BEHAVIOUR OF p–LAPLACIAN PROBLEMS WITH NEUMANN BOUNDARY CONDITIONS WHEN p GOES TO 1

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Abstract. We consider the solution u_p to the Neumann problem for the p–Laplacian equation with the normal component of the flux across the boundary given by $g \in L^\infty(\partial\Omega)$. We study the behaviour of u_p as p goes to 1 showing that they converge to a measurable function u and the gradients $|\nabla u_p|^{p-2}\nabla u_p$ converge to a vector field z.

We prove that z is bounded and that the properties of u depend on the size of g measured in a suitable norm: if g is small enough, then u is a function of bounded variation (it vanishes on the whole domain, when g is very small) while if g is large enough, then u takes the value ∞ on a set of positive measure. We also prove that in the first case, u is a solution to a limit problem that involves the 1−Laplacian. Finally, explicit examples are shown.

1. Introduction

In this paper we deal with the limit as p goes to 1 of solutions to the p–Laplacian with non-homogeneous Neumann boundary conditions. To be more precise, consider the following problem:

$$\begin{cases} -\mathrm{div}\big(|\nabla u_p|^{p-2}\nabla u_p\big) = 0, & \text{in } \Omega, \\ |\nabla u_p|^{p-2}\dfrac{\partial u_p}{\partial \nu} = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$ and ν denotes the unit outward normal to Ω. As far as the datum g is concerned, it belongs to $L^\infty(\partial\Omega)$, and verifies the compatibility condition

$$\int_{\partial\Omega} g \, d\mathcal{H}^{N-1} = 0. \tag{1.2}$$

In order to obtain a unique solution we impose the normalization

$$\int_{\partial\Omega} u_p \, d\mathcal{H}^{N-1} = 0. \tag{1.3}$$

Our aim is to study the behaviour as p goes to 1 of the solutions u_p. Thus, we may assume without loss of generality $p < N$. If we argue formally the limit $\lim_{p\to 1} u_p = u$ should be a solution to the following limit problem that involves the 1–Laplacian,

$$\begin{cases} -\mathrm{div}\Big(\dfrac{Du}{|Du|}\Big) = 0, & \text{in } \Omega\,, \\ \Big[\dfrac{Du}{|Du|}, \nu\Big] = g, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

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Note that one of the major difficulties to define a solution to this problem is to give a sense to $\frac{Du}{|Du|}$ when $Du = 0$. This difficulty was tackled for homogeneous Neumann boundary conditions in [2] (where the correct concept of solution is introduced) and also in [4], where the authors deal with a nonlinear boundary condition: $-\frac{Du}{|Du|} \cdot \nu \in \beta(u)$. However, up to our knowledge, this is the first time that inhomogeneous Neumann boundary conditions are studied. For the equation in (1.4) with Dirichlet boundary conditions and a nontrivial right hand side, see [7], [9], [10] and the book [3].

Our main result states that the functions u_p converge pointwise to a measurable function u whose features depend on the size of g. More precisely, there exists $\|\cdot\|_*$ a norm in $L^\infty(\partial\Omega)$ (see Definition 2.3, this norm is actually equivalent to the usual one in $L^\infty(\partial\Omega)$), such that,

If $\|g\|_* < 1$, then $u(x) = 0$ for all $x \in \Omega$.

If $\|g\|_* = 1$, then $u \in BV(\Omega)$ is a solution to the limit problem.

If $\|g\|_* > 1$, then $|u| = \infty$ on a set of positive measure.

We point out that, as in in the case of Dirichlet problem (see [7], [6], [9]) our methods can also be applied to study the behaviour of solutions of the problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u_p|^{p-2}\nabla u_p\big) = f, & \text{in } \Omega, \\ |\nabla u_p|^{p-2}\dfrac{\partial u_p}{\partial \nu} = g, & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^N(\Omega)$ (or f in the Marcinkiewicz space $L^{N,\infty}(\Omega)$) and $g \in L^\infty(\partial\Omega)$, but then we have to consider a quantity that depends on the size of f and g. We just restrict ourselves to (1.1) for the sake of simplicity.

This paper is organized as follows: next section is devoted to fix our notation and introduce the precise norm that measures the size of g. The behaviour of the solutions u_p is studied in Section 3, we prove that u_p converge to a measurable function u whose main features depend on the size of g. In Section 4 we analyze conditions under which the limit function u is solution of the limit problem. Finally, in Section 5, we compute explicit examples of solutions u_p and their limit.

2. Notation and auxiliary results

Throughout this paper Ω will denote an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Thus, there exists a unit vector defined on $\partial\Omega$ that is outward normal to Ω: it will be denoted by ν. This vector field is defined for $\mathcal{H}^{N-1}$–almost every point of $\partial\Omega$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure.

The energy space to study problems (1.1) is the Sobolev space $W^{1,p}(\Omega)$, while the natural energy space for considering the limit problem is the space of functions of bounded variation $BV(\Omega)$. We refer to [1] for information concerning functions of bounded variation and their features.

Along this paper we will always assume that a weak solution u_p to (1.1) is normalized according to (1.3). So we begin by proving the existence and uniqueness of such a solution. Let us recall the definition of weak solution to problem (1.1)

Definition 2.1. *We say that $u \in W^{1,p}(\Omega)$ is a weak solution to* (1.1) *if there holds*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = \int_{\partial\Omega} g\varphi \, d\mathcal{H}^{N-1}$$

for every $\varphi \in W^{1,p}(\Omega)$.

Theorem 2.2. *Assume that* g *verifies* (1.2)*, then there exists a weak solution to* (1.1)*. Moreover, the solution is unique if we normalize it according to* (1.3)*.*

Proof. The proof is standard. The result can be obtained minimizing the functional

$$F(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \int_{\partial\Omega} gu \, d\mathcal{H}^{N-1}$$

in the space

$$S_p = \left\{ u \in W^{1,p}(\Omega) \ : \ \int_{\partial\Omega} u \, d\mathcal{H}^{N-1} = 0 \right\}.$$

Just recall that the usual norm of $W^{1,p}(\Omega)$ is equivalent to the norm $\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)}$ in S_p. Hence, we have the compact Sobolev trace embedding $S_p \hookrightarrow L^r(\partial\Omega)$ when $r = p(N-1)/(N-p)$, therefore $\int_{\partial\Omega} gu \, d\mathcal{H}^{N-1}$ is well defined for $g \in L^\infty(\partial\Omega)$ and $u \in S_p$. ■

As for S_p, let us introduce

$$S_1 = \left\{ u \in W^{1,1}(\Omega) \ : \ \int_{\partial\Omega} u \, d\mathcal{H}^{N-1} = 0 \right\}.$$

We recall that in S_1 the norm $\|\nabla u\|_{L^1(\Omega;\mathbb{R}^N)}$ turns to be equivalent to the usual norm of $W^{1,1}(\Omega)$. We will use this space to define the norm $\|\cdot\|_*$ in $L^\infty(\partial\Omega)$.

Definition 2.3. *For every* $g \in L^\infty(\partial\Omega)$*, we define*

$$\|g\|_* = \sup_{u \in S_1 \setminus \{0\}} \left\{ \frac{\int_{\partial\Omega} gu \, d\mathcal{H}^{N-1}}{\int_\Omega |\nabla u| \, dx} \right\}.$$

Now, our goal is to show that $\|\cdot\|_*$ is equivalent to $\|\cdot\|_{L^\infty(\partial\Omega)}$. To this aim we need the following lemma.

Lemma 2.4. *There exists a constant* $\Lambda = \Lambda(\Omega) \geq 1$ *satisfying*

$$\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \leq \Lambda \int_\Omega |\nabla u| \, dx$$

for every $u \in S_1$.

Proof. By the continuity of the trace operator

$$S_1 \hookrightarrow L^1(\partial\Omega),$$

we already know that this constant is finite and positive. We will prove $\Lambda \geq 1$.

Let $u \in W^{1,1}(\Omega)$ satisfy $u \neq 0$ on $\partial\Omega$ and $\int_{\partial\Omega} u \, d\mathcal{H}^{N-1} = 0$. Applying a result in [5], we may find a sequence $(w_n)_n$ in $W^{1,1}(\Omega)$ satisfying

$$w_n\big|_{\partial\Omega} = u\big|_{\partial\Omega}, \qquad \text{and} \qquad \int_\Omega |\nabla w_n| \, dx \leq \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} + \frac{1}{n}.$$

Hence,

$$\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \leq \Lambda \int_\Omega |\nabla w_n| \, dx \leq \Lambda \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} + \frac{\Lambda}{n}.$$

Letting $n \to \infty$, we deduce $\Lambda \geq 1$. ■

REMARK 2.5. One may wonder if $\Lambda = 1$ could happen. This actually occurs in one dimensional domains. Indeed, set $\Omega =]a, b[$. Assuming, for definiteness, that $0 < u(b) = -u(a)$, we have

$$\Lambda = \sup_{u \in S_1} \left\{ \frac{|u(b)| + |u(a)|}{\int_a^b |u'|} \right\} = \sup_{u \in S_1} \left\{ \frac{u(b) - u(a)}{\int_a^b u'} \ : \ u \text{ increasing} \right\} .$$

By Lebesgue's version of Barrow's rule, every quotient in the above expression is equal to one. Therefore, $\Lambda = 1$.

This no longer happens in higher dimensions where Λ can be as large as we want. A simple example in $\mathbb{R}^2$ is as follows. Let $\Omega =]-L, L[\times]0, 1[$ and consider the function defined by

$$u(x, y) = \begin{cases} -\frac{1}{2}\,, & \text{if } x < -\frac{1}{2}\,; \\ x\,, & \text{if } -\frac{1}{2} < x < \frac{1}{2}\,; \\ \frac{1}{2}\,, & \text{if } x > \frac{1}{2}\,. \end{cases}$$

Then

$$\int_{\partial\Omega} u \, d\mathcal{H}^1 = 0\,, \quad \int_\Omega |\nabla u| \, dx = 1 \text{ and } \int_{\partial\Omega} |u| \, d\mathcal{H}^1 = L + \frac{1}{2}\,.$$

Hence,

$$\Lambda \geq \frac{\int_{\partial\Omega} |u| \, d\mathcal{H}^1}{\int_\Omega |\nabla u| \, dx} = L + \frac{1}{2}\,.$$

Proposition 2.6. *$\| \cdot \|_*$ is a norm on $L^\infty(\partial\Omega)$ such that*

$$\|g\|_{L^\infty(\partial\Omega)} \leq \|g\|_* \leq \Lambda \|g\|_{L^\infty(\partial\Omega)}\,,$$

for all $g \in L^\infty(\partial\Omega)$. In other words, $\| \cdot \|_*$ and $\| \cdot \|_{L^\infty(\partial\Omega)}$ are equivalent norms.

Proof. Fixed $g \in L^\infty(\partial\Omega)$, we define $T : S_1 \to \mathbb{R}$ by

$$T(u) = \int_{\partial\Omega} gu \, d\mathcal{H}^{N-1}\,.$$

So that $\|T\| = \|g\|_*$. We claim that

$$\|g\|_{L^\infty(\partial\Omega)} \leq \|T\| \leq \Lambda \|g\|_{L^\infty(\partial\Omega)}.$$

Applying Lemma 2.4, we obtain

$$|T(u)| \leq \|g\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \leq \Lambda \|g\|_{L^\infty(\partial\Omega)} \int_\Omega |\nabla u| \, dx\,,$$

so that $\|T\| \leq \Lambda \|g\|_{L^\infty(\partial\Omega)}$. To see the other inequality, fix $h \in L^1(\partial\Omega)$ such that $\|h\|_1 \leq 1$. Applying the same result in [5] as in Lemma 2.4, we may find a sequence $(w_n)_n$ in $W^{1,1}(\Omega)$ satisfying

$$w_n\big|_{\partial\Omega} = h\,, \qquad \text{and} \qquad \int_\Omega |\nabla w_n| \, dx \leq \int_{\partial\Omega} |h| \, d\mathcal{H}^{N-1} + \frac{1}{n}\,.$$

Thus,

$$\left| \int_{\partial\Omega} gh \, d\mathcal{H}^{N-1} \right| = |T(w_n)| \leq \|T\| \int_\Omega |\nabla w_n| \, dx \leq \|T\| \left(\int_{\partial\Omega} |h| \, d\mathcal{H}^{N-1} + \frac{1}{n} \right)\,,$$

that is,

$$\left| \int_{\partial\Omega} gh \, d\mathcal{H}^{N-1} \right| \leq \|T\| \, \|h\|_1 \leq \|T\|\,.$$

By duality,

$$\|g\|_{L^\infty(\partial\Omega)} = \sup\left\{\left|\int_{\partial\Omega} gh\, d\mathcal{H}^{N-1}\right| \,:\, \int_{\partial\Omega} |h|\, d\mathcal{H}^{N-1} \le 1\right\} \le \|T\|\,.$$

■

3. Convergence of u_p as p goes to 1

In what follows, abusing of the terminology, we will say that u_p is a sequence and we will consider subsequences of it, as p goes to 1.

We begin by establishing the following fundamental estimate:

Lemma 3.1. *Let u_p denote a weak solution to* (1.1)*. Then the following estimate holds*

$$\int_\Omega |\nabla u_p|\, dx \le \|g\|_*^{\frac{1}{p-1}} |\Omega|\,. \tag{3.1}$$

Proof. Taking u_p as test function in (1.1), we obtain

$$\int_\Omega |\nabla u_p|^p\, dx = \int_{\partial\Omega} g u_p \le \|g\|_* \|\nabla u_p\|_{L^1(\Omega;\mathbb{R}^N)}\,.$$

Theorem 2.4 and Hölder's inequality yield

$$\int_\Omega |\nabla u_p|\, dx \le \Big(\int_\Omega |\nabla u_p|^p\, dx\Big)^{1/p} |\Omega|^{(p-1)/p} \le \|g\|_*^{1/p} \|\nabla u_p\|^{1/p}_{L^1(\Omega;\mathbb{R}^N)} |\Omega|^{(p-1)/p}\,.$$

Therefore,

$$\left(\int_\Omega |\nabla u_p|\, dx\right)^{1-\frac{1}{p}} \le \|g\|_*^{1/p} |\Omega|^{(p-1)/p}\,,$$

and we conclude that (3.1) holds true. ■

Next, we study the behaviour of u_p in the case where the datum g is small, that is $\|g\|_* \le 1$.

Proposition 3.2. *Assume that $g \in L^\infty(\partial\Omega)$ satisfies $\|g\|_* \le 1$. Then there exists $u \in BV(\Omega)$ and a subsequence of u_p, not relabelled, satisfying*

$$\nabla u_p \rightharpoonup Du \quad \text{*–weakly in the sense of measures;} \tag{3.2}$$

$$u_p \to u \quad \text{a.e. in } \Omega\,; \tag{3.3}$$

$$u_p \to u \quad \text{in } L^1(\Omega). \tag{3.4}$$

Proof. It is a straightforward consequence of the previous Lemma, since then

$$\int_\Omega |\nabla u_p|\, dx \le |\Omega| \qquad \text{for all } p\,.$$

■

Corollary 3.3. *If $\|g\|_* < 1$ then*

$$u_p \to 0$$

in the same topologies used in the previous result and strongly in $L^1(\partial\Omega)$.

Proof. It is a consequence of (3.1) and the fact that

$$\int_{\partial\Omega} |u_p|\, d\mathcal{H}^{N-1} \le \Lambda \int_\Omega |\nabla u_p|\, dx$$

using the lower–semicontinuity of the functional

$$u \to \int_\Omega |\nabla u|\, dx + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1}\,.$$

■

Let us denote by T_k the truncation at level k, that is,

$$T_k(s) = \begin{cases} k & s > k, \\ s & -k \le s \le k, \\ -k & s < -k. \end{cases}$$

Theorem 3.4. *Let u_p be the solution to problem* (1.1)*, then there exists a measurable function u such that*

$$T_k(u) \in BV(\Omega)\,, \quad \text{for all } k > 0\,,$$

and, up to subsequences,

$$u_p \to u \quad a.e. \ in\ \Omega\,.$$

Proof. Following [11], consider $\Psi(s) = s/(1+|s|)$, which is a strictly increasing and bounded real function. Moreover

$$\Big| \int_0^{u_p} (\Psi'(s))^p\, ds \Big| \le \int_0^{|u_p|} \Psi'(s)\, ds = \Psi(|u_p|) \le 1.$$

So that if we take

$$\phi(x) = \int_0^{u_p(x)} (\Psi'(s))^p\, ds,$$

as test function in (1.1), then

$$\int_\Omega \Psi'(u_p)^p |\nabla u_p|^p\, dx = \int_{\partial\Omega} g\phi\, d\mathcal{H}^{N-1}\,.$$

In other words

$$\int_\Omega |\nabla \Psi(u_p)|^p\, dx \le \int_{\partial\Omega} |g|\, d\mathcal{H}^{N-1}\,.$$

Thus, Hölder's inequality implies that the sequence $\big(\Psi(u_p)\big)_p$ is bounded in $W^{1,1}(\Omega)$ and so a subsequence, also denoted by $\big(\Psi(u_p)\big)_p$, converges *-weakly in $BV(\Omega)$. As a consequence, it also converges strongly in $L^1(\Omega)$ and a.e. Since Ψ is strictly increasing, the sequence $(u_p)_p$ tends a.e. to a measurable function u. We point out that, when $\lim_{p\to 1} \Psi(u_p) = \pm 1$, we have $u = \pm\infty$.

On the other hand, taken $T_k(u_p)$ as test function in (1.1), we have that

$$\int_\Omega |\nabla T_k(u_p)|^p \le \int_{\partial\Omega} g T_k(u_p)\, d\mathcal{H}^{N-1} \le k \int_{\partial\Omega} |g|\, d\mathcal{H}^{N-1}\,. \tag{3.5}$$

Young's inequality implies that $T_k(u_p)$ is bounded in $W^{1,1}(\Omega)$ and, by the pointwise convergence $T_k(u_p) \to T_k(u)$, we obtain

$$T_k(u_p) \rightharpoonup T_k(u) \quad \text{*–weakly in } BV(\Omega)$$

Thus, $T_k(u) \in BV(\Omega)$. ■

To finish this section we study the convergence of the gradients. In the statement of the next result, we deal with the weak trace on $\partial\Omega$ of the normal component of z, which will be denoted by $[z,\nu]$. It is a function belonging to $L^\infty(\partial\Omega)$ whose existence is guaranteed by the theory of bounded divergence–measure vector fields of Anzellotti [5]. It is proved in [5] that if $v \in W^{1,1}(\Omega)$ and $z \in L^\infty(\Omega;\mathbb{R}^N)$ satisfies $\operatorname{div} z \in L^N(\Omega)$, then the following Green formula holds

$$\int_\Omega v \operatorname{div} z\,dx + \int_\Omega z\cdot\nabla v\,dx = \int_{\partial\Omega}[z,\nu]\,v\;d\mathcal{H}^{N-1}\,. \tag{3.6}$$

Theorem 3.5. *Let u_p be the solution to problem* (1.1)*, then there exists a vector field $z \in L^\infty(\Omega;\mathbb{R}^N)$ such that, up to subsequences,*

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \textit{weakly in } L^s(\Omega) \textit{ for all } 1 \le s < +\infty\,, \tag{3.7}$$

$$-\operatorname{div} z = 0 \quad \textit{in } \mathcal{D}'(\Omega)\,, \tag{3.8}$$

$$\|z\|_{L^\infty(\Omega;\mathbb{R}^N)} = \|g\|_*\,, \tag{3.9}$$

$$[z,\nu] = g \quad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega\,. \tag{3.10}$$

Proof. We will follow the arguments of Proposition 4.1 in [9].

Step 1: Proof of (3.7). Arguing as in the proof of Theorem 3.1, we obtain the inequality (3.1),

$$\int_\Omega |\nabla u_p|\,dx \le \|g\|_*^{\frac{1}{p-1}}|\Omega|\,.$$

Then for every s, $1 \le s < p'$, we have

$$\begin{aligned}\int_\Omega |\nabla u_p|^{(p-1)s}\,dx &\le \Big(\int_\Omega |\nabla u_p|^p\,dx\Big)^{(p-1)s/p}|\Omega|^{1-\frac{(p-1)s}{p}}\\ &\le |\Omega|^{\frac{(p-1)s}{p}}\|g\|_*^{p'\frac{(p-1)s}{p}}|\Omega|^{1-\frac{(p-1)s}{p}}\\ &= |\Omega|\|g\|_*^s\,.\end{aligned} \tag{3.11}$$

This implies that, for any $s>1$ fixed, the sequence $|\nabla u_p|^{p-2}\nabla u_p$ is bounded in $L^s(\Omega;\mathbb{R}^N)$ and then there exists $z_s \in L^s(\Omega;\mathbb{R}^N)$ such that, up to subsequences,

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z_s \quad \text{in } L^s(\Omega;\mathbb{R}^N) \quad \text{for all } 1 \le s < +\infty\,.$$

Moreover, by a diagonal argument we can find a limit z that does not depend on s, that is

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \text{in } L^s(\Omega;\mathbb{R}^N) \quad \text{for all } 1 \le s < +\infty\,. \tag{3.12}$$

Now by (3.11) we deduce

$$\||\nabla u_p|^{p-2}\nabla u_p\|_{L^s(\Omega;\mathbb{R}^N)} \le |\Omega|^{1/s}\|g\|_* \quad \text{for } 1 \le s < +\infty \text{ and for } p \in]1, s'[\,.$$

Therefore, by lower semicontinuity of the norm, we have

$$\|z\|_{L^s(\Omega;\mathbb{R}^N)} \le |\Omega|^{1/s}\|g\|_* \quad \text{for all } 1 \le s < +\infty\,.$$

Letting $s \to \infty$, we get that $z \in L^\infty(\Omega;\mathbb{R}^N)$ and

$$\|z\|_{L^\infty(\Omega;\mathbb{R}^N)} \le \|g\|_* \quad \text{for all } 1 \le s < +\infty\,. \tag{3.13}$$

Step 2: Proof of (3.8). Since u_p is a distributional solution to problem (1.1), it follows that

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi\, dx = 0, \qquad \forall\varphi \in C_0^\infty(\Omega).$$

Hence, using (3.12) we obtain

$$\int_\Omega z\cdot\nabla\varphi\, dx = 0\,, \qquad \forall\varphi\in C_0^\infty(\Omega)\,,$$

that is (3.8).

Step 3: Proof of (3.10). Let $1 < p < 2$ and consider $v \in W^{1,2}(\Omega)$ as test function, then we get

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla v\, dx = \int_{\partial\Omega} gv\, d\mathcal{H}^{N-1}\,.$$

By letting p to 1, we obtain

$$\int_\Omega z\cdot\nabla v\, dx = \int_{\partial\Omega} gv\, d\mathcal{H}^{N-1}\,.$$

By density, it follows that

$$\int_\Omega z\cdot\nabla v\, dx = \int_{\partial\Omega} gv\, d\mathcal{H}^{N-1}\,,$$

for every $v \in W^{1,1}(\Omega)$. Having $\operatorname{div} z = 0$ in mind, we apply Green's formula (3.6) to the left hand side and obtain,

$$\int_{\partial\Omega} v[z,\nu]\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} gv\, d\mathcal{H}^{N-1}\,,$$

for every $v \in W^{1,1}(\Omega)$, hence

$$\int_{\partial\Omega} h[z,\nu]\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} gh\, d\mathcal{H}^{N-1}\,,$$

for every $h \in L^1(\partial\Omega)$. We conclude that $[z,\nu] = g$ $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$.

Step 4: Proof of (3.9). We already know that $\|z\|_{L^\infty(\Omega;\mathbb{R}^N)} \le \|g\|_*$, by (3.13). The reverse inequality follows applying Green's formula (3.6). Indeed, given $v \in S_1$ and having in mind $\operatorname{div} z = 0$, we have

$$\int_{\partial\Omega} vg\, d\mathcal{H}^{N-1} = \int_{\partial\Omega} v[z,\nu]\, d\mathcal{H}^{N-1} = \int_\Omega z\cdot\nabla v\, dx \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}\int_\Omega |\nabla v|\, dx\,.$$

Since

$$\frac{\int_{\partial\Omega} vg\, d\mathcal{H}^{N-1}}{\int_\Omega |\nabla v|\, dx} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}\,, \quad \text{for all } v\in S_1\backslash\{0\}\,,$$

we obtain $\|g\|_* \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$. Therefore (3.9) is proved. ■

4. Existence of solutions to the limit problem

In this section, we consider the limit problem to 1.1, that is

$$\begin{cases} -\mathrm{div}\Big(\dfrac{Du}{|Du|}\Big) = 0, & \text{in } \Omega\,, \\ \Big[\dfrac{Du}{|Du|},\nu\Big] = g, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Firstly we need a notion of solution to (4.1). To understand the meaning of being a solution to (4.1), we have to begin by giving a sense to the quotient $\frac{Du}{|Du|}$. This can be done using the theory of L^∞–divergence–measure vector fields developed by Anzellotti [5].

Given $z \in L^\infty(\Omega;\mathbb{R}^N)$ with distributional divergence $\mathrm{div}\, z \in L^N(\Omega)$ and $u \in BV(\Omega)$, we define the following distribution on Ω: for every $\varphi \in C_0^\infty(\Omega)$, we write

$$\langle (z,Du),\varphi\rangle = -\int_\Omega u\,\varphi\,\mathrm{div}\, z\,dx - \int_\Omega uz\cdot\nabla\varphi\,dx\,.$$

In [5] (see also [3, Corollary C.7, C.16]) it is proved the following result.

Proposition 4.1. *The distribution (z,Du) is actually a Radon measure with finite total variation. The measures (z,Du), $|(z,Du)|$ satisfy*

$$\left|\int_B (z,Du)\right| \le \int_B |(z,Du)| \le \|z\|_{L^\infty(U)}\int_B |Du|$$

for all Borel sets B and for all open sets U such that $B\subset U\subset\Omega$.

Denoting by $\theta(z,Du,\cdot):\Omega\to\mathbb{R}$ the Radon–Nikodým derivative of (z,Du) with respect to $|Du|$, it follows that

$$\int_B (z,Du) = \int_B \theta(z,Du,x)\,|Du| \qquad \text{for all Borel sets } \ B\subset\Omega$$

and

$$\|\theta(z,Du,\cdot)\|_{L^\infty(\Omega,|Du|)} \le \|z\|_\infty.$$

Moreover, if $F:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous increasing function, then

$$\theta(z,D(F\circ u),x) = \theta(z,Du,x), \qquad |Du|-\text{a.e. in } \ \Omega \tag{4.2}$$

Moreover, the following *Green's formula*, relating the bounded function $[z,\nu]$ and the measure (z,Du) is established

$$\int_\Omega u\ \mathrm{div}\,(z)\ dx + \int_\Omega (z,Du) = \int_{\partial\Omega}[z,\nu]u\ d\mathcal{H}^{N-1}. \tag{4.3}$$

Now we are ready to introduce our notion of solution to problem (4.1).

Definition 4.2. *We say that $u\in BV(\Omega)$ is a solution to* (1.1) *if the following hold: There exists $z\in L^\infty(\Omega;\mathbb{R}^N)$ satisfying*

$$\|z\|_{L^\infty(\Omega;\mathbb{R}^N)} \le 1\,; \tag{4.4}$$

$$-\mathrm{div}\, z = 0 \quad \textit{in } \mathcal{D}'(\Omega)\,; \tag{4.5}$$

$$[z,\nu] = g \quad \mathcal{H}^{N-1}\textit{-a.e. on } \ \partial\Omega\,; \tag{4.6}$$

$$(z,\,Du) = |Du| \quad \textit{as Radon measures}\,. \tag{4.7}$$

By applying Green's formula, one can easily deduce that the following variational formulation,

$$\int_\Omega |Du| - \int_\Omega (z, Dv) = \int_{\partial\Omega} g(u - v),$$

holds for every $v \in BV(\Omega)$.

We point out that if u is a solution to (4.1) and we add a constant, then u and $u+C$ have the same gradient and so we obtain another solution to (4.1). Therefore, adding a constant if necessary, we may always assume that our solution satisfies the normalization condition $\int_{\partial\Omega} u = 0$. However this normalization does not imply uniqueness as the following result shows.

Theorem 4.3. *Given u a solution to* (4.1) *and F a Lipschitz continuous and increasing function, then $F(u)$ is also a solution to* (4.1).

Proof. The same vector field z will do the job. Indeed, the only condition that remains to check is $(z, DF(u)) = |DF(u)|$ as measures. Since $(z, Du) = |Du|$, the Radon–Nikodým derivative of (z, Du) with respect to $|Du|$ is identically 1. By (4.2), the Radon–Nikodým derivative of $(z, DF(u))$ with respect to $|DF(u)|$ is also identically 1 $|Du|$–a.e. Hence, we deduce $(z, DF(u)) = |DF(u)|$ as measures. ■

As far as the existence concerns, we prove that problem (4.1) has a solution if $\|g\|_* \le 1$; such a solution is the limit function of u_p. In contrast, if $\|g\|_*$ is large (4.1) has not a solution, since in this case the limit function of u_p is not in $BV(\Omega)$.

Theorem 4.4. *Assume that $g \in L^\infty(\partial\Omega)$ with $\|g\|_* \le 1$. Then there exists, at least, a solution u to problem* (4.1). *In particular if $\|g\|_* < 1$, then $u \equiv 0$.*

Proof. By applying Theorem 3.2 and Theorem 3.5, we obtain u and z satisfying (4.4), (4.5) and (4.6) in Definition 4.2. We proceed to prove (4.7), the last condition of Definition 4.2.

We now choose $u_p\, \varphi$, with $\varphi \in C_0^\infty(\Omega)$ and $\varphi \ge 0$, as test function in (1.1). Then

$$\int_\Omega \varphi |\nabla u_p|^p \, dx + \int_\Omega u_p |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla\varphi \, dx = 0\,.$$

We apply Young's inequality and let p goes to 1 to obtain

$$\int_\Omega \varphi |Du| + \int_\Omega u z \cdot \nabla\varphi \le 0\,.$$

It follows from Green's formula that

$$\int_\Omega \varphi |Du| \le \langle (z, Du), \varphi \rangle\,,$$

for all $\varphi \in C_0^\infty(\Omega)$ satisfying $\varphi \ge 0$. Hence,

$$|Du| \le (z, Du) \qquad \text{as measures.}$$

Equality follows since

$$(z, Du) \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)} |Du| \le |Du|\,.$$

■

Theorem 4.5. *Assume that $g \in L^\infty(\partial\Omega)$ with $\|g\|_* > 1$ and, for each $k > 0$, denote $z_k = z\chi_{\{|u|<k\}}$. Then the following conditions hold*

$$\|z_k\|_{L^\infty(\Omega;\mathbb{R}^N)} \leq 1\,; \tag{4.8}$$

$$-\operatorname{div} z_k = (z, D\chi_{\{|u|\geq k\}}) \quad \text{in } \mathcal{D}'(\Omega)\,. \tag{4.9}$$

An immediate consequence of this theorem is that the limit u cannot be finite a.e when $\|g\|_* > 1$.

Corollary 4.6. *If $\|g\|_* > 1$ then $|u| = +\infty$ on a set of positive measure. Hence, $u \notin BV(\Omega)$.*

Proof. Since $\|z\|_{L^\infty(\Omega;\mathbb{R}^N)} = \|g\|_* > 1$ and $\|z\chi_{\{|u|<+\infty\}}\|_{L^\infty(\Omega;\mathbb{R}^N)} \leq 1$, it follows from the previous theorem $|u| = +\infty$ on a set of positive measure. ■

Proof of Theorem 4.5. Here we use arguments from [2] (see also [9]). By (3.11), for any fixed $k > 0$, the sequence $(|\nabla u_p|^{p-1}\ \chi_{\{|u_p|<k\}})_p$ is bounded in $L^s(\Omega, \mathbb{R}^N)$. Thus, as p goes to 1, we have

$$|\nabla u_p|^{p-2}\nabla u_p\ \chi_{\{|u_p|<k\}} \rightharpoonup w_k \quad \text{weakly in} \quad L^1(\Omega;\mathbb{R}^N), \tag{4.10}$$

for some vector field $w_k \in L^1(\Omega;\mathbb{R}^N)$. For every fixed $k > 0$, $h > 0$ and $p > 1$, we denote

$$B_{p,h,k} = \{x \in \Omega : |\nabla T_k(u_p)| > h\}.$$

Applying again (3.11), as p goes to 1, we have (up to subsequences)

$$|\nabla u_p|^{p-2}\nabla u_p\ \chi_{B_{p,h,k}\cap\{|u_p|<k\}} \rightharpoonup g_{h,k} \quad \text{weakly in} \quad L^1(\Omega, \mathbb{R}^N), \tag{4.11}$$

and

$$|\nabla u_p|^{p-2}\nabla u_p\ \chi_{(\Omega\setminus B_{p,h,k})\cap\{|u_p|<k\}} \rightharpoonup f_{h,k} \quad \text{weakly in} \quad L^1(\Omega, \mathbb{R}^N), \tag{4.12}$$

for some $g_{h,k} \in L^1(\Omega;\mathbb{R}^N)$ and $f_{h,k} \in L^1(\Omega;\mathbb{R}^N)$. On the other hand, by (3.5) the following inequality holds true

$$|B_{p,h,k}| \leq \frac{1}{h^p}\int_\Omega |\nabla T_k(u_p)|^p \leq \frac{k}{h^p}\|g\|_{L^1(\partial\Omega)}\,. \tag{4.13}$$

Therefore, by Hölder's inequality, (3.5) and (4.13), for any $\Phi \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\|\Phi\|_{L^\infty(\Omega;\mathbb{R}^N)} \leq 1$, we have

$$\begin{aligned}\Big|\int_{B_{p,h,k}\cap\{|u_p|<k\}} |\nabla u_p|^{p-2}\nabla u_p\cdot\Phi\,dx\Big| &\leq \left(\int_\Omega |\nabla T_k u_p|^p\,dx\right)^{(p-1)/p} |B_{p,h,k}|^{1/p}\\ &\leq \left(k\|g\|_{L^1(\partial\Omega)}\right)^{(p-1)/p}\left(\frac{k\|g\|_{L^1(\partial\Omega)}}{h^p}\right)^{1/p} = \frac{k\|g\|_{L^1(\partial\Omega)}}{h}.\end{aligned}$$

By (4.11), for any fixed $k > 0$ and $h > 0$, this implies

$$\Big|\int_\Omega g_{h,k}\cdot\Phi\,dx\Big| \leq \frac{k\|g\|_{L^1(\partial\Omega)}}{h},$$

for any $\Phi \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\|\Phi\|_{L^\infty(\Omega;\mathbb{R}^N)} \leq 1$. By duality, we deduce the following estimate for g_{hk}

$$\int_\Omega |g_{h,k}|\,dx \leq \frac{k\|g\|_{L^1(\partial\Omega)}}{h},$$

for any fixed $h > 0$ and $k > 0$. Moreover, by definition of the set $B_{p,h,k}$ we have

$$\left||\nabla u_p|^{p-2}\nabla u_p \; \chi_{(\Omega\setminus B_{p,h,k})\cap\{|u_p|<k\}}\right| \le h^{p-1} \qquad \text{a.e. in } \Omega.$$

This implies the following pointwise estimate for $f_{h,k}$

$$|f_{h,k}| \le \lim_{p\to 1} h^{p-1} = 1, \qquad \text{a.e. in } \Omega\,.$$

For any fixed $h > 0$ and $k > 0$, we have

$$w_k = f_{h,k} + g_{h,k}$$

with

$$\|f_{h,k}\|_{L^\infty(\Omega;\mathbb{R}^N)} \le 1 \qquad \text{and} \qquad \int_\Omega |g_{h,k}|\,dx \le \frac{M}{h}.$$

Therefore, letting $h \to \infty$, we obtain

$$\|w_k\|_{L^\infty(\Omega;\mathbb{R}^N)} \le 1, \tag{4.14}$$

for all $k > 0$. Now observe that, since $|\Omega| < +\infty$, the set of the values k such that $|\{|u| = k\}| > 0$ is countable. So it follows from $\lim_{p\to1} u_p(x) = u(x)$ almost everywhere in Ω, that

$$\chi_{\{|u_p|<k\}} \to \chi_{\{|u|<k\}}, \quad \text{strongly in} \quad L^\rho(\Omega), \quad \text{for every } 1 \le \rho < +\infty\,,$$

for almost all $k > 0$. Therefore, by (3.7) and (4.10), we conclude

$$w_k = z\,\chi_{\{|u|<k\}} = z_k\,,$$

for almost all $k > 0$. Observe that, from

$$\lim_{k\to+\infty} w_k = \lim_{k\to+\infty} z\chi_{\{|u|<k\}} = z\chi_{\{|u|<+\infty\}}, \qquad \text{a.e. in } \Omega,$$

and (4.14), we deduce $\|z\ \chi_{\{|u|<k\}}\|_{L^\infty(\Omega;\mathbb{R}^N)} \le \|z\ \chi_{\{|u|<+\infty\}}\|_{L^\infty(\Omega;\mathbb{R}^N)} \le 1$ for all $k > 0$. This proves (4.8).

We still have to prove (4.9). This is a consequence of the following computations (see [8, equation (2.7)]):

$$-\mathrm{div}\,(z\chi_{\{|u|<k\}}) = -\mathrm{div}\,(z)\llcorner_{\{|u|<k\}} - (z, D\chi_{\{|u|<k\}}) = (z, D\chi_{\{|u|\ge k\}})\,.$$

■

5. Examples

In this section we will compute explicit examples of solutions to our problem (4.1) as limit of solutions to (1.1).

5.1. Dimension 1. Set $\Omega =]-1,1[$ and $g(\pm1) = \pm A$, with $A > 0$. The normalized solution of

$$\begin{cases} -(|u_p'|^{p-2}u_p')' = 0\,, & \text{in }]-1,1[\,; \\ \pm|u_p'(\pm1)|^{p-2}u_p'(\pm1) = g(\pm1)\,; \end{cases}$$

is given by $u_p(x) = A^{1/(p-1)}x$. Letting p go to 1, we obtain three possibilities.

(1) When $0 < A < 1$, $\lim_{p\to1} u_p(x) = 0$.
(2) When $A = 1$, $\lim_{p\to1} u_p(x) = x$.
(3) When $A > 1$,

$$\lim_{p\to1} u_p(x) = \begin{cases} +\infty\,, & \text{if } x > 0\,; \\ 0\,, & \text{if } x = 0\,; \\ -\infty\,, & \text{if } x < 0\,. \end{cases}$$

Observe that in any case

$$z(x) = \lim_{p\to 1} |u_p'(x)|^{p-2} u_p'(x) = A\,.$$

5.2. **Dimension 2.** Take now $\Omega = B_1(0)$ in $\mathbb{R}^2$ and let

$$g(\cos\theta, \sin\theta) = \frac{A}{\sqrt{2}}(\cos\theta + \sin\theta)\,, \quad \text{with } A > 0\,.$$

The normalized solution to

$$\begin{cases} -\Delta_p u_p = 0\,, & \text{in } \Omega\,; \\ |\nabla u_p|^{p-2}\nabla u_p \cdot (\cos\theta, \sin\theta) = g(\cos\theta, \sin\theta)\,, & \theta \in [0, 2\pi[\,; \end{cases}$$

is defined by

$$u_p(x,y) = \frac{A^{1/(p-1)}}{\sqrt{2}}(x+y)\,.$$

As in the above example, we have to distinguish three possibilities.

(1) When $0 < A < 1$, $\lim_{p\to 1} u_p(x,y) = 0$.
(2) When $A = 1$, $\lim_{p\to 1} u_p(x,y) = \frac{\sqrt{2}}{2}(x+y)$.
(3) When $A > 1$,

$$\lim_{p\to 1} u_p(x,y) = \begin{cases} +\infty\,, & \text{if } x+y > 0\,; \\ 0\,, & \text{if } x+y = 0\,; \\ -\infty\,, & \text{if } x+y < 0\,. \end{cases}$$

Finally,

$$z(x,y) = \lim_{p\to 1} |\nabla u_p(x,y)|^{p-2}\nabla u_p(x,y) = A\Big(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\Big)\,.$$

We point out that

$$|z(x,y)| \le 1 \text{ for all } (x,y) \in \Omega \Leftrightarrow \lim_{p\to 1} |u_p(x,y)| < \infty \text{ for all } (x,y) \in \Omega\,.$$

5.3. **Dimension N.** Consider $A > 0$ and $0 < R_2 < R_1$. Let Ω be the annulus between the surfaces $\partial B_{R_2}(0)$ and $\partial B_{R_1}(0)$. Let g_i denote the flux through $\partial B_{R_i}(0)$, $i = 1, 2$: we take

$$g_1 = A R_1^{-(N-1)} \qquad \text{and} \qquad g_2 = -A R_2^{-(N-1)}\,.$$

We remark that with this choice,

$$\int_{\partial B_{R_1}(0)} g_1\, d\mathcal{H}^{N-1} + \int_{\partial B_{R_2}(0)} g_2\, d\mathcal{H}^{N-1} = 0\,.$$

We look for radial normalized solutions to

$$\begin{cases} -\Delta_p u_p = 0\,, & \text{in } \Omega\,; \\ |\nabla u_p|^{p-1}\nabla u_p \cdot \nu_i = g_i\,, & \text{on } \partial B_{R_i}(0)\,; \end{cases}$$

so that $u_p(x) = \varphi(|x|)$, with φ regular enough and nondecreasing. It is easy to see that then

$$\varphi'(r) = C_p\, r^{-\frac{N-1}{p-1}}\,,$$

and the Neumann condition implies that $C_p = A^{1/(p-1)}$. Therefore,

$$\varphi(r) = K_p - A^{1/(p-1)}\frac{p-1}{N-p} r^{-\frac{N-p}{p-1}}\,.$$

The value of K_p can be computed by the normalization condition. Indeed, it follows from

$$\int_{\partial B_{R_1}(0)} \varphi(R_1)\, d\mathcal{H}^{N-1} + \int_{\partial B_{R_2}(0)} \varphi(R_2)\, d\mathcal{H}^{N-1} = 0$$

that

$$K_p = A^{1/(p-1)} \frac{p-1}{N-p} \frac{R_1^{N-1-\frac{N-p}{p-1}} + R_2^{N-1-\frac{N-p}{p-1}}}{R_1^{N-1} + R_2^{N-1}}\,.$$

Hence,

$$\begin{aligned} u_p(x) &= A^{1/(p-1)} \frac{p-1}{N-p} \left[\frac{R_1^{N-1-\frac{N-p}{p-1}} + R_2^{N-1-\frac{N-p}{p-1}}}{R_1^{N-1} + R_2^{N-1}} - |x|^{-\frac{N-p}{p-1}} \right] \\ &= \left(AR_2^{-(N-1)}\right)^{1/(p-1)} R_2 \frac{p-1}{N-p} \left[\frac{R_1^{N-1}\left(\frac{R_2}{R_1}\right)^{\frac{N-p}{p-1}} + R_2^{N-1}}{R_1^{N-1} + R_2^{N-1}} - \left(\frac{R_2}{|x|}\right)^{\frac{N-p}{p-1}} \right]. \end{aligned}$$

Having in mind

$$\lim_{p\to 1} \left(\frac{R_2}{R_1}\right)^{\frac{N-p}{p-1}} = \lim_{p\to 1} \left(\frac{R_2}{|x|}\right)^{\frac{N-p}{p-1}} = 0\,,$$

it is straightforward to let p goes to 1, and then we get

$$\lim_{p\to 1} u_p(x) = \begin{cases} 0\,, & \text{if } A \le R_2^{N-1}\,; \\ +\infty\,, & \text{if } A > R_2^{N-1}\,. \end{cases}$$

On the other hand,

$$z(x) = \lim_{p\to 1} |\nabla u_p(x)|^{p-2} \nabla u_p(x) = \lim_{p\to 1} \varphi'(|x|)^{p-1} \frac{x}{|x|} = A \frac{x}{|x|^N}\,.$$

Observe that

$$\begin{aligned} |z(x)| \le 1 \text{ for all } x \in \Omega &\Leftrightarrow A \le |x|^{N-1} \text{ for all } x \in \Omega \\ &\Leftrightarrow A \le R_2^{N-1} \Leftrightarrow \lim_{p\to 1} u_p(x) = 0 \text{ for all } x \in \Omega\,. \end{aligned}$$

REMARK 5.1. Throughout this paper, a normalization condition have been imposed, namely

$$\int_{\partial\Omega} u_p\, d\mathcal{H}^{N-1} = 0\,.$$

This is not the only possible normalization. One may wonder if the behaviour of the sequence u_p as p goes to 1 depends or not on this condition.

Assume that we change the normalization condition and impose

$$\int_\Omega u_p\, dx = 0\,. \tag{5.1}$$

Note that in the first and second examples 5.1 and 5.2, our approximate solutions u_p do satisfy this condition, but they do not hold in the third example 5.3. So we are going to rewrite the computations in example 5.3. Observe that, in the last example, (5.1) implies

$$0 = \int_{R_1}^{R_2} \varphi(r) r^{N-1}\, dr = \int_{R_1}^{R_2} K_p r^{N-1}\, dr - \int_{R_1}^{R_2} A^{1/(p-1)} \frac{p-1}{N-p} r^{-\frac{N-p}{p-1}+N-1}\, dr\,.$$

Hence

$$K_p\left(\frac{R_2^N}{N}-\frac{R_1^N}{N}\right)=A^{1/(p-1)}\frac{p-1}{N-p}\frac{1}{(-\frac{N-p}{p-1}+N)}\left(R_2^{-\frac{N-p}{p-1}+N}-R_1^{-\frac{N-p}{p-1}+N}\right),$$

and it follows that

$$\varphi(r)=A^{1/(p-1)}\frac{p-1}{N-p}\times \left[\left(\frac{R_2^N}{N}-\frac{R_1^N}{N}\right)^{-1}\frac{1}{(-\frac{N-p}{p-1}+N)}\left(R_2^{-\frac{N-p}{p-1}+N}-R_1^{-\frac{N-p}{p-1}+N}\right)-r^{-\frac{N-p}{p-1}}\right].$$

Thus,

$$\varphi(r)=\frac{p-1}{N-p}\times \left[\frac{\left(\frac{R_2^N}{N}-\frac{R_1^N}{N}\right)^{-1}}{(-\frac{N-p}{p-1}+N)}\left(\left(\frac{A}{R_2^{N-p}}\right)^{\frac{1}{p-1}}R_2^N-\left(\frac{A}{R_1^{N-p}}\right)^{\frac{1}{p-1}}R_1^N\right)-\left(\frac{A}{r^{N-p}}\right)^{\frac{1}{p-1}}\right].$$

Hence again with this new normalization we get that

$$\lim_{p\to 1}u_p(x)=\begin{cases}0\,, & \text{if } A\le R_2^{N-1}\,;\\ +\infty\,, & \text{if } A>R_2^{N-1}\,.\end{cases}$$

Therefore, in this case, the critical value of A does not depend on the normalization.

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