,\infty}(\Omega)$ and the case where the datum f belongs to the dual space $W^{-1,\infty}(\Omega)$.

We are interested in finding the pointwise limit of u_p as p goes to 1 and in proving that such a limit is a solution to the "limit equation" of (1.1), namely:

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right) = f \qquad \text{in } \Omega \tag{1.2}$$

with homogeneous Dirichlet boundary condition. Hence, firstly we study the behaviour of u_p when p goes to 1, finding a limit function u, and secondly we prove that such a

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limit function u is a solution to (1.2). Both aspects of our study have been investigated by several authors. The interest in studying such a case comes from an optimal design problem in the theory of torsion and related geometrical problems (see also [17]). The behaviour of u_p in the case where the datum f is constant has been studied in [16] by Kawohl, where the author proved that under a suitable smallness assumption on the domain, it results

$$\lim_{p\to 1} u_p = 0, \tag{1.3}$$

while under the assumption that the domain is large enough one has

$$\lim_{p\to 1} u_p = +\infty\,. \tag{1.4}$$

The behaviour of u_p in the case where f is not constant and it belongs to the Lebesgue space $L^N(\Omega)$ (or to the Lorentz space $L^{N,\infty}(\Omega)$) is studied in [10]. In such a paper the authors prove again (1.3) under the assumption that the $L^N(\Omega)$–norm (or $L^{N,\infty}(\Omega)$–norm) of the datum f is small enough.

As just pointed out, the second aim of our study consists in proving that the limit function $u = \lim_{p\to 1} u_p$ is a solution to problem (1.2). Notions of solution to the limit equation (1.2) have been introduced by various authors (see for instance [5], [6], [9], [11], [12] and references there in). Motivations for such an interest are found in the variational approach to image restoration introduced by L. Rudin, S. Osher and E. Fatemi[1]. The definitions of solutions to equation (1.2) typically consider a datum in $L^N(\Omega)$ or $L^N_{\rm loc}(\Omega)$; moreover such solutions are functions belonging to the space $BV(\Omega)$, which guaranties the existence of a distributional gradient, well defined as a Radon measure. In order to give a meaning to $\dfrac{Du}{|Du|}$ in the limit equation, any definition of solution to (1.2) relies on the existence of a vector field $z : \Omega \to \mathbb{R}^N$, which belongs to $L^\infty(\Omega;\mathbb{R}^N)$, with $\|z\|_\infty \le 1$. Moreover z satisfies the equation $-{\rm div}\, z = f$ in the distributional sense and $z \cdot Du = |Du|$. The boundary condition may be included as $z\cdot\nu \in$ sign $(-u)$ a.e. on $\partial\Omega$. The expressions $z \cdot Du$ and $z\cdot\nu$ have sense thanks to the Anzellotti theory (see [4] or [7]) which defines a Radon measure (z, Du), provides the definition of a weakly trace on $\partial\Omega$ to the normal component of z, denoted by $[z, \nu]$, and guaranties a Green's formula. Roughly speaking, z plays the role of $\frac{Du}{|Du|}$.

In this paper we consider problem (1.2) with data belonging to the Lorentz space $L^{N,\infty}(\Omega)$ and to the dual space $W^{-1,\infty}(\Omega)$. Let us now explain the reason for which we consider such types of data. The embedding $W^{-1,\infty}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ for all $p > 1$ ensures the existence of an unique weak solution $u_p \in W^{1,p}_0(\Omega)$ to problem (1.1) (see [18]). Smallness assumption on the data allows us to prove the existence a limit function $u = \lim_{p\to 1} u_p$ which belongs to the space $BV(\Omega)$. On the other hand, we prove the existence of a vector field $z \in L^\infty(\Omega,\mathbb{R}^N)$ satisfying $-$ div $z = f$ in the sense of distributions. This implies that

$$|< f\,, \varphi >| = \left|\int_\Omega z\cdot\nabla\varphi\, dx\right| \le \|z\|_\infty \int_\Omega |\nabla\varphi|\, dx$$

[1]For a review on the development of variational models in image processing and a deep study of the Minimizing Total Variation Flow, see [7].

for all $\varphi \in C_0^\infty(\Omega)$, and hence $f \in W^{-1,\infty}(\Omega)$.

Observe that we take a datum belonging to $W^{-1,\infty}(\Omega)$ and we find a solution in $BV(\Omega)$, even if $W^{-1,\infty}(\Omega)$ is not the dual space of $BV(\Omega)$. This fact yields some difficulties which we completely solve only in the case where the datum f belongs to specific subspaces of $W^{-1,\infty}(\Omega)$.

Let us observe that, by the improvement of Sobolev embedding (see for example [4]) and duality arguments, the Lorentz space $L^{N,\infty}(\Omega)$ is a subset of the dual space $W^{-1,\infty}(\Omega)$. We will consider data belonging to $L^{N,\infty}(\Omega)$ which are radially symmetric, without smallness assumptions. Indeed such symmetries allow us to write the expression of the solutions u_p and to handle it in order to study the behaviour of u_p.

The case where the data do not belong to $W^{-1,\infty}(\Omega)$ have been considered in [5], [6], [9]. However, in such papers the solutions to the "limit equation" equation (1.2) are not obtained as limit of u_p, except in the case where the data are smooth enough, and the methods employed do not apply in our framework. We explicitly remark that the asymptotic behaviour of u_p when the datum f belongs to $L^1(\Omega)$ will be studied by the authors in the forthcoming paper [19], where a notion of solution to the equation (1.2) is introduced.

Finally we summarize the contents of the present paper. After introducing our notation (see Section 2), we begin by studying the case where f is a radially decreasing function defined in a ball Ω and belonging to $L^{N,\infty}(\Omega)$, without assuming any smallness condition on its $L^{N,\infty}(\Omega)$–norm (see Section 3). Next we study the case where f belongs to the dual space $W^{-1,\infty}(\Omega)$ and we prove that (1.3) holds true under the assumption $\|f\|_{W^{-1,\infty}(\Omega)} < 1$. We also prove that if $\|f\|_{W^{-1,\infty}(\Omega)} = 1$, then u_p tends to a BV–function, and if $\|f\|_{W^{-1,\infty}(\Omega)} > 1$, then there is not any BV–function which is the pointwise limit of u_p (see Section 4). In general we are not able to prove that u_p tends to a solution to problem (1.2) when $\|f\|_{W^{-1,\infty}(\Omega)} = 1$. In such a case we have to assume that f belongs to the predual space of $BV(\Omega)$ (see Subsection 4.2).

We conclude with few words on the Appendix. First we present some properties of the predual space of $BV(\Omega)$ and we prove that its norm as a subspace of the dual of $BV(\Omega)$ is exactly the same as the norm of $W^{-1,\infty}(\Omega)$. Secondly, we adapt Anzellotti's theory in the framework of the predual of $BV(\Omega)$.

2. Notation

In this Section we will introduce some notation which will be used throughout this paper. We will denote by Ω a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. Thus there exists a unit vector field (denoted by ν) normal to $\partial\Omega$ and exterior to Ω, defined $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$. Here $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure. Here and in the sequel, $|E|$ denotes the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$.

Let $u : \Omega \to \mathbb{R}$ be a measurable function. We denote by μ_u the distribution function of u, that is the function $\mu_u : [0, +\infty[\to [0, +\infty[$ defined by

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}| , \quad t \geq 0 .$$

The decreasing rearrangement of u is the function $u^* :]0, |\Omega|] \to \mathbb{R}^+$ defined by

$$u^*(s) = \sup\{t > 0 : \mu_u(t) > s\}\,, \qquad s \in]0, |\Omega|].$$

For $1 < q < \infty$, the Lorentz space $L^{q,\infty}(\Omega)$, also known as Marcinkiewicz or weak–Lebesgue, is the space of Lebesgue measurable functions u such that

$$\sup_{t>0} t\,\mu(t)^{1/q} < +\infty\,. \tag{2.1}$$

It is endowed with the norm

$$\|u\|_{q,\infty} = \sup_{0<s<|\Omega|} s^{\frac{1}{q}} u^{**}(s)\,,$$

where $u^{**}(s) = \dfrac{1}{s}\displaystyle\int_0^s u^*(\sigma)\,d\sigma$. For $1 < q < \infty$, the Lorentz space $L^{q,1}(\Omega)$ is the space of all Lebesgue measurable functions u such that

$$\|u\|_{q,1} = \int_0^\infty t^{\frac{1}{q}-1} u^*(t)\,dt < +\infty, \tag{2.2}$$

endowed with the norm (2.2). It is well–known (cf. [15], [21]) that the following inclusions hold

$$L^{q+\epsilon}(\Omega) \hookrightarrow L^{q,1}(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^{q,\infty}(\Omega) \hookrightarrow L^{q-\epsilon}(\Omega),$$

for every $\epsilon > 0$. Finally we recall that the Marcinkiewicz space $L^{N,\infty}(\Omega)$ is the dual space of $L^{\frac{N}{N-1},1}(\Omega)$.

We define $\mathcal{M}(\Omega)$ as the space of all Radon measures with bounded total variation on Ω and we denote by $|\mu|$ the total variation of $\mu \in \mathcal{M}(\Omega)$. The space of all functions of finite variation, that is the space of those $u \in L^1(\Omega)$ whose distributional gradient belongs to $\mathcal{M}(\Omega)$, is denoted by $BV(\Omega)$. It is endowed with the norm defined by $\|u\|_{BV(\Omega)} = \displaystyle\int_\Omega |u|\,dx + |Du|(\Omega)$, for any $u \in BV(\Omega)$. Since Ω has Lipschitz boundary, if u belongs to $BV(\Omega)$, then the function

$$u_0 = \begin{cases} u, & \text{in } \Omega; \\ 0, & \text{in } \mathbb{R}^N \setminus \Omega; \end{cases}$$

belongs to $BV(\mathbb{R}^N)$ and $|Du_0|(\mathbb{R}^N) = \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} + |Du|(\Omega)$. We explicitly point out that $|Du_0|(\mathbb{R}^N)$ defines an equivalent norm on $BV(\Omega)$, which we will use in the sequel. Through the paper, with an abuse of notation, we still denote u_0 by u.

We will denote by $S_{N,p}$ the best constant in the Sobolev inequality (cf. [22]), that is,

$$\|u\|_{p^*} \le S_{N,p}\|\nabla u\|_p\,, \quad \text{for all } u \in W^{1,p}(\Omega)\,.$$

We will also write S_N instead of $S_{N,1}$. It is well–known (cf. [22]), that

$$\lim_{p\to 1} S_{N,p} = S_N\,. \tag{2.3}$$

Sobolev's inequality can be improved in the context of Lorentz spaces (cf. [1]) and, furthermore, by an approximation argument may be extended to BV–functions (see for instance [25]); as a consequence we obtain the continuous embedding

$$BV(\Omega) \hookrightarrow L^{\frac{N}{N-1},1}(\Omega)\,. \tag{2.4}$$

We will denote by $W^{-1,q'}(\Omega)$ the dual space of $W_0^{1,q}(\Omega)$, $1 \le q < \infty$. Here we just recall that the norm in $W^{-1,\infty}(\Omega)$ is given by

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup\left\{ <\mu,\varphi>_{W^{-1,\infty}(\Omega),W_0^{1,1}(\Omega)} \colon \int_\Omega |\nabla\varphi|\,dx \le 1 \right\}. \tag{2.5}$$

It is worth pointing out some remarkable subspaces of $W^{-1,\infty}(\Omega)$. One of these is $\mathcal{M}(\Omega)\cap W^{-1,\infty}(\Omega)$ whose elements are named Guy David measures in [20]. Another subspace is the so–called predual of $BV(\Omega)$. Indeed, the space $BV(\Omega)$ is the dual of a separable space which will be denoted by $\Gamma(\Omega)$; its elements can be written as $f - \operatorname{div} F$, with $(f,F) \in C_0(\Omega;\mathbb{R}^{N+1})$ (see [14], and also [20] and [3]). Since the elements of $W^{-1,\infty}(\Omega)$ may be written as $f - \operatorname{div} F$, with $(f,F) \in L^\infty(\Omega;\mathbb{R}^{N+1})$, we deduce that $\Gamma(\Omega) \subset W^{-1,\infty}(\Omega)$. Recall that in the 1–dimensional case we have

$$W^{-1,\infty}(a,b) = \{ f' \,:\, f \in L^\infty(a,b)\, \}$$

$$\mathcal{M}(a,b)\cap W^{-1,\infty}(\Omega) = \{ f' \,:\, f \in BV(a,b)\, \}$$

$$\Gamma(a,b) = \{ f' \,:\, f \in C(a,b) \text{ and } f(a) = f(b) = 0\, \}\,.$$

So that it is easy to find examples in any dimension that show all these spaces are different.

Moreover, we will denote by $BV(\Omega)^*$ the dual space of $BV(\Omega)$. The norm in $BV(\Omega)^*$ is given by

$$||\mu||_{BV(\Omega)^*} = \sup\{ <\mu,\varphi>_{BV(\Omega)^*,BV(\Omega)} \colon |D\varphi|(\Omega) + \int_{\partial\Omega} |\varphi|\,d\mathcal{H}^{N-1} \le 1\}\,.$$

Of course, $\Gamma(\Omega) \hookrightarrow BV(\Omega)^*$; we will prove in Appendix below that in the space $\Gamma(\Omega)$ the norms as subset of $BV(\Omega)^*$ and as subset of $W^{-1,\infty}(\Omega)$ coincide.

Finally we recall that a sequence $(\mu_n)_n$ in $\mathcal{M}(\Omega)$ weakly* converges to μ if

$$\lim_{n\to\infty}\int_\Omega f\,d\mu_n = \int_\Omega f\,d\mu$$

for every $f \in C_0(\Omega)$. We will say that a sequence $(u_n)_n$ weakly* converges to u in $BV(\Omega)$ if it strongly converges in $L^1(\Omega)$ and $(Du_n)_n$ weakly* converges to Du in $\mathcal{M}(\Omega)$. At least for sufficiently regular domains, this notion corresponds to weak* convergence in the usual sense: that is with respect to $\sigma(\Gamma(\Omega), BV(\Omega))$.

3. The radial case

In this Section we consider problem (1.1) in the case where the domain Ω is a ball centered at the origin, i.e. $\Omega \equiv B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and the datum f is a nonnegative radially decreasing function belonging to the Lorentz space $L^{N,\infty}(B_R)$.

Since both the domain and the datum are radially symmetric, it is well–known (see for instance [23]) that the weak solution u_p is given by

$$u_p(x) = \frac{1}{N^{p'} C_N^{p'/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{p'}{N}-p'} \left(\int_0^s f^*(\sigma)d\sigma \right)^{\frac{1}{p-1}} ds, \tag{3.1}$$

for almost every $x \in B_R$.

Our aim is to describe the behaviour of u_p and the behaviour of $|\nabla u_p|^{p-2}\nabla u_p$ as p goes to 1.

We begin by introducing some notation. In what follows we will denote by

$$\|f\|_s = \sup_{s\le\sigma<C_N R^N} \sigma^{\frac{1}{N}} f^{**}(\sigma), \quad \text{for every } s \in [0, C_N R^N[. \tag{3.2}$$

Clearly $\|f\|_0 \equiv ||f||_{L^{N,\infty}}$. Moreover we will denote by

$$s_1 = \inf\{s \ge 0 : \|f\|_s \le N C_N^{\frac{1}{N}}\}, \quad s_2 = \inf\{s \ge 0 : \|f\|_s < N C_N^{\frac{1}{N}}\}, \tag{3.3}$$

and by r_1 and r_2 the radii of the balls centered in the origin having measure s_1 and s_2 respectively:

$$C_N r_1^N = s_1 \qquad \text{and} \qquad C_N r_2^N = s_2.$$

We set $s_1 = C_N R^N$ if the set $\{s \ge 0 : \|f\|_s \le N C_N^{\frac{1}{N}}\}$ is empty, and $s_2 = C_N R^N$ if $\{s \ge 0 : \|f\|_s < N C_N^{\frac{1}{N}}\}$ is empty. We explicitly remark that it results $s_1 \le s_2$ and hence $r_1 \le r_2 \le R$. Therefore the balls B_{r_1} and B_{r_2} centered at the origin and radii r_1 and r_2 respectively are both contained in B_R.

In general the limit of solutions u_p, as p goes to 1, is finite a.e. in Ω when the datum f is small enough. For instance, if the datum f is constant, that is $f^*(s) = \lambda > 0$, then

$$\lim_{p\to 1} u_p = 0\,, \qquad \text{if } \lambda \le \frac{N}{R}\,;$$
$$\lim_{p\to 1} u_p = +\infty\,, \qquad \text{if } \lambda > \frac{N}{R}\,;$$

(cf. [16]). In this section we analyze the behaviour of u_p in a more general case, where f is not constant, without any dependence on the smallness of the datum. Indeed Theorem 3.1 below states that, as p goes to 1, u_p diverges in the ball B_{r_1}, that it has a finite non–negative limit (for which we give an upper bound) in the annulus $\overline{B_{r_2}}\backslash B_{r_1}$ of radii r_1 and r_2 and finally that it converges to zero in the annulus $B_R\backslash\overline{B_{r_2}}$ of radii r_2 and R.

Theorem 3.1. *Let u_p be the solution to problem* (1.1). *Then*

$$\lim_{p\to 1} u_p(x) = +\infty, \qquad \textit{for almost all } x \in B_{r_1}, \tag{3.4}$$

$$0 \le \lim_{p\to 1} u_p(x) \le R - |x| \qquad \textit{for almost all } x \in \overline{B_{r_2}}\backslash B_{r_1}, \tag{3.5}$$

$$\lim_{p\to 1} u_p(x) = 0, \qquad \textit{for almost all } x \in B_R\backslash\overline{B_{r_2}}. \tag{3.6}$$

Remark 3.1. We explicitly observe that Theorem 3.1 improves the result proved in [10] where the behaviour of u_p is studied just under a smallness assumption on f, i.e. $||f||_{L^{N,\infty}} \leq NC_N^{1/N}$. Indeed if $||f||_{L^{N,\infty}} < NC_N^{1/N}$, then $s_1 = s_2 = 0$ and therefore we deduce that

$$\lim_{p\to 1} u_p(x) = 0 \quad \text{a.e. in } B_R;$$

if $||f||_{L^{N,\infty}} = NC_N^{1/N}$, then $s_1 = 0$ and hence the limit function is a.e. finite in B_R, as in [10]. □

Remark 3.2. We point out that the values r_1 and r_2 (or equivalently s_1 and s_2) may be different. Indeed consider the function $f : B_R \to \mathbb{R}$ defined by $f(x) = \frac{N-1}{|x|}$. Then we have $f^*(\sigma) = C_N^{1/N}(N-1)\sigma^{-1/N}$ and $f^{**}(\sigma) = NC_N^{1/N}\sigma^{-\frac{1}{N}}$ for all $\sigma \in]0, C_N R^N[$. Consequently $\|f\|_s = NC_N^{\frac{1}{N}}$ for every $s \in]0, C_N R^N[$; therefore, $s_1 = 0$ and $s_2 = C_N R^N$. This example allows also to show that the value $R - |x|$ in (3.5) may be attained, i.e. then $\lim_{p\to 1} u_p(x) = R - |x|$ (see Remark 3.2 in [10]). □

Proof of Theorem 3.1: From (3.1) we can write, almost everywhere in B_{r_1},

$$u_p(x) = U_p(x) + I, \tag{3.7}$$

where

$$U_p(x) = \frac{1}{N^{p'} C_N^{p'/N}} \int_{C_N|x|^N}^{C_N r_1^N} s^{\frac{p'}{N}-p'} \left(\int_0^s f^*(\sigma)d\sigma \right)^{\frac{1}{p-1}} ds, \qquad \text{a.e. } x \in B_{r_1}, \tag{3.8}$$

$$I = \frac{1}{N^{p'} C_N^{p'/N}} \int_{C_N r_1^N}^{C_N R^N} s^{\frac{p'}{N}-p'} \left(\int_0^s f^*(\sigma)d\sigma \right)^{\frac{1}{p-1}} ds. \tag{3.9}$$

Since I is a nonnegative constant, by (3.7) it follows that

$$u_p(x) \geq U_p(x), \tag{3.10}$$

for almost all $x \in B_{r_1}$.

Now we evaluate $U_p(x)$, for almost every fixed $x \in B_{r_1}$. Denote by

$$\sigma_x = C_N |x|^N.$$

Since $|x| < r_1$, it follows that $\sigma_x < s_1$. Thus, by definition (3.3) of s_1, we deduce that

$$\|f\|_{\sigma_x} > NC_N^{\frac{1}{N}}.$$

By (3.2), there exists a constant $\hat{\sigma}_x$ such that

$$0 < \sigma_x \leq \hat{\sigma}_x < C_N R^N$$

and

$$\hat{\sigma}_x^{\frac{1}{N}} f^{**}(\hat{\sigma}_x) > NC_N^{\frac{1}{N}}.$$

Note also that (3.3) implies $\hat\sigma_x < s_1$. Hence, by the continuity of the function $g(\sigma) = \sigma^{\frac{1}{N}} f^{**}(\sigma)$ and by $g(\hat\sigma_x) > NC_N^{\frac{1}{N}}$, it yields

$$g(s) = s^{\frac{1}{N}} f^{**}(s) > NC_N^{\frac{1}{N}}, \tag{3.11}$$

for every $s \in]\hat\sigma_x - \delta, \hat\sigma_x + \delta[$, for a suitable $\delta > 0$.

Therefore,

$$\begin{aligned} U_p(x) =& \frac{1}{N^{p'} C_N^{p'/N}} \int_{C_N|x|^N}^{\hat\sigma_x} s^{\frac{p'}{N}-p'} \left(\int_0^s f^*(\sigma) d\sigma \right)^{\frac{1}{p-1}} ds + \\ &+ \frac{1}{N^{p'} C_N^{p'/N}} \int_{\hat\sigma_x}^{\hat\sigma_x+\delta} s^{\frac{p'}{N}-p'} \left(\int_0^s f^*(\sigma) d\sigma \right)^{\frac{1}{p-1}} ds + \\ &+ \frac{1}{N^{p'} C_N^{p'/N}} \int_{\hat\sigma_x+\delta}^{C_N r_1^N} s^{\frac{p'}{N}-p'} \left(\int_0^s f^*(\sigma) d\sigma \right)^{\frac{1}{p-1}} ds. \end{aligned} \tag{3.12}$$

Since the first and the third integral in (3.12) are non–negative, we obtain

$$\begin{aligned} U_p(x) \geq & \frac{1}{N^{p'} C_N^{p'/N}} \int_{\hat\sigma_x}^{\hat\sigma_x+\delta} s^{\frac{p'}{N}-p'} \left(\int_0^s f^*(\sigma) d\sigma \right)^{\frac{1}{p-1}} ds \\ =& \frac{1}{NC_N^{1/N}} \int_{\hat\sigma_x}^{\hat\sigma_x+\delta} s^{\frac{1}{N}-1} \left(\frac{1}{NC_N^{1/N}} s^{\frac{1}{N}-1} \int_0^s f^*(\sigma) d\sigma \right)^{\frac{1}{p-1}} ds. \end{aligned} \tag{3.13}$$

By (3.11) and definition of f^{**}, we have

$$\frac{1}{NC_N^{1/N}} s^{\frac{1}{N}-1} \int_0^s f^*(\sigma) d\sigma > 1, \quad \forall s \in]\hat\sigma_x, \hat\sigma_x + \delta[\,.$$

Thus, the right–hand side of (3.13) tends to $+\infty$ when p goes to 1 and by (3.10) we deduce (3.4)

Now we prove (3.5). Using (3.1), we have

$$u_p(x) = \frac{1}{NC_N^{1/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{1}{N}-1} \left(\frac{s^{\frac{1}{N}-1}}{NC_N^{1/N}} \int_0^s f^*(\sigma) d\sigma \right)^{\frac{1}{p-1}} ds\,, \tag{3.14}$$

for almost all $x \in \overline{B_{r_2}} \setminus B_{r_1}$, where, by (3.3),

$$\frac{s^{\frac{1}{N}-1}}{NC_N^{1/N}} \int_0^s f^*(\sigma) d\sigma \leq \frac{\|f\|_s}{NC_N^{1/N}} \leq 1 \quad s \in [C_N|x|^N, C_N R^N[. \tag{3.15}$$

Hence immediately follows that u_p is nonincreasing with respect to p; then $\lim_{p\to 1} u_p$ exists and it is non–negative. Furthermore, by (3.14) and (3.15), we get

$$u_p(x) \leq \frac{1}{NC_N^{1/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{1}{N}-1} \left(\frac{\|f\|_s}{NC_N^{1/N}} \right)^{\frac{1}{p-1}} ds \leq \frac{1}{NC_N^{1/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{1}{N}-1} \, ds\,.$$

Therefore $\lim_{p\to 1} u_p(x) \leq R - |x|$, which gives (3.5).

Now we prove (3.6). By the definition of s_2 in (3.3) we deduce that

$$\|f\|_s < NC_N^{\frac{1}{N}}, \qquad \forall s \in]s_2, C_N R^N[.$$

Let $\hat{s} \in]s_2, C_N R^N[$ be fixed. Since $\|f\|_{\hat{s}} < NC_N^{1/N}$, we may write $\|f\|_{\hat{s}} = (1-\epsilon)NC_N^{1/N}$ for some $\epsilon > 0$. Thus,

$$s^{\frac{1}{N}-1} \int_0^s f^*(\sigma)\, d\sigma \le (1-\epsilon)NC_N^{1/N},$$

for all $s \in [\hat{s}, R[$. Therefore by (3.1), we have

$$u_p(x) \le \frac{((1-\varepsilon)NC_N^{1/N})^{\frac{1}{p-1}}}{N^{p'}C_N^{p'/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{1}{N}-1} ds = \frac{(1-\varepsilon)^{\frac{1}{p-1}}}{NC_N^{1/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{1}{N}-1} ds,$$

for almost every $x \in B_{\hat{s}}$. Since

$$\lim_{p\to 1} \frac{(1-\varepsilon)^{\frac{1}{p-1}}}{NC_N^{1/N}} = 0,$$

we deduce that $\lim_{p\to 1} u_p(x) = 0$ uniformly on $B_{\hat{s}}$. Thus (3.6) holds true. □

Next we turn to describe the behaviour of $|\nabla u_p|^{p-2}\nabla u_p$ as p goes to 1.

Proposition 3.1. *Let u_p be the solution to problem* (1.1)*. Denote by z the vector field*

$$z = -\frac{f^{**}(C_N|x|^N)}{N} x\,. \tag{3.16}$$

Then we have

$$|\nabla u_p(x)|^{p-2}\nabla u_p(x) = z(x)\,, \quad \textit{for every } p > 1 \textit{ and for almost every } x \in B_R. \tag{3.17}$$

Moreover it results

$$\|z\|_{L^\infty(B_{r_1};\mathbb{R}^N)} > 1\,, \tag{3.18}$$

$$\|z\|_{L^\infty(\overline{B_{r_2}}\setminus B_{r_1};\mathbb{R}^N)} \le 1\,, \tag{3.19}$$

$$\|z\|_{L^\infty(B_R\setminus\overline{B_{r_2}};\mathbb{R}^N)} < 1\,. \tag{3.20}$$

Proof: By (3.1) it follows that, for almost every $x \in B_R$,

$$\begin{aligned} \nabla u_p(x) &= -\left(\frac{|x|}{N}\frac{1}{C_N|x|^N}\int_0^{C_N|x|^N} f^*(\sigma)d\sigma\right)^{\frac{1}{p-1}} \frac{x}{|x|} \\ &= -\left(\frac{|x|}{N} f^{**}(C_N|x|^N)\right)^{\frac{1}{p-1}} \frac{x}{|x|}\,. \end{aligned} \tag{3.21}$$

Hence, the vector field

$$|\nabla u_p(x)|^{p-2}\nabla u_p(x) = -\frac{|x|}{N} f^{**}(C_N|x|^N)\frac{x}{|x|}, \tag{3.22}$$

does not depend on p and this implies (3.17). Moreover, since

$$z(x) = -\frac{(C_N|x|^N)^{1/N}}{NC_N^{1/N}} f^{**}(C_N|x|^N)\frac{x}{|x|}\,,$$

(3.18), (3.19) and (3.20) follow in a straightforward way from (3.2) and (3.3). □

Remark 3.3. Let us point out that in the above proof, we have obtained

$$\|z\|_{L^\infty(B_R\setminus B_r;\mathbb{R}^N)} = \frac{\|f\|_{C_N r^N}}{NC_N^{1/N}}, \qquad \text{for all } \ r \in]0,R[\,.$$

Therefore we deduce

$$\|z\|_\infty = \sup_{0\le\sigma<C_N R^N} \frac{\sigma^{1/N}}{NC_N^{1/N}} f^{**}(\sigma) = \frac{\|f\|_{N,\infty}}{NC_N^{1/N}}\,. \quad \square \tag{3.23}$$

Remark 3.4. As a straightforward consequence of the definition of z, (3.1) becomes

$$u_p(x) = \frac{1}{NC_N^{1/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{1}{N}-1}\Big|z\Big(\frac{s^{1/N}}{C_N^{1/N}}\Big)\Big|^{\frac{1}{p-1}} ds$$

for almost all $x \in B_R$. Thus, if $\|f\|_{L^{N,\infty}(\Omega)} \le NC_N^{1/N}$, then

$$\lim_{p\to 1} u_p(x) = \frac{1}{NC_N^{1/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{1}{N}-1}\chi_{\{|z(s^{1/N}/C_N^{1/N})|=1\}}(s)\,ds$$

and, changing the integration variable, we finally obtain that the above limit is equal to the measure of the set $\{\,|z(x)| = 1\,\} \cap [\,|x|, R]$. □

The above results, Theorem 3.1 and Proposition 3.1, allow to prove a stability type result for the "limit equation" (1.2). More precisely we will prove that if the norm of f satisfies suitable smallness assumptions, then $u = \lim_{p\to 1} u_p$ is a solution to the following "limit problem" in the sense of the definition given in [5], [6], [7].

$$\begin{cases} -\text{div}\left(\dfrac{Du}{|Du|}\right) = f & \text{in } B_R, \\ u \in BV(\mathbb{R}^N), & \\ u = 0 & \text{in } \mathbb{R}^N \setminus B_R. \end{cases} \tag{3.24}$$

Theorem 3.2. *Let $f \in L^{N,\infty}(B_R)$ with $||f||_{L^{N,\infty}} \leq NC_N^{1/N}$ and let u_p be the solution to problem (1.1), for any $1 < p < \infty$. Then u_p converges a.e. in B_R to a function $u \in W_0^{1,1}(B_R)$ and there exists a vector field $z : B_R \to \mathbb{R}^N$ such that*

$$z \in L^\infty(B_R; \mathbb{R}^N) \text{ with } \|z\|_\infty \leq 1;$$

$$-\text{div} z = f \quad \text{in } \mathcal{D}'(B_R);$$

$$z \cdot \nu \leq 0 \quad \mathcal{H}^{N-1}\text{–a.e on } \partial B_R,$$

where ν denotes the outer normal to ∂B_R;

$$z \cdot \nabla u = |\nabla u| \quad \text{as measures in } B_R.$$

Remark 3.5. We explicitly observe that the function u given by Theorem 3.2 is a solution to (3.24) in the sense of the definition given in [5], [6], [7] (see Definition 4.1 and Remark 4.3 below). □

Proof of Theorem 3.2: Since $\|f\|_s \leq NC_N^{1/N}$ for all $s \in]0, C_N R^N[$, we have $r_1 = 0$ and by Theorem 3.1 we have

$$0 \leq u(x) = \lim_{p \to 1} u_p(x) \leq R - |x|, \qquad \text{a.e. in } B_R. \tag{3.25}$$

Moreover, from Proposition 3.1, we deduce that the vector field z defined in (3.16) satisfies the conditions $\|z\|_\infty \leq 1$ and

$$\int_{B_R} |\nabla u_p|^p \, dx \leq \Big(\frac{\|f\|_{N,\infty}}{NC_N^{1/N}} \Big)^{p'} |B_R|.$$

Since $\|f\|_{N,\infty} \leq NC_N^{1/N}$, Sobolev's inequality for BV–functions implies that

$$\nabla u_p \rightharpoonup Du \qquad \text{weakly* in the sense of measures.}$$

On the other hand, by (3.16), (3.21) and (3.25), we have $u \in W_0^{1,1}(B_R)$ and

$$\nabla u(x) = 0, \quad \text{if } |z(x)| < 1\,,$$

$$\nabla u(x) = z(x) = -\tfrac{x}{|x|} \quad \text{if } |z(x)| = 1.$$

Moreover from (3.17), since u_p is a solution to problem (1.1), it follows that $-\text{div } z = f$ in $\mathcal{D}'(B_R)$. Furthermore, by definition of z, it results

$$z(x) = -|z(x)| \frac{x}{|x|}\,.$$

If $|x| = R$, then

$$z \cdot \nu(x) = -|z(x)| \frac{x}{R} \cdot \frac{x}{R} = -|z(x)| < 0.$$

Finally, a straightforward calculation shows that, for every Borel set $B \subset B_R$,

$$\int_B z \cdot \nabla u = \int_{\{|z(x)|=1\} \cap B} z \cdot \nabla u = |\{|z(x)| = 1\} \cap B| = \int_B |\nabla u|.$$

This yields the conclusion. □

Remark 3.6. One could think from the results in this Section that the set where $|z| \le 1$ is the same as the set where $|u| < +\infty$; this is not the case as the following example shows.
Consider problem (1.1) with datum $f(C_N|x|^N)$ defined by

$$f(s) = \begin{cases} 0, & \text{if } 0 \le s \le C_N\left(\frac{R}{2}\right)^N; \\ \frac{\lambda}{s^{1/N}}, & \text{if } C_N\left(\frac{R}{2}\right)^N < s < C_N R^N. \end{cases} \tag{3.26}$$

It is not difficult to check that the solution is given by

$$u_p(x) = \frac{1}{NC_N^{1/N}} \int_{C_N \max\{|x|, \frac{R}{2}\}^N}^{C_N R^N} s^{\frac{1}{N}-1} g(s)^{\frac{1}{p-1}} ds; \tag{3.27}$$

with

$$g(s) = \frac{\lambda}{(N-1)C_N^{1/N}} \left(1 - \left(\frac{C_N R^N}{s2^N}\right)^{1-\frac{1}{N}}\right), \quad C_N \frac{R^N}{2^N} < s < C_N R^N.$$

Observe that g is an increasing function; thus, if $\lambda \le \dfrac{(N-1)C_N^{1/N} 2^{N-1}}{2^{N-1}-1}$, then $g(s) < 1$ for all $s < C_N R_N$ and so $u_p(x) \to 0$ everywhere in B_R. On the other hand, if $\lambda > \dfrac{(N-1)C_N^{1/N} 2^{N-1}}{2^{N-1}-1}$, then $g(s) > 1$ in an interval $]s_0, C_N R^N[$, so that $u_p(x) \to +\infty$ everywhere.

Let us compute the vector field z: Since

$$\nabla u_p(x) = \begin{cases} 0, & \text{if } 0 \le |x| \le \frac{R}{2}; \\ -\frac{x}{|x|} g(C_N|x|^N)^{1/(p-1)}, & \text{if } \frac{R}{2} < |x| < R; \end{cases}$$

and so $|\nabla u_p|^{p-2}\nabla u_p$ does not depend on p, it follows that

$$z(x) = \begin{cases} 0, & \text{if } 0 \le |x| \le \frac{R}{2}; \\ -\frac{x}{|x|} g(C_N|x|^N), & \text{if } \frac{R}{2} < |x| < R; \end{cases}$$

If $\lambda \le \dfrac{(N-1)C_N^{1/N} 2^{N-1}}{2^{N-1}-1}$, then $|z| < 1$ everywhere while if $\lambda > \dfrac{(N-1)C_N^{1/N} 2^{N-1}}{2^{N-1}-1}$, then $|z| > 1$ only in a neighborhood of the boundary that does not intersect $B_{R/2}(0)$.

Hence, $B_{R/2}(0) \subset \{|z| \le 1\}$ but $\{|u| < +\infty\} = \emptyset$ for all $\lambda > \frac{(N-1)C_N^{1/N} 2^{N-1}}{2^{N-1}-1}$. Therefore, $\{|u| < +\infty\} \subsetneq \{|z| \le 1\}$ for these values. Finally, we observe that when the limit function blows up at the boundary, it blows up everywhere. □

4. Stability results with $W^{-1,\infty}(\Omega)$ data

Consider the nonlinear elliptic problems

$$\text{(4.1)} \qquad \begin{cases} -\mathrm{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = \mu\,, & \text{in } \Omega\,; \\ u_p = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

where $p > 1$ and the datum μ belongs to $W^{-1,\infty}(\Omega)$. Since by duality arguments, $W^{-1,\infty}(\Omega)$ is included in $W^{-1,p'}(\Omega)$ for every $p > 1$, then the existence and uniqueness of the solution $u_p \in W_0^{1,p}(\Omega)$ to problem (4.1) can be proved by classical methods (see, for instance, [18]). In this Section we will study the behaviour, as p goes to 1, of these solutions u_p to problems (4.1). We prove that, if the norm in $W^{-1,\infty}(\Omega)$ of the datum μ is less than 1, then u_p converges to the function $u \equiv 0$; if the norm in $W^{-1,\infty}(\Omega)$ of the datum μ is equal to 1, then u_p converges to a function u belonging to $BV(\Omega)$. Moreover we prove that u_p does not converge to any BV–function if the norm in $W^{-1,\infty}(\Omega)$ of the datum μ is greater than 1 (see Subsection 4.1 below). As in the previous Section, we deduce a stability result for the limit problem (1.2) when the norm in $W^{-1,\infty}(\Omega)$ of the datum μ is less than or equal to 1. Actually we does not give a stability result in the case where $\|\mu\|_{W^{-1,\infty}(\Omega)} = 1$ for all $\mu \in W^{-1,\infty}(\Omega)$. We study such a case when μ belongs to a subspace of $W^{-1,\infty}(\Omega)$, namely $\Gamma(\Omega)$, the predual space of $BV(\Omega)$ (see Subsection 4.2 below).

From now on, abusing of the terminology, we will say that u_p is a sequence and we will consider subsequences of it, as p goes to 1.

4.1. General data in $W^{-1,\infty}(\Omega)$

The main Theorem of this Subsection is the following.

Theorem 4.1. *Let u_p be the solution to problem* (4.1).
If $\|\mu\|_{W^{-1,\infty}(\Omega)} < 1$, then, as p goes to 1, u_p converges to 0 *in $L^q(\Omega)$, with $q < \frac{N}{N-1}$.*
If $\|\mu\|_{W^{-1,\infty}(\Omega)} = 1$, then, up to a subsequence, u_p converges in $L^q(\Omega)$, with $q < \frac{N}{N-1}$.
If $\|\mu\|_{W^{-1,\infty}(\Omega)} > 1$, then

$$\lim_{p\to 1} \int_\Omega |\nabla u_p|\,dx = +\infty\,,$$

and hence there is not any $u \in BV(\Omega)$ which is the weak limit of u_p.*

Proof: Since u_p is a weak solution to problem (4.1), the following inequalities hold true

$$\text{(4.2)} \qquad \begin{aligned} \int_\Omega |\nabla u_p|^p dx =&< \mu, u_p >_{W^{-1,\infty}(\Omega), W_0^{1,1}(\Omega)} \le \|\mu\|_{W^{-1,\infty}(\Omega)} \int_\Omega |\nabla u_p| dx \le \\ &\le \|\mu\|_{W^{-1,\infty}(\Omega)} |\Omega|^{1/p'} \Big(\int_\Omega |\nabla u_p|^p dx\Big)^{1/p}. \end{aligned}$$

Therefore, one always has

$$\text{(4.3)} \qquad \int_\Omega |\nabla u_p|^p dx \le \|\mu\|^{p'}_{W^{-1,\infty}(\Omega)} |\Omega|\,.$$

Assume first that $\|\mu\|_{W^{-1,\infty}(\Omega)} < 1$. Thus, we can write $\|\mu\|_{W^{-1,\infty}(\Omega)} = 1-\epsilon$, for a suitable $\epsilon > 0$. By Young inequality, we have

$$\int_\Omega |\nabla u_p|\,dx \le \left[\frac{1}{p}(1-\epsilon)^{\frac{p}{p-1}} + \frac{p-1}{p}\right]|\Omega| \le |\Omega|. \tag{4.4}$$

This estimate implies that there exists $u \in BV(\Omega)$ and a subsequence, still denoted by u_p, satisfying $u_p \to u$ in $L^q(\Omega)$, with $q < \frac{N}{N-1}$, and

$$|Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} \le \liminf_{p\to 1} \int_\Omega |\nabla u_p|\,dx\,.$$

By (4.4), letting $p \to 1$, we obtain

$$|Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} = 0,$$

and therefore $u = 0$. Since $u \equiv 0$ is the unique limit point, by Sobolev inequality, we actually obtain that $\lim_{p\to 1} u_p = 0$ in $L^q(\Omega)$, with $q < \frac{N}{N-1}$.

Let us now assume that $\|\mu\|_{W^{-1,\infty}(\Omega)} = 1$. Then (4.3) becomes $\int_\Omega |\nabla u_p|^p\,dx \le |\Omega|$ and by Young's inequality we obtain

$$\int_\Omega |\nabla u_p|\,dx \le |\Omega|\,. \tag{4.5}$$

So that from Sobolev's inequality for BV–functions we deduce the existence of a function $u \in BV(\Omega)$ such that, up to subsequences,

$$\begin{cases} \nabla u_p \rightharpoonup Du & \text{weakly* in the sense of measures,} \\ u_p \to u & \text{strongly in } L^q(\Omega)\,, \quad 1 \le q < \frac{N}{N-1}\,, \\ u_p \to u & \text{a.e. in } \Omega\,. \end{cases}$$

.

Finally let us assume that $\|\mu\|_{W^{-1,\infty}(\Omega)} > 1$. Since $\|\mu\|_{W^{-1,\infty}(\Omega)} = \lim_{p\to 1} \|\mu\|_{W^{-1,p'}(\Omega)}$, we may take $\epsilon > 0$ and $p_0 > 1$ such that $\|\mu\|_{W^{-1,p'}(\Omega)} > 1+\epsilon$, for all $p \le p_0$.

On the other hand, if $\varphi \in W_0^{1,p}(\Omega)$ with $\|\nabla\varphi\|_{L^p(\Omega;\mathbb{R}^N)} \le 1$, then

$$< \mu, \varphi >_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \int_\Omega |\nabla u_p|^{p-1}\nabla u_p \cdot \nabla\varphi\,dx \le \left(\int_\Omega |\nabla u_p|^p\,dx\right)^{\frac{p-1}{p}}. \tag{4.6}$$

Since by definition we have

$$\|\mu\|_{W^{-1,p'}(\Omega)} = \sup\left\{ < \mu, \varphi >_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} : \int_\Omega |\nabla\varphi|^p\,dx \le 1 \right\},$$

then (4.6) implies

$$\|\mu\|_{W^{-1,p'}(\Omega)}^{\frac{p}{p-1}} \le \int_\Omega |\nabla u_p|^p\,dx.$$

Therefore,

$$(1+\epsilon)^{\frac{p}{p-1}} \le \int_\Omega |\nabla u_p|^p\,dx$$

for $p \le p_0$. This implies

$$\lim_{p\to 1}\int_\Omega |\nabla u_p|^p\,dx = +\infty.$$

Since for a suitable $g \in L^\infty(\Omega)$, we have $\mu =$ div g, the conclusion follows from

$$\int_\Omega |\nabla u_p|^p\,dx =<\mu, u_p>_{W^{-1,\infty}(\Omega),W_0^{1,1}(\Omega)}= \int_\Omega g\cdot\nabla u_p \le \|g\|_\infty \int_\Omega |\nabla u_p|\,dx\,. \quad \square$$

Let us prove the following result which describes the behaviour of $|\nabla u_p|^{p-2}\nabla u_p$

Proposition 4.1. *Let u_p be the solution to problem* (4.1)*, then there exists a vector field $z \in L^\infty(\Omega;\mathbb{R}^N)$ such that, up to subsequences,*

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \textit{weakly in } L^q(\Omega) \textit{ for all } 1\le q<+\infty\,, \tag{4.7}$$

$$-\mathrm{div}\, z = \mu \quad \textit{in } \mathcal{D}'(\Omega)\,, \tag{4.8}$$

$$\|z\|_\infty = \|\mu\|_{W^{-1,\infty}(\Omega)}\,. \tag{4.9}$$

Proof:

Step 1: Proof of (4.7)

Arguing as in the proof of Theorem 4.1, we obtain inequality (4.3). Then for every q, $1\le q<p'$, we have

$$\begin{aligned}\int_\Omega |\nabla u_p|^{(p-1)q} &\le \Big(\int_\Omega |\nabla u_p|^p\Big)^{(p-1)q/p}|\Omega|^{1-\frac{(p-1)q}{p}}\\ &\le |\Omega|^{\frac{(p-1)q}{p}}\|\mu\|_{W^{-1,\infty}(\Omega)}^{p'\frac{(p-1)q}{p}}|\Omega|^{1-\frac{(p-1)q}{p}}\\ &= |\Omega|\|\mu\|^q_{W^{-1,\infty}(\Omega)}\,.\end{aligned}\tag{4.10}$$

It yields that, for any q fixed, the sequence $|\nabla u_p|^{p-2}\nabla u_p$ is bounded in $L^q(\Omega;\mathbb{R}^N)$ and then there exists $z_q \in L^q(\Omega;\mathbb{R}^N)$ such that, up to subsequences,

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z_q \quad \text{in } L^q(\Omega) \quad \text{for all } 1\le q<+\infty\,.$$

Moreover, by a diagonal argument we can find a limit z that does not depend on q, that is

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \text{in } L^q(\Omega) \quad \text{for all } 1\le q<+\infty\,. \tag{4.11}$$

Now by (4.10) we deduce

$$\||\nabla u_p|^{p-2}\nabla u_p\|_q \le |\Omega|^{1/q}\|\mu\|_{W^{-1,\infty}(\Omega)} \quad \text{for } 1\le q<+\infty \text{ and for } p\in]1,q'[\,.$$

Therefore, by lower semicontinuity of the norm, we have

$$\|z\|_q \le |\Omega|^{1/q}\|\mu\|_{W^{-1,\infty}(\Omega)} \quad \text{for all } 1\le q<+\infty\,. \tag{4.12}$$

Step 2: Proof of (4.8)

Since u_p is a distributional solution to problem (4.1), it follows that

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx = <\mu,\varphi>_{W^{-1,\infty}(\Omega),W_0^{1,1}(\Omega)},\qquad \forall\varphi\in C_0^\infty(\Omega)\,.$$

Hence, using (4.11) we obtain

$$\int_\Omega z\cdot\nabla\varphi\,dx = <\mu,\varphi>_{W^{-1,\infty}(\Omega),W_0^{1,1}(\Omega)}\,,\qquad \forall\varphi\in C_0^\infty(\Omega)\,,$$

that is (4.8).

Step 3: Proof of (4.9)

For any fixed $h>0$ and $p>1$, we denote

$$D_{p,h}=\{x\in\Omega : |\nabla u_p(x)|^{p-1}>h\}.$$

By (4.10) for $q=1$, as p goes to 1 we have

$$|\nabla u_p|^{p-2}\nabla u_p\chi_{D_{p,h}}\rightharpoonup g_h\quad\text{weakly in } L^1(\Omega;\mathbb{R}^N)\,,\tag{4.13}$$

$$|\nabla u_p|^{p-2}\nabla u_p\chi_{\Omega\setminus D_{p,h}}\rightharpoonup f_h\quad\text{weakly in } L^1(\Omega;\mathbb{R}^N)\,,\tag{4.14}$$

for some $g_h\in L^1(\Omega)$ and $f_h\in L^1(\Omega)$. On the other hand by (4.10) with $q=1$

$$|D_{p,h}|\le\frac{1}{h}\int_\Omega|\nabla u_p|^{p-1}\le\frac{|\Omega|\,\|\mu\|_{W^{-1,\infty}(\Omega)}}{h}\,.\tag{4.15}$$

Therefore by Hölder's inequality and (4.15) for every fixed $\Phi\in L^\infty(\Omega;\mathbb{R}^N)$, with $||\Phi||_\infty\le 1$, we have

$$\begin{aligned}\Big|\int_{D_{p,h}}|\nabla u_p|^{p-2}\nabla u_p\cdot\Phi\Big| &\le\|\Phi\|_\infty\int_{D_{p,h}}|\nabla u_p|^{p-1}\\ &\le|D_{p,h}|^{1-\frac1q}\Big(\int_\Omega|\nabla u_p|^{(p-1)q}\Big)^{\frac1q}\\ &\le\frac{1}{h^{1-\frac1q}}|\Omega|^{1-\frac1q}\|\mu\|^{1-\frac1q}_{W^{-1,\infty}(\Omega)}|\Omega|^{\frac1q}\|\mu\|_{W^{-1,\infty}(\Omega)}\\ &\le\frac{1}{h^{1-\frac1q}}|\Omega|\,\|\mu\|^{2-\frac1q}_{W^{-1,\infty}(\Omega)}\,.\end{aligned}\tag{4.16}$$

By (4.13) and (4.16), for any fixed $h>0$ we deduce

$$\Big|\int_\Omega g_h\cdot\Phi\Big|\le\frac{1}{h^{1-\frac1q}}|\Omega|\,\|\mu\|^{2-\frac1q}_{W^{-1,\infty}(\Omega)},$$

for every $\Phi\in L^\infty(\Omega;\mathbb{R}^N)$ such that $||\Phi||_\infty\le 1$. By duality we deduce the following estimate for g_h

$$\int_\Omega|g_h|\le\frac{1}{h^{1-\frac1q}}|\Omega|\,\|\mu\|^{2-\frac1q}_{W^{-1,\infty}(\Omega)}\,,$$

for any fixed $h>0$. Moreover by definition of the set $D_{p,h}$,

$$|\chi_{\Omega\setminus D_{p,h}}|\nabla u_p|^{p-2}\nabla u_p|\le h,\qquad\text{a.e. in }\Omega.$$

This implies, using the inequality contained in [24], p. 337, the following pointwise estimate for f_h

$$|f_h| \le h, \qquad \text{a.e. in } \Omega,$$

for any fixed $h > 0$. Therefore $f_h \in L^\infty(\Omega;\mathbb{R}^N)$. Applying once again (4.10), we have

$$\int_{\Omega\setminus D_{p,h}} |\nabla u_p|^{q(p-1)} \le \int_\Omega |\nabla u_p|^{q(p-1)} \le |\Omega|\, \|\mu\|^q_{W^{-1,\infty}(\Omega)}\,,$$

that is, for some q_0,

$$\|\chi_{\Omega\setminus D_{p,h}} |\nabla u_p|^{p-2}\nabla u_p\|_q \le |\Omega|^{1/q}\, \|\mu\|_{W^{-1,\infty}(\Omega)} \le 2\|\mu\|_{W^{-1,\infty}(\Omega)} \text{ for all } q \ge q_0.$$

This implies

$$\|f_h\|_q \le 2\|\mu\|_{W^{-1,\infty}(\Omega)} \text{ for all } q \ge q_0.$$

Since $f_h \in L^\infty(\Omega;\mathbb{R}^N)$, it yields

$$\|f_h\|_\infty \le 2\|\mu\|_{W^{-1,\infty}(\Omega)}.$$

Therefore, for every $h > 0$, we have

$$z = f_h + g_h,$$

with

$$\int_\Omega |g_h| \le \frac{C_q}{h^{1-\frac{1}{q}}}$$

and

$$\|f_h\|_\infty \le 2\|\mu\|_{W^{-1,\infty}(\Omega)}.$$

The above condition on g_h gives

$$\lim_{h\to\infty} g_h = 0 \quad \text{ in } L^1(\Omega)$$

and hence

$$\lim_{h\to\infty} f_h = \lim_{h\to\infty} z - g_h = z \quad \text{ in } L^1(\Omega).$$

Since $\|f_h\|_\infty \le 2\|\mu\|_{W^{-1,\infty}(\Omega)}$ for all $h > 0$, we obtain that $z \in L^\infty(\Omega;\mathbb{R}^N)$. Then (4.12) implies

$$\|z\|_\infty \le \|\mu\|_{W^{-1,\infty}(\Omega)}\,.$$

From (4.8) and the definition of the norm $\|\mu\|_{W^{-1,\infty}(\Omega)}$, since we have

$$< \mu, \varphi >_{W^{-1,\infty}(\Omega),W_0^{1,1}(\Omega)} = \int_\Omega z\cdot\nabla\varphi \le \|z\|_\infty \int_\Omega |\nabla\varphi|\,,$$

the reverse inequality follows, and therefore (4.9) is proved. □

Remark 4.1. In Theorem 4.1, when $\|\mu\|_{W^{-1,\infty}(\Omega)} > 1$, we did not state which is the pointwise limit of u_p. Nevertheless, an "a posteriori" argument can be done to obtain some kind of limit of the solutions. In fact, we can prove the following claim.

There exists a function v satisfying, up to subsequences,

(4.17) $$|u_p|^{p-1} \rightharpoonup v \quad \textit{weakly in } L^q(\Omega) \quad \textit{and} \quad \|v\|_q \le |\Omega|^{1/q}\|\mu\|_{W^{-1,\infty}(\Omega)}\,,$$

for all $1 \le q < +\infty$.

To prove this claim, we must carefully apply Sobolev's inequality. It is well–known (see, for instance, [25] p. 57 or p. 82) that a straightforward argument yields a simple connection between $S_{N,p}$ and S_N, namely: $S_{N,p} \le \dfrac{(N-1)p}{N-p} S_N$, $(1 \le p < N)$ and so

$$S_{N,p} \le 2(N-1)S_N \qquad \text{for } \ 1 \le p \le \frac{2N}{3}\,.$$

From (4.10), we deduce

$$\Big(\int_\Omega |\nabla u_p|^{q(p-1)} \Big)^{\frac{1}{q(p-1)}} \le |\Omega|^{\frac{1}{q(p-1)}} \|\mu\|_{W^{-1,\infty}(\Omega)}^{\frac{1}{p-1}}\,.$$

Consider r such that $1 \le r(p-1) \le \frac{2N}{3}$, by applying Sobolev's inequality, we get

$$\begin{aligned} \bigg(\int_\Omega \Big(|u_p|^{r(p-1)} \Big)^{\frac{N}{N-r(p-1)}} \bigg)^{\frac{N-r(p-1)}{Nr(p-1)}} &\le S_{N,r(p-1)} \Big(\int_\Omega |\nabla u_p|^{r(p-1)} \Big)^{\frac{1}{r(p-1)}} \\ &\le S_{N,r(p-1)} |\Omega|^{\frac{1}{r(p-1)}} \|\mu\|_{W^{-1,\infty}(\Omega)}^{\frac{1}{p-1}}\,. \end{aligned}$$

Since $\frac{N}{N-r(p-1)} > 1$, Hölder's inequality implies

$$\begin{aligned} \Big(\int_\Omega |u_p|^{r(p-1)} \Big)^{\frac{1}{r(p-1)}} &\le |\Omega|^{\frac{1}{N}} \bigg(\int_\Omega \Big(|u_p|^{r(p-1)} \Big)^{\frac{N}{N-r(p-1)}} \bigg)^{\frac{N-r(p-1)}{Nr(p-1)}} \\ &\le S_{N,r(p-1)} |\Omega|^{\frac{1}{r(p-1)}+\frac{1}{N}} \|\mu\|_{W^{-1,\infty}(\Omega)}^{\frac{1}{p-1}}\,. \end{aligned}$$

Therefore, taking $S_{N,r(p-1)} \le 2(N-1)S_N$ into account, we obtain

$$\|\,|u_p|^{p-1}\|_r \le \Big(2(N-1)S_N |\Omega|^{\frac{1}{r(p-1)}+\frac{1}{N}} \Big)^{p-1} \|\mu\|_{W^{-1,\infty}(\Omega)}\,.$$

for all r satisfying $1 \le r(p-1) \le \frac{2N}{3}$.

Now let q satisfy $1 \le q \le \frac{1}{p-1} < r$ and apply Hölder's inequality, then

$$\|\,|u_p|^{p-1}\|_q \le |\Omega|^{\frac{1}{q}-\frac{1}{r}} \Big(\int_\Omega |u_p|^{r(p-1)} \Big)^{\frac{1}{r}} \le |\Omega|^{\frac{1}{q}} \Big(2(N-1)S_N |\Omega|^{\frac{1}{N}} \Big)^{p-1} \|\mu\|_{W^{-1,\infty}(\Omega)}\,.$$

Since $p \to 1$, it follows that we may consider any q such that $1 \le q < +\infty$; moreover, $|u_p|^{p-1}$ is bounded in $L^q(\Omega)$ and its bound tends to $|\Omega|^{\frac{1}{q}} \|\mu\|_{W^{-1,\infty}(\Omega)}$ as $p \to 1$. Therefore, (4.17) is a consequence of a diagonal argument.

We also remark that there is some connection between the functions $u = \lim_{p\to 1} u_p$ and v. Indeed, on the set $\{u = 0\}$ it yields $v \le \limsup_{p\to 1} e^{(p-1)\log|u_p|} \le 1$ a.e., while on $\{|u| = +\infty\}$ we have $v \ge \liminf_{p\to 1} e^{(p-1)\log|u_p|} \ge 1$ a.e. Finally, up a null set, we obtain $v = 1$ on $\{0 < |u| < +\infty\}$. $\square$

Remark 4.2. Since the Marcinkiewicz space $L^{N,\infty}(B_R)$ is included in $W^{-1,\infty}(B_R)$, if Ω is the ball B_R and the datum μ belongs to $L^{N,\infty}(B_R)$, then Proposition 4.1 and Theorem 4.1 holds true. Taking into account Proposition 4.1 and Remark 3.3, we may deduce that for every $f \in L^{N,\infty}(B_R)$,

$$\|f\|_{W^{-1,\infty}(B_R)} = \sup_{0\le\sigma<C_N R^N} \frac{\sigma^{1/N}}{NC_N^{1/N}} f^{**}(\sigma) = \frac{\|f\|_{N,\infty}}{NC_N^{1/N}}\,.$$

Observe that Theorem 3.1 implies the existence of a finite limit if and only if

$$\|f\|_{N,\infty} = \sup_{0\le\sigma<C_N R^N} \sigma^{\frac{1}{N}} f^{**}(\sigma) \le NC_N^{\frac{1}{N}}\,,$$

that is, when $\|f\|_{W^{-1,\infty}(B_R)} \le 1$. □

As in the previous section, the study of the behaviour, as p goes to 1, of u_p and $|\nabla u_p|^{p-2}\nabla u_p$ allows to deduce a stability result to the "limit problem"

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = \mu & \text{in } \Omega, \\ u \in BV(\mathbb{R}^N), & \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.18}$$

Let us begin by recalling the definition of solution to this problem (see [5], [7], [9] and [12]). To this aim, we need to introduce the following distribution (cf. [4]):

Let u be a function belonging to $BV(\Omega)$ and let z be a vector field belonging to $L^\infty(\Omega;\mathbb{R}^N)$ such that $\mathrm{div}\, z$, in the sense of distributions, belongs to $BV(\Omega)^*$, i.e.

$$< \mathrm{div}\, z, \varphi >= -\int_\Omega z\cdot\nabla\varphi\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$. Then we define the distribution

$$(z, Du) : C_0^\infty(\Omega) \to \mathbb{R}$$

by

$$< (z, Du), \varphi >= -\int_\Omega u\, z\cdot\nabla\varphi\, dx - < \mathrm{div}\, z, u\,\varphi >_{BV(\Omega)^*, BV(\Omega)}\,, \tag{4.19}$$

for every $\varphi \in C_0^\infty(\Omega)$. Since $u \in BV(\Omega) \subset L^{N',1}(\Omega)$, $\varphi \in C_0^\infty(\Omega)$, $z \in L^\infty(\Omega;\mathbb{R}^N)$ and $\mathrm{div}\, z \in BV(\Omega)^*$, all terms in (4.19) make sense.

Moreover as in [7] (pp. 126–127) we may define the weak trace of the exterior normal component of z, which will be denoted by $[z,\nu]$.

Definition 4.1. A function $u : \Omega \to \mathbb{R}$ is a solution to (4.18) if the following conditions hold true

$$u \in BV(\Omega);$$

there exists a vector field $z:\Omega\to\mathbb{R}^N$ such that

$$z\in L^\infty(\Omega;\mathbb{R}^N) \text{ with } \|z\|_\infty\le 1; \tag{4.20}$$

$$-\operatorname{div} z=\mu \quad \text{in } \mathcal{D}'(\Omega); \tag{4.21}$$

$$[z,\nu]\in \operatorname{sign}(-u) \quad \mathcal{H}^{N-1}\text{–a.e on } \partial\Omega; \tag{4.22}$$

$$(z,Du) \text{ is a Radon measure and} \tag{4.23}$$

$$(z,Du)=|Du| \quad \text{as measures in } \Omega. \tag{4.24}$$

Remark 4.3. Let us observe that if the function u in the previous definition belongs to $W^{1,1}(\Omega)$, then the measure (z,Du) coincides with $z\cdot\nabla u$. As already observed, this means that the function u and the vector field z whose existence has been proved in the previous Section yields a solution to problem (3.24). □

The announced stability result it is now an easy consequence of Theorem 4.1 and Proposition 4.1.

Theorem 4.2. *Let u_p be the solution to problem* (4.1)*.*
If $\|\mu\|_{W^{-1,\infty}(\Omega)}<1$, then, as p goes to 1, u_p converges to $u\equiv 0$ solution to problem (4.18) *in the sense of Definition 4.1.*
If $\|\mu\|_{W^{-1,\infty}(\Omega)}>1$, then there is not solution to problem (4.18) *in the sense of Definition 4.1.*

Remark 4.4. We explicitly remark that by Theorem 4.1, if $\|\mu\|_{W^{-1,\infty}(\Omega)}=1$, then u_p converges, up to a subsequence, to a function $u\in BV(\Omega)$. Nevertheless we are not able to prove, in general, that u is a solution to problem (4.18) in the sense of Definition 4.1 (cf. Examples 4.1 and 4.2 below) □

We conclude this Subsection by showing some examples with datum belonging to $W^{-1,\infty}(\Omega)$. The following example gives the explicit expression of the limit function u when $\|\mu\|_{W^{-1,\infty}(\Omega)}=1$.

Example 4.1. Let Ω be an open subset in $\mathbb{R}^N$ containing $B_R(0)$ and consider a vector field $g:\Omega\to\mathbb{R}^2$ defined by

$$g(x)=\begin{cases} -\frac{\sqrt{N}}{N}(\operatorname{sign} x_1,\dots,\operatorname{sign} x_N), & \text{if } |x_1|+\cdots+|x_N|\le R;\\ 0, & \text{if } |x_1|+\cdots+|x_N|>R.\end{cases}$$

Since $|g|\le 1$, $\operatorname{div} g\in W^{-1,\infty}(\Omega)$ and $\|\operatorname{div}(g)\|_{W^{-1,\infty}(\Omega)}=1$. We point out that $\operatorname{div} g\notin L^1(\Omega)$ since evaluating this divergence some measures appear. The solution to

$$\begin{cases} -\Delta_p u_p= \operatorname{div} g, & \text{in } \Omega;\\ \\ u_p=0, & \text{on } \partial\Omega;\end{cases}$$

is given by

$$u_p(x)=\begin{cases} \frac{\sqrt{N}R}{N}-\frac{\sqrt{N}}{N}(|x_1|+\cdots+|x_N|), & \text{if } |x_1|+\cdots+|x_N|\le R;\\ 0, & \text{if } |x_1|+\cdots+|x_N|>R.\end{cases}$$

Hence, u_p does not depend on p and

$$u(x)=\lim_{p\to 1}u_p(x)=\begin{cases}\frac{\sqrt{N}\,R}{N}-\frac{\sqrt{N}}{N}(|x_1|+\cdots+|x_N|)\,, & \text{if } |x_1|+\cdots+|x_N|\le R\,;\\ 0, & \text{if } |x_1|+\cdots+|x_N|> R\,.\end{cases}$$

It is easy to prove that u is a non trivial solution to the limit problem. □

Example 4.2. Let us consider the problem

$$\begin{cases}-\Delta_p u_p = \operatorname{div} g\,, & \text{in } B_R\,;\\ \\ u_p=0\,, & \text{on } \partial B_R\,;\end{cases}$$

where $g\in L^\infty(B_R;\mathbb{R}^N)$ is a radial and bounded vector field. It is well–known, see [23], that the solution of this problem is given by

$$u_p(x)=\frac{1}{NC_N^{1/N}}\int_{C_N|x|^N}^{C_NR^N}G(t)^{\frac{p'}{p}}\,t^{\frac{1}{N}-1}\,dt\,, \tag{4.25}$$

where G is a non–negative function satisfying

$$\int_0^{C_NR^N}G(t)^{p'}dt=\int_{B_R}|g(x)|^{p'}dx$$

(see [2]). Thus, applying Hölder's inequality and performing easy computations, it follows that

$$\begin{aligned}u_p(x)&\le\frac{1}{NC_N^{1/N}}\Big(\int_0^{C_NR^N}G(t)^{p'}dt\Big)^{1/p}\Big(\int_{C_N|x|^N}^{C_NR^N}t^{\frac{p'}{N}-p'}dt\Big)^{1/p'}\\ &\le\frac{1}{NC_N^{1/N}}C_N^{1/p}R^{N/p}\|g\|_\infty^{\frac{1}{p-1}}\left(\frac{N(p-1)}{N-p}\Big(\big(C_N|x|^N\big)^{-\frac{N-p}{N(p-1)}}-\big(C_NR^N\big)^{-\frac{N-p}{N(p-1)}}\Big)\right)^{1/p'}\\ &=\frac{R^{N/p}}{N}\|g\|_\infty^{\frac{1}{p-1}}\frac{1}{|x|^{\frac{N-p}{p}}}\Big(\frac{N(p-1)}{N-p}\Big)^{(p-1)/p}\Big(1-\Big(\frac{|x|}{R}\Big)^{\frac{N-p}{p-1}}\Big)^{(p-1)/p}\\ &\le\frac{R^{N/p}}{N}\|g\|_\infty^{\frac{1}{p-1}}\frac{1}{|x|^{\frac{N-p}{p}}}\Big(\frac{N(p-1)}{N-p}\Big)^{(p-1)/p}.\end{aligned}$$

Therefore, if $\|g\|_\infty<1$, then $\|g\|_\infty^{1/(p-1)}$ goes to 0 and so

$$\lim_{p\to1}u_p(x)=0\,.$$

On the other hand, if $\|g\|_\infty=1$, then

$$0\le\lim_{p\to1}u_p(x)\le\frac{R^N}{N}\frac{1}{|x|^{N-1}}\,.$$

(Observe that this estimate is worse than (3.5).) □

4.2. Data in the predual space of $BV(\Omega)$

In the previous Subsection, we have stated a stability result when $\|\mu\|_{W^{-1,\infty}(\Omega)} \neq 1$. The case $\|\mu\|_{W^{-1,\infty}(\Omega)} = 1$ is the most interesting since then $\lim_{p\to\infty} u_p$ defines non–trivial solutions to the "limit problem" (4.18). To check that $\lim_{p\to\infty} u_p$ is indeed a solution to (4.18), apart from passing to the limit, some extension of Anzellotti's theory is required. We refer to the definition of solution to (4.18) given in Definition 4.1.

We are able to extend the Anzellotti theory in some distinguished subspaces of $W^{-1,\infty}(\Omega)$. The case of the space of all Guy David measures is cumbersome and will be provided in a forthcoming paper. The case of the predual space $\Gamma(\Omega)$ of $BV(\Omega)$ is shown in the Appendix. We point out (see Theorem 5.1 in the Appendix) that if $z \in L^\infty(\Omega;\mathbb{R}^N)$ is a vector field such that its divergence in the sense of distributions belongs to $\Gamma(\Omega)$, then the distribution defined in (4.19) is always a Radon measure.

In this Subsection we completely analyze the case where the datum μ belongs to $\Gamma(\Omega)$. The main Theorem of this Subsection is the following

Theorem 4.3. *Let $\mu \in \Gamma(\Omega)$ and let u_p be the solution to problem* (4.1)*. Then the following statement are equivalent:*

1) Up to subsequences, u_p converges a.e., as p goes to 1*, to a measurable function u which is a solution to problem* (4.18) *in the sense of Definition 4.1.*

2) $||\mu||_{W^{-1,\infty}(\Omega)} \le 1$.

Remark 4.5. Since it is well–known that uniqueness does not hold to problem (4.18) (we refer, for instance, to [8], p. 485), we cannot deduce that, when $\|\mu\|_{W^{-1,\infty}(\Omega)} = 1$, there exists $\lim_{p\to 1} u_p$, but just that a "subsequence" converges, as stated. □

Proof of Theorem 4.3: We start by assuming that $\|\mu\|_{W^{-1,\infty}(\Omega)} \le 1$. By Proposition 4.1, there exists a vector field $z \in L^\infty(\Omega;\mathbb{R}^N)$ such that $\|z\|_\infty \le 1$ and

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \text{weakly in } L^q(\Omega) \text{ for all } 1 \le q < +\infty\,. \tag{4.26}$$

Arguing as in the proof of Theorem 4.1, we obtain inequality (4.5) and then a function $u \in BV(\Omega)$ such that, up to subsequences,

$$\begin{cases} \nabla u_p \rightharpoonup Du & \text{weakly* in the sense of measures,} \\ u_p \to u & \text{strongly in } L^q(\Omega)\,, \quad 1 \le q < \frac{N}{N-1}\,, \\ u_p \to u & \text{a.e. in } \Omega\,. \end{cases}$$

As a consequence, $u_p \rightharpoonup u$ weakly* in $BV(\Omega)$ and therefore

$$< \mu, u_p >_{\Gamma(\Omega),BV(\Omega)} \to < \mu, u >_{\Gamma(\Omega),BV(\Omega)}\,. \tag{4.27}$$

Since u_p is a weak solution to problem (4.1) then, choosing as test function $u_p\varphi$ with $\varphi \in C_0^\infty(\Omega)$ and $\varphi \ge 0$, we get

$$\int_\Omega |\nabla u_p|^p \varphi\, dx = < \mu, \varphi u_p >_{\Gamma(\Omega),BV(\Omega)} - \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi\, u_p\, dx.$$

Passing to the limit in the right hand side, since $\mu = -\mathrm{div}\, z$, we get

$$\lim_{p\to 1} \int_\Omega |\nabla u_p|^p \varphi\, dx =< (z, Du), \varphi > \ . \tag{4.28}$$

On the other hand, by Young's inequality

$$\int_\Omega |\nabla u_p| \varphi\, dx \le \frac{1}{p} \int_\Omega |\nabla u_p|^p \varphi\, dx + \frac{p-1}{p} \int_\Omega \varphi\, dx$$

and as a consequence,

$$\int_\Omega |Du|\varphi \le \liminf_{p\to 1} \int_\Omega |\nabla u_p|\varphi\, dx \le \liminf_{p\to 1} \int_\Omega |\nabla u_p|^p \varphi\, dx$$

Therefore, by (4.28), it yields

$$\int_\Omega |Du|\varphi \le \int_\Omega (z, Du)\varphi$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$. Hence, $|Du| \le (z, Du)$ and $\|z\|_\infty \le 1$ implies $|Du| = (z, Du)$, thus (4.24) is done.

Taking now u_p as test function in the weak formulation of problem (4.1), it follows that

$$\int_\Omega |\nabla u_p|^p\, dx =< \mu, u_p >_{W^{-1,\infty}(\Omega), W_0^{1,p}(\Omega)} \ .$$

Applying Young's inequality we get

$$\begin{aligned} \int_\Omega |\nabla u_p|\, dx &\le \frac{1}{p} \int_\Omega |\nabla u_p|^p\, dx + \frac{p-1}{p}|\Omega| \\ &= \frac{1}{p} < \mu, u_p >_{\Gamma(\Omega), BV(\Omega)} + \frac{p-1}{p}|\Omega|\, . \end{aligned} \tag{4.29}$$

Now we let p goes to 1 in (4.29), by (4.27), it yields

$$|Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \le < \mu, u >_{\Gamma(\Omega), BV(\Omega)}\, ,$$

or equivalently,

$$- < \mu, u >_{\Gamma(\Omega), BV(\Omega)} + \int_\Omega (z, Du) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \le 0\, .$$

By Green's formula (5.4) in the Appendix below, we get

$$\int_{\partial\Omega} [z, \nu]\, u\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \le 0\, .$$

Since $\|[z,\nu]\|_\infty \le \|z\|_\infty \le 1$, it follows that

$$[z, \nu]\, u + |u| = 0 \qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega\, ,$$

which proves (4.22).

Now let us assume that $\|\mu\|_{W^{-1,\infty}(\Omega)} > 1$. By Theorem 4.1 we deduce that there is not solution to problem (4.18). □

Remark 4.6. We point out that the solution whose existence is proved in Theorem 4.3 satisfies a variational formulation, namely:

$$
\begin{aligned}
&|Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + < \mu, w >_{BV(\Omega)^*,BV(\Omega)} \leq \\
&\leq \int_{\partial\Omega} |w|\, d\mathcal{H}^{N-1} + \int_\Omega (z, Dw) + < \mu, u >_{BV(\Omega)^*,BV(\Omega)} ,
\end{aligned}
\tag{4.30}
$$

for every $w \in BV(\Omega)$.

Next, we will only sketch the proof of this fact, which is "inspired" in [7], pp. 133–134 and pp 139–140.

Let $w \in W_0^{1,2}(\Omega)$ and take $w - u_p$ as test function in the weak formulation of problem (4.1) for $1 < p \leq 2$, it follows that

$$\int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w - \int_\Omega |\nabla u_p|^p = < \mu, w - u_p >_{BV(\Omega)^*,BV(\Omega)} .$$

In order to take limit when $p \to 1$, apply Young's inequality and have in mind (4.26) and (4.27), to get

$$|Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + < \mu, w >_{BV(\Omega)^*,BV(\Omega)} \leq \int_\Omega z \cdot \nabla w + < \mu, u >_{BV(\Omega)^*,BV(\Omega)} ,$$

for all $w \in W_0^{1,2}(\Omega)$. By approximating, this inequality holds for all $w \in W_0^{1,1}(\Omega)$. In the following step assume that $w \in W^{1,1}(\Omega)$, take into account $(w_n)_n$ (the sequence of Lemma 5.1), pass to the limit as $n \to \infty$ and obtain

$$
\begin{aligned}
&|Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + < \mu, w >_{BV(\Omega)^*,BV(\Omega)} \leq \\
&\leq \int_{\partial\Omega} |w|\, d\mathcal{H}^{N-1} + \int_\Omega z \cdot \nabla w \, dx + < \mu, u >_{BV(\Omega)^*,BV(\Omega)} .
\end{aligned}
$$

Finally, given $w \in BV(\Omega)$, by Proposition 5.1, consider $w_n \in W^{1,1}(\Omega)$ satisfying $w_n|_{\partial\Omega} = w|_{\partial\Omega}$ for all $n \in \mathbb{N}$,

$$w_n \to w \qquad \text{in } L^1(\Omega) ,$$

$$\int_\Omega |\nabla w_n|\, dx \to |Dw|(\Omega) .$$

Next apply the above inequality to each w_n and let n goes to ∞ taking into account $< \mu, w_n >_{BV(\Omega)^*,BV(\Omega)} \to < \mu, w >_{BV(\Omega)^*,BV(\Omega)}$ and $\int_\Omega z \cdot \nabla w_n \, dx \to \int_\Omega (z, Dw)$. Then we arrive to the desired inequality (4.30). □

Remark 4.7. The same scheme followed in Theorem 4.3 can be adapted when the datum lives in $L^N(\Omega)$. Indeed, if $\mu = f \in L^N(\Omega)$ and $\|f\|_{W^{-1,\infty}(\Omega)} \le 1$, then (4.5) holds true and so we may find $u \in BV(\Omega)$ such that, up to subsequences,

$$\begin{cases} \nabla u_p \rightharpoonup Du & \text{weakly* in the sense of measures,} \\ \\ u_p & \text{is bounded in } L^{\frac{N}{N-1}}(\Omega)\,, \\ \\ u_p \to u & \text{a.e in } \Omega\,. \end{cases}$$

These two last properties imply that $u_p \rightharpoonup u$ weakly in $L^{\frac{N}{N-1}}(\Omega)$, so that (4.27) becomes

$$\int_\Omega f u_p \, dx \to \int_\Omega f u \, dx\,.$$

Once we have obtained this convergence, we may follow the same proof of the above theorem and get that u_p converges to a solution to problem (4.18) in the sense of Definition 4.1 if and only if $\|f\|_{W^{-1,\infty}(\Omega)} \le 1$.

We point out that the above reasoning cannot be extended to all data belonging to the Marcinkiewicz space $L^{N,\infty}(\Omega)$ since, in general, u_p bounded in $L^{\frac{N}{N-1},1}(\Omega)$ and $u_p \to u$ a.e. in Ω does not imply $u_p \rightharpoonup u$ weakly in $L^{\frac{N}{N-1},1}(\Omega)$. Nevertheless, we can handle every datum that belongs to $L^{N,\infty}(\Omega)$ by using truncations: see [19]. $\square$

Example 4.3. In this example we give an element of the predual $\Gamma(\Omega)$ which does not belong to $L^{N,\infty}(\Omega)$.

Let $h \in C_0(]-1,1[^{N-1})$ satisfy $0 \le h \le 1$ and $h|_{[-1/2,1/2]^{N-1}} \equiv 1$. We consider $\Omega =]-1,1[^N$ and $g : \Omega \to \mathbb{R}^N$ given by

$$g(x_1, x_2, \dots, x_N) = \Big(\frac{\lambda}{1-\alpha}(1-|x_1|)^{1-\alpha} h(x_2, \dots, x_N), 0, 0, \dots, 0\Big)\,,$$

with $\lambda \in \mathbb{R}$ and $\frac{1}{N} < \alpha < 1$, and we set $F \equiv \operatorname{div} g$.

Since $g \in C_0(\Omega; \mathbb{R}^N)$, it follows that $F \in \Gamma(\Omega)$. On the other hand,

$$F(x_1, x_2, \dots, x_N) = \frac{-\lambda \text{ sign } x_1}{(1-|x_1|)^\alpha} h(x_2, \dots, x_N)\,.$$

We now prove that $F \notin L^{N,\infty}(\Omega)$. Indeed, given $t > 0$, one has

$$\begin{aligned} &\{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N \,:\, \frac{\lambda}{(1-|x_1|)^\alpha} h(x_2, \dots, x_N) > t\} \\ &\quad \supset \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N \,:\, \frac{\lambda}{(1-|x_1|)^\alpha} > t \text{ and } h(x_2, \dots, x_N) = 1\} \\ &\qquad \supset \{x_1 \in \mathbb{R} \,:\, \frac{\lambda}{(1-|x_1|)^\alpha} > t\} \times \Big[\frac{-1}{2}, \frac{1}{2}\Big]^{N-1}. \end{aligned}$$

Now, it follows from

$$|\{x_1 \in \mathbb{R} \,:\, \frac{\lambda}{(1-|x_1|)^\alpha} > t\}| = 4\Big(\frac{\lambda}{t}\Big)^{1/\alpha}$$

that

$$|\{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N \,:\, |Fx_1, x_2, \dots, x_N)| > t\}| \geq \frac{C_\lambda}{t^{1/\alpha}}\,,$$

with $1 < \frac{1}{\alpha} < N$. By (2.1), we deduce that $F \notin L^{N,\infty}(\Omega)$. □

5. Appendix

This Appendix contains some simple facts on $\Gamma(\Omega)$ which we prove for the sake of completeness.

5.1. Norm of $\Gamma(\Omega)$

We need the following result, which is stated in [4] (see also [7]).

Lemma 5.1. *Given $u \in BV(\Omega)$, and so $u|_{\partial\Omega} \in L^1(\partial\Omega)$, there exists a sequence $(w_n)_n$ in $W^{1,1}(\Omega) \cap C(\Omega)$ satisfying*

(1) $w_n|_{\partial\Omega} = u|_{\partial\Omega}\,.$

(2) $\displaystyle\int_\Omega |\nabla w_n|\,dx \leq \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} + \frac{1}{n}\,.$

(3) $\displaystyle\int_\Omega |w_n|\,dx \leq \frac{1}{n}\,.$

(4) $w_n(x) = 0, \qquad$ *if $dist(x, \partial\Omega) > \frac{1}{n}$.*

A straightforward consequence of conditions (2) and (3) is

$$w_n \rightharpoonup 0 \qquad \textit{weakly* in} \quad BV(\Omega)\,. \tag{5.1}$$

Proposition 5.1. *If $\mu \in \Gamma(\Omega)$, then*

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \|\mu\|_{BV(\Omega)^*}\,. \tag{5.2}$$

Proof: Since $\|\mu\|_{W^{-1,\infty}(\Omega)} \leq \|\mu\|_{BV(\Omega)^*}$, we only have to see the inequality $\|\mu\|_{BV(\Omega)^*} \leq \|\mu\|_{W^{-1,\infty}(\Omega)}$. To this end, let $u \in BV(\Omega)$ be such that

$$|Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} \leq 1$$

and fix $\epsilon > 0$. Given $u \in BV(\Omega)$, consider the sequence $(w_n)_n$ of Lemma 5.1; observe that (5.1) implies $< \mu, w_n > \to 0$. Let $n \in \mathbb{N}$ satisfy

$$|< \mu, w_n >_{\Gamma(\Omega), BV(\Omega)}| + \frac{1}{n} < \epsilon\,.$$

Since n is already fixed and $(u - w_n)|_{\partial\Omega} = 0$, there exists a sequence $(\varphi_k)_k$ in $C_0^\infty(\Omega)$ such that $\varphi_k \rightharpoonup u - w_n$ weakly* in $BV(\Omega)$ and $\int_\Omega |\nabla\varphi_k| \leq \int_\Omega |D(u - w_n)|$. Hence,

$$< \mu, \varphi_k >_{\Gamma(\Omega), BV(\Omega)} \to < \mu, u - w_n >_{\Gamma(\Omega), BV(\Omega)} \qquad \text{as } k \to \infty$$

and

$$\int_\Omega |\nabla \varphi_k|\,dx \le |Du|(\Omega) + \int_\Omega |\nabla w_n| \le |Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} + \frac{1}{n} \le 1+\epsilon\,.$$

Then

$$\begin{aligned} | < \mu, u >_{\Gamma(\Omega),BV(\Omega)} | \ &\le | < \mu, w_n >_{\Gamma(\Omega),BV(\Omega)} | + | < \mu, u - w_n >_{\Gamma(\Omega),BV(\Omega)} | \\ &\le \epsilon + \lim_{k\to\infty} | < \mu, \varphi_k >_{\Gamma(\Omega),BV(\Omega)} | \\ &\le \epsilon + \lim_{k\to\infty} | < \mu, \varphi_k >_{W^{-1,\infty}(\Omega),W_0^{1,1}(\Omega)} | \\ &\le \epsilon + \liminf_{k\to\infty} \|\mu\|_{W^{-1,\infty}(\Omega)} \textstyle\int_\Omega |\nabla\varphi_k|\,dx \\ &\le \epsilon + \|\mu\|_{W^{-1,\infty}(\Omega)}(1+\epsilon)\,. \end{aligned}$$

Since ϵ is arbitrary, the conclusion follows. □

5.2. The Anzellotti Theory in $\Gamma(\Omega)$

In this second part of the Appendix, we adapt Anzellotti's theory to the case $\operatorname{div}(z) \in \Gamma(\Omega)$. Recalling that $[z,\nu]$, the weak trace of the exterior normal component of z is already considered (see [7] pp.126–127), we have to define (z, Du), see that is a Radon measure for all $u \in BV(\Omega)$ and prove a Green's formula. We only show the proofs corresponding to (z, Du) in Theorem 5.1 bellow, the others follow the same schema of [4] adapted in the same way than the proof of Theorem 5.1.

Theorem 5.1. *Let $z \in L^\infty(\Omega,\mathbb{R}^N)$ be a vector field such that its divergence in the sense of distributions $\xi = \operatorname{div}(z)$ belongs to $\Gamma(\Omega)$. Then* (4.19) *defines a Radon measure on Ω such that for every open set $U \subset \Omega$ and for every $\varphi \in C_0^\infty(U)$, we have*

$$| < (z, Du), \varphi > | \le \|\varphi\|_\infty\, \|z\|_{L^\infty(U)}\, |Du|(U). \tag{5.3}$$

Proof: Since $u \in BV(\Omega)$, we may find a sequence $(u_n)_n$ in $C^\infty(\Omega) \cap BV(\Omega)$ such that

$$u_n \to u \qquad \text{in} \quad L^1(\Omega)\,,$$

$$\lim_{n\to\infty} \textstyle\int_V |\nabla u_n| = |Du|(V)\,,$$

for all open set $V \subset\subset \Omega$ satisfying $|Du|(\partial V) = 0$.

Now, given $\varphi \in C_0^\infty(U)$ take an open set V such that $\operatorname{supp}(\varphi) \subset V \subset\subset U$ and $|Du|(\partial V) = 0$. We point out that $(u_n\varphi)_n$ is a sequence in $BV(\Omega)$ that weakly* converges to $u\varphi$. It follows from $\operatorname{div}(z) = \xi \in \Gamma(\Omega)$ that

$$< \xi, u_n\,\varphi >_{\Gamma(\Omega),BV(\Omega)} \to < \xi, u\,\varphi >_{\Gamma(\Omega),BV(\Omega)}\ .$$

Observe that $\int_\Omega z\cdot\nabla u_n\,\varphi\,dx = -\int_\Omega u_n\, z\cdot\nabla\varphi\,dx - < \xi, u_n\,\varphi >_{\Gamma(\Omega),BV(\Omega)}$ for all $n \in \mathbb{N}$, and so the sequence $\left(\int_\Omega z\cdot\nabla u_n\,\varphi\right)_n$ converges to $< (z, Du), \varphi >$. Since

$$\Big|\int_\Omega z\cdot\nabla u_n\,\varphi\Big|\,dx \le \|\varphi\|_\infty \|z\|_{L^\infty(U)} \int_V |\nabla u_n|\,dx\,,$$

letting n goes to $+\infty$, we have

$$|< (z, Du), \varphi >| \leq \|\varphi\|_\infty \|z\|_{L^\infty(U)} |Du|(V) \leq \|\varphi\|_\infty \|z\|_{L^\infty(U)} |Du|(U). \quad \square$$

It is straightforward consequence of the above arguments that

$$\Big| \int_B (z, Du) \Big| \leq \int_B |(z, Du)| \leq \|z\|_{L^\infty(\Omega)} |Du|(B)$$

for all Borel sets $B \subset \Omega$.

Finally, we state the Green formula that may be proved in our case.

Theorem 5.2. *Let $z \in L^\infty(\Omega, \mathbb{R}^N)$ be a vector field such that its divergence in the sense of distributions $\xi = \operatorname{div}(z)$ belongs to $\Gamma(\Omega)$. If $u \in BV(\Omega)$, then*

$$< \operatorname{div}(z), u >_{\Gamma(\Omega), BV(\Omega)} + \int_\Omega (z, Du) = \int_{\partial\Omega} [z, \nu]\, u \, d\mathcal{H}^{N-1}. \tag{5.4}$$

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