Stability for solutions to p-Laplacian equations with L^1 data as p goes to 1

A Mercaldo (*), S. Segura de León(**) and C. Trombetti (*)

(*) Dipartimento di Matematica e Applicazioni "R. Caccioppoli",
Università di Napoli "Federico II",
Complesso Monte S. Angelo, Via Cintia, 80126 Napoli, Italy

(**) Departament d'Anàlisi Matemàtica - Universitat de València
Dr. Moliner 50, 46100 Burjassot, València, Spain

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Abstract

In the present paper we study the behaviour, as p goes to 1, of the renormalized solutions to the problems

$$\begin{cases} -\mathrm{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = f & \text{in } \Omega \\ \\ u_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $p > 1$, Ω is a bounded open set of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary and f belongs to $L^1(\Omega)$. We prove that these renormalized solutions pointwise converge, up to "subsequences", to a function u. With a suitable definition of solution we also prove that u is a solution to a "limit problem". Moreover we analyze the situation occurring when more regular data f are considered.

Key words: Nonlinear elliptic equations, renormalized solutions, 1–Laplace operator, L^1–data.

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1 Introduction

In the present paper we study the behaviour, when p goes to 1, of the renormalized solutions to the problems

$$\begin{cases} -\mathrm{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = f & \text{in } \Omega \\ \\ u_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$, Ω is a bounded open set of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary and f belongs to $L^1(\Omega)$.

The notion of renormalized solution was introduced in order to extend the classical setting of monotone operators (see [32]) and so be able to define a notion of solution to problems whose data do not belong to the dual space $W^{-1,p'}(\Omega)$ (as, for instance, the case of L^1–data). The main interest is not to get a solution to (1.1) in the sense of distributions but to have a concept which allows to obtain existence (see [10] and [11] to this end) and uniqueness. Renormalized solutions were adapted to second order elliptic problems by P.–L. Lions and F. Murat in [33] (see also [36] or [37]); both existence and uniqueness of such a solution are proved if the datum f belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. In [19] and [20] such a notion has been extended to the case where the right–hand side is a Radon measure with bounded total variation; the authors proved an existence result and a partial uniqueness result. We refer to [20] for an exhaustive treatment of renormalized solutions. An equivalent notion, the concept of entropy solution, was introduced in [9] (see also [12]). For such a solution both existence and uniqueness have been proved when f belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. Other approaches to define suitable generalized solutions can be found in [21] and [38] (see also [1] where symmetrization techniques are used).

Our purpose is to study the renormalized solutions u_p with two objectives. First, we will study the behaviour of u_p when p goes to 1, proving that, up to a subsequence (considering that u_p is a sequence),

$$\begin{aligned} &u_p \to u \qquad \text{pointwise in } \Omega\,, \\ &|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \text{in } L^q(\Omega)\,, \quad 1 \leq q < \frac{N}{N-1}\,. \end{aligned}$$

Second we prove that this function u is a solution to the "limit equation" of (1.1), namely:

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = f\,, & \text{in } \Omega\,; \\[2ex] \qquad\qquad u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

To this end, we need to introduce a precise formulation of such a solution. At least, we will achieved a new point of view of the above issues which enable us to a better understanding of what happens when more regular data are taken (see Theorem 4.2 below). A suitable notion of solution to (1.2) was introduced in [4] while dealing with the equation

$$u - \mathrm{div}\left(\frac{Du}{|Du|}\right) = f \in L^1(\Omega)\,, \tag{1.3}$$

and a general Dirichlet boundary condition; equipped with such a notion of solution, the authors are able to prove existence and uniqueness for such a problem. Their notion of solution gives sense to the quotient $\dfrac{Du}{|Du|}$ (recall that, in general, Du is not a function but a Radon measure) through a vector field z satisfying

- $z \in L^\infty(\Omega;\mathbb{R}^N)$ with $\|\,|z|\,\|_\infty \leq 1$.
- $-\mathrm{div}\, z = f$ in $\mathcal{D}'(\Omega)$.

- $(z, Du) = |Du|$.

Observe that, formally, $\| |z| \|_\infty \le 1$ and $(z, Du) = |Du|$ imply $z = \dfrac{Du}{|Du|}$. The meaning of (z, Du) relies on the theory of L^∞–divergence–measure vector fields due to G. Anzellotti [3] and to G.–Q. Chen and H. Frid [14] (their approaches, however, are very different). This theory defines the pairing (z, Du) as a Radon measure, where $z \in \mathcal{DM}^\infty(\Omega)$ (see Section 2 for its Definition) and u is a certain BV–function; it also provides the definition of a weakly trace on $\partial\Omega$ to the normal component of z, denoted by $[z, \nu]$, and guaranties a Green's formula. Following [4] (see also [6]), we will use $[z, \nu]$ to include (in a very weak sense) the boundary condition in the concept of solution to (1.2).

As it was mentioned, our second aim in the present paper is to show that the limit function u, is actually a solution to (1.2). Hence, we will consider problem (1.2) with data belonging to $L^1(\Omega)$; so that, in some sense, we are covering the stage from regular data to L^1–data, in the same order of ideas of [10], [9] or [33]. However, in our situation there is not hope of finding a unique solution, and so we are not looking for every solution of (1.2), just those solutions which are pointwise limits of u_p (see also Remark 4.6). We remark that in previous works, authors have considered problem (1.2) with a datum in $W^{-1,\infty}(\Omega)$, typically in $L^N(\Omega)$ or $L^N_{\rm loc}(\Omega)$. Indeed, although in [4] L^1–data are considered in equation (1.3), the regularity enjoyed because of the lower term, allow the authors to get a solution such that $f - u \in W^{-1,\infty}(\Omega)$.

Let us briefly describe some features involved in the study of the limit equation with L^1–data. First of all we need a definition of solution to equation (1.2), which should be an extension of the definition given in [4] (see also [34]) when the datum is more regular. Of course, as in problem (1.1), we cannot expect that such a solution u belongs to the energy space $BV(\Omega)$: only the truncations of the solution $T_k(u)$ are there. Nevertheless, there are other difficulties arisen in our study that we spell out below.

1. Not only the function limit does not lies in $BV(\Omega)$, but it is typically infinite on a set of positive measure, as was already shown in [34], Theorem 3.1, by means of radial solutions (see also Exemple 4.1 below).

2. Unless the data are in the dual space $W^{-1,\infty}(\Omega)$, the vector field z no longer belongs to $L^\infty(\Omega; \mathbb{R}^N)$, as happens when the equation (1.3) is studied (see Remark 4.4 below). Instead, we have to consider a family of "local vector fields" $z_k = z\chi_{\{|u|<k\}}$ such that $z_k \in L^\infty(\Omega; \mathbb{R}^N)$ with $\| |z_k| \|_\infty \le 1$.

3. One of the main difficulties in our investigation is just to find the equation satisfied by z_k. In the same way that $v = T_k(u_p)$ solves equation

$$-\Delta_p v = f\chi_{\{|u_p|<k\}} + \lambda_p \,,$$

where λ_p is a Radon measure concentrated on $\{|u_p| = k\}$ (see [18] and [19]), we see that z_k satisfies

$$-\operatorname{div} z_k = f\chi_{\{|u|<k\}} + (z, D\chi_{\{|u|>k\}}) \,, \quad \text{in } \mathcal{D}'(\Omega) \,,$$

where $(z, D\chi_{\{|u|>k\}})$ is a Radon measure defined in (4.5) below (see also Step 4 in the proof of Theorem 4.1 and Proposition 6.3, where it is seen that it is concentrated on $\{|u| = k\}$).

4. Under these conditions, we already have the approach by G.–Q. Chen and H. Frid to make sense of the pairing $(z_k, DT_k(u))$. However, we will need to apply the inequality
$$|(z_k, DT_k(u))| \le \| \, |z_k| \, \|_\infty |DT_k(u)| \tag{1.4}$$
(see Proposition 5.4 below) while in [14] it is only shown that the Radon measure $(z_k, DT_k(u))$ is absolutely continuous with respect to $|DT_k(u)|$. On the contrary, in Anzellotti's approach [3] (see also [6]) the above inequality is proved, but only when $T_k(u)$ is a continuous function. Hence, we will need to extend the Anzellotti approach.

5. Actually, we will extend even further this Anzellotti's theory to give meaning to pairings such as $(z, DT_k(u))$ or $(z, D\chi_{\{|u|>k\}})$ (recall that the vector field z is not bounded). We explicitly remark that the theory of divergence–measure fields has been generalized to more general vector fields in [39] and [16] (see also the survey [15] and references therein), but these generalizations cannot be applied to our purposes.

Now we briefly mention some articles that deal with issues similar to those studied here. The asymptotic behaviour have been considered by [28], [17] and [34] (see also [30] and [27]). In turn, several authors have focused their research on finding solutions to the limit problem (1.2), the list includes [4], [5], [7], [8], [22], [23], and references therein. Other related works are [31] and [26]. The interest in this framework comes out, on the one hand, from an optimal design problem in the theory of torsion and related geometrical problems (see [29]) and, on the other, from the variational approach to image restoration (see [6]).

The plan of this paper is as follows. After introducing our notation (see next Section), we begin by studying the asymptotic behaviour of (u_p) in Theorem 3.1. In Section 4 we introduce our concept of solution and prove in Theorem 4.1 that the limit function that was found in Section 3 satisfies its formulation. We also see in Theorem 4.2 some consequences of Theorem 4.1 that illustrate the situation when regular data are considered. Lastly, two appendix are included. In the first one, the Anzellotti approach to the theory of divergence–measure fields is extended to cover the case that the vector field belongs to $\mathcal{DM}^\infty(\Omega)$ and the function lies in $BV(\Omega) \cap L^\infty(\Omega)$. Appendix 2 is devoted to show some properties of $(z, DT_k(u))$ and $(z, D\chi_{\{|u|>k\}})$.

2 Notation

In this Section we will introduce some notation which will be used throughout this paper.

As it was stated in the Introduction, our aim is to study the convergence of u_p as p goes to 1. From now on, abusing terminology, we will say that u_p is a sequence and we will consider subsequences of it.

In the present paper, $|z|$ will denote the Euclidean norm of $z \in \mathbb{R}^N$. We will denote by Ω a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. Thus there exists a unit vector field (denoted by ν) normal to $\partial\Omega$ and exterior to Ω, defined $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$. Here $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure. Here and in the sequel, $|E|$ denotes the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$.

For $1 < q < \infty$, the Lorentz space $L^{q,\infty}(\Omega)$, also known as Marcinkiewicz or weak–Lebesgue, is the space of Lebesgue measurable functions u such that

$$\sup_{t>0} t\, |\{x \in \Omega \,:\, |u(x)| > t\}|^{1/q} < +\infty\,. \tag{2.1}$$

We define $\mathcal{M}(\Omega)$ as the space of all Radon measures with bounded total variation on Ω and we denote by $|\mu|$ the total variation of $\mu \in \mathcal{M}(\Omega)$. The space of all functions of finite variation, that is the space of those $u \in L^1(\Omega)$ whose distributional gradient belongs to $\mathcal{M}(\Omega)$, is denoted by $BV(\Omega)$. It is endowed with the norm defined by $\|u\|_{BV(\Omega)} = \displaystyle\int_\Omega |u| + |Du|(\Omega)$, for any $u \in BV(\Omega)$. Since Ω has Lipschitz boundary, if u belongs to $BV(\Omega)$, then the function

$$u_0 = \begin{cases} u, & \text{in } \Omega; \\ 0, & \text{in } \mathbb{R}^N \setminus \Omega; \end{cases}$$

belongs to $BV(\mathbb{R}^N)$ and $|Du_0|(\mathbb{R}^N) = \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} + |Du|(\Omega)$. We explicitly point out that $|Du_0|(\mathbb{R}^N)$ defines an equivalent norm on $BV(\Omega)$, which we will use in the sequel. Through the paper, with an abuse of notation, we still denote u_0 by u.

We will denote by $S_{N,p}$ the best constant in the Sobolev inequality (cf. [40]), that is,

$$\|u\|_{p^*} \le S_{N,p} \|\, |\nabla u\,| \|_p\,, \quad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

We will also write S_N instead of $S_{N,1}$. It is well–known (cf. [40]), that

$$\lim_{p\to 1} S_{N,p} = S_N\,. \tag{2.2}$$

We will denote by $W^{-1,\infty}(\Omega)$ the dual space of $W_0^{1,1}(\Omega)$, its norm is given by

$$\|\mu\|_{W^{-1,\infty}(\Omega)} = \sup\left\{ \langle \mu, \varphi \rangle_{W^{-1,\infty}(\Omega), W_0^{1,1}(\Omega)} \,:\, \int_\Omega |\nabla\varphi| \le 1 \right\}\,. \tag{2.3}$$

Following [14] we define $\mathcal{DM}^\infty(\Omega)$ as the space of all vector fields $z \in L^\infty(\Omega; \mathbb{R}^N)$ whose divergence in the sense of distribution is a Radon measure, i.e.,

$$z \in \mathcal{DM}^\infty(\Omega) \Leftrightarrow \operatorname{div} z \in \mathcal{M}(\Omega) \cap W^{-1,\infty}(\Omega)\,.$$

Then $\mu = \operatorname{div} z$ satisfy the following condition: there exists a constant $C > 0$ such that

$$|\mu(B)| \le CR^{N-1} \qquad \text{for all (open or closed) balls } B \subset \Omega \text{ with radius } R\,. \tag{2.4}$$

It is well known that if $|\mu|(B)| \leq CR^{N-1}$ for all balls $B \subset \Omega$ with radius R, then μ can be extended from $W_0^{1,1}(\Omega)$ to $BV(\Omega)$, see [42], Theorem 5.12.4. (While the extension of a functional is not necessarily unique, a particular extension to $BV(\Omega)$ will be singularized: namely, that given by the integral, with respect to μ, of the precise representative of each $u \in BV(\Omega)$, this representative is mentioned in Appendix 1 below.) These measures are called David measures in [35].

3 The asymptotic behaviour

Consider the nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = f, & \text{in } \Omega, \\ \\ u_p = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary, p is a real number $p > 1$ and f is a function belonging to $L^1(\Omega)$.
In this Section, we will study the behaviour, as p goes to 1, of renormalized solutions u_p to problem (3.1).

For $k > 0$, denote by $T_k : \mathbb{R} \to \mathbb{R}$ the usual truncation at level k, that is

$$T_k(s) = \begin{cases} s & |s| \leq k, \\ k \ \mathrm{sign}(s) & |s| > k, \end{cases} \qquad \forall s \in \mathbb{R}.$$

We may extend this definition to infinite values: $T_k(\pm\infty) = \pm k$.

Consider a measurable function $u : \Omega \to \overline{\mathbb{R}}$ which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists (see e.g. [9], Lemma 2.1) a unique measurable function $v : \Omega \to \overline{\mathbb{R}}^N$ such that

$$\nabla T_k(u) = v\chi_{\{|u|\leq k\}} \quad \text{almost everywhere in} \quad \Omega, \quad \forall k > 0. \tag{3.2}$$

Remark 3.1 We point out that although truncations can be applied to functions that are infinite on a set of positive measure, its gradient cannot be defined by the above expression.

Definition 3.1 Assume that $1 < p < N$. Let $u_p : \Omega \to \overline{\mathbb{R}}$ be measurable and almost everywhere finite on Ω. We say that u_p is a renormalized solution of (3.1) if it satisfies the following conditions:

$$T_k(u_p) \in W_0^{1,p}(\Omega), \quad \forall k > 0; \tag{3.3}$$

$$|u_p| \in L^{\frac{N(p-1)}{N-p},\infty}(\Omega); \tag{3.4}$$

the gradient ∇u_p introduced in (3.2), satisfies:

$$|\nabla u_p| \in L^{\frac{N(p-1)}{N-1},\infty}(\Omega), \tag{3.5}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le|u_p|<2n\}} |\nabla u_p|^p = 0; \tag{3.6}$$

and finally

$$\int_\Omega |\nabla u_p|^p \, h'(u_p)\phi + \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\phi \, h(u_p) = \int_\Omega f h(u_p)\phi, \tag{3.7}$$

for every $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, for all $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, which are such that $h(u_p)\phi \in W_0^{1,p}(\Omega)$.

Remark 3.2 By standard arguments the following estimate for the truncations of the renormalized solution u_p to problem (3.1) holds true (see, for example, [9], [36] or [20])

$$\int_\Omega |\nabla T_k(u_p)|^p \le k\|f\|_{L^1(\Omega)} \quad \text{for any } k > 0\,. \tag{3.8}$$

Remark 3.3 If u_p is a renormalized solution to problem (3.1), then u_p is also a distributional solution in the sense that it satisfies the equality (see, for instance, [20])

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\phi = \int_\Omega f\phi, \quad \text{for any } \phi \in C_0^\infty(\Omega)\,.$$

The main result of this Section is given by the following Theorem

Theorem 3.1 *For every fixed $p \in]1, N[$, let u_p denote the renormalized solution to problem* (3.1)*. Then, there exist a measurable function u and a vector field z belonging to $L^{\frac{N}{N-1},\infty}(\Omega;\mathbb{R}^N)$ such that, up to a subsequence,*

$$u_p \to u \quad a.e. \ in \quad \Omega, \tag{3.9}$$

and

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad weakly\ in \quad L^q(\Omega;\mathbb{R}^N), \quad for\ every\ 1 \le q < \frac{N}{N-1}\,. \tag{3.10}$$

Proof: *Step 1: A priori estimates*

The first step consists in proving a priori estimates: the sequence $(|\nabla u_p|^{p-1})_p$ is bounded in the Marcinkiewicz space $L^{\frac{N}{N-1},\infty}(\Omega)$. Such a proof is well-known and contained in [9] (see also [20]). Here we need to include it in order to make explicit the dependence on p.

We begin by estimate the sequence $(|u_p|^{p-1})_{p>1}$. For every fixed $k > 0$, denote $h = k^{1/(p-1)}$. Then, Sobolev's embedding Theorem and (3.8) imply

$$|\{|u_p|^{p-1} \ge k\}| = |\{|u_p| \ge k^{1/(p-1)}\}| \le \int_\Omega \frac{|T_h(u_p)|^{p^*}}{k^{p^*/(p-1)}} \le$$

$$\le \frac{S_{N,p}^{p^*}}{k^{p^*/(p-1)}} \|\, |\nabla T_h(u_p)|\, \|_p^{p^*} \le \frac{S_{N,p}^{p^*}}{k^{p^*/(p-1)}} h^{\frac{p^*}{p}} \|f\|_{L^1(\Omega)}^{\frac{p^*}{p}} \le S_{N,p}^{p^*} \left(\frac{\|f\|_{L^1(\Omega)}}{k}\right)^{\frac{N}{N-p}},$$

where $S_{N,p}$ denotes the best constant in the Sobolev embedding Theorem and $p^* = \frac{Np}{N-p}$. Therefore

$$|\{|u_p|^{p-1} \geq k\}| \leq S_{N,p}^{p^*} \left(\frac{\|f\|_{L^1(\Omega)}}{k} \right)^{\frac{N}{N-p}} . \tag{3.11}$$

Now we go on in proving the boundedness of the sequence $(|\nabla u_p|^{p-1})_{p>1}$ in the Marcinkiewicz space $L^{\frac{N}{N-1},\infty}(\Omega)$. Indeed, since for every fixed $k > 0$ and $\eta > 0$, we have

$$\{|\nabla u_p|^{p-1} \geq \eta\} \subset \{|u_p| \geq k\} \cup \{|\nabla T_k(u_p)|^{p-1} \geq \eta\}.$$

Using (3.11) and (3.8), it yields

$$\begin{aligned} |\{|\nabla u_p|^{p-1} \geq \eta\}| &\leq |\{|u_p| \geq k\}| + |\{|\nabla T_k(u_p)| \geq \eta^{1/(p-1)}\}| \\ &\leq S_{N,p}^{p^*} \left(\frac{\|f\|_{L^1(\Omega)}}{k^{p-1}} \right)^{\frac{N}{N-p}} + \int_\Omega \frac{|\nabla T_k(u_p)|^p}{\eta^{p/(p-1)}} \\ &\leq S_{N,p}^{p^*} \|f\|_{L^1(\Omega)}^{\frac{N}{N-p}} \frac{1}{k^{\frac{N(p-1)}{N-p}}} + \frac{k\|f\|_{L^1(\Omega)}}{\eta^{\frac{p}{p-1}}} . \end{aligned}$$

Now choosing

$$k = S_{N,p}^{\frac{N}{N-1}} \|f\|_{L^1(\Omega)}^{\frac{1}{N-1}} \eta^{\frac{N-p}{(N-1)(p-1)}}$$

in the previous inequality, we obtain

$$|\{|\nabla u_p|^{p-1} \geq \eta\}| \leq 2 \left(\frac{S_{N,p}\|f\|_{L^1(\Omega)}}{\eta} \right)^{\frac{N}{N-1}} , \tag{3.12}$$

for any $\eta > 0$. From (3.12), since $S_N = \lim_{p\to 1} S_{N,p}$, it follows that

$$|\{|\nabla u_p|^{p-1} \geq \eta\}| \leq 2 \left(\frac{(S_N + 1)\|f\|_{L^1(\Omega)}}{\eta} \right)^{\frac{N}{N-1}} , \tag{3.13}$$

for p close to 1. For each $1 \leq q < \frac{N}{N-1}$, by estimate (3.13), we deduce that, up to subsequences, there exists a vector field z_q belonging to $L^q(\Omega;\mathbb{R}^N)$ such that

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z_q \text{ weakly in } L^q(\Omega;\mathbb{R}^N) .$$

Finally, by a diagonal argument we may find a limit that does not depend on q; hence (3.10) is proved. Observe also that (3.13) and (3.10) imply $z \in L^{\frac{N}{N-1},\infty}(\Omega;\mathbb{R}^N)$.

Step 2: Pointwise convergence of $(u_p)_p$

We will prove that, up to a subsequence,

$$u_p \to u \quad \text{a.e. in } \Omega, \tag{3.14}$$

where u is a measurable function in Ω.

Following [38], first consider $\Psi(s) = s/(1+|s|)$, which is a strictly increasing and bounded real function. Moreover

$$\Big| \int_0^{u_p} (\Psi'(s))^p \, ds \Big| \leq \int_0^{|u_p|} \Psi'(s) \, ds = \Psi(|u_p|) \leq 1.$$

So that if, for each $k > 0$, we take

$$\phi(x) = \int_0^{T_k(u_p(x))} (\Psi'(s))^p \, ds,$$

and

$$h_n(s) = \begin{cases} 1, & \text{if } |s| \leq n\,; \\ -\frac{1}{n}|s| + 2, & \text{if } n \leq |s| \leq 2n\,; \\ 0, & \text{if } |s| \geq 2n\,; \end{cases}$$

in (3.7), then

$$\frac{-1}{n} \int_{\{n \leq |u_p| \leq 2n\}} |\nabla u_p|^p \phi \ \text{sign}\ (u_p) + \int_\Omega \Psi'(T_k(u_p))^p |\nabla T_k(u_p)|^p h_n(u_p)$$
$$= \int_\Omega f h_n(u_p) \phi \leq \int_\Omega |f|\,.$$

By letting n go to infinity and applying (3.6), we get

$$\int_\Omega |\nabla \Psi(T_k(u_p))|^p = \int_\Omega \Psi'(T_k(u_p))^p |\nabla T_k(u_p)|^p \leq \int_\Omega |f|\,.$$

By Fatou's Lemma, when k goes to infinity we obtain

$$\int_\Omega |\nabla \Psi(u_p)|^p \leq \int_\Omega |f|\,.$$

Thus, Hölder's inequality implies that the sequence $\big(\Psi(u_p)\big)_p$ is bounded in $W_0^{1,1}(\Omega)$ and so a subsequence, also denoted by $\big(\Psi(u_p)\big)_p$, converges *-weakly in $BV(\Omega)$. As a consequence, it also converges strongly in $L^1(\Omega)$ and a.e. Since Ψ is strictly increasing, the sequence $(u_p)_p$ tends a.e. to a measurable function u. We point out that, when $\lim_{p\to 1} \Psi(u_p) = \pm 1$, we have $u = \pm\infty$

Remark 3.4 We remark that when the datum f is more regular, we may find better regularity on z. Indeed it is well-known that, if $f \in L^m(\Omega)$, with $1 < m < N$, then the sequence $(|\nabla u_p|^{p-2}\nabla u_p)_p$ is bounded in $L^{m^*}(\Omega;\mathbb{R}^N)$ and so $z \in L^{m^*}(\Omega;\mathbb{R}^N)$. Observe also the regularity enjoyed by z in Example 4.1 below.

4 The limit problem

In this Section we will show that the limit function u whose existence has been proved in the previous section is a solution (in the sense of Definition 3.1 below) to a boundary value problem associated to the "limit equation" of equation in (3.1), which can formally be written

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|\,Du\,|}\right) = f, & \text{in } \Omega\,; \\ \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $f \in L^1(\Omega)$. We begin by introducing the notion of solution to such a problem, which needs some preliminaries.

Let $u : \Omega \to \overline{\mathbb{R}}$ be a measurable function on Ω, such that $T_k(u) \in BV(\Omega)$ for any $k > 0$. Let $z \in L^{\frac{N}{N-1},\infty}(\Omega;\mathbb{R}^N)$ be a vector field satisfying

$$\text{- div}\,(z) = f \qquad \text{in } \mathcal{D}'(\Omega)\,,$$

and

$$z\chi_{\{|u|<k\}} \in \mathcal{DM}^\infty(\Omega)\,, \qquad \text{for all } k > 0\,;$$

i.e., denoting $z_k = z\chi_{\{|u|<k\}}$ and $\mu_k = \mathrm{div}\,(z_k)$ it holds

$$z_k \;\in L^\infty(\Omega;\mathbb{R}^N) \quad \text{and} \quad \mu_k \quad \text{is a Radon measure.}$$

With the above notation, we define the distributions

$$(z_k, DT_k(u)) : C_0^\infty(\Omega) \to \mathbb{R} \tag{4.2}$$

$$(z, D\chi_{\{|u|>k\}}) : C_0^\infty(\Omega) \to \mathbb{R} \tag{4.3}$$

by

$$\langle (z_k, DT_k(u)), \phi\rangle = -\int_\Omega T_k(u)\,\phi\,d\mu_k \;-\int_\Omega T_k(u) z_k\cdot\nabla\phi, \tag{4.4}$$

$$\langle (z, D\chi_{\{|u|>k\}}), \phi\rangle = \int_{\{|u|>k\}} f\,\phi - \int_{\{|u|>k\}} z\cdot\nabla\phi, \tag{4.5}$$

for any $\phi \in C_0^\infty(\Omega)$. Since $T_k(u) \in L^\infty(\Omega)\cap BV(\Omega) \subset L^1(\Omega,\mu_k)$ (cf. Appendix 1 below, we point out that a singular extension of μ_k to bounded BV–functions has been chosen), $z_k \in L^\infty(\Omega;\mathbb{R}^N)$, $f \in L^1(\Omega)$, $z \in L^{\frac{N}{N-1},\infty}(\Omega;\mathbb{R}^N)$ and $\phi \in C_0^\infty(\Omega)$ all terms in (4.4) and (4.5) make sense.

Definition 4.1 We say that a measurable function $u : \Omega \to \overline{\mathbb{R}}$ is a solution to problem (4.1) if the following conditions hold

$$T_k(u) \in BV(\Omega), \quad \text{for all } k > 0; \tag{4.6}$$

there exists a vector field $z \in L^{\frac{N}{N-1},\infty}(\Omega;\mathbb{R}^N)$ such that

$$-\mathrm{div}\, z = f \text{ in } \mathcal{D}'(\Omega); \tag{4.7}$$

for each $k > 0$, the distribution $(z, D\chi_{\{|u|>k\}})$ is a Radon measure and the vector field $z_k = z\chi_{\{|u|<k\}}$ satisfies

$$\|\,|z_k|\,\|_\infty \le 1, \tag{4.8}$$

$$-\operatorname{div} z_k = f\chi_{\{|u|<k\}} + (z, D\chi_{\{|u|>k\}}) \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{4.9}$$

$$(z_k, DT_k(u)) \quad \text{is a Radon measure, and}$$

$$(z_k, DT_k(u)) = |DT_k(u)| \quad \text{as measures in } \Omega\,; \tag{4.10}$$

$$\int_{\partial\Omega} |T_k(u)|\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} [z\chi_{\{|u|<\infty\}}, \nu] T_k(u)\, d\mathcal{H}^{N-1} + k\int_{\partial\Omega} [z\chi_{\{u=+\infty\}}, \nu]\, d\mathcal{H}^{N-1} - k\int_{\partial\Omega} [z\chi_{\{u=-\infty\}}, \nu]\, d\mathcal{H}^{N-1} \le 0\,. \tag{4.11}$$

As a consequence of (4.11), in the case where u is finite on $\partial\Omega$, the following condition holds true:

$$[z, \nu] = [z\chi_{\{|u|<\infty\}}, \nu] \in \operatorname{sign}(-u) \quad \text{on} \quad \partial\Omega\,. \tag{4.12}$$

Remark 4.1 Observe that, as a consequence of (4.8), the vector field $z\chi_{\{|u|<\infty\}}$ satisfies $\|\,|z\chi_{\{|u|<\infty\}}|\,\|_\infty \le 1$, so that the weak trace on $\partial\Omega$ of its normal component is well defined by the results in [3] (or [14]) and $|[z\chi_{\{|u|<\infty\}}, \nu]| \le 1$, $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$. A definition of $[z, \nu]$, $[z\chi_{\{u=+\infty\}}, \nu]$ and $[z\chi_{\{u=-\infty\}}, \nu]$ can be found in Appendix 3 below.

We remark that we have defined

$$\int_{\partial\Omega} [z, \nu]\, v\, d\mathcal{H}^{N-1}, \quad \int_{\partial\Omega} [z\chi_{\{u=+\infty\}}, \nu]\, v\, d\mathcal{H}^{N-1} \quad \text{and} \quad \int_{\partial\Omega} [z\chi_{\{u=-\infty\}}, \nu]\, v\, d\mathcal{H}^{N-1}$$

for $v \in W^{1-\frac{1}{q},q}(\partial\Omega) \cap L^\infty(\partial\Omega)$, with $q > N$. If we extended those expressions to every $v \in L^\infty(\partial\Omega)$, then (4.11) would be written as

$$\int_{\partial\Omega} |T_k(u)|\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} [z, \nu] T_k(u)\, d\mathcal{H}^{N-1} \le 0\,.$$

Remark 4.2 Let us observe that Definition 4.1 coincides, when $u \in BV(\Omega)$, with the definition given in [4] (see also [34] Definition 4.1) for regular enough data (see also Theorem 4.2 below).

Remark 4.3 Roughly speaking, it follows from (4.8) and (4.10) that z_k coincides with the vector field $\frac{Du}{|Du|}$ on the set $\{|u| < k\}$ for all $k > 0$, and so the vector field z plays the role of $\frac{Du}{|Du|}$ on the set $\{|u| < +\infty\}$.

Remark 4.4 Let us observe that we cannot expect that in general the vector field z belongs to $L^\infty(\Omega, \mathbb{R}^N)$ as the following simple argument shows. Consider $f \in L^1(\Omega)$ and assume that u is a solution to (4.1) with $z \in L^\infty(\Omega, \mathbb{R}^N)$. Then we have

$$\left|\int_\Omega f\phi\right| = \left|\int_\Omega z\cdot\nabla\phi\right| \le \|\,|z|\,\|_\infty \int_\Omega |\nabla\phi| \qquad \text{for all } \phi \in C_0^\infty(\Omega)\,.$$

This distribution generated by f can uniquely be extended to $\phi \in W_0^{1,1}(\Omega)$; thus $f \in W^{-1,\infty}(\Omega)$, which is impossible for a general $f \in L^1(\Omega)$ when $N \ge 2$ (see also Theorem 4.2 and Example 4.1 below).

The main result of this section is

Theorem 4.1 *The limit function u given by Theorem 3.1 is a solution to problem* (4.1) *in the sense of Definition 4.1.*

Proof: We proceed dividing the proof in several steps.
Step 1: $T_k(u) \in BV(\Omega)$ for all $k > 0$.

It follows from (3.14) that $T_k(u_p) \to T_k(u)$ a.e for all $k > 0$. On the other hand, from Hölder's inequality and (3.8), we deduce

$$\int_\Omega |\nabla T_k(u_p)| \le |\Omega|^{1-\frac{1}{p}} \left(\int_\Omega |\nabla T_k(u_p)|^p \right)^{\frac{1}{p}} \le |\Omega|^{1-\frac{1}{p}} \|f\|_{L^1(\Omega)}^{\frac{1}{p}} k^{\frac{1}{p}}. \tag{4.13}$$

Therefore, once k is chosen, $\big(T_k(u_p)\big)_p$ is bounded in $W_0^{1,1}(\Omega)$, and consequently

$$T_k(u_p) \rightharpoonup T_k(u) \quad \text{*-weakly in} \quad BV(\Omega); \tag{4.14}$$

so that $T_k(u) \in BV(\Omega)$ for all $k > 0$ and (4.6) holds true.

Step 2: $-\operatorname{div} z = f$ in the sense of distributions.

Since u_p is a solution in the sense of distributions to problem (3.1) (see Remark 3.3), we have

$$\int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \phi = \int_\Omega \phi f, \qquad \forall \phi \in C_0^\infty(\Omega).$$

By Theorem 3.1, letting p goes to 1, we get

$$\int_\Omega z \cdot \nabla \phi = \int_\Omega \phi f, \qquad \forall \phi \in C_0^\infty(\Omega),$$

or, equivalently, $-\operatorname{div} z = f$ in the sense of distributions.

Step 3: the vector field $z_k = z\chi_{\{|u|<k\}}$ belongs to $L^\infty(\Omega; \mathbb{R}^N)$ and $\| \, |z_k| \, \|_\infty \le 1$

Here we repeat the same arguments used in [4] (see also [34]). For any fixed $k > 0$, the sequence $(|\nabla u_p|^{p-1} \, \chi_{\{|u_p|<k\}})_p$ is bounded in $L^{\frac{N}{N-1},\infty}(\Omega, \mathbb{R}^N)$, by (3.13). Thus, as p goes to 1, we have

$$|\nabla u_p|^{p-2} \nabla u_p \; \chi_{\{|u_p|<k\}} \rightharpoonup w_k \quad \text{weakly in} \quad L^1(\Omega; \mathbb{R}^N), \tag{4.15}$$

for some vector field $w_k \in L^1(\Omega; \mathbb{R}^N)$. For every fixed $k > 0$, $h > 0$ and $p > 1$, we denote

$$B_{p,h,k} = \{x \in \Omega : |\nabla T_k(u_p)| > h\}. \tag{4.16}$$

By (3.13), as p goes to 1, we have (up to subsequences)

$$|\nabla u_p|^{p-2} \nabla u_p \; \chi_{B_{p,h,k} \cap \{|u_p|<k\}} \rightharpoonup g_{h,k} \quad \text{weakly in} \quad L^1(\Omega, \mathbb{R}^N), \tag{4.17}$$

$$|\nabla u_p|^{p-2} \nabla u_p \; \chi_{(\Omega \setminus B_{p,h,k}) \cap \{|u_p|<k\}} \rightharpoonup f_{h,k} \quad \text{weakly in} \quad L^1(\Omega, \mathbb{R}^N), \tag{4.18}$$

for some $g_{h,k} \in L^1(\Omega;\mathbb{R}^N)$ and $f_{h,k} \in L^1(\Omega;\mathbb{R}^N)$. On the other hand, by (3.8) the following inequality holds true

$$|B_{p,h,k}| \leq \frac{1}{h^p} \int_\Omega |\nabla T_k(u_p)|^p \leq \frac{k}{h^p} \|f\|_{L^1(\Omega)} \,. \tag{4.19}$$

Therefore, by Hölder's inequality, (3.8) and (4.19), for any $\Phi \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\| \, |\Phi| \, \|_\infty \leq 1$, we have

$$\begin{aligned} \Big| \int_{B_{p,h,k} \cap \{|u_p|<k\}} |\nabla u_p|^{p-2} \nabla u_p \cdot \Phi \Big| &\leq \left(\int_\Omega |\nabla T_k u_p|^p \right)^{(p-1)/p} |B_{p,h,k}|^{1/p} \leq \\ &\leq \left(k \|f\|_{L^1(\Omega)} \right)^{(p-1)/p} \left(\frac{k\|f\|_{L^1(\Omega)}}{h^p} \right)^{1/p} = \frac{k\|f\|_{L^1(\Omega)}}{h}. \end{aligned}$$

By (4.17), for any fixed $k > 0$ and $h > 0$, this implies

$$\Big| \int_\Omega g_{h,k} \cdot \Phi \Big| \leq \frac{k\|f\|_{L^1(\Omega)}}{h}$$

for any $\Phi \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\| \, |\Phi| \, \|_\infty \leq 1$. By duality, we deduce the following estimate for g_{hk}

$$\int_\Omega |g_{h,k}| \leq \frac{k\|f\|_{L^1(\Omega)}}{h},$$

for any fixed $h > 0$ and $k > 0$. Moreover, by definition of the set $B_{p,h,k}$ we have

$$\big| |\nabla u_p|^{p-2} \nabla u_p \ \chi_{(\Omega \setminus B_{p,h,k}) \cap \{|u_p|<k\}} \big| \leq h^{p-1} \qquad \text{a.e. in } \Omega.$$

This implies the following pointwise estimate for $f_{h,k}$

$$|f_{h,k}| \leq \lim_{p \to 1} h^{p-1} = 1, \qquad \text{a.e. in } \Omega \,.$$

For any fixed $h > 0$ and $k > 0$,

$$w_k = f_{h,k} + g_{h,k} \tag{4.20}$$

with

$$\|f_{h,k}\|_\infty \leq 1 \qquad \text{and} \qquad \int_\Omega |g_{h,k}| \leq \frac{M}{h}.$$

Therefore, we obtain (see [4], and also Step 3 of Proposition 4.1 in [34])

$$\| \, |w_k| \, \|_\infty \leq 1, \tag{4.21}$$

for all $k > 0$. Since $\lim_{p \to 1} u_p(x) = u(x)$ almost everywhere in Ω, it follows that

$$\chi_{\{|u_p|<k\}} \to \chi_{\{|u|<k\}}, \quad \text{strongly in} \quad L^\rho(\Omega), \quad \text{for every } 1 \leq \rho < +\infty \,,$$

for almost all $k > 0$. We point out that, since $|\Omega| < +\infty$, the set of the values k such that $|\{|u| = k\}| > 0$ is countable. Therefore, by (3.10) and (4.15), we conclude

$$w_k = z \, \chi_{\{|u|<k\}} = z_k \,,$$

for almost all $k > 0$. Observe that, by applying

$$\lim_{k\to+\infty} w_k = \lim_{k\to+\infty} z\chi_{\{|u|<k\}} = z\chi_{\{|u|<+\infty\}}, \qquad \text{a.e. in } \Omega$$

and (4.21), we deduce $\| \, |z \ \chi_{\{|u|<k\}}| \, \|_\infty \le \| \, |z \ \chi_{\{|u|<+\infty\}}| \, \|_\infty \le 1$ for all $k > 0$. This proves (4.8).

Step 4: Proof of $(z, D\chi_{\{|u|>k\}})$ *is a Radon measure and* (4.9) holds.

Let us consider $h_{k\epsilon}(u_p)\phi$ as test function in (3.7), where $\phi \in C_0^\infty(\Omega)$ and $h_{k\epsilon}(s)$ is defined by

$$h_{k\epsilon}(s) = \begin{cases} 0, & |s| \ge k+\epsilon; \\ 1, & |s| \le k; \\ \frac{k+\epsilon-|s|}{\epsilon}, & k < |s| < k+\epsilon. \end{cases}$$

Then we have

$$-\frac{1}{\epsilon}\int_{\{k\le|u_p|<k+\epsilon\}} |\nabla u_p|^p \phi \text{ sign } (u_p) + \int_\Omega h_{k\epsilon}(u_p)|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\phi = \int_\Omega h_{k\epsilon}(u_p) f\phi\,. \quad (4.22)$$

Letting p go to 1, we get

$$\int_\Omega h_{k\epsilon}(u) z\cdot\nabla\phi = \int_\Omega h_{k\epsilon}(u) f\phi + \lim_{p\to1}\frac{1}{\epsilon}\int_{\{k\le|u_p|<k+\epsilon\}} |\nabla u_p|^p\phi \text{ sign } (u_p)$$

and therefore letting ϵ go to zero

$$\int_{\{|u|\le k\}} z\cdot\nabla\phi = \int_{\{|u|\le k\}} f\phi + \lim_{\epsilon\to0}\lim_{p\to1}\frac{1}{\epsilon}\int_{\{k\le|u_p|<k+\epsilon\}} |\nabla u_p|^p\phi \text{ sign } (u_p). \quad (4.23)$$

Hence, since (4.7) holds, we have

$$\lim_{\epsilon\to0}\lim_{p\to1}\frac{1}{\epsilon}\int_{\{k\le|u_p|<k+\epsilon\}} |\nabla u_p|^p\phi \text{ sign } (u_p) = \int_{\{|u|>k\}} f\phi - \int_{\{|u|>k\}} z\cdot\nabla\phi\,,$$

that is, by Definition 4.5

$$\lim_{\epsilon\to0}\lim_{p\to1}\frac{1}{\epsilon}\int_{\{k\le|u_p|<k+\epsilon\}} |\nabla u_p|^p\phi \text{ sign } (u_p) = \langle (z, D\chi_{\{|u|>k\}}), \phi\rangle\,. \quad (4.24)$$

Since

$$\left|\frac{1}{\epsilon}\int_{\{k\le|u_p|<k+\epsilon\}} |\nabla u_p|^p\phi \text{ sign } (u_p)\right| \le \frac{\|\phi\|_\infty}{\epsilon}\int_{\{k\le|u_p|<k+\epsilon\}} |\nabla u_p|^p = \frac{\|\phi\|_\infty}{\epsilon}\int_\Omega f\, T_\epsilon(u_p - T_k(u_p)) \le \|\phi\|_\infty\int_\Omega |f|$$

for all $p > 1$ and all $\epsilon > 0$, we deduce from (4.24) that $(z, D\chi_{\{|u|>k\}})$ is actually a Radon measure satisfying

$$|(z, D\chi_{\{|u|>k\}})|(\Omega) \le \int_\Omega |f| \,.$$

On the other hand, (4.23) becomes

$$\int_\Omega z_k \cdot \nabla\phi = \int_\Omega f\,\chi_{\{|u|<k\}}\phi + \langle (z, D\chi_{\{|u|>k\}}), \phi\rangle \,,$$

and it yields (4.9). We explicitely observe that, since the right-hand side in (4.9) is a Radon measure, we deduce that $-\mathrm{div}(z_k)$ is a Radon measure in the dual space $W^{-1,\infty}(\Omega)$. Moreover, since the measure $(z, D\chi_{\{|u|>k\}})$ belongs to $L^1(\Omega)+W^{-1,\infty}(\Omega)$ and therefore (see Proposition 5.2 in Appendix 1) it is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{N-1}$, then the precise representative (as mentioned in Appendix 1) of every $v \in BV(\Omega)\cap L^\infty(\Omega)$ belongs to $L^1(\Omega, (z, D\chi_{\{|u|>k\}}))$.

Step 5: Study of $(z_k, DT_k(u))$

As pointed out in the previous step, $-\mathrm{div} z_k$ is a Radon measure. Therefore by Appendix 2 Proposition 5.4,

$$|(z_k, DT_k(u))| \le \|\, |z_k|\, \|_\infty |DT_k(u)|$$

and, since $\|\, |z_k|\, \|_\infty \le 1$, we have

$$(z_k, DT_k(u)) \le |DT_k(u)| \,, \quad \text{as measures in } \Omega \,. \tag{4.25}$$

Now we prove that in fact equality holds in (4.25). Denote, for every $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \langle (z, D\chi_{\{u>k\}}), \phi\rangle &= \int_{\{u>k\}} f\phi - \int_{\{u>k\}} z\cdot\nabla\phi \\ \langle (z, D\chi_{\{-u>k\}}), \phi\rangle &= \int_{\{-u>k\}} f\phi - \int_{\{-u>k\}} z\cdot\nabla\phi \,. \end{aligned} \tag{4.26}$$

By Proposition 6.3 in Appendix 2, these distributions are Radon measure concentrated in $\{u = k\}$ and $\{-u = k\}$, respectively. Therefore, by (4.4) and (4.9), we obtain

$$\langle (z_k, DT_k(u)), \phi\rangle = \int_{\{|u|<k\}} fT_k(u)\phi + \int_\Omega T_k(u)\phi\, d(z, D\chi_{\{|u|>k\}}) - \int_{\{|u|<k\}} T_k(u) z\cdot\nabla\phi \,.$$

Since

$$\int_{\{|u|<k\}} fT_k(u)\phi = \int_\Omega fT_k(u)\phi - k\int_{\{u>k\}} f\phi + k\int_{\{-u>k\}} f\phi \,,$$

$$\int_\Omega T_k(u)\phi\, d(z, D\chi_{\{|u|>k\}}) = k\langle (z, D\chi_{\{u>k\}}), \phi\rangle - k\langle (z, D\chi_{\{-u>k\}}), \phi\rangle$$

and

$$\int_{\{|u|\le k\}} T_k(u) z\cdot\nabla\phi = \int_\Omega T_k(u) z\cdot\nabla\phi - k\int_{\{u>k\}} z\cdot\nabla\phi + k\int_{\{-u>k\}} z\cdot\nabla\phi \,;$$

it follows that

$$\langle (z_k, DT_k(u)), \phi\rangle = \int_\Omega fT_k(u)\phi - \int_\Omega T_k(u)z\cdot\nabla\phi\,. \tag{4.27}$$

Now, we denote

$$\langle (z, DT_k(u)), \phi\rangle = \int_\Omega fT_k(u)\phi - \int_\Omega T_k(u)z\cdot\nabla\phi \quad \phi\in C_0^\infty(\Omega). \tag{4.28}$$

In Appendix 2, Proposition 6.1, we prove that the distribution defined by the above expression is actually a Radon measure. From (4.27) we deduce that

$$(z_k, DT_k(u)) = (z, DT_k(u)) \quad \text{for all } k>0\,, \tag{4.29}$$

and thefore by (4.25)

$$(z, DT_k(u)) \le |DT_k(u)| \quad \text{as measures in } \Omega\,, \tag{4.30}$$

Now we prove that

$$|DT_k(u)| \le (z, DT_k(u)) \quad \text{ as measures in } \Omega\,. \tag{4.31}$$

Denote for $n>k$

$$h_{kn}(s) = \begin{cases} 0\,, & |s|\ge k+2n\,;\\[2mm] \dfrac{(k+2n-|s|)k \text{ sign } s}{n}\,, & k+n<|s|<k+2n\,;\\[2mm] T_k(s)\,, & |s|\le k+n\,. \end{cases}$$

Obviously h_{kn} tends to $T_k(s)$ as $n\to+\infty$. Let ϕ be a nonnegative function belonging to $C_0^\infty(\Omega)$. By choosing $h_{kn}(u_p)\phi$ as test function in (3.7) and letting n go to infinity, we get

$$\int_\Omega |\nabla T_k(u_p)|^p\,\phi + \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\phi\;T_k(u_p) = \int_\Omega T_k(u_p)\phi f.$$

By Young's inequality we have

$$\int_\Omega |\nabla T_k(u_p)|\,\phi \le \frac{1}{p}\int_\Omega |\nabla T_k(u_p)|^p\,\phi + \frac{p-1}{p}\int_\Omega \phi$$

$$= \frac{1}{p}\int_\Omega T_k(u_p)\,\phi f - \frac{1}{p}\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\phi\;T_k(u_p) + \frac{p-1}{p}\int_\Omega \phi.$$

This implies

$$\int_\Omega |\nabla T_k(u_p)|\,\phi + \frac{1}{p}\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\phi\,T_k(u_p) \le$$

$$\le \frac{1}{p}\int_\Omega T_k(u_p)\,\phi f + \frac{p-1}{p}\int_\Omega \phi.$$

Now we let p go to 1 and we obtain

$$\int_\Omega \phi \, d|DT_k(u)| + \int_\Omega z\cdot\nabla\phi \; T_k(u) \le \int_\Omega T_k(u)\phi f,$$

for every nonnegative $\phi \in C_0^\infty(\Omega)$. On the other hand, by (4.28), it follows that

$$\langle |DT_k(u)|, \phi\rangle \;\le\; \langle (z, DT_k(u)), \phi\rangle,$$

for every $\phi \in C_0^\infty(\Omega)$ with $\phi \ge 0$. This yields (4.31), and by (4.29) and (4.30) we arrive to (4.10).

We explicitly observe that the previous arguments imply

$$\lim_{p\to 1}\int_\Omega |\nabla T_k(u_p)|^p\phi = \langle (z, DT_k(u)), \phi\rangle = \langle |DT_k(u)|, \phi\rangle, \quad \forall\, \phi \in C_0^\infty(\Omega).$$

Step 6: Proof of (4.11).

We begin by observing that, since $z\chi_{\{|u|<+\infty\}} \in \mathcal{DM}^\infty(\Omega)$, by [3], $[z\chi_{\{|u|<+\infty\}}, \nu]$ is well–defined and satisfies $|[z\chi_{\{|u|<+\infty\}}, \nu]| \le \|\, |z\chi_{\{|u|<+\infty\}}|\,\|_\infty \le 1$. On the other hand, the weak traces of $[z,\nu]$, $[z\chi_{\{u=+\infty\}}, \nu]$ and $[z\chi_{\{u=-\infty\}}, \nu]$ will be introduced in Appendix 3 below.

Fixed $k>0$, our starting point is the equality

$$\int_\Omega |\nabla T_k(u_p)|^p = \int_\Omega f T_k(u_p)$$

that holds for all $p>1$. Applying Young's inequality, we obtain

$$\int_\Omega |\nabla T_k(u_p)| \le \frac{1}{p}\int_\Omega |\nabla T_k(u_p)|^p + \frac{p-1}{p}|\Omega| = \frac{1}{p}\int_\Omega f T_k(u_p) + \frac{p-1}{p}|\Omega| \,.$$

Now the inferior semicontinuity implies

$$|DT_k(u)|(\Omega) + \int_{\partial\Omega} |T_k(u)|\, d\mathcal{H}^{N-1} \le \int_\Omega f T_k(u) = -\int_\Omega (\operatorname{div} z)\, T_k(u)\,. \tag{4.32}$$

Taking into account that $z = z\chi_{\{|u|<+\infty\}} + z\chi_{\{u=+\infty\}} + z\chi_{\{u=-\infty\}}$, we may split the right hand of (4.32) into three parts. By the Gauss–Green formula (5.1), we deduce

$$\begin{aligned} -\int_\Omega &\big(\operatorname{div}(z\chi_{\{|u|<+\infty\}})\big)\, T_k(u) \\ &= \int_\Omega (z\chi_{\{|u|<+\infty\}}, DT_k(u)) - \int_{\partial\Omega} [z\chi_{\{|u|<+\infty\}}, \nu] T_k(u)\, d\mathcal{H}^{N-1}\,. \end{aligned} \tag{4.33}$$

In order to compute $(z\chi_{\{|u|<+\infty\}}, DT_k(u))$ we have to perform some computations are needed. For every $\phi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} &\langle (z\chi_{\{|u|<+\infty\}}, DT_k(u)), \phi\rangle \\ &\quad = \int_{\{|u|<+\infty\}} f T_k(u)\phi + \int_\Omega T_k(u)\phi\, d(z, D\chi_{\{|u|=+\infty\}}) - \int_{\{|u|<+\infty\}} T_k(u) z\cdot\nabla\phi\,. \end{aligned}$$

Performing similar manipulations to those done in Step 5, we obtain

$$\langle (z\chi_{\{|u|<+\infty\}}, DT_k(u)), \phi\rangle = \int_\Omega fT_k(u)\phi - \int_\Omega T_k(u) z\cdot\nabla\phi\,.$$

Thus, $\int_\Omega (z\chi_{\{|u|<+\infty\}}, DT_k(u)) = \int_\Omega (z, DT_k(u)) = |DT_k(u)|(\Omega)$. From here and (4.33), one deduces

$$-\int_\Omega \big(\mathrm{div}\,(z\chi_{\{|u|<+\infty\}})\big)\,T_k(u) \\ = |DT_k(u)|(\Omega) - \int_{\partial\Omega} [z\chi_{\{|u|<+\infty\}},\nu] T_k(u)\, d\mathcal{H}^{N-1}\,. \quad (4.34)$$

The other two parts are easier of handling. Observe that, choosing $v\equiv 1$ in the definition of $[z\chi_{\{u=+\infty\}},\nu]$, we obtain

$$\int_{\partial\Omega} [z\chi_{\{u=+\infty\}},\nu]\, d\mathcal{H}^{N-1} = \int_{\{u=+\infty\}} \big(\mathrm{div}\,(z\chi_{\{u=+\infty\}})\big)\,,$$

so that

$$\begin{aligned}-\int_\Omega \big(\mathrm{div}\,(z\chi_{\{u=+\infty\}})\big)\,T_k(u) &= \int_{\{u=+\infty\}} fT_k(u) - \int_\Omega T_k(u)\, d(z, D\chi_{\{u=+\infty\}}) \\ &= k\Big[\int_{\{u=+\infty\}} f - \int_\Omega d(z, D\chi_{\{u=+\infty\}})\Big] \\ &= -k\int_\Omega \mathrm{div}\,(z\chi_{\{u=+\infty\}}) = -k\int_{\partial\Omega} [z\chi_{\{u=+\infty\}},\nu]\, d\mathcal{H}^{N-1}\,. \quad (4.35)\end{aligned}$$

Analogously, we have

$$-\int_\Omega \big(\mathrm{div}\,(z\chi_{\{u=-\infty\}})\big)\,T_k(u) = k\int_{\partial\Omega} [z\chi_{\{u=-\infty\}},\nu]\, d\mathcal{H}^{N-1}\,. \quad (4.36)$$

Having in mind (4.34), (4.35) and (4.36), inequality (4.32) becomes

$$\begin{aligned}|DT_k(u)|(\Omega) &+ \int_{\partial\Omega} |T_k(u)|\, d\mathcal{H}^{N-1} \\ &\le |DT_k(u)|(\Omega) - \int_{\partial\Omega} [z\chi_{\{|u|<+\infty\}},\nu] T_k(u)\, d\mathcal{H}^{N-1} \\ &\quad - k\int_{\partial\Omega} [z\chi_{\{u=+\infty\}},\nu]\, d\mathcal{H}^{N-1} + k\int_{\partial\Omega} [z\chi_{\{u=-\infty\}},\nu]\, d\mathcal{H}^{N-1}\,,\end{aligned}$$

from where (4.11) follows.

Finally assume that u is $\mathcal{H}^{N-1}$–a.e. finite on $\partial\Omega$. Then

$$\int_{\partial\Omega} [z\chi_{\{u=+\infty\}},\nu]\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} [z\chi_{\{u=-\infty\}},\nu]\, d\mathcal{H}^{N-1} = 0$$

and so, by (4.11),

$$\int_{\partial\Omega} |T_k(u)|\, d\mathcal{H}^{N-1} \le -\int_{\partial\Omega} [z\chi_{\{|u|<+\infty\}},\nu]\, T_k(u)\, d\mathcal{H}^{N-1}\,.$$

Since $\| [z\chi_{\{|u|<+\infty\}}, \nu] \|_\infty \le \| \, |z\chi_{\{|u|<+\infty\}}| \, \|_\infty \le 1$, this implies

$$|T_k(u)| = -[z\chi_{\{|u|<+\infty\}}, \nu]\, T_k(u) \quad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega \,.$$

Therefore, $[z\chi_{\{|u|<+\infty\}}, \nu] \in$ sign $(-u)$, $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$, and (4.12) is proved.

Remark 4.5 In [34], (Theorems 4.2 and 4.3), we have shown that, when the norm of a datum belonging to $W^{-1,\infty}(\Omega)$ is small enough, we find a solution to problem (4.1) which is a function belonging to $BV(\Omega)$. This situation does not hold for general L^1-data. Indeed, observe that $\| \, |z\, \chi_{\{|u|<\infty\}}| \, \|_\infty \le 1$ and so, by the same argument in Remark 4.4, $z\, \chi_{\{|u|<\infty\}} = z$ only when the datum belongs to $W^{-1,\infty}(\Omega)$ and its norm is small enough. We explicitly point out that if $f \in L^1(\Omega) \backslash W^{-1,\infty}(\Omega)$, or $f \in W^{-1,\infty}(\Omega)$ with $\|f\|_{W^{-1,\infty}(\Omega)} > 1$, then the set $\{|u| = +\infty\}$ has positive measure. This feature is stated more precisely in Theorem 4.2 below and illustrated in Example 4.1, where data which do not belong to $W^{-1,\infty}(\Omega)$ are considered.

Theorem 4.2 *For every fixed $p > 1$ let u_p denote the renormalized solution to problem* (3.1). *If u is the pointwise limit of u_p, and z is the weak limit of $|\nabla u_p|^{p-2}\nabla u_p$, as in Theorem 3.1, then the following statements are equivalent*

(1) $f \in W^{-1,\infty}(\Omega)$ with $\|f\|_{W^{-1,\infty}(\Omega)} \le 1$;

(2) $u \in BV(\Omega)$;

(3) u is almost everywhere finite in Ω;

(4) $z \in L^\infty(\Omega; \mathbb{R}^N)$ with $\| \, |z| \, \|_\infty \le 1$.

Sketch of Proof: (1) $\Rightarrow$ (2) It is a consequence of the estimate $\int_\Omega |\nabla u_p| \le |\Omega|$ for all $p > 1$ (see [34], Theorem 4.1).

(2) $\Rightarrow$ (3) It is straightforward.

(3) $\Rightarrow$ (4) Theorem 4.1 yields $z\chi_{\{|u|<\infty\}} \in L^\infty(\Omega; \mathbb{R}^N)$ with $\| \, |z\chi_{\{|u|<\infty\}}| \, \|_\infty \le 1$. So that $|u(x)| < \infty$ a.e. implies condition (4).

(4) $\Rightarrow$ (1) The argument is contained in Remark 4.4.

Example 4.1 For every $0 < \lambda < +\infty$ and $1 < q < N$, we consider the problem

$$\begin{cases} -\Delta_p u_p = f \,, & \text{in } B_R(0)\,; \\[2ex] \quad u_p = 0 \,, & \text{on } \partial B_R(0)\,; \end{cases}$$

where $f(x) = \frac{\lambda}{C_N^{q/N}|x|^q}$. Since f is a radial function and its decreasing rearrangement is defined by $f^*(s) = \frac{\lambda}{s^{q/N}}$, the solution of our problem is given by (see [41])

$$\begin{aligned} u_p(x) \;&= \tfrac{1}{N^{p'} C_N^{p'/N}} \int_{C_N|x|^N}^{C_N R^N} s^{\frac{p'}{N}-p'} \Big(\int_0^s f^*(\sigma)\, d\sigma \Big)^{1/(p-1)} ds = \\ &= \tfrac{\lambda^{1/(p-1)}}{C_N^{p'/N}(N-q)^{1/(p-1)}} \tfrac{p-1}{q-p} \Big(\big(C_N|x|^N\big)^{-\frac{q-p}{N(p-1)}} - \big(C_N R^N\big)^{-\frac{q-p}{N(p-1)}} \Big) = \\ &= \Big(\tfrac{\lambda}{C_N^{q/N}(N-q)|x|^{q-p}} \Big)^{1/(p-1)} \tfrac{p-1}{q-p} \Big(1 - \Big(\tfrac{|x|}{R}\Big)^{\frac{q-p}{p-1}} \Big)\,. \end{aligned}$$

Letting p go to 1, we point out that $|x| < \left(\frac{\lambda}{C_N^{q/N}(N-q)}\right)^{1/(q-1)}$ implies $u_p(x) \to +\infty$ and $|x| \geq \left(\frac{\lambda}{C_N^{q/N}(N-q)}\right)^{1/(q-1)}$ yields $u_p(x) \to 0$. Hence, for all λ, the limit blows up in a ball of positive measure. Furthermore, when $\lambda \geq (N-q)C_N^{q/N}R^{q-1}$, the limit blows up everywhere.

In this example, we may also compute the vector field z. Indeed,

$$|\nabla u_p|^{p-2}\nabla u_p = -\frac{\lambda}{(N-q)C_N^{q/N}}\frac{x}{|x|^q}$$

for all $p > 1$, so that

$$|z(x)| = \frac{\lambda}{C_N^{q/N}(N-q)|x|^{q-1}}\,.$$

Therefore, this vector field is not bounded, and it belongs to $L^{\frac{N}{q-1},\infty}(B_R;\mathbb{R}^N)$. On the other hand, we also point out that

$$\begin{aligned} &\text{if} \quad |z| \leq 1\,, \qquad \text{then} \quad u_p \to 0\,; \\ &\text{if} \quad |z| > 1\,, \qquad \text{then} \quad u_p \to +\infty\,. \end{aligned} \tag{4.37}$$

Example 4.2 For every $0 < \lambda < +\infty$, we consider

$$\begin{cases} -\Delta_p u_p = \lambda\,\delta_0\,, & \text{in } B_R(0)\,; \\ \quad u_p = 0\,, & \text{on } \partial B_R(0)\,; \end{cases}$$

where δ_0 denotes the delta function concentrated on $\{0\}$.

The solution to this problem is given by

$$u_p(x) = \frac{p-1}{N-p}\,\frac{\lambda^{1/(p-1)}}{(NC_N)^{1/(p-1)}}\left(\frac{1}{|\,x|^{(N-p)/(p-1)}} - \frac{1}{R^{(N-p)/(p-1)}}\right).$$

Thus, if $|\,x| < \left(\frac{\lambda}{NC_N}\right)^{1/(N-1)}$, then $u_p(x) \to +\infty$. Hence, for all λ, the limit blows up in a set of positive measure. Furthermore, when $\lambda \geq NC_NR^{N-1}$, the limit blows up everywhere. On the other hand, if $|x| \geq \left(\frac{1}{NC_N}\right)^{1/(N-1)}$, then $u_p(x) \to 0$.
We also remark that

$$|\nabla u_p|^{p-2}\nabla u_p = \frac{\lambda}{NC_N}\,\frac{x}{|x|^N} \qquad \text{for all} \quad p > 1\,,$$

and consequently $|z| = \frac{\lambda}{NC_N}\,\frac{1}{|x|^{N-1}}$. Hence,the vector field z and the limit function u are linked as in (4.37).

Remark 4.6 As far as uniqueness is concerned, if u is a regular solution to (4.1) and $h \in C^1(\mathbb{R},\mathbb{R})$ is strictly increasing, then $h(u)$ is also a solution to (4.1) (for instance, $\arctan(u)$ is a solution). Hence uniqueness in general does not hold. Indeed we remark that only a "subsequence" converges.

However, the limit points are not general solutions to (4.1) since we have got $|u| = +\infty$ in a subset of positive measure (unless $\|f\|_{W^{-1,\infty}(\Omega)} \leq 1$). Therefore, not every solution to problem (4.1) is the limit of a "subsequence" of $(u_p)_{p>1}$, in other words, our limit points to $(u_p)_{p>1}$ are some specific solutions to (4.1).

5 Appendix 1: L^∞–divergence–measure fields

In this Appendix we will study some properties involving divergence–measure vector fields and functions of bounded variation. We will prove basic approximation results, and we will give an extension of the Anzellotti's theory proved in [3] (see also [6], [31], [34]).

5.1 Approximation results

Let us begin by stating two basic results. In the first one every BV–function will be approximated by smooth ones; its proof is a simple combination of Lemma 5.1 in [3] and [2] p. 175.

Proposition 5.1 *Let $u \in BV(\Omega)$ and let $(\rho_\epsilon)_\epsilon$ be a family of positive symmetric mollifiers. Define*

$$u_\epsilon(x) = \begin{cases} u * \rho_\epsilon(x), & \text{if } x \in \Omega; \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

Then,

(1) u_ϵ *pointwise converge* $\mathcal{H}^{N-1}$*–a.e. to the precise representative of* u

(2) $\displaystyle\int_A |\nabla u_\epsilon| \to |Du|(A) \quad$ *if* $A \subset \Omega$ *is open and* $|Du|(\partial A) = 0$.

(3) *If* $u \in BV(\Omega) \cap L^\infty(\Omega)$, *then* $|u_\epsilon(x)| \le \|u\|_\infty \quad \mathcal{H}^{N-1}$*–a.e.*

The second basic fact is the following proposition, which is proved in [14]. It can also be proved as a consequence of (2.4) and inequality in [25] p. 171 relating Hausdorff measure and Hausdorff spherical measure.

Proposition 5.2 *For every $z \in \mathcal{DM}^\infty(\Omega)$, the measure $\mu = \operatorname{div} z$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$. As a consequence, $|\mu| \ll \mathcal{H}^{N-1}$.*

Consider now $\mu = \operatorname{div} z$ with $z \in \mathcal{DM}^\infty(\Omega)$ and let $u \in BV(\Omega)$. Since the precise representative of u is equal $\mathcal{H}^{N-1}$–a.e. to the Borel function $\lim_{\epsilon\to 0} u_\epsilon$, then one deduces from Proposition 5.2, that (the precise representative of) u is equal μ–a.e. to a Borel function. From now on, given $u \in BV(\Omega)$, we also denote by u its precise representative; and we will say that every BV–function is μ–measurable. Moreover, $u \in BV(\Omega) \cap L^\infty(\Omega)$ implies $u \in L^\infty(\Omega, \mu) \subset L^1(\Omega, \mu)$. On the other hand, it also yields

$$\int_\Omega u\, d\mu = -\int_\Omega z \cdot \nabla u = \langle \mu, u \rangle_{W^{-1,\infty}(\Omega), W_0^{1,1}(\Omega)},$$

for every $u \in W_0^{1,1}(\Omega)$.

Proposition 5.3 *Let $\mu = \operatorname{div} z$ with $z \in \mathcal{DM}^\infty(\Omega)$. For each $u \in BV(\Omega) \cap L^\infty(\Omega)$ there exists a sequence (u_n) in $W^{1,1}(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ such that*

(1) $u_n \to u$ *in* $L^1(\Omega, |\mu|)$

(2) $\displaystyle\int_\Omega |\nabla u_n| \to |Du|(\Omega)$.

(3) $u_n|_{\partial\Omega} = u|_{\partial\Omega}$ *for all* $n \in \mathbb{N}$.
(*Here* $u|_{\partial\Omega}$ *denotes the trace of* u *and not the trace of the extension* u_0.)

(4) $|u_n(x)| \le \|u\|_\infty$ $|\mu|$*–a.e. and for all* $n \in \mathbb{N}$, *and* $u_n \to u$ *in* $L^\infty -$ *weak**.

Moreover, if $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, *then*

(2)' $u_n \to u$ *in* $W^{1,1}(\Omega)$.

instead of (2).

Proof: Fixed $\delta > 0$, we prove the existence of a function $u_\delta \in BV(\Omega) \cap C^\infty(\Omega)$ such that

$$\int_\Omega |u - u_\delta|\, d|\mu| < \delta\,, \quad \int_\Omega |u - u_\delta| < \delta \quad \text{and} \quad \int_\Omega |\nabla u_\delta| \le |Du|(\Omega) + \delta\,,$$

and if $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, it also satisfies

$$\int_\Omega |\nabla u_\delta - \nabla u| \le \delta\,.$$

In order to prove such a claim, we denote by Ω_k a sequence of open sets defined as in the proof of the Meyers–Serrin theorem (see for instance [2], p. 122). Consider also a partition of unity subordinate to this covering: $\phi_k \in C_0^\infty(\Omega)$ such that $\operatorname{supp} \phi_k \subset \Omega_k$, $0 \le \phi \le 1$ and $\sum_{k=0}^\infty \phi_k(x) = 1$ for all $x \in \Omega$. Moreover, let $(\rho_n)_n$ be a sequence of positive symmetric mollifiers. Finally, let $(\delta_k)_k$ be a sequence of positive numbers satisfying $\sum_{k=1}^\infty \delta_k < \delta$.

Since $\lim_{n\to\infty} \big(\rho_n * (\phi_k u)\big)(x) = \phi_k(x)u(x)$ for $\mathcal{H}^{N-1}$–almost all point x, one deduces from Proposition 5.2 that $\lim_{n\to\infty} \big(\rho_n * (\phi_k u)\big)(x) = \phi_k(x)u(x)$ for $|\mu|$–almost all point x and so

$$\lim_{n\to\infty} \int_\Omega |\rho_n * (\phi_k u) - \phi_k u|\, d|\mu| = 0 \quad \text{for all } k \in \mathbb{N}\,.$$

Thus, for each $k \in \mathbb{N}$, we can find $\epsilon_k > 0$ such that

$$\operatorname{supp} \rho_{\epsilon_k} * (\phi_k u) \subset \Omega_k \,,$$

$$\int_\Omega |\rho_{\epsilon_k} * (\phi_k u) - \phi_k u| \, d|\mu| < \delta_k \,,$$

$$\int_\Omega |\rho_{\epsilon_k} * (\phi_k u) - \phi_k u| < \delta_k \quad \text{and}$$

$$\int_\Omega |\rho_{\epsilon_k} * (u\nabla\phi_k) - u\nabla\phi_k| < \delta_k \,.$$

When $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ we may consider $\epsilon_k > 0$ satisfying

$$\int_\Omega |\rho_{\epsilon_k} * \nabla(u\phi_k) - \nabla(u\phi_k)| < \delta_k \,.$$

Defining $u_\delta = \sum_{k=0}^\infty \rho_{\epsilon_k} * (u\nabla\phi_k)$, we may next follow the proof of [2], p. 123.

■

5.2 The pairing (z, Du) and the Gauss–Green formula

In this Section we define a measure (z, Du) when $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $z \in \mathcal{DM}^\infty(\Omega)$. Denoting $\mu = \operatorname{div} z$, we first define a distribution by the following expression

$$\langle (z, Du), \phi \rangle = -\int_\Omega u\phi \, d\mu - \int_\Omega uz \cdot \nabla\phi, \quad \phi \in C_0^\infty(\Omega) \,.$$

Every term is well defined since $u \in BV(\Omega) \cap L^\infty(\Omega) \subset L^1(\Omega, \mu) \cap L^1(\Omega)$ and $z \in L^\infty(\Omega, \mathbb{R}^N)$.

Proposition 5.4 *Let z, μ and u be as above. Then the distribution (z, Du) is a Radon measure on Ω such that, for every open $U \subset \Omega$ and every $\phi \in C_0^\infty(U)$, we have*

$$|\langle (z, Du), \phi \rangle| \leq \|\phi\|_\infty \, \| \, |z| \, \|_{L^\infty(U)} \, |Du|(U).$$

As a consequence,

$$\Big| \int_B (z, Du) \Big| \leq \int_B |(z, Du)| \leq \| \, |z| \, \|_{L^\infty(\Omega)} \, |Du|(B)$$

for every Borel set $B \subset \Omega$.

Proof: Since $u \in BV(\Omega) \cap L^\infty(\Omega)$, by Proposition 5.1 and Proposition 5.2, we can find a sequence $(u_n)_n$ in $C^\infty(\Omega) \cap W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that

$$u_n \to u \qquad |\mu| - \text{a.e.},$$

$$\lim_{n\to\infty} \int_V |\nabla u_n| = |Du|(V)$$

$$|u_n(x)| \leq \|u\|_\infty \quad |\mu|\text{–a.e.},$$

for every open $V \subset\subset \Omega$ satisfying $|Du|(\partial V) = 0$.

Let $\phi \in C_0^\infty(U)$ be fixed and consider an open set V such that $\operatorname{supp}(\phi) \subset V \subset\subset U$ and $|Du|(\partial V) = 0$. Since $(u_n\phi)_n$ is a sequence in $L^1(\Omega, \mu)$ that converges to $u\phi$ $|\mu|$–a.e. and $|u_n(x)\phi(x)| \le \|\phi\|_\infty \|u\|_\infty$ $|\mu|$–a.e., it follows from Lebesgue's Theorem that

$$\int_\Omega u_n \, \phi \, d\mu \to \int_\Omega u \, \phi \, d\mu \,.$$

On the other hand $u_n\phi \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$ implies

$$\int_\Omega z \cdot \nabla u_n \, \phi = -\int_\Omega u_n \, z \cdot \nabla\phi - \int_\Omega u_n \, \phi \, d\mu \qquad \text{for all } n \in \mathbb{N} \,,$$

and so the sequence $\left(\int_\Omega z \cdot \nabla u_n \, \phi\right)_n$ tends to $\langle (z, Du), \phi\rangle$. Since

$$\Big| \int_\Omega z \cdot \nabla u_n \, \phi \Big| \le \|\phi\|_\infty \| \, |z| \, \|_{L^\infty(U)} \int_V |\nabla u_n| ,$$

taking the limit as n goes to $+\infty$, we obtain

$$|\langle (z, Du), \phi\rangle| \le \|\phi\|_\infty \| \, |z| \, \|_{L^\infty(U)} \, |Du|(V) \le \|\phi\|_\infty \| \, |z| \, \|_{L^\infty(U)} \, |Du|(U) \,. \qquad \blacksquare$$

Lemma 5.1 *Let $z \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. If $(u_n)_n$ is a sequence in $BV(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ that converges to u as in Proposition 5.3, then*

$$\int_\Omega z \cdot \nabla u_n \to \int_\Omega (z, Du) \,.$$

Proof: The proof is similar to that in [3], Lemma 1.8. ■

In order to get the generalized Gauss–Green formula, an easy step is still needed. By [3], we already have the normal trace $[z, \nu]$ defined for all $z \in \mathcal{DM}^\infty(\Omega)$. Moreover, it holds

$$\int_\Omega u \, d\mu + \int_\Omega z \cdot \nabla u = \int_{\partial\Omega} [z, \nu] u \, d\mathcal{H}^{N-1}$$

for every $u \in W^{1,1}(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$.

Theorem 5.1 *For every $z \in \mathcal{DM}^\infty(\Omega)$ and every $u \in BV(\Omega) \cap L^\infty(\Omega)$, it holds*

$$\int_\Omega u \, d\mu + \int_\Omega z \cdot \nabla u = \int_{\partial\Omega} [z, \nu] u \, d\mathcal{H}^{N-1} \,,$$

where $\mu = \operatorname{div} z$.

Proof: It is enough consider a sequence $(u_n)_n$ as in Proposition 5.3 and take the limit in

$$\int_\Omega u_n \, d\mu + \int_\Omega z \cdot \nabla u_n = \int_{\partial\Omega} [z, \nu] u_n \, d\mathcal{H}^{N-1} \,,$$

as n goes to ∞, by applying Lemma 5.1. ■

6 Appendix 2: Properties of measures $(z_k, DT_k(u))$ and $(z, D\chi_{\{|u|>k\}})$

In this Appendix we study the properties of the distributions defined in (4.4), (4.5) and (4.28); our main result is given by Proposition 6.3.

Let us recall the distribution defined in (4.28), i.e.

$$\langle (z, DT_k(u)), \phi \rangle = \int_\Omega f T_k(u)\phi - \int_\Omega T_k(u)\, z \cdot \nabla\phi \,, \quad \forall \phi \in C_0^\infty(\Omega)\,;$$

here u denotes the limit function and z the vector field whose existence have been proved in Theorem 3.1. We begin by proving the following result.

Proposition 6.1 *The distribution $(z, DT_k(u))$ is a Radon measure.*

Proof: Let u_p be the renormalized solution to problem (3.1). As in Step 5 of the proof of Theorem 4.1, we obtain

$$\int_{\{|u_p|<k\}} |\nabla u_p|^p \phi + \int_\Omega T_k(u_p)|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\phi = \int_\Omega T_k(u_p) f\phi \,.$$

Therefore letting p go to 1, we get

$$\lim_{p\to 1} \int_{\{|u_p|<k\}} |\nabla u_p|^p \phi = \int_\Omega f T_k(u)\phi - \int_\Omega T_k(u) z \cdot \nabla\phi. = \langle (z, DT_k(u)), \phi \rangle \,. \qquad (6.1)$$

Moreover it results:

$$\left| \int_{\{|u_p|<k\}} |\nabla u_p|^p \phi \right| \le \|\phi\|_\infty \int_\Omega |\nabla T_k(u_p)|^p = \|\phi\|_\infty \int_\Omega f T_k(u_p) \le \|\phi\|_\infty k \int_\Omega |f| \,,$$

and therefore by (6.1)

$$|\langle (z, DT_k(u)), \phi \rangle| \le \|\phi\|_\infty k \int_\Omega |f| \,.$$

This yields the conclusion.

Remark 6.1 Actually in the same way we can define the distributions $(z, Dh(u))$ for all Lipschitz function h such that the support of its derivative is compact, that is

$$\langle (z, Dh(u)), \phi \rangle = \int_\Omega f h(u)\phi - \int_\Omega h(u)\, z \cdot \nabla\phi \,, \quad \forall \phi \in C_0^\infty(\Omega).$$

With the same arguments used in the proof of Proposition 6.1, we can prove that $(z, Dh(u))$ is a Radon measure. In this way, for every $k > 0$ and $\eta \ge 0$, we obtain that $(z, DT_k(u - T_\eta(u))^+)$ and $(z, DT_k(u - T_\eta(u)^-)$ are Radon measures satisfying

$$|\langle (z, DT_k(u - T_\eta(u))^+), \phi \rangle| \le \|\phi\|_\infty k \int_{\{u \ge \eta\}} |f| \,,$$

and

$$|\langle (z, DT_k(u - T_\eta(u))^-), \phi \rangle| \le \|\phi\|_\infty k \int_{\{-u \ge \eta\}} |f| \,,$$

respectively.

Now we prove that the measure $(z, DT_k(u))$ is concentrated on the set $\{|u| \le k\}$. To this aim we need some preliminaries.

Proposition 6.2 *The Radon measure $(z, DT_k(u - T_\eta(u))^+)$ is concentrated on the set $\{\eta \le u \le k + \eta\}$. Analogously the Radon measure $(z, DT_k(u - T_\eta(u))^-)$ is concentrated on the set $\{\eta \le -u \le k + \eta\}$. In particular when $\eta = 0$, the Radon measure $(z, DT_k(u))$ is concentrated on the set $\{|u| \le k\}$.*

Proof: We only prove that the Radon measure $(z, DT_k(u - T_\eta(u))^+)$ is concentrated on the set $\{\eta \le u \le k + \eta\}$ since the second part of the proposition is obtained by the same arguments. To this aim we have to prove that

$$(z, DT_k(u - T_\eta(u))^+)(\omega \cap \{u > k + \eta\}) = (z, DT_k(u - T_\eta(u))^+)(\omega \cap \{u < \eta\}) = 0$$

for any ω open subset such that $\omega \subset\subset \Omega$.

Let us fix $\omega \subset\subset \Omega$ and consider a sequence of mollifiers $(\rho_n)_n$. Denote by $z_n = \rho_n * z$ and $f_n = \rho_n * f$. Then $f_n = -\mathrm{div}\ z_n$ in ω, for n large enough, and moreover

$$z_n \to z \qquad \text{in } L^1(\omega; \mathbb{R}^N) \qquad \text{and} \qquad f_n \to f \qquad \text{in } L^1(\omega)\,.$$

By the results proved in [3], we have

$$\int_{\omega \cap \{u > k+h\}} |(z_n, DT_k(u - T_\eta(u))^+)| \le \|\, |z_n|\, \|_\infty |DT_k(u - T_\eta(u))^+|(\omega \cap \{u > k + \eta\})$$

Thus, since $|DT_k(u - T_\eta(u))^+|(\{u > k + \eta\}) = 0$, we obtain

$$\int_{\omega \cap \{u > k+\eta\}} (z_n, DT_k(u - T_\eta(u))^+) = 0\,, \quad \forall n.$$

In an analogous way we also obtain

$$\int_{\omega \cap \{u < \eta\}} (z_n, DT_k(u - T_\eta(u))^+) = 0\,, \quad \forall n.$$

On the other hand, for any $\phi \in C_0^\infty(\omega)$, we have

$$\Big| \int_\omega T_k(u - T_\eta(u))^+ z \cdot \nabla\phi \Big| \le \Big| \langle (z, DT_k(u - T_\eta)^+), \phi \rangle \Big| + \Big| \int_\omega T_k(u - T_\eta(u))^+ f\phi \Big| \le 2\|\phi\|_\infty k \int_{\{u \ge \eta\}} |f|\,.$$

Therefore for n large enough, it yields

$$\Big| \int_\omega T_k(u - T_\eta(u))^+ z_n \cdot \nabla\phi \Big| \le 3\|\phi\|_\infty k \int_{\{u \ge \eta\}} |f|.$$

We deduce that

$$
\begin{aligned}
&|\langle(z_n, DT_k(u-T_\eta(u))^+),\phi\rangle| \\
&\qquad\qquad \le 3\|\phi\|_\infty k\int_{\{u\ge\eta\}}|f| + \Big|\int_\omega T_k(u-T_\eta(u))^+ f_n\phi\Big| \\
&\qquad\qquad\qquad\qquad \le 4\|\phi\|_\infty k\int_{\{u\ge\eta\}}|f|\,.
\end{aligned}
$$

Moreover, since

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\omega T_k(u-T_\eta(u))^+ f_n\phi - \int_\omega T_k(u-T_\eta(u))^+ z_n\cdot\nabla\phi \\
&\qquad = \int_\omega T_k(u-T_\eta(u))^+ f\phi - \int_\omega T_k(u-T_\eta(u))^+ z\cdot\nabla\phi\,,
\end{aligned}
$$

we get

$$
(z_n, DT_k(u-T_\eta(u))^+)\llcorner\omega \to (z, DT_k(u-T_\eta(u))^+)\llcorner\omega \qquad \text{weakly-* as measures.}
$$

Therefore

$$
\int_{\omega\cap\{u>k+\eta\}}(z, DT_k(u-T_\eta(u))^+) = \lim_{n\to\infty}\int_{\omega\cap\{u>k+\eta\}}(z_n, DT_k(u-T_\eta(u))^+) = 0
$$

and

$$
\int_{\omega\cap\{u<\eta\}}(z, DT_k(u-T_\eta(u))^+) = \lim_{n\to\infty}\int_{\omega\cap\{u<\eta\}}(z_n, DT_k(u-T_\eta(u))^+) = 0\,.
$$

Corollary 6.1 *For any $\epsilon>0$, the measure $(z, DT_\epsilon(u-T_k(u))^+)$ is concentrated on $\{k\le u\le k+\epsilon\}$ and the measure $(z, DT_\epsilon(u-T_k(u))^-)$ is concentrated on $\{k\le -u\le k+\epsilon\}$.*

Proposition 6.3 *Let $(z, D\chi_{\{u>k\}})$ and $(z, D\chi_{\{-u>k\}})$ be the Radon measures defined in* (4.26)*. Then, for every $\phi\in C_0^\infty(\Omega)$,*

$$
\begin{aligned}
\langle(z, D\chi_{\{u>k\}}),\phi\rangle &= \lim_{\epsilon\to0}\frac{1}{\epsilon}\langle\big(z, DT_\epsilon(u-T_k(u))^+\big),\phi\rangle \\
\langle(z, D\chi_{\{-u>k\}}),\phi\rangle &= \lim_{\epsilon\to0}\frac{-1}{\epsilon}\langle\big(z, DT_\epsilon(u-T_k(u))^-\big),\phi\rangle\,.
\end{aligned}
$$

Moreover, $(z, D\chi_{\{u>k\}})$ is concentrated on $\{u=k\}$, while $(z, D\chi_{\{-u>k\}})$ is concentrated on $\{u=-k\}$.

As a consequence, the Radon measure $(z, D\chi_{\{|u|>k\}})$ is concentrated on $\{|u|=k\}$.

Proof: Let $\phi \in C_0^\infty(\Omega)$ and let u_p be the renormalized solution to problem (3.1). Arguing as in Step 4 of the proof of Theorem 4.1, we obtain

$$\frac{1}{\epsilon}\int_{\{k\leq u_p<k+\epsilon\}} |\nabla u_p|^p \phi = \tag{6.2}$$

$$= \frac{1}{\epsilon}\int_\Omega f T_\epsilon(u_p - T_k(u_p))^+ \phi - \frac{1}{\epsilon}\int_\Omega T_\epsilon(u_p - T_k(u_p))^+ |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\phi\,.$$

Therefore, letting p go to 1 and ϵ to zero, we have

$$\lim_{\epsilon\to 0}\lim_{p\to 1}\frac{1}{\epsilon}\int_{\{k\leq u_p<k+\epsilon\}} |\nabla u_p|^p \phi = \langle (z, D\chi_{\{u>k\}}), \phi\rangle\,. \tag{6.3}$$

It follows from (6.2) that

$$\lim_{p\to 1}\frac{1}{\epsilon}\int_{\{k\leq u_p<k+\epsilon\}} |\nabla u_p|^p \phi = \frac{1}{\epsilon}\langle (z, DT_\epsilon(u - T_k(u))^+), \phi\rangle.$$

By (6.3) and the above equality, we get

$$\langle (z, D\chi_{\{u>k\}}), \phi\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle (z, DT_\epsilon(u - T_k(u))^+), \phi\rangle.$$

Therefore, by Corollary 6.1, we deduce that the measure $(z, D\chi_{\{u>k\}})$ is concentrated on the set $\cap_{n=1}^\infty\{k \leq u \leq k + \frac{1}{n}\} = \{u = k\}$.

In the same way we can prove the assertions concerning the measure $(z, D\chi_{\{-u>k\}})$. The last statement is a consequence of $(z, D\chi_{\{|u|>k\}}) = (z, D\chi_{\{u>k\}}) + (z, D\chi_{\{-u>k\}})$.

7 Appendix 3: Weak trace on $\partial\Omega$ of the normal component of z.

The aim of this Appendix is define $[z, \nu]$, the weak trace on $\partial\Omega$ of the normal component of z, z denoting the vector field found in Theorem 3.1. Recall that it satisfies $z \in L^{\frac{N}{N-1},\infty}(\Omega; \mathbb{R}^N)$ and $-\operatorname{div} z = f$ in $\mathcal{D}'(\Omega)$.

Let $v \in W^{1-\frac{1}{q},q}(\partial\Omega) \cap L^\infty(\partial\Omega)$ for some $q > N$. Then there exists $w \in W^{1,q}(\Omega) \cap L^\infty(\Omega)$ such that $w|_{\partial\Omega} = v$. We define

$$\langle z, v\rangle_{\partial\Omega} = \int_\Omega z \cdot \nabla w - \int_\Omega f w\,. \tag{7.1}$$

The following result can be proved using similar arguments that those in [3] (see also [14]).

Proposition 7.1 *The value $\langle z, v\rangle_{\partial\Omega}$, defined in* (7.1)*, does not depend on the chosen function w and the expression $\langle z, \cdot\rangle_{\partial\Omega}$ defines a linear map on $W^{1-\frac{1}{q},q}(\partial\Omega) \cap L^\infty(\partial\Omega)$ which is continuous in the space $W^{1-\frac{1}{q},q}(\partial\Omega)$, for all $q > N$.*

We will write $\int_{\partial\Omega}[z,\nu]\, v\, d\mathcal{H}^{N-1}$ instead of $\langle z, v\rangle_{\partial\Omega}$.

To define $\int_{\partial\Omega}[z\chi_{\{u=+\infty\}},\nu]\, v\, d\mathcal{H}^{N-1}$ and $\int_{\partial\Omega}[z\chi_{\{u=-\infty\}},\nu]\, v\, d\mathcal{H}^{N-1}$, we need to know an expression to $-\mathrm{div}\,(z\chi_{\{u=+\infty\}})$ and $-\mathrm{div}\,(z\chi_{\{u=-\infty\}})$, respectively. It is easy to check that

$$\begin{aligned} -\mathrm{div}\left(z\chi_{\{u=+\infty\}}\right) &= f\chi_{\{u=+\infty\}} - (z, D\chi_{\{u=+\infty\}})\,, \\ -\mathrm{div}\left(z\chi_{\{u=-\infty\}}\right) &= f\chi_{\{u=-\infty\}} - (z, D\chi_{\{u=-\infty\}})\,, \\ -\mathrm{div}\left(z\chi_{\{|u|<+\infty\}}\right) &= f\chi_{\{|u|<+\infty\}} + (z, D\chi_{\{|u|=+\infty\}})\,, \end{aligned}$$

holds in the sense of distributions. Thus, we may write

$$\begin{aligned} &\int_{\partial\Omega}[z\chi_{\{u=+\infty\}},\nu]\, v\, d\mathcal{H}^{N-1} \\ &\qquad\qquad = \int_\Omega z\chi_{\{u=+\infty\}}\cdot\nabla w - \int_{\{u=+\infty\}} fw + \int_\Omega w\, d(z, D\chi_{\{u=+\infty\}})\,, \end{aligned}$$

and

$$\begin{aligned} &\int_{\partial\Omega}[z\chi_{\{u=-\infty\}},\nu]\, v\, d\mathcal{H}^{N-1} \\ &\qquad\qquad = \int_\Omega z\chi_{\{u=-\infty\}}\cdot\nabla w - \int_{\{u=-\infty\}} fw + \int_\Omega w\, d(z, D\chi_{\{u=-\infty\}})\,, \end{aligned}$$

where $v \in W^{1-\frac{1}{q},q}(\partial\Omega)\cap L^\infty(\partial\Omega)$, for some $q > N$, and $w \in W^{1,q}(\Omega)\cap L^\infty(\Omega)$ satisfies $w|_{\partial\Omega} = v$. Moreover, it yields

$$\begin{aligned} \int_{\partial\Omega}[z,\nu]\, v\, d\mathcal{H}^{N-1} &= \int_{\partial\Omega}[z\chi_{\{|u|<\infty\}},\nu]\, v\, d\mathcal{H}^{N-1} \\ &\quad + \int_{\partial\Omega}[z\chi_{\{u=+\infty\}},\nu]\, v\, d\mathcal{H}^{N-1} + \int_{\partial\Omega}[z\chi_{\{u=-\infty\}},\nu]\, v\, d\mathcal{H}^{N-1}\,, \end{aligned}$$

for every $v \in W^{1-\frac{1}{q},q}(\partial\Omega)\cap L^\infty(\partial\Omega)$, with $q > N$.

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References

[1] A. Alvino, A. Mercaldo: "Nonlinear elliptic problems with L^1 data : an approach via symmetrization methods" *Mediterr. J. Math., to appear.*

[2] L. Ambrosio, N. Fusco and D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press, Oxford, 2000.

[3] G. Anzellotti: "Pairings between measures and bounded functions and compensated compactness" *Ann. Mat. Pura Appl.* **135** (1983), 293–318

[4] F. Andreu, C. Ballester, V, Caselles and J.M. Mazón: "The Dirichlet problem for the total variation flow" *J. Func. Anal.* **180** (2001), 347–403.

[5] F. Andreu-Vaillo, V. Caselles, J.M. Mazón and J.S. Moll: "The Minimizing Total Variation Flow with Measure Initial Conditions" *Comm. Contem. Math.* **6** (2004), 431–494.

[6] F. Andreu-Vaillo, V. Caselles and J.M. Mazón: *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals* Birkhäuser, Basel-Boston-Berlin, 2004.

[7] G. Bellettini, V. Caselles and M. Novaga: "The Total Variation Flow in $\mathbb{R}^N$" *J. Differential Equations* **184** (2002), 475–525.

[8] G. Bellettini, V. Caselles and M. Novaga: "Explicit solutions of the eigenvalued problem $-\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) = u$ in $\mathbb{R}^2$" *SIAM J. Math. Anal.* **36** (2005), 1095–1129.

[9] Ph. Bénilan, L. Boccardo, Th. Gallouët, R Gariepy, M. Pierre and J.L. Vázquez: "An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations" *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **22** (1995), 241–273.

[10] L. Boccardo and Th. Gallouët: "Nonlinear elliptic and parabolic equations involving measure data" *J. Funct. Anal.* **87** (1989), 149–169.

[11] L. Boccardo and Th. Gallouët: "Nonlinear elliptic equations with right hand side measures" *Comm. Partial Differential Equations* **17** (1992), 641–655.

[12] L. Boccardo, Th. Gallouët and L. Orsina: "Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data" *Ann. Inst. H. Poincaré (Non-Lineaire)* **13** (1996), 539–551.

[13] L. Boccardo, D. Giachetti, J.I. Díaz and F. Murat: "Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms" *J. Differential Equations* **106** (1993), 215–237.

[14] G.–Q. Chen and H. Frid: "Divergence-measure fields and hyperbolic conservation laws." *Arch. Ration. Mech. Anal.* **147** (1999), no. 2, 89–118.

[15] G.–Q. Chen and H. Frid: "On the theory of divergence-measure fields and its applications." Dedicated to Constantine Dafermos on his 60th birthday. *Bol. Soc. Brasil. Mat. (N.S.)* **32** (2001), no. 3, 401–433.

[16] G.–Q. Chen and H. Frid: "Extended divergence-measure fields and the Euler equations for gas dynamics." *Comm. Math. Phys.* **236** (2003), no. 2, 251–280.

[17] M. Cicalese and C. Trombetti: "Asymptotic behaviour of solutions to p-Laplacian equation" *Asymptot. Anal.* **35** (2003), 27–40.

[18] G. Dal Maso and A. Malusa: "Some properties of reachable solutions of nonlinear elliptic equations with measure data" Dedicated to Ennio De Giorgi. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **25** (1997), no. 1-2, 375–396 (1998).

[19] G. Dal Maso, F. Murat, L. Orsina and A. Prignet: "Definition and existence of renormalized solutions of elliptic equations with general measure data" *C. R. Acad. Sci. Paris Série I* **325** (1997), 481–486.

[20] G. Dal Maso, F. Murat, L. Orsina and A. Prignet: "Renormalized solutions of elliptic equations with general measure data" *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **28** (1999), 741–808.

[21] A. Dall'Aglio: "Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations", *Ann. Mat. Pura Appl.* (4) **170** (1996), 207–240.

[22] F. Demengel: "On some nonlinear equation involving the 1-Laplacian and trace map inequalities" *Nonlinear Anal. T.M.A.*, **48** (2002) Ser. A : Theory, Methods, Applications, 1151–1163.

[23] F. Demengel: "Functions locally almost 1-harmonic" *Appl. Anal.* **83** (2004), 865–896.

[24] L.C. Evans and R.F. Gariepy: *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[25] H. Federer: *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969.

[26] G. Huisken and T. Ilmanen: "The inverse mean curvature flow and the Riemannian Penrose inequality" *J. Differential Geom.*, **59** (2001), 353–437.

[27] P. Juutinen: "p-Harmonic approximation of functions of least gradient." *Indiana Univ. Math. J.* **54** (2005), no. 4, 1015–1029.

[28] B. Kawohl: "On a family of torsional creep problems" *J. Reine Angew. Math.*, **410** (1990), 1–22.

[29] B. Kawohl: "From p-Laplace to mean curvature operator and related questions" Progress in partial differential equations: the Metz surveys, 40–56, Pitman Res. Notes Math. Ser., 249, Longman Sci. Tech., Harlow, 1991.

[30] B. Kawohl and V. Fridman: "Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant" *Comment. Math. Univ. Carolin.* **44** (2003), 659–667.

[31] B. Kawohl and F. Schuricht: "Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem" *Commun. Contemp. Math.* **9** (2007), 515–543.

[32] J. Leray and J. L. Lions: "Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder" *Bull. Soc Math. France* **93** (1965), 97–107.

[33] P. L. Lions and F. Murat: "Sur les solutions renormalisées d'équations elliptiques non linéaires" Manuscript.

[34] A. Mercaldo, S. Segura de León and C. Trombetti: "On the behaviour of the solutions to p-Laplacian equations as p goes to 1", *Publ. Mat.* **52** (2008), 377–411.

[35] Y. Meyer: *Oscillating patterns in image processing and nonlinear evolution equations*, The fifteenth Dean Jacqueline B. Lewis memorial lectures. University Lecture Series, 22. American Mathematical Society, Providence, RI, 2001.

[36] F. Murat: "Soluciones renormalizadas de EDP elípticas no lineales" Preprint 93023, Laboratoire d'Analyse Numérique de l'Université Paris VI (1993).

[37] F. Murat: "Équations elliptiques non linéaires avec second membre L^1 ou mesure" *Actes du 26ème Congrés National d'Analyse Numérique,* Les Karellis, France (1994), A12-A24.

[38] J.M. Rakotoson: "Generalized solutions in a new type of sets for problems with measure as data" *Differential Integral Equations,* **6** (1993), 27–36.

[39] J.-F. Rodrigues: *Obstacle Problems in Mathematical Physics,* North-Holland Mathematics Studies, 134, Elsevier Science Publishers B. V. 1987.

[40] G. Talenti: "Best constant in Sobolev inequality" *Ann. Mat. Pura Appl.* **110** (1976), 353–372.

[41] G. Talenti: "Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces" *Ann. Mat. Pura Appl.* **120** (1979), 160–184.

[42] W.P. Ziemer: *Weakly differentiable functions*, Springer, Berlin, 1989.