Anisotropic 1-Laplacian problems with unbounded weights

JUAN C. ORTIZ CHATA [1], MARCOS T. O. PIMENTA[2] AND SERGIO SEGURA DE LEÓN[3*]

1. Departamento de Matemática
Inst. de Biociências, Letras e Ciências Exatas, Universidade Estadual Paulista - UNESP
15054-000 - São José do Rio Preto - SP, Brazil,
e-mail address: hacermate@outlook.com
2. Departamento de Matemática e Computação
Fac. de Ciências e Tecnologia, Universidade Estadual Paulista - UNESP
19060-900 - Presidente Prudente - SP, Brazil,
e-mail address: marcos.pimenta@unesp.br
ORCID ID: 0000-0003-4961-3038
3. Departament d'Anàlisi Matemàtica
Universitat de València
46100 - Burjassot, Valencia, Spain
e-mail address: sergio.segura@uv.es
ORCID ID: 0000-0002-8515-7108

Key Words. 1-Laplacian operator, Caffarelli-Kohn-Nirenberg inequality, weighted L^∞–divergence–measure vector fields, weighted quasilinear elliptic problems
AMS Classification. 35J60, 35J20, 35J92.

Acknowledgment

Juan C. Ortiz Chata is supported by FAPESP 2017/06119-0 and 2019/13503-7. Marcos T. O. Pimenta is supported by FAPESP 2019/14330-9 and CNPq 303788/2018-6. Sergio Segura de León is supported by the Spanish Ministerio de Ciencia, Innovación y Universidades and FEDER, under project PGC2018–094775–B–I00.

* Corresponding author

1. Introduction

In the celebrated paper [13], Caffarelli, Kohn and Nirenberg established an interpolation inequality involving weighted Lebesgue norms of functions and their first derivatives. This inequality, in turn, allows one to show continuous and compact embeddings theorems dealing with weighted Sobolev spaces. Furthermore, this inequality and the connected embeddings have been applied to analyze several elliptic and parabolic problems involving weighted Laplacian and p-Laplacian operators (for elliptic problems, see for instance [1, 2, 9, 14, 12, 36] and the references therein).

Regarding anisotropic problems involving the 1-Laplacian operator, we refer to [32] as the first paper which studies the existence and uniqueness of the anisotropic total variation flow. On the other hand, in [29], the author finds the Euler-Lagrange equation for the anisotropic least gradient problem

$$\inf\left\{\int_\Omega \phi(x,Du) : \, u\in BV(\Omega),\, u|_{\partial\Omega}=f\right\}. \tag{1.1}$$

We could also cite [37], where the author studies questions about the existence and the regularity of minimizers of (1.1), where $\phi(x,Du)=a(x)|Du|$ and the weight function $a(\cdot)$ is a smooth bounded function.

As a common hypothesis in all of these articles, we have the fact that the weight w satisfies $0<\alpha\le w(x)\le\beta<\infty$. This assumption implies that the natural space to analyze the corresponding problem is BV, the space of functions of bounded variation (as in the isotropic case).

The aim of this paper is to consider some anisotropic problems with unbounded weights related to the Caffarelli-Kohn-Nirenberg inequality. More precisely, we study existence of positive solutions to the following problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{1}{|x|^a}\dfrac{Du}{|Du|}\right) &= \dfrac{1}{|x|^b}f(u) \quad \text{in } \Omega,\\ \qquad\qquad\qquad u &= 0 \qquad\qquad\quad \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

where Ω is a bounded open set in $\mathbb{R}^N$ (with $N\ge 2$) containing the origin and having Lipschitz boundary $\partial\Omega$, and the two parameters satisfy $0<a<N-1$ and $a<b<a+1$. Hypotheses on function $f:\mathbb{R}\to\mathbb{R}$ will be listed further below.

To the best of our knowledge, this work is the first attempt to deal with anisotropic problems having unbounded weights. In this situation, $BV(\Omega)$ is unsuitable and it cannot be the natural space to analyze this problem. Now, the energy space turns out to be a weighted BV–space. In the first step this weighted space, denoted by $BV_a(\Omega)$, is introduced. Since our weights are related to the Caffarelli–Kohn–Nirenberg inequality, one of our main endeavors is to adapt this inequality to our setting. More specifically, we prove the following result.

Theorem 1.1. *Let* $0 < a < N-1$, $0 < \theta \leq 1$ *and* $a < b < a+1$. *Then there exists a constant* $\mathfrak{C}_{CKN} > 0$ *such that*

$$\left(\int_\Omega \frac{1}{|x|^{\alpha r_\theta}}|u|^{r_\theta}dx\right)^{\frac{1}{r_\theta}}$$
$$\leq \mathfrak{C}_{CKN}\left(\int_\Omega \frac{1}{|x|^a}|Du| + \int_{\partial\Omega}\frac{1}{|x|^a}|u|d\mathcal{H}^{N-1}\right)^\theta \left(\int_\Omega |u|dx\right)^{1-\theta} \qquad (1.3)$$

holds for all $u \in BV_a(\Omega)$, *where* $\alpha = \theta b$ *and* $r_\theta = \frac{N}{N-\theta(1+a-b)}$. *Here* $BV_a(\Omega)$ *denotes the appropriate weighted* BV*–space, which was introduced in* [10] *(see Subsection* 2.3 *below).*

The concept of solution to problems involving the 1–Laplacian operator lies on the theory of L^∞–divergence–measure vector fields (see [7, 18]). It provides tools to handle bounded vector fields and gradients of BV–functions, including a Green's formula. Since in our context this theory can no longer be used, it follows that we must extend it to establish the necessary tools to deal with it. This extension is far from being trivial, since the weight which is included in the vector field is unbounded. Using this tool, we may introduce the concept of solution to problem (1.2) (see Definition 4.9 below) and broach its study.

Before stating our main result in this paper, we list the assumptions on function f in problem (1.2):

(f_1) $f \in C^0([0,+\infty),\mathbb{R})$;
(f_2) $f(0) = 0$;
(f_3) There exist constants $c_1, c_2 > 0$ and $1 < q < \frac{N}{N-(1+a-b)}$, such that
$$|f(s)| \leq c_1 + c_2 s^{q-1}, \quad s \in [0,+\infty);$$
(f_4) There exist $\mu > 1$ and $s_0 > 0$ such that
$$0 < \mu F(s) \leq f(s)s, \quad \forall\, s \geq s_0,$$
where $F(t) = \int_0^t f(s)ds$;
(f_5) f is increasing on $[0,+\infty)$.

Remark 1.2. Some consequences of (f_4) are in order. It is not difficult to deduce from (f_4) that there exist two positive constants d_1 and d_2 satisfying

$$F(s) \geq d_1 s^\mu - d_2$$

for all $s > 0$. Applying (f_4) again, we get $f(s) \geq \mu(d_1 s^{\mu-1} - d_2 s^{-1})$ for all $s \geq s_0$ and so, having in mind $\mu > 1$, it yields

$$\lim_{s\to+\infty} f(s) = +\infty\,.$$

Remark 1.3. Since we are looking for nonnegative solutions, we may (and will) extend $f(s)$ as usual defining $f(s) = 0$ if $s < 0$. As a consequence, we have $F(s) = 0$ for all $s < 0$.

Our main result is the following.

Theorem 1.4. *Suppose that f satisfies conditions $(f_1)-(f_4)$. Then there exists a nontrivial nonnegative solution to problem* (1.2)*. This solution is actually a ground-state solution (i.e., that solution which has the lowest energy among all nontrivial ones) if we further require condition (f_5).*

Two different approaches will be used to prove this result. In each case a suitable variant of Mountain Pass Theorem (see [3]) is applied. In the first of them, we consider approximate solutions to problems involving the p–Laplacian operator and next we let p go to 1. Then we find a hindrance due to the assumptions on the function f which are needed to find solutions to p–problems. Indeed, in the literature on the p–Laplacian setting, our assumption (f_2) is too general to get a solution and a hypothesis as $\displaystyle\lim_{s\to 0}\frac{f(s)}{|s|^{p-1}} = 0$ is required. The difficulty is overcome by modifying the reaction term in the p–problems and then control the convergence process. In the second, we work by using variational methods applied to the problem itself defined in $BV_a(\Omega)$. We apply a version of Mountain Pass Theorem suitable for functionals defined on this sort of spaces. In addittion, by using this approach, we are able also to show that this mountain pass solution is in fact a ground-state solution of the problem, i. e., its energy level is the lowest one among all the nontrivial solutions.

We briefly explain the plan of this paper. In Section 2 we present some preliminary results and define the space $BV_a(\Omega)$. In Section 3 we set the Caffarelli-Kohn-Nirenberg inequality in the space $BV_a(\Omega)$. In Section 4 we extend the Anzellotti pairing theory to include unbounded vector fields and also define the sense of solution we deal with. Section 5 is devoted to prove Theorem 1.4 by using the approximation method by problems involving weighted p-Laplacian problems. Finally, in Section 6 we present the proof of Theorem 1.4 by using the purely variational approach.

2. Preliminaries

We denote by $\mathcal{H}^{N-1}(E)$ the $(N-1)$-dimensional Hausdorff measure of a set E while $|E|$ stands for its N-dimensional Lebesgue measure. We will usually handle an auxiliary function: the truncation function at level $\pm k$ defined by

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\dfrac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{2.4}$$

In what follows, $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ is an open and bounded set such that $0 \in \Omega$. Moreover, its boundary $\partial\Omega$ is Lipschitz–continuous. Thus, an outward normal unit vector $\nu(x)$ is defined for $\mathcal{H}^{N-1}$–almost every $x \in \partial\Omega$.

From now on, we denote:

- $C_c^1(\Omega)$, stands for the space of functions with compact support which are continuously differentiable on Ω
- $C_c^\infty(\Omega)$, denotes the space of all functions with compact support having derivatives of all orders

We will make use of the usual Lebesgue and Sobolev spaces. Lebesgue spaces with respect to a measure μ will be written as $L^q(\Omega, \mu)$. The measure will be deleted when it is Lebesgue measure.

Sometimes we will need to use convolution with mollifiers. We will denote by $\rho \in C_c^\infty(\mathbb{R}^N)$ a symmetric mollifier whose support is $\overline{B(0,1)}$ and its associated approximation to the identity by $\rho_\epsilon(x) := \frac{1}{\epsilon^N}\rho\left(\frac{x}{\epsilon}\right)$, for $\epsilon > 0$. The main properties of approximation to identity can be found, for instance, in [4] or [11].

We explicitly remark that, if not otherwise specified, we will denote by C several positive constants whose value may change from line to line. These values will only depend on the data but they will never depend on p or other indexes we will introduce.

2.1. Weighted spaces

Our objective in this subsection is to study spaces having a weight of the form $x \mapsto |x|^{-a}$, with $a > 0$. We refer to [26, 25, 28] as sources for a more extensive study on weights and weighted spaces. We begin by introducing some features of these weights.

Recall that w, a nonnegative locally integrable function on $\mathbb{R}^N$, belongs to Muckenhoupt's class A_1 if there exists a constant $C_w > 0$ such that

$$\fint_B w\,dx \le C_w \operatorname*{ess\,inf}_B w, \qquad \text{for all ball } B \subset \mathbb{R}^N, \tag{2.5}$$

where $\fint_B f dx = \frac{1}{|B|}\int_B f dx$.

It is well-known that the weight function $w(x) = \frac{1}{|x|^a}$ belongs to Muckenhoupt's class A_1 if and only if $0 < a < N$, so that in this case there exists a constant $C_a > 0$ such that

$$\fint_{B(x,r)} \frac{1}{|y|^a} dy \leq C_a \inf_{y \in B(x,r)} \frac{1}{|y|^a}, \tag{2.6}$$

for all $B(x,r) \subset \mathbb{R}^N$. We point out that this fact implies an inequality connecting mollifiers and this weight. Indeed,

$$(\rho_\epsilon * w)(x) = \frac{1}{\epsilon^N}\int_{B(x,\epsilon)} \rho\left(\frac{x-y}{\epsilon}\right)\frac{1}{|y|^a}\, dy \leq \frac{\|\rho\|_\infty |B(x,1)|}{|B(x,\epsilon)|}\int_{B(x,\epsilon)} \frac{1}{|y|^a}\, dy$$

and, as a consequence of belonging to A_1,

$$(\rho_\epsilon * w)(x) \leq C_a \|\rho\|_\infty |B(0,1)| \inf_{y \in B(x,\epsilon)} \frac{1}{|y|^a} \leq \frac{C}{|x|^a} \tag{2.7}$$

holds for all $x \in \Omega$.

Given $a > 0$ and $s \geq 1$, let us denote by $L^s_a(\Omega)$ the set of measurable functions u such that

$$\left(\int_\Omega \frac{1}{|x|^a}|u|^s dx\right)^{\frac{1}{s}} < \infty.$$

Remark 2.1. Since Ω is a bounded set, it follows that

$$m_a := \inf_{x \in \Omega}\left\{\frac{1}{|x|^a}\right\}$$

is positive. We note that this implies that the embedding $L^s_a(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for all $s \geq 1$.

Definition 2.2. *Let $p \geq 1$ and fix $0 < a < \frac{N-p}{p}$. The weighted Sobolev space $\mathcal{D}^{1,p}_a(\Omega)$ is defined as the completion of restrictions of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm given by*

$$\|u\|_{p,a} = \left(\int_\Omega \frac{1}{|x|^{ap}}|u|^p dx + \int_\Omega \frac{1}{|x|^{ap}}|\nabla u|^p dx\right)^{\frac{1}{p}}$$

Observe that functions in this space belong to

$$W^{1,p}(\Omega, |x|^{-ap}) = \{u \in L^p_{ap}(\Omega);\ \nabla u \in L^p_{ap}(\Omega; \mathbb{R}^N)\}.$$

Reasoning as in Remark 2.1, we deduce that there is a continuous embedding $\mathcal{D}^{1,p}_a(\Omega) \hookrightarrow W^{1,p}(\Omega)$.

Remark 2.3. In [27] is proved that the space $W^{1,p}(\Omega, |x|^{-ap})$ is equal to the closure of $\{\varphi \in C^\infty(\Omega);\ \|u\|_{p,a} < \infty\}$.

The Sobolev space $\mathcal{D}^{1,p}_{0,a}(\Omega)$ is defined as the completion of $C^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_{p,a}$. Notice that there is a continuous embedding $\mathcal{D}^{1,p}_{0,a}(\Omega) \hookrightarrow W^{1,p}_0(\Omega)$. A Poincaré type inequality implies that this norm is equivalent in $\mathcal{D}^{1,p}_{0,a}(\Omega)$ to the norm given by

$$\|u\| = \left(\int_\Omega \frac{1}{|x|^{ap}}|\nabla u|^p dx\right)^{\frac{1}{p}}. \tag{2.8}$$

This will be the norm we will use in what follows.

For more information on weighted Sobolev spaces, we refer to [27] (see also [2, 36]).

2.2. The space $BV(\Omega)$

In this subsection, we just introduce some properties of the space of functions of bounded variation. As mentioned in the introduction, it is the natural space to study problems involving the 1–Laplacian operator. This space is defined as

$$BV(\Omega) = \left\{u \in L^1(\Omega)\ :\ Du \text{ is a finite Radon measure }\right\}$$

where $Du : \Omega \to \mathbb{R}^N$ denotes the distributional gradient of u. Henceforth, we denote the distributional gradient by ∇u when it belongs to $L^1(\Omega; \mathbb{R}^N)$.

We recall that the space $BV(\Omega)$ endowed with the norm

$$\|u\|_{BV(\Omega)} = \int_\Omega |Du| + \int_\Omega |u| dx$$

is a Banach space which is non reflexive and non separable. On the other hand, the notion of a trace on the boundary can be extended to functions $u \in BV(\Omega)$, so that we may write $u\big|_{\partial\Omega}$. Indeed, there exists a continuous linear operator $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$ extending the boundaries values of functions in $C(\overline{\Omega})$. As a consequence, an equivalent norm on $BV(\Omega)$ can be defined:

$$\|u\|_{BV(\Omega),1} = \int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1}.$$

We will often use this norm in what follows.

In addition, the following continuous embeddings hold

$$BV(\Omega) \hookrightarrow L^m(\Omega)\,, \quad \text{for every } 1 \le m \le \frac{N}{N-1}\,, \tag{2.9}$$

which are compact for $1 \le m < \frac{N}{N-1}$.

For further properties of functions of bounded variations, we refer to [4] and [21].

2.3. The space $BV_a(\Omega)$

In this subsection, we study the definition and main properties of the space $BV_a(\Omega)$, which is our energy space. We mainly follow [10] to where we refer for a wider analysis.

Let us define $\mathrm{var}_a u(\Omega)$ as

$$\mathrm{var}_a u(\Omega) := \sup\left\{ \int_\Omega u \,\mathrm{div}\, \phi dx;\ \phi \in C_c^1(\Omega, \mathbb{R}^N), \text{ s.t. } |\phi(x)| \le \frac{1}{|x|^a} \right\}.$$

We remark that the Riesz representation Theorem implies that $\mathrm{var}_a u(\Omega)$ defines a Radon measure (see, for instance, [21, Section 1.8]).

We point out that the function

$$x \mapsto \frac{1}{|x|^a}, \qquad 0 < a < N-1,$$

is continuous in $\Omega\setminus\{0\}$, and hence it is lower semicontinuous. Then, appealing to [10, Theorem 4.1], we obtain the next result.

Theorem 2.4. *The following statements are equivalent:*

a) $var_a u(\Omega) < \infty$*;*
b) $u \in BV(\Omega)$ *and* $\frac{1}{|x|^a} \in L^1(\Omega, |Du|)$.

Moreover,

$$var_a u(\Omega) = \int_\Omega \frac{1}{|x|^a}|Du|.$$

Definition 2.5. *Let* $BV_a(\Omega)$ *be the space of functions* $u \in L^1(\Omega)$ *such that* $|\cdot|^{-a}|Du|$ *is a finite Radon measure, i.e.,*

$$BV_a(\Omega) = \left\{ u \in L^1(\Omega) \ :\ \int_\Omega \frac{1}{|x|^a}|Du| < +\infty \right\}.$$

The space $BV_a(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV_a(\Omega)} := \int_\Omega \frac{1}{|x|^a}|Du| + \int_\Omega |u| dx.$$

Moreover, note that $m_a \int_\Omega |Du| \le \int_\Omega \frac{1}{|x|^a}|Du|$ (m_a as in Remark 2.1), so that

$$BV_a(\Omega) \hookrightarrow BV(\Omega).$$

Then

$$BV_a(\Omega) \hookrightarrow L^1(\partial\Omega)$$

and so every $u \in BV_a(\Omega)$ has a trace on $\partial\Omega$.

We point out that the functional given by

$$u \mapsto \mathrm{var}_a u(\Omega) = \int_\Omega \frac{1}{|x|^a}|Du|$$

is lower semicontinuous with respect to the L^1–convergence since each $u \mapsto \int_\Omega u \,\mathrm{div}\, \phi dx$ is so. Furthermore, similar arguments lead to the lower semicontinuity of the functional

$$u \mapsto \int_\Omega \frac{1}{|x|^a}|Du| + \int_{\partial\Omega} \frac{1}{|x|^a}|u|\, d\mathcal{H}^{N-1} \tag{2.10}$$

We also need to use the lower semicontinuity of another functional. For a fixed nonnegative $\varphi \in C_c^\infty(\Omega)$, consider

$$u \mapsto \int_\Omega \varphi \frac{1}{|x|^a}|Du|\,, \tag{2.11}$$

As a consequence of [10, Theorem 3.3], we may write

$$\int_\Omega \varphi \frac{1}{|x|^a}|Du| = \sup\left\{ \int_\Omega u\,\mathrm{div}\,(\varphi\Phi)\, dx \;:\; \Phi \in C_c^1(\Omega)^N \quad |\Phi| \le \frac{1}{|x|^a} \right\} \tag{2.12}$$

from where the desired lower semicontinuity follows.

We end this subsection by showing that just like in the space $BV(\Omega)$, we can have an equivalent norm in $BV_a(\Omega)$ which involves an integral over $\partial\Omega$. Its proof is a consequence of being equivalent $\|\cdot\|_{BV(\Omega)}$ and $\|\cdot\|_{BV(\Omega),1}$, and using that the positive quantities

$$M_a = \sup_{x\in\partial\Omega}\left\{\frac{1}{|x|^a}\right\} \qquad \text{and} \qquad m_a = \inf_{x\in\partial\Omega}\left\{\frac{1}{|x|^a}\right\}$$

are finite.

Proposition 2.6. *The norm $\|\cdot\|_{BV_a}$ is equivalent to the norm given by*

$$\|u\|_{BV_a(\Omega),1} = \int_\Omega \frac{1}{|x|^a}|Du| + \int_{\partial\Omega} \frac{1}{|x|^a}|u| d\mathcal{H}^{N-1}.$$

Proposition 2.7. *Let $u, v \in BV_a(\Omega)$, then* $\max\{u,v\}, \min\{u,v\} \in BV_a(\Omega)$ *and the following inequality is valid*

$$\|\max\{u,v\}\|_{BV_a(\Omega),1} + \|\min\{u,v\}\|_{BV_a(\Omega),1} \le \|u\|_{BV_a(\Omega),1} + \|v\|_{BV_a(\Omega),1}. \tag{2.13}$$

In particular, choosing $v = 0$, we have that $u^+ := \max\{u,0\}$, $u^- = \min\{u,0\} \in BV_a(\Omega)$, with $u = u^+ + u^-$, and it holds

$$\|u\|_{BV_a(\Omega),1} = \|u^+\|_{BV_a(\Omega),1} + \|u^-\|_{BV_a(\Omega),1}. \tag{2.14}$$

3. The Caffarelli-Kohn-Nirenberg inequality in $BV_a(\Omega)$

In this section we are going to present a version of the Caffarelli-Kohn-Nirenberg inequality [13] in the space $BV_a(\Omega)$. We do not prove it in its full generality, but just introduce those cases to be applied. In particular, we employ them to prove embeddings involving $BV_a(\Omega)$.

First of all we state the particular cases of the Caffarelli-Kohn-Nirenberg inequality we are interested in.

Lemma 3.1. *Let $p \geq 1$ and consider parameters satisfying $0 < a < \frac{N-p}{p}$, $0 < \theta \leq 1$ and $a < b < a+1$. Then there exists a constant $\mathfrak{C}_{CKN} > 0$ such that the following inequality holds for all $u \in C_c^\infty(\mathbb{R}^N)$:*

$$\left(\int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha r_\theta}} |u|^{r_\theta} dx\right)^{\frac{1}{r_\theta}} \leq \mathfrak{C}_{CKN} \left(\int_{\mathbb{R}^N} \frac{1}{|x|^{ap}} |\nabla u|^p dx\right)^{\frac{\theta}{p}} \left(\int_{\mathbb{R}^N} |u| dx\right)^{1-\theta},$$

where $\alpha = \theta b$ and $r_\theta = \frac{Np}{\theta N - p[\theta(1+a-b) - N(1-\theta)]}$.

Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $u \in BV_a(\Omega)$ and consider its extension to $\mathbb{R}^N$ defined by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

We remark that $D\tilde{u} = Du + u|_{\partial\Omega} \cdot \mathcal{H}^{N-1} \llcorner_{\partial\Omega}$ (see [4, Theorem 3.87]).

Note also that $\tilde{u} * \rho_\epsilon \in C_c^\infty(\mathbb{R}^N)$ and so we may apply Lemma 3.1 for $p = 1$ (so that $r_\theta = \frac{N}{N-\theta(1+a-b)}$). Thus, for every $\epsilon > 0$, we get

$$\left(\int_{\mathbb{R}^N} \frac{1}{|x|^{\alpha r_\theta}} |\tilde{u} * \rho_\epsilon|^{r_\theta} dx\right)^{\frac{1}{r_\theta}} \leq \mathfrak{C}_{CKN} \left(\int_{\mathbb{R}^N} \frac{1}{|x|^a} |\nabla(\tilde{u} * \rho_\epsilon)| dx\right)^{\theta} \left(\int_{\mathbb{R}^N} |\tilde{u} * \rho_\epsilon| dx\right)^{1-\theta}. \tag{3.15}$$

We will separately take the limit as $\epsilon \to 0$ in each integral.

We begin by analyzing the gradient term. Thanks to [4, Proposition 3.2(c)], we write

$$\int_{\mathbb{R}^N} \frac{1}{|x|^a} |\nabla(\tilde{u} * \rho_\epsilon)| dx \leq \int_{\mathbb{R}^N} \left(\frac{1}{|x|^a} * \rho_\epsilon\right) |D\tilde{u}|$$

Moreover, by the continuity of our weight,

$$\frac{1}{|x|^a} * \rho_\epsilon \to \frac{1}{|x|^a} \quad \text{pointwise in } \mathbb{R}^N \setminus \{0\} \tag{3.16}$$

and this fact, jointly with (2.7), allows us to apply the Dominated Convergence Theorem and obtain

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^a}*\rho_\epsilon\right)|D(\tilde{u})| = \int_{\mathbb{R}^N}\frac{1}{|x|^a}|D(\tilde{u})| = \int_\Omega \frac{1}{|x|^a}|Du| + \int_{\partial\Omega}\frac{1}{|x|^a}|u|\, d\mathcal{H}^{N-1}\,. \tag{3.17}$$

On the other hand, since

$$\rho_\epsilon * \tilde{u} \to \tilde{u} \text{ in } L^1(\mathbb{R}^N), \tag{3.18}$$

it follows that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^N}|\tilde{u}*\rho_\epsilon|dx = \int_{\mathbb{R}^N}|\tilde{u}|dx = \int_\Omega |u|dx\,. \tag{3.19}$$

Furthermore, we deduce from

$$\rho_\epsilon * \tilde{u}(x) \to \tilde{u}(x) \quad \text{a. e. in } \mathbb{R}^N,$$

and Fatou's Lemma that

$$\int_\Omega \frac{1}{|x|^{\alpha r_\theta}}|u|^{r_\theta}dx = \int_{\mathbb{R}^N}\frac{1}{|x|^{\alpha r_\theta}}|\tilde{u}|^{r_\theta}dx \le \liminf_{\epsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{|x|^{\alpha r_\theta}}|\tilde{u}*\rho_\epsilon|^{r_\theta}dx\,. \tag{3.20}$$

Therefore, using (3.17), (3.19) and (3.20), we may pass to the limit in (3.15) and obtain the desired result. □

In the following results, we denote $C_\Omega = \sup\{|x| \,:\, x\in\Omega\}$, which is finite since Ω is bounded.

Theorem 3.2. *Let $a < b < a+1$ and $r = \frac{N}{N-(1+a-b)}$. Then for all $q\in\mathbb{R}$, $1\le q\le r$, the embedding*

$$BV_a(\Omega) \hookrightarrow L^q_b(\Omega)$$

is continuous.

Proof. In this proof, we consider several cases. All of them are consequence of some manipulations involving Hölder's inequality and the version of Caffarelli-Kohn-Nirenberg's inequality given in Theorem 1.1.

First of all, let us consider the case $q = 1$. We apply the mentioned inequalities to get

$$\begin{aligned}\int_\Omega \frac{1}{|x|^b}|u|dx &\le \left(\int_\Omega \frac{1}{|x|^{br}}|u|^r\right)^{\frac{1}{r}}|\Omega|^{\frac{r-1}{r}}\\ &\le |\Omega|^{\frac{r-1}{r}}\mathfrak{C}_{CKN}\left(\int_\Omega\frac{1}{|x|^a}|Du| + \int_{\partial\Omega}\frac{1}{|x|^a}|u|d\mathcal{H}^{N-1}\right).\end{aligned}$$

Now consider $1 < q < r$. In this case, arguing as above, we obtain

$$\begin{aligned}\int_\Omega \frac{1}{|x|^b}|u|^q dx &= \int_\Omega \frac{1}{|x|^{b-bq}}\frac{1}{|x|^{bq}}|u|^q dx\\ &\le C_\Omega^{bq-b}\int_\Omega \frac{1}{|x|^{bq}}|u|^q dx\\ &\le C_\Omega^{bq-b}|\Omega|^{\frac{r-q}{r}}\left(\int_\Omega \frac{1}{|x|^{br}}|u|^r dx\right)^{\frac{q}{r}}\\ &\le C_\Omega^{bq-b}|\Omega|^{\frac{r-q}{r}}\mathfrak{C}_{CKN}^q\left(\int_\Omega \frac{1}{|x|^a}|Du| + \int_{\partial\Omega}\frac{1}{|x|^a}|u|d\mathcal{H}^{N-1}\right)^q.\end{aligned}$$

Finally, the case $q = r$ follows from a similar argument.

Therefore, in any case, there exists $C > 0$ such that

$$\left(\int_\Omega \frac{1}{|x|^b}|u|^q dx\right)^{\frac{1}{q}} \le C\left(\int_\Omega \frac{1}{|x|^a}|Du| + \int_{\partial\Omega}\frac{1}{|x|^a}|u|d\mathcal{H}^{N-1}\right)$$

holds for every $u \in BV_a(\Omega)$ and we are done. □

Theorem 3.3. *Let $a < b < a+1$ and $r = \frac{N}{N-(1+a-b)}$. Then for all q, $1 \le q < r$ the embedding*

$$BV_a(\Omega) \hookrightarrow L_b^q(\Omega)$$

is compact.

Proof. Let (u_n) be a bounded sequence in $BV_a(\Omega)$ and note that, since $BV_a(\Omega) \hookrightarrow BV(\Omega)$, (u_n) is also bounded in $BV(\Omega)$. Then, by the compact embedding in $BV(\Omega)$, there exist a subsequence (not relabeled) and $u \in BV(\Omega)$ such that

$$u_n \to u \quad \text{ in } L^1(\Omega)\,. \tag{3.21}$$

Let $1 < q < r$. Note that there exists $\theta \in (0,1)$ such that

$$\frac{1}{\theta} < q < \frac{N}{N-\theta(1+a-b)}.$$

Then, using first Hölder's inequality and then (1.3) we get

$$\begin{aligned}\int_\Omega \frac{1}{|x|^b}|u_n-u|^q dx &= \int_\Omega \frac{1}{|x|^{b-\theta bq}}\frac{1}{|x|^{\theta bq}}|u_n-u|^q dx\\ &\le C_\Omega^{\theta bq-b}\int_\Omega \frac{1}{|x|^{\theta bq}}|u_n-u|^q dx\\ &\le C_\Omega^{\theta bq-b}|\Omega|^{\frac{r_\theta-q}{r_\theta}}\left(\int_\Omega \frac{1}{|x|^{\theta br_\theta}}|u_n-u|^{r_\theta} dx\right)^{\frac{q}{r_\theta}}\\ &\le C_\Omega^{\theta bq-b}|\Omega|^{\frac{r_\theta-q}{r_\theta}}\mathfrak{C}_{CKN}^q\|u_n-u\|^{q\theta}_{BV_a(\Omega),1}\left(\int_\Omega |u_n-u|dx\right)^{(1-\theta)q},\end{aligned}$$

which tends to 0 as $n \to \infty$. Here we have used that $(u_n)_n$ is bounded in $BV_a(\Omega)$ and (3.21).

It remains to consider $q = 1$. Note that there exists $0 < \bar\theta < 1$ such that $\bar\theta(1 + a) > b$. Performing similar manipulations, we get

$$\begin{aligned}\int_\Omega \frac{1}{|x|^b}|u_n - u|dx &= \int_\Omega \frac{1}{|x|^{b-\bar\theta b}}\frac{1}{|x|^{\bar\theta b}}|u_n - u|dx \\ &\le \left(\int_\Omega \left(\frac{1}{|x|^{b-\bar\theta b}}\right)^{\frac{r_{\bar\theta}}{r_{\bar\theta}-1}}\right)^{\frac{r_{\bar\theta}-1}{r_{\bar\theta}}} \left(\int_\Omega \frac{1}{|x|^{\bar\theta b r_{\bar\theta}}}|u_n - u|^{r_{\bar\theta}}dx\right)^{\frac{1}{r_{\bar\theta}}}\end{aligned}$$

Observe that, since $\bar\theta(a + 1) > b$, it follows that $b(1 - \bar\theta)\frac{r_{\bar\theta}}{r_{\bar\theta}-1} < N$, so that

$$A = \left(\int_\Omega \left(\frac{1}{|x|^{b-\bar\theta b}}\right)^{\frac{r_{\bar\theta}}{r_{\bar\theta}-1}}\right)^{\frac{r_{\bar\theta}-1}{r_{\bar\theta}}} < +\infty\,.$$

Hence, applying (1.3), it yields

$$\begin{aligned}\int_\Omega \frac{1}{|x|^b}|u_n - u|dx &\le A\left(\int_\Omega \frac{1}{|x|^{\bar\theta b r_{\bar\theta}}}|u_n - u|^{r_{\bar\theta}}dx\right)^{\frac{1}{r_{\bar\theta}}} \\ &\le A\mathfrak{C}_{CKN}\|u_n - u\|^{\bar\theta}_{BV_a(\Omega),1}\left(\int_\Omega |u_n - u|dx\right)^{1-\bar\theta},\end{aligned}$$

which tends to 0 as above. □

4. Extension of the Anzellotti theory

In this Section, we extend the Anzellotti theory to a setting which involves unbounded vector fields. To begin with, we recall this theory. Not only these results will be applied, but they will also serve us as a guide for its broadening.

4.1. Remainder of Anzellotti's theory

We recall the notion of weak trace on $\partial\Omega$ of the normal component defined in [7] for every $z \in L^\infty(\Omega;\mathbb{R}^N)$ such that its distributional divergence $\operatorname{div} z$ is a Radon measure having finite total variation. This trace is a function $[z,\nu] : \partial\Omega \to \mathbb{R}$ satisfying $[z,\nu] \in L^\infty(\partial\Omega)$ and $\|\,[z,\nu]\,\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, being $\nu(\cdot)$ the outer normal unitary vector on $\partial\Omega$.

In [7], it was also introduced a distribution $(z, Du) : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$ defined by

$$\langle (z, Du), \varphi\rangle = -\int_\Omega u\,\varphi \operatorname{div} z - \int_\Omega u\, z\cdot\nabla\varphi dx\,, \tag{4.22}$$

where

$$u \in BV(\Omega)\cap L^\infty(\Omega) \quad \text{and} \quad \operatorname{div} z \in L^1(\Omega)\,, \tag{4.23}$$

among other possible pairings. It is then proved

$$|\langle (z, Du), \varphi\rangle| \le \|\varphi\|_\infty \|z\|_{L^\infty(U)} \int_U |Du| \tag{4.24}$$

for all open sets $U \subset \Omega$ such that $\operatorname{supp} \varphi \subset U$. As a consequence, (z, Du) is a Radon measure whose total variation satisfies

$$|(z, Du)| \le \|z\|_\infty |Du|\,. \tag{4.25}$$

Finally, a Green formula involving the measure (z, Du) and the weak trace $[z, \nu]$ is established in [7], namely:

$$\int_\Omega (z, Du) + \int_\Omega u \operatorname{div} z = \int_{\partial\Omega} u\,[z,\nu]\, d\mathcal{H}^{N-1} \tag{4.26}$$

being z and u as in (4.23).

4.2. Weighted theory

In this subsection, we consider weights $w(x) = |x|^{-a}$, with $0 < a < N-1$. Nevertheless, We point out that most of the results holds for more general weights.

We define the space

$$\mathcal{DM}_a(\Omega) = \left\{ z \in L^\infty(\Omega, \mathbb{R}^N);\, \operatorname{div}\left(\frac{1}{|x|^a} z\right) \in L^1(\Omega) \right\}.$$

Note that, for every $z \in \mathcal{DM}_a(\Omega)$, the following equalities are valid in the sense of distributions

$$\operatorname{div}\left(T_k\left(\frac{1}{|x|^a}\right) z\right) = T_k\left(\frac{1}{|x|^a}\right)\operatorname{div}(z) + z\cdot\nabla T_k\left(\frac{1}{|x|^a}\right), \quad \forall k > 0. \tag{4.27}$$

Hence, letting $k \to \infty$, it also holds

$$\operatorname{div}\left(\frac{1}{|x|^a} z\right) = \frac{1}{|x|^a}\operatorname{div}(z) + z\cdot\nabla\left(\frac{1}{|x|^a}\right), \tag{4.28}$$

in the sense of distributions. Since $\operatorname{div}\left(\frac{1}{|x|^a} z\right)$ and $z\cdot\nabla\left(\frac{1}{|x|^a}\right)$ belong to $L^1(\Omega)$, this last identity implies that

$$\frac{1}{|x|^a}\operatorname{div}(z) \in L^1(\Omega) \tag{4.29}$$

and so, taking into account that Ω is bounded,

$$\operatorname{div}(z) \in L^1(\Omega)\,. \tag{4.30}$$

Then Anzellotti's theory supplies us with the weak trace $[z, \nu]$ on $\partial\Omega$ and the Radon measure (z, Du) for every $u \in BV(\Omega) \cap L^\infty(\Omega)$ (and so for every $u \in BV_a(\Omega) \cap L^\infty(\Omega)$).

It is easy to compare $\left[\frac{1}{|x|^a} z, \nu\right]$ and $\frac{1}{|x|^a}[z,\nu]$. To see that they are equal, we just employ the inequality

$$\frac{1}{|x|^a} \le M_a, \qquad \text{for all } x \in \partial\Omega.$$

for certain finite constant M_a.

Lemma 4.1. *For every $z \in \mathcal{DM}_a(\Omega)$ we have that*

$$\left[\frac{1}{|x|^a} z, \nu\right] = \frac{1}{|x|^a}[z,\nu] \quad \mathcal{H}^{N-1} - \ a.\ e.\ \partial\Omega.$$

Proof. For each $k > 0$, by the Proposition 2 of [15], we obtain

$$\left[T_k\left(\frac{1}{|x|^a}\right) z, \nu\right] = T_k\left(\frac{1}{|x|^a}\right)[z,\nu] \quad \mathcal{H}^{N-1} - \ \text{a. e. } \partial\Omega$$

Now it is enough to take $k \ge M_a$ to get our result. □

4.3. Measures $\left(\frac{1}{|\cdot|^a} z, Du\right)$ and $\frac{1}{|\cdot|^a}(z, Du)$

In this subsection, we take $z \in \mathcal{DM}_a(\Omega)$ and $u \in BV_a(\Omega) \cap L^\infty(\Omega)$, and introduce two distributions $\left(\frac{1}{|x|^a} z, Du\right)$ and $\frac{1}{|x|^a}(z, Du)$, which turn out to be equal. Finally, we will prove a Green's formula that connects them to traces $\left[\frac{1}{|x|^a} z, \nu\right] = \frac{1}{|x|^a}[z,\nu]$.

We begin by observing that $\left(T_k\left(\frac{1}{|x|^a}\right) z, Du\right) = T_k\left(\frac{1}{|x|^a}\right)(z, Du)$ as measures for all $k > 0$. In order to do so, first notice that $\operatorname{div}\left(\frac{1}{|x|^a} z\right) \in L^1(\Omega)$. Then $\left(T_k\left(\frac{1}{|x|^a}\right) z, Du\right)$ is defined as in (4.22) by

$$\begin{aligned}&\left\langle \left(T_k\left(\frac{1}{|x|^a}\right) z, Du\right), \varphi\right\rangle \\ &\qquad = -\int_\Omega u\varphi \operatorname{div}\left(T_k\left(\frac{1}{|x|^a}\right) z\right) dx - \int_\Omega u T_k\left(\frac{1}{|x|^a}\right) z \cdot \nabla\varphi \, dx\,.\end{aligned}$$

On the other hand, $T_k\left(\frac{1}{|x|^a}\right)(z, Du)$ is such that

$$\left\langle T_k\left(\frac{1}{|x|^a}\right)(z, Du), \varphi\right\rangle = \int_\Omega T_k\left(\frac{1}{|x|^a}\right)\varphi\,(z, Du)\ .$$

It is not difficult to connect both distributions. To this end, denote $w(x) = T_k\left(\frac{1}{|x|^a}\right)$ and consider the mollification of φw. Then

$$\begin{aligned}\rho_\epsilon * (\varphi w) \to \varphi w \quad &\text{uniformly in } \Omega \\ \nabla\left(\rho_\epsilon * (\varphi w)\right) \to \nabla(\varphi w) \quad &\text{strongly in } L^1(\Omega; \mathbb{R}^N)\end{aligned}$$

and so

$$\int_\Omega T_k\left(\frac{1}{|x|^a}\right)\varphi\,(z,Du)=\lim_{\epsilon\to 0}\int_\Omega \rho_\epsilon*(\varphi w)\,(z,Du)$$
$$=-\lim_{\epsilon\to 0}\int_\Omega u(\rho_\epsilon*(\varphi w))\mathrm{div}\,z\,dx-\lim_{\epsilon\to 0}\int_\Omega uz\cdot\nabla\left(\rho_\epsilon*(\varphi w)\right)\,dx$$
$$=-\int_\Omega u\varphi T_k\left(\frac{1}{|x|^a}\right)\mathrm{div}\,z\,dx-\int_\Omega uz\cdot\nabla\left(\varphi T_k\left(\frac{1}{|x|^a}\right)\right)\,dx\,.$$

We stress that (4.27) implies that both distributions are equal. So, we have proved the following lemma.

Lemma 4.2. *For every $z\in\mathcal{DM}_a(\Omega)$ and $u\in BV_a(\Omega)\cap L^\infty(\Omega)$, we have that*

$$\left(T_k\left(\frac{1}{|x|^a}\right)z,Du\right)=T_k\left(\frac{1}{|x|^a}\right)(z,Du)\qquad \textit{as Radon measures in } \Omega, \forall k>0.$$

We define the weighted pairings as the limit of the above functionals.

Definition 4.3. *Let $z\in\mathcal{DM}_a(\Omega)$ and $u\in BV_a(\Omega)\cap L^\infty(\Omega)$. Then we define the functional $\left(\frac{1}{|x|^a}z,Du\right):C_c^\infty(\Omega)\to\mathbb{R}$ as*

$$\left\langle\left(\frac{1}{|x|^a}z,Du\right),\varphi\right\rangle=-\int_\Omega u\varphi\mathrm{div}\left(\frac{1}{|x|^a}z\right)dx-\int_\Omega\frac{1}{|x|^a}uz\cdot\nabla\varphi dx$$

Lemma 4.4. *For every $z\in\mathcal{DM}_a(\Omega)$ and $u\in BV_a(\Omega)\cap L^\infty(\Omega)$, we have that*

$$\left(\frac{1}{|x|^a}z,Du\right)=\frac{1}{|x|^a}(z,Du)\qquad\textit{as distributions.}$$

As a consequence, since $\frac{1}{|x|^a}(z,Du)$ is a Radon measure in Ω, so is $\left(\frac{1}{|x|^a}z,Du\right)$.

Proof. We point out that $\frac{1}{|x|^a}\in L^1(\Omega,(z,Du))$, since $|(z,Du)|\le\|z\|_\infty|Du|$ and $u\in BV_a(\Omega)\cap L^\infty(\Omega)$. Moreover, we have

$$\left\langle\frac{1}{|x|^a}(z,Du),\varphi\right\rangle=\int_\Omega\varphi\frac{1}{|x|^a}(z,Du)$$
$$=-\int_\Omega u\varphi\frac{1}{|x|^a}\mathrm{div}\;(z)\;dx-\int_\Omega u\varphi z\cdot\nabla\left(\frac{1}{|x|^a}\varphi\right)\,dx.$$

Thus, having in mind (4.28), both distributions are equal. □

Theorem 4.5. *Let $z\in\mathcal{DM}_a(\Omega)$ and $u\in BV_a(\Omega)\cap L^\infty(\Omega)$. For all open sets $U\subset\Omega$ and for all functions $\varphi\in C_c^\infty(U)$, it yields*

$$\left|\left\langle\left(\frac{1}{|x|^a}z,Du\right),\varphi\right\rangle\right|\le\|\varphi\|_\infty\|z\|_{L^\infty(U)}\int_U\frac{1}{|x|^a}|Du|.$$

Proof. Note that, from (4.24) we have that

$$\begin{aligned}\left|\left\langle\left(\frac{1}{|x|^a}z,Du\right),\varphi\right\rangle\right| &= \left|\int_U \frac{1}{|x|^a}\varphi(z,Du)\right| &(4.31)\\ &\le \int_U \frac{1}{|x|^a}|\varphi|\,|(z,Du)| &(4.32)\\ &\le \|\varphi\|_\infty\|z\|_{L^\infty(U)}\int_U \frac{1}{|x|^a}|Du|dx, &(4.33)\end{aligned}$$

what proves the result. ∎

Corollary 4.6. *The measures* $\left(\frac{1}{|x|^a}z,Du\right)$ *and* $\left|\left(\frac{1}{|x|^a}z,Du\right)\right|$ *are absolutely continuous with respect to the measure* $\frac{1}{|x|^a}|Du|$ *and the inequality*

$$\left|\int_B\left(\frac{1}{|x|^a}z,Du\right)\right| \le \int_B\left|\left(\frac{1}{|x|^a}z,Du\right)\right| \le \|z\|_{L^\infty(U)}\int_B\frac{1}{|x|^a}|Du|$$

holds for all Borel sets B and for all open sets U such that $B\subset U\subset\Omega$.

Theorem 4.7. *Let $z\in\mathcal{DM}_a(\Omega)$ and $u\in BV_a(\Omega)\cap L^\infty(\Omega)$. Then we have*

$$\int_\Omega u\mathrm{div}\left(\frac{1}{|x|^a}z\right)dx+\int_\Omega\frac{1}{|x|^a}(z,Du)=\int_{\partial\Omega}\frac{1}{|x|^a}[z,\nu]\,u\,d\mathcal{H}^{N-1}.$$

Proof. It follows from (4.26), jointly with Lemmas 4.1 and 4.2, that

$$\begin{aligned}\int_\Omega u\,\mathrm{div}\left(T_k\left(\frac{1}{|x|^a}\right)z\right)dx+\int_\Omega T_k\left(\frac{1}{|x|^a}\right)(z,Du)\\ =\int_{\partial\Omega}T_k\left(\frac{1}{|x|^a}\right)[z,\nu]\,u\,d\mathcal{H}^{N-1}. \quad (4.34)\end{aligned}$$

for all $k>0$. Since $x\mapsto\frac{1}{|x|^a}$ is a bounded function on $\partial\Omega$, then for k large enough, $T_k\left(\frac{1}{|x|^a}\right)=\frac{1}{|x|^a}$. Hence,

$$\lim_{k\to+\infty}\int_{\partial\Omega}T_k\left(\frac{1}{|x|^a}\right)[z,\nu]\,u\,d\mathcal{H}^{N-1}=\int_{\partial\Omega}\frac{1}{|x|^a}[z,\nu]\,u\,d\mathcal{H}^{N-1}. \quad (4.35)$$

On the left hand side of (4.34), we will apply the Dominated Convergence Theorem. In the first term, we may pass to the limit as in the proof of the Theorem 4.5, taking into account (4.27), (4.29) and $\nabla\left(\frac{1}{|x|^a}\right)\in L^1(\Omega)$. On the other hand, we denote by $\theta(z,Du)$ the Radon–Nikodým derivative of (z,Du) with respect to $|Du|$, so that $|\theta(z,Du)|\le\|z\|_\infty$. Then

$$T_k\left(\frac{1}{|x|^a}\right)(z,Du)=T_k\left(\frac{1}{|x|^a}\right)\theta(z,Du)|Du|$$

and

$$\left|T_k\left(\frac{1}{|x|^a}\right)\theta(z,Du)\right|\le\frac{\|z\|_\infty}{|x|^a}.$$

Owing to $\frac{1}{|x|^a} \in L^1(\Omega, |Du|)$, we are allowed to use the Dominated Convergence Theorem. Therefore, when $k \to \infty$, identity (4.34) becomes

$$\int_\Omega u\mathrm{div}\left(\frac{1}{|x|^a} z\right) dx + \int_\Omega \frac{1}{|x|^a}(z, Du) = \int_{\partial\Omega} \frac{1}{|x|^a}[z,\nu]\, u\, d\mathcal{H}^{N-1}$$

as desired. □

Remark 4.8. Note that, by Lemmas 4.1 and 4.4, the last identity can also be written as

$$\int_\Omega u\mathrm{div}\left(\frac{1}{|x|^a} z\right) dx + \int_\Omega \left(\frac{1}{|x|^a} z, Du\right) = \int_{\partial\Omega} \left[\frac{1}{|x|^a} z, \nu\right] u\, d\mathcal{H}^{N-1}.$$

4.4. Concept of solution to problem (1.2)

Once we have the weighted theory available, we may introduce the definition of solution to problem (1.2).

Definition 4.9. *We say that $u \in BV_a(\Omega) \cap L^\infty(\Omega)$ is a solution of problem* (1.2) *if there exists a vector field $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \le 1$ and such that*

$$\begin{aligned} &(1) \quad -\mathrm{div}\left(\tfrac{1}{|x|^a} z\right) &&= \frac{1}{|x|^b} f(u), \ \text{in } \mathcal{D}'(\Omega),\\ &(2) \quad \left(\tfrac{1}{|x|^a} z, Du\right) &&= \frac{1}{|x|^a}|Du| \ \text{as measures on } \Omega,\\ &(3) \quad [z,\nu] &&\in \mathrm{sign}\,(-u) \ \text{on } \partial\Omega. \end{aligned}$$

We will need a variational formulation of our concept of solution. We begin with the following equivalence, whose proof in the non weighted setting can be found in [6, Proposition 2].

Proposition 4.10. *For $u \in BV_a(\Omega) \cap L^\infty(\Omega)$, the following assertions are equivalent.*

a) *u is a solution to problem* (1.2).
b) *there exists a vector field $z \in L^\infty(\Omega, \mathbb{R}^N)$ satisfying $\|z\|_\infty \le 1$,*

$$-\mathrm{div}\left(\frac{1}{|x|^a} z\right) = \frac{1}{|x|^b} f(u), \ \text{in } \mathcal{D}'(\Omega),$$

and

$$\int_\Omega \frac{1}{|x|^b} f(u)(v-u)dx = \int_\Omega \frac{1}{|x|^a}(z, Dv) - \int_{\partial\Omega} \frac{1}{|x|^a} v[z,\nu]\, d\mathcal{H}^{N-1} - \|u\|_{BV(\Omega),1} \tag{4.36}$$

for all $v \in BV_a(\Omega) \cap L^\infty(\Omega)$.

Proof. To see that $(a) \Rightarrow (b)$, just take $v \in BV_a(\Omega) \cap L^\infty(\Omega)$, multiply the equality (1) of Definition 4.9 by $v - u$ and apply Green's formula and conditions (2) and (3).
The reverse implication $(b) \Rightarrow (a)$ is deduced by taken $v = u$ in (4.36). Indeed, we obtain

$$\|u\|_{BV(\Omega),1} \leq \int_\Omega \frac{1}{|x|^a}(z, Du) - \int_{\partial\Omega} \frac{1}{|x|^a} u[z,\nu]\, d\mathcal{H}^{N-1}$$

and conditions (2) and (3) follow since $\|z\|_\infty \leq 1$. □

Corollary 4.11. *If u is a solution to problem* (1.2)*, then*

$$\int_\Omega \frac{1}{|x|^b} f(u)(v-u)dx \leq \|v\|_{BV(\Omega),1} - \|u\|_{BV(\Omega),1} \,. \tag{4.37}$$

holds for every $v \in BV_a(\Omega)$.

Proof. When $v \in BV_a(\Omega) \cap L^\infty(\Omega)$, it is an easy consequence of Proposition 4.10 and the condition $\|z\|_\infty \leq 1$. For a general $v \in BV_a(\Omega)$, apply this inequality to $T_k(v)$ to get

$$\begin{aligned}\int_\Omega \frac{1}{|x|^b} f(u)(T_k(v)-u)dx &\leq \|T_k(v)\|_{BV(\Omega),1} - \|u\|_{BV(\Omega),1} \\ &\leq \|v\|_{BV(\Omega),1} - \|u\|_{BV(\Omega),1} \,. \end{aligned} \tag{4.38}$$

Now, on account of Theorem 3.2, $v \in L^1_b(\Omega)$ and so we may let k go to ∞ on the left hand side of (4.38). □

Corollary 4.12. *Every solution to problem* (1.2) *is nonnegative.*

Proof. Let u be a solution to problem (1.2). By Proposition 2.7, we may take $v = u^+$ in Corollary 4.11 obtaining

$$\int_\Omega \frac{1}{|x|^b} f(u)(-u^-) \leq \|u^+\|_{BV(\Omega),1} - \|u\|_{BV(\Omega),1} = -\|u^-\|_{BV(\Omega),1} \,.$$

On the left hand side, the integrand vanishes (recall that $f(s) = 0$ for all $s \leq 0$) and we get

$$\int_\Omega \frac{1}{|x|^b} f(u)(-u^-) = \int_{\{u<0\}} \frac{1}{|x|^b} f(u)(-u) = 0 \,.$$

Therefore, $\|u^-\|_{BV(\Omega),1} \leq 0$ and so $u = u^+ \geq 0$. □

To characterize the sub-differential of the norm, we could try to adapt the proof of [6, Section 5] to our weighted framework. Nevertheless, for our purposes, the following result will be enough.

Proposition 4.13. *Let $h \in L^1(\Omega)$ and assume that problem*

$$\begin{cases} -\mathrm{div}\left(\dfrac{1}{|x|^a}\dfrac{Du}{|Du|}\right) = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.39}$$

has a bounded solution w. (By a solution to problem (4.39) we mean that w satisfies Definition 4.9 with the obvious replacement of $\frac{1}{|x|^b}f(w)$ by h.)
If $u \in BV_a(\Omega) \cap L^\infty(\Omega)$ and $h \in \partial\|u\|_{BV(\Omega),1}$, then u is also a solution to problem (4.39).

Proof. Let $w \in BV_a(\Omega) \cap L^\infty(\Omega)$ be a solution to problem (4.39). Then there exists a vector field $z \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\|z\|_\infty \le 1$ and

$$-\mathrm{div}\left(\frac{1}{|x|^a} z\right) = h \qquad \text{in } \mathcal{D}'(\Omega)$$

jointly with conditions (2) and (3). Taken $w - u$ as test function, it yields

$$\begin{aligned} &\int_\Omega h(w-u)\,dx \\ &\qquad = \int_\Omega \frac{1}{|x|^a}(z, Dw) - \int_\Omega \frac{1}{|x|^a}(z, Du) \\ &\qquad - \int_{\partial\Omega} \frac{1}{|x|^a} w[z,\nu]\,d\mathcal{H}^{N-1} + \int_{\partial\Omega} \frac{1}{|x|^a} u[z,\nu]\,d\mathcal{H}^{N-1} \\ &\qquad = \|w\|_{BV(\Omega),1} - \int_\Omega \frac{1}{|x|^a}(z, Du) + \int_{\partial\Omega} \frac{1}{|x|^a} u[z,\nu]\,d\mathcal{H}^{N-1} \end{aligned} \tag{4.40}$$

On the other hand, assumption $h \in \partial\|u\|_{BV(\Omega),1}$ implies

$$\int_\Omega h(w-u)\,dx \le \|w\|_{BV(\Omega),1} - \|u\|_{BV(\Omega),1}\,. \tag{4.41}$$

Hence, gathering (4.40) and (4.41), it follows that

$$-\int_\Omega \frac{1}{|x|^a}(z, Du) + \int_{\partial\Omega} \frac{1}{|x|^a} u[z,\nu]\,d\mathcal{H}^{N-1} \le -\|u\|_{BV(\Omega),1}$$

and the result is a consequence of being $\|z\|_\infty \le 1$. □

5. Proof of Theorem 1.4 through p–Laplacian problems

This section is devoted to prove Theorem 1.4 by an approximating approach. We first consider problems involving the p-Laplacian operator and, following the arguments of [31], we prove a priori estimates which allows us to find the solution w of Problem (1.2) as $p \to 1^+$.

5.1. Approximating problems involving p-Laplacian operators

First of all, we consider $1 < \bar{p} < 2$ and so $\bar{p} < N < \frac{N}{1+a-b}$. Since $0 < a < N-1$, $\mu > 1$ and $1 < q < \frac{N}{N-(1+a-b)}$, we may assume that $\bar{p}$ also satisfies

$$a < \frac{N-\bar{p}}{\bar{p}} < N-1, \quad \mu > \bar{p} \quad \text{and} \quad \bar{p} < q < q+\bar{p}-1 < \frac{N}{N-(1+a-b)}.$$

This implies that, denoting $\bar{q} = q+\bar{p}-1$,

$$p < \frac{N}{1+a-b}, \quad a < \frac{N-p}{p}, \quad \mu > p \quad \text{and} \quad p < \bar{q} < \frac{Np}{N-p(1+a-b)}.$$

for every $1 < p \leq \bar{p}$. Now, for each $1 < p \leq \bar{p}$, we consider the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{1}{|x|^{ap}}|\nabla u|^{p-2}\nabla u\right) & = \dfrac{1}{|x|^{b}} f_p(u) \quad \text{in } \Omega, \\ u & = 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{5.42}$$

where $f_p(s) = f(s)|s|^{p-1}$. Observe that, as a consequence of $(f_1)-(f_4)$, the function f_p satisfies:

(f_{1p}) $f_p \in C^0([0,+\infty),\mathbb{R})$;

(f_{2p}) $\displaystyle\lim_{s\to 0^+} \frac{f_p(s)}{|s|^{p-1}} = 0$;

(f_{3p}) There exist constants $c_1, c_2 > 0$ and $p < \bar{q} < \frac{Np}{N-p(1+a-b)}$, such that

$$|f_p(s)| \leq c_1 + c_2 s^{\bar{q}-1} \qquad \text{for all } s \in [0,+\infty).$$

(f_{4p}) There exists $\mu > p$ such that

$$0 < \mu F_p(s) \leq f_p(s)s, \quad \forall\, s \geq s_0,$$

where $F_p(t) = \int_0^t f_p(s)ds$.

Remark 5.1. *The conditions $(f_{1p})-(f_{3p})$ are straightforward to check. To prove the condition (f_{4p}), just integrate by parts to obtain*

$$\frac{f_p(s)s}{F_p(s)} = \frac{f(s)|s|^{p-1}s}{F(s)|s|^{p-1} - (p-1)\int_0^s \frac{F(\sigma)}{|\sigma|^{2-p}}\,d\sigma} = \frac{f(s)s}{F(s) - (p-1)\frac{1}{|s|^{p-1}}\int_0^s \frac{F(\sigma)}{|\sigma|^{2-p}}\,d\sigma}$$

when $s > 0$. It follows from $\lim_{s\to+\infty} F(s) = +\infty$ and L'Hôpital's rule that

$$\lim_{s\to+\infty} (p-1)\frac{1}{|s|^{p-1}} \int_0^s \frac{F(\sigma)}{|\sigma|^{2-p}}\,d\sigma = +\infty,$$

so that it is positive for s large enough. Hence,

$$\frac{f_p(s)s}{F_p(s)} \geq \frac{f(s)s}{F(s)} \geq \mu$$

for s large enough.

Problem (5.42) has been studied in [12] using the lower and upper–solutions method. Nevertheless, we need to obtain a solution applying the Mountain Pass Theorem to get estimates independent of p and thus be able to pass to the limit as $p \to 1$.

In order to get a nontrivial solution to (5.42), we work in the space $\mathcal{D}^{1,p}_{0,a}(\Omega)$ that is defined in Subsection 2.1. Moreover, the functions of this space satisfy the following Caffarelli-Kohn-Nirenberg inequality.

Theorem 5.2. *Let* $0 < a < \frac{N-\bar{p}}{\bar{p}}$, $0 < \theta \leq 1$ *and* $a < b < a+1$. *Then there exists a constant* $\mathfrak{C}_{CKN} > 0$ *such that the following inequality holds for all* $u \in \mathcal{D}^{1,p}_{0,a}(\Omega)$

$$\left(\int_\Omega \frac{1}{|x|^{\alpha r_{\theta p}}}|u|^{r_{\theta p}}dx\right)^{\frac{1}{r_{\theta p}}} \leq \mathfrak{C}_{CKN}\left(\int_\Omega \frac{1}{|x|^{ap}}|\nabla u|^p dx\right)^{\frac{\theta}{p}}\left(\int_\Omega |u|dx\right)^{1-\theta},$$

where $\alpha = \theta b$ *and* $r_{\theta p} = \frac{Np}{\theta N - p[\theta(1+a-b)-N(1-\theta)]}$.

Proof. The proof follows as that one of Theorem 3.1, with the difference that

$$\rho_\epsilon * \tilde{u} \to \tilde{u} \text{ in } \mathcal{D}^{1,p}_{0,a}(\Omega) \quad \text{as } \epsilon \to 0, \tag{5.43}$$

as showed in Theorem 2.5 of [27]. □

Thanks to this version of the Caffarelli-Kohn-Nirenberg inequality and using the arguments of the proofs of Theorems 3.2 and 3.3, we can show the following embedding result. Probably this result already has been proved in the literature (for a related result, see [36, Theorem 2.1]). However, we state it here for the sake of completeness.

Theorem 5.3. *Let* $0 < a < \frac{N-p}{p}$, $a < b < a+1$ *and* $r_p = r_{1p} = \frac{Np}{N-p(1+a-b)}$. *Then the embedding*

$$\mathcal{D}^{1,p}_{0,a}(\Omega) \hookrightarrow L^q_b(\Omega)$$

is continuous for all $q \in [1, r_p]$ *and compact for all* $q \in [1, r_p)$.

The functional associated to problem (5.42) is given by

$$J_p(u) = \frac{1}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla u|^p dx - \int_\Omega \frac{1}{|x|^b}F_p(u)dx \quad \text{for all } u \in \mathcal{D}^{1,p}_{0,a}(\Omega).$$

By the conditions (f_{2p}), (f_{3p}), (f_{4p}) and the Theorem 5.3, the functional J_p satisfies the geometric conditions of the Mountain Pass Theorem (see [35]), which imply that there exists a $(PS)_c$ sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{D}^{1,p}_{0,a}(\Omega)$, i.e.,

$$J_p(w_n) \to c_p \quad \text{and} \quad J'_p(w_n) \to 0, \quad \text{as } n \to \infty,$$

where

$$c_p = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_p(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], \mathcal{D}^{1,p}_{0,a}(\Omega));\ \gamma(0) = 0,\ J_p(\gamma(1)) < 0\}.$$

Well–known arguments can be used to show that $(w_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{D}^{1,p}_{0,a}(\Omega)$ and consequently, that there exists $w_p \in \mathcal{D}^{1,p}_{0,a}(\Omega)$ in such a way that

$$w_n \to w_p \quad \text{in } \mathcal{D}^{1,p}_{0,a}(\Omega),\ \text{as } n \to \infty.$$

Since $J_p \in C^1(\mathcal{D}^{1,p}_{0,a}(\Omega))$ the previous convergence implies that

$$J_p(w_p) = c_p \quad \text{and} \quad J_p'(w_p) = 0$$

and consequently w_p is a nontrivial solution in $\mathcal{D}^{1,p}_{0,a}(\Omega)$ to problem (5.42).

Once we have got the family of approximate solutions $(w_p)_{1<p\le\bar p}$, our main concern is to get bounds of this family which do not depend on p. To this end, let us consider the functional $I_p : \mathcal{D}^{1,p}_{0,a}(\Omega) \to \mathbb{R}$ defined by

$$I_p(u) = \frac{1}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla u|^p dx + \frac{p-1}{p}|\Omega|.$$

It is straightforward to see that $p \mapsto I_p(u)$ is a nondecreasing function, for every $u \in W^{1,\bar p}_0(\Omega, |x|^{-a})$. Indeed, let $1 < p_1 < p_2 < \bar p$ and note that, by Young's inequality,

$$\begin{aligned} I_{p_1}(u) &= \frac{1}{p_1}\int_\Omega \frac{1}{|x|^{ap_1}}|\nabla u|^{p_1} dx + \frac{p_1-1}{p_1}|\Omega| \\ &\le \frac{1}{p_1}\left(\frac{p_1}{p_2}\int_\Omega \frac{1}{|x|^{ap_2}}|\nabla u|^{p_2} dx + \frac{p_2-p_1}{p_2}|\Omega|\right) + \frac{p_1-1}{p_1}|\Omega| \\ &= I_{p_2}(u). \end{aligned}$$

Moreover, the critical points of J_p are the same of those of $u \mapsto I_p(u) - \int_\Omega \frac{1}{|x|^b} F_p(u) dx$.

Next, we show that there exists $e \in C^\infty_c(\Omega)$ such that

$$J_p(e) < 0, \quad \text{for all } 1 < p \le \bar p.$$

Fix a nontrivial $\phi \in C^\infty_c(\Omega)$ such that $\phi \ge 0$ and $\|\phi\|_\infty \le 1$. This fact leads to

$$\int_\Omega \frac{1}{|x|^b}|\phi|^{\bar p} dx \le \int_\Omega \frac{1}{|x|^b}|\phi|^p dx \le \int_\Omega \frac{1}{|x|^b}|\phi|\, dx \tag{5.44}$$

for every $1 < p \le \bar p$. Moreover, the Lebesgue Dominated Convergence Theorem implies

$$\lim_{p\to 1^+} \int_\Omega \frac{1}{|x|^b}|\phi|^p dx = \int_\Omega \frac{1}{|x|^b}|\phi|\, dx$$

and, as a consequence, we may assume that

$$\frac{1}{2}\int_\Omega \frac{1}{|x|^b}|\phi|\,dx < \int_\Omega \frac{1}{|x|^b}|\phi|^{\bar p}dx\,. \tag{5.45}$$

Analogously, there is no loss of generality in assuming that

$$\int_\Omega \frac{1}{|x|^{ap}}|\nabla\phi|^p dx < 2\int_\Omega \frac{1}{|x|^a}|\nabla\phi|\,dx \tag{5.46}$$

for every $1 < p \le \bar p$.

Now let $t > 1$. Then, owing to $\lim_{s\to+\infty} f(s) = +\infty$, given

$$K = 16\frac{\int_\Omega \frac{1}{|x|^a}|\nabla\phi|\,dx}{\int_\Omega \frac{1}{|x|^b}|\phi|\,dx}, \tag{5.47}$$

we can find $M > 0$ such that $f(s) > K$, and consequently $f_p(s) > Ks^{p-1}$, for all $s > M$. Hence, if $s > M$, then

$$F_p(s) > \int_M^s f(s)|s|^{p-1}ds > K\frac{s^p}{p} - K\frac{M^p}{p} > K\frac{s^p}{p} - K(1+M)^{\bar p}\,.$$

Denoting $K_1 = K(1+M)^{\bar p}\int_\Omega \frac{1}{|x|^b}\,dx$ and taking t large enough such that

$$\frac{1}{2}\int_\Omega \frac{1}{|x|^b}|\phi|^{\bar p}dx < \int_{\{\phi>M/t\}} \frac{1}{|x|^b}|\phi|^{\bar p}dx\,, \tag{5.48}$$

we deduce

$$\begin{aligned}\int_\Omega \frac{1}{|x|^b}F_p(t\phi)\,dx &\ge \int_{\{\phi>M/t\}} \frac{1}{|x|^b}F_p(t\phi)\,dx > K\frac{t^p}{p}\int_{\{\phi>M/t\}} \frac{1}{|x|^b}|\phi|^p dx - K_1\\ &\ge K\frac{t^p}{p}\int_{\{\phi>M/t\}} \frac{1}{|x|^b}|\phi|^{\bar p}dx - K_1 \ge K\frac{t^p}{2p}\int_\Omega \frac{1}{|x|^b}|\phi|^{\bar p}dx - K_1\\ &> K\frac{t^p}{4p}\int_\Omega \frac{1}{|x|^b}|\phi|\,dx - K_1 = 4\frac{t^p}{p}\int_\Omega \frac{1}{|x|^a}|\nabla\phi|\,dx - K_1\,,\end{aligned}$$

where have also used (5.47). Therefore, from (5.46)

$$\begin{aligned}I_p(t\phi) &- \int_\Omega \frac{1}{|x|^b}F_p(t\phi)dx\\ &\le \frac{t^p}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla\phi|^p\,dx - 4\frac{t^p}{p}\int_\Omega \frac{1}{|x|^a}|\nabla\phi|\,dx + K_1\\ &\le K_1 - 2\frac{t^p}{p}\int_\Omega \frac{1}{|x|^a}|\nabla\phi|\,dx\\ &\le K_1 - t\int_\Omega \frac{1}{|x|^a}|\nabla\phi|\,dx,\end{aligned}$$

since $p < 2$ and $t > 1$. Thus, choosing t large enough, we find $e = t\phi$ satisfying

$$J_p(e) \le I_p(e) - \int_\Omega \frac{1}{|x|^b}F_p(e)dx < 0, \qquad \text{for all } 1 < p \le \bar p. \tag{5.49}$$

Since e does not depend on p, thanks to the Mountain Pass Theorem, we know that w_p satisfies

$$I_p(w_p) - \int_\Omega \frac{1}{|x|^b} F_p(w_p)dx = \inf_{\gamma\in\Gamma_p} \max_{t\in[0,1]} \left(I_p(\gamma(t)) - \int_\Omega \frac{1}{|x|^b} F_p(\gamma(t))dx \right),$$

where

$$\Gamma_p = \{\gamma \in C([0,1], \mathcal{D}^{1,p}_{0,a}(\Omega)) : \gamma(0) = 0,\, \gamma(1) = e\}.$$

5.2. Estimate of the family $\{w_p\}$

We claim that the sequence $\left(I_p(w_p) - \int_\Omega \frac{1}{|x|^b} F_p(w_p)dx\right)_{1<p<\bar{p}}$ is bounded by a constant which does not depend on p. Indeed, let $1 < p_1 < p_2 < \bar{p}$ and let us apply the monotonicity of I_p and the fact that $\Gamma_{p_2} \subset \Gamma_{p_1}$ (because $\mathcal{D}^{1,p_2}_{0,a}(\Omega) \subset \mathcal{D}^{1,p_1}_{0,a}(\Omega)$). Then

$$\begin{aligned} I_{p_1}(w_{p_1}) - \int_\Omega \frac{1}{|x|^b} F_{p_1}(w_{p_1})dx &= \inf_{\gamma\in\Gamma_{p_1}} \left(\max_{t\in[0,1]} I_{p_1}(\gamma(t)) - \int_\Omega \frac{1}{|x|^b} F_{p_1}(\gamma(t))dx \right) \\ &\le \inf_{\gamma\in\Gamma_{p_2}} \max_{t\in[0,1]} \left(I_{p_1}(\gamma(t)) - \int_\Omega \frac{1}{|x|^b} F_{p_1}(\gamma(t))dx \right) \\ &\le \inf_{\gamma\in\Gamma_{p_2}} \max_{t\in[0,1]} \left(I_{p_2}(\gamma(t)) - \int_\Omega \frac{1}{|x|^b} F_{p_1}(\gamma(t))dx \right). \end{aligned}$$

It yields

$$\begin{aligned} & I_{p_1}(w_{p_1}) - \int_\Omega \frac{1}{|x|^b} F_{p_1}(w_{p_1})dx \\ &\quad \le \inf_{\gamma\in\Gamma_{p_2}} \max_{t\in[0,1]} I_{p_2}(\gamma(t)) - \int_\Omega \frac{1}{|x|^b} F_{p_2}(\gamma(t))dx \\ &\quad + \int_\Omega \frac{1}{|x|^b} F_{p_2}(\gamma(t))dx - \int_\Omega \frac{1}{|x|^b} F_{p_1}(\gamma(t))dx \\ &\quad \le I_{p_2}(w_{p_2}) - \int_\Omega \frac{1}{|x|^b} F_{p_2}(w_{p_2}) \\ &\quad + \inf_{\gamma\in\Gamma_{p_2}} \max_{t\in[0,1]} \left(\int_\Omega \frac{1}{|x|^b} |F_{p_2}(\gamma(t))|dx + \int_\Omega \frac{1}{|x|^b} |F_{p_1}(\gamma(t))|dx \right) \\ &\quad \le I_{p_2}(w_{p_2}) - \int_\Omega \frac{1}{|x|^b} F_{p_2}(w_{p_2}) \\ &\quad + \max_{t\in[0,1]} \left(\int_\Omega \frac{1}{|x|^b} |F_{p_2}(\gamma_0(t))|)dx + \int_\Omega \frac{1}{|x|^b} |F_{p_1}(\gamma_0(t))|dx \right), \end{aligned}$$

where $\gamma_0(t) = te$. Now, for $1 < p < \bar{p}$, it is straightforward to see that

$$\int_\Omega \frac{1}{|x|^b} F_p(te)dx \le \int_\Omega \frac{1}{|x|^b} F(te)|te|^{p-1}dx \le (\|e\|_\infty + 1)^{\bar{p}-1} \int_\Omega \frac{1}{|x|^b} F(te)dx$$

and so

$$\max_{t\in[0,1]}\left(\int_\Omega \frac{1}{|x|^b}|F_{p_2}(\gamma_0(t))|)dx+\int_\Omega \frac{1}{|x|^b}|F_{p_1}(\gamma_0(t))|dx\right)$$
$$\leq 2(1+\|e\|_\infty)^{\bar{p}-1}\max_{t\in[0,1]}\int_\Omega \frac{1}{|x|^b}F(te)dx.$$

It follows that if $1<p<\bar{p}$, then

$$I_p(w_p)-\int_\Omega \frac{1}{|x|^b}F_p(w_p)dx$$
$$\leq I_{\bar{p}}(w_{\bar{p}})-\int_\Omega \frac{1}{|x|^b}F_{\bar{p}}(w_{\bar{p}})dx+2(1+\|e\|_\infty)^{\bar{p}-1}\max_{t\in[0,1]}\int_\Omega \frac{1}{|x|^b}F(te)dx$$

and the claim is proved. Thus, there exists $C>0$ such that

$$J_p(w_p)=\frac{1}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p|^p dx-\int_\Omega \frac{1}{|x|^b}F_p(w_p)dx\leq C,\quad \text{for all } p\in(1,\bar{p}) \tag{5.50}$$

where the constant C is independent of p.

Let $\Omega_p=\{x\in\Omega : |w_p(x)|\leq s_0\}$, for any $p\in(1,\bar{p})$. Then, by (f_{3p}), we have

$$|F_p(s)|\leq\left|\int_0^s |f_p(\sigma)|\,d\sigma\right|\leq c_1|s|+\frac{c_2}{\bar{q}}|s|^{\bar{q}}$$

and so

$$\int_{\Omega_p}\frac{1}{|x|^b}F_p(w_p)dx\leq c_1\int_{\Omega_p}\frac{1}{|x|^b}|w_p|dx+\frac{c_2}{\bar{q}}\int_{\Omega_p}\frac{1}{|x|^b}|w_p|^{\bar{q}}dx \tag{5.51}$$
$$\leq c_1\int_\Omega\frac{1}{|x|^b}s_0dx+\frac{c_2}{\bar{q}}\int_\Omega\frac{1}{|x|^b}s_0^{\bar{q}}dx=C_1.$$

By the condition (f_{4p}) and since w_p is a solution of (5.42), it holds

$$\int_{\Omega\setminus\Omega_p}\frac{1}{|x|^b}F_p(w_p)\,dx\leq\frac{1}{\mu}\int_{\Omega\setminus\Omega_p}\frac{1}{|x|^b}f_p(w_p)w_p\,dx$$
$$=\frac{1}{\mu}\int_\Omega\frac{1}{|x|^{ap}}|\nabla w_p|^p\,dx-\frac{1}{\mu}\int_{\Omega_p}\frac{1}{|x|^b}f_p(w_p)w_p\,dx. \tag{5.52}$$

On the other hand, note that condition (f_{3p}) also implies

$$-\int_{\Omega_p}\frac{1}{|x|^b}f_p(w_p)w_pdx\leq c_1s_0\int_\Omega\frac{1}{|x|^b}dx+c_2s_0^{\bar{q}}\int_\Omega\frac{1}{|x|^b}dx=C_2 \tag{5.53}$$

Thus, by (5.52) and (5.53), we get

$$\int_{\Omega\setminus\Omega_p}\frac{1}{|x|^b}F_p(w_p)dx\leq\frac{1}{\mu}\int_\Omega\frac{1}{|x|^{ap}}|\nabla w_p|^pdx+C_2. \tag{5.54}$$

Gathering together (5.50), (5.51) and (5.52), we have

$$\left(\frac{1}{p}-\frac{1}{\mu}\right)\int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p|^p dx \le C + C_1 + C_2, \quad \forall p\in(1,\bar p).$$

Moreover, since $1 < p \le \bar p < \mu$ by the last inequality we have that there exists $\tilde C > 0$ independent of p such that

$$\int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p|^p dx \le \tilde C, \quad \forall p\in(1,\bar p). \tag{5.55}$$

Now, using the previous estimate, Young and Hölder's inequalities we have

$$\begin{aligned}\int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p| dx &\le \frac{1}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p|^p dx + \frac{p-1}{p}\int_\Omega \frac{1}{|x|^{ap}}dx\\ &\le \frac{1}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p|^p dx + \left(\int_\Omega \frac{1}{|x|^{a\bar p}}dx\right)^{\frac{p}{\bar p}}|\Omega|^{\frac{\bar p - p}{\bar p}}\\ &\le \tilde C + \left(\int_\Omega \frac{1}{|x|^{a\bar p}}dx + 1\right)(|\Omega|+1) = \hat C\,,\end{aligned} \tag{5.56}$$

where $\hat C$ is a constant independent of p.

5.3. Convergence of $(w_p)_p$

Recalling that $w_p\big|_{\partial\Omega} = 0$, it follows from (5.56) that the sequence $\{w_p\}_{1<p<\bar p}$ is bounded in $BV_a(\Omega)$. Then, up to a subsequence, there exists w such that, by Theorem 3.3,

$$w_p \to w \quad \text{in } L^q_b(\Omega), \tag{5.57}$$

for all $q \in \left[1, \frac{N}{N-(1+a-b)}\right)$ as well as, by (2.9),

$$w_p \to w \quad \text{in } L^s(\Omega), \tag{5.58}$$

for all $s \in \left[1, \frac{N}{N-1}\right)$. Up to a further subsequence, by [11, Theorem 4.9], we may also assume

$$w_p(x) \to w(x) \quad \text{a. e. } x\in\Omega\,. \tag{5.59}$$

and that there exists $g \in L^q_b(\Omega)$, $1 \le q < \frac{N}{N-(1+a-b)}$, such that

$$|w_p(x)| \le g(x) \quad \text{a. e. } x\in\Omega \tag{5.60}$$

holds for all $p \in (1,\bar p]$. Finally, the lower semicontinuity of the functional $u \mapsto \int_\Omega \frac{1}{|x|^a}|Du|$ guarantees that $w \in BV_a(\Omega)$.

5.4. Boundedness of the limit

Let $k \geq 0$ and let $w_p \in \mathcal{D}^{1,p}_{0,a}(\Omega)$ be a solution of problem (5.42). Define

$$A_{k,p} = \{x \in \Omega;\ |w_p(x)| \geq k \text{ a. e. in } \Omega\}.$$

Lemma 5.4. *Let $p > 1$ be small enough. For each $\epsilon > 0$ there exists $k_0 > 0$ (which does not depend on p) such that*

$$\int_{A_{k,p}} \frac{1}{|x|^b}(1 + |w_p|^{\bar{q}-1})^{\frac{N}{1+a-b}}\,dx < \epsilon \quad \textit{for all } k \geq k_0,$$

where $\bar{q}$ is as in (f_{3p}).

Proof. Note that

$$\int_{A_{k,p}} \frac{1}{|x|^b}dx \leq \frac{1}{k^{\frac{N}{N-(1+a-b)}}} \int_{A_{k,p}} \frac{1}{|x|^b}|w_p|^{\frac{N}{N-(1+a-b)}}\,dx. \tag{5.61}$$

Now we denote $\alpha = \frac{(\bar{q}-1)[N-(1+a-b)]}{1+a-b}$ and $l = \frac{N}{1+a-b}$, which satisfy $0 < \alpha < 1$ and $l > 1$. Using (5.61) and Hölder's inequality, we obtain

$$\int_{A_{k,p}} \frac{1}{|x|^b}(1 + |w_p|^{\bar{q}-1})^l dx \leq 2^{l-1}\left(\int_{A_{k,p}} \frac{1}{|x|^b}dx + \int_{A_{k,p}} \frac{1}{|x|^b}|w_p|^{(\bar{q}-1)l}dx\right)$$

$$\leq 2^{l-1}\left(\int_{A_{k,p}} \frac{1}{|x|^b}dx + \left(\int_{A_{k,p}} \frac{1}{|x|^b}|w_p|^{\frac{N}{N-(1+a-b)}}\,dx\right)^{\alpha}\left(\int_{A_{k,p}} \frac{1}{|x|^b}dx\right)^{1-\alpha}\right)$$

$$\leq 2^{l-1}\left(\frac{1}{k^{\frac{N}{N-(1+a-b)}}}\int_{A_{k,p}} \frac{1}{|x|^b}|w_p|^{\frac{N}{N-(1+a-b)}}\,dx\right)$$

$$+2^{l-1}\left(\int_{A_{k,p}} \frac{1}{|x|^b}|w_p|^{\frac{N}{N-(1+a-b)}}\,dx\right)^{\alpha}\left(\frac{1}{k^{\frac{N}{N-(1+a-b)}}}\int_{A_{k,p}} \frac{1}{|x|^b}|w_p|^{\frac{N}{N-(1+a-b)}}\,dx\right)^{1-\alpha}$$

$$\leq 2^{l-1}\left(\frac{1}{k^{\frac{N}{N-(1+a-b)}}} + \frac{1}{k^{\frac{N(1-\alpha)}{N-(1+a-b)}}}\right)\int_{A_{k,p}} \frac{1}{|x|^b}|w_p|^{\frac{N}{N-(1+a-b)}}\,dx$$

Hence, we have got

$$\int_{A_{k,p}} \frac{1}{|x|^b}(1 + |w_p|^{\bar{q}-1})^l dx \leq \omega(k)\int_{\Omega} \frac{1}{|x|^b}|w_p|^{\frac{N}{N-(1+a-b)}}\,dx \tag{5.62}$$

where $\omega(k)$ stands for a quantity independent on p that tends to 0 as $k \to +\infty$. On the other hand, by the Caffarelli-Kohn-Nirenberg inequality, the Hölder inequality and the estimate (5.55) we obtain

$$\int_{\Omega} \frac{1}{|x|^b} |w_p|^{\frac{N}{N-(1+a-b)}} dx \leq \mathfrak{C}_{CKN}^{\frac{N}{N-(1+a-b)}} \left(\int_{\Omega} \frac{1}{|x|^a} |\nabla w_p| dx \right)^{\frac{N}{N-(1+a-b)}}$$
$$\leq \mathfrak{C}_{CKN}^{\frac{N}{N-(1+a-b)}} \tilde{C}^{\frac{N}{N-(1+a-b)}} \tag{5.63}$$

due to (5.56).

Therefore using (5.63) in (5.62) we get

$$\int_{A_{k,p}} \frac{1}{|x|^b} (1 + |w_p|^{\bar{q}-1})^l dx \leq \omega(k) \mathfrak{C}_{CKN}^{\frac{N}{N-(1+a-b)}} \tilde{C}^{\frac{N}{N-(1+a-b)}},$$

which tends to 0 as $k \to \infty$. □

Now, let us deduce from Lemma 5.4 that $w \in L^\infty(\Omega)$. To this end, given $k > 0$, we define the auxiliary function $G_k : \mathbb{R} \to \mathbb{R}$ as

$$G_k(s) = \begin{cases} s - k & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ s + k & \text{if } s < -k. \end{cases} \tag{5.64}$$

Choosing $G_k(w_p)$ as a test function in problem (5.42), we get

$$\int_{\Omega} \frac{1}{|x|^{ap}} |\nabla G_k(w_p)|^p dx = \int_{\Omega} \frac{1}{|x|^b} f_p(w_p) G_k(w_p) dx.$$

Set $1_a^* = \frac{N}{N-(1+a-b)}$. Then the previous identity, Caffarelli-Kohn-Nirenberg's, Young's and Hölder's inequalities and the condition (f_{3p}) lead to

$$\left(\int_\Omega \frac{1}{|x|^b}|G_k(w_p)|^{1_a^*}dx\right)^{\frac{1}{1_a^*}} \le C_\Omega^{\frac{b(1_a^*-1)}{1_a^*}} \left(\int_\Omega \frac{1}{|x|^{b1_a^*}}|G_k(w_p)|^{1_a^*}dx\right)^{\frac{1}{1_a^*}}$$
$$\le C\int_\Omega \frac{1}{|x|^a}|\nabla G_k(w_p)|dx$$
$$\le \frac{C}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla G_k(w_p)|^p dx + \frac{C(p-1)}{p}|\Omega|$$
$$= \frac{C}{p}\int_\Omega \frac{1}{|x|^b} f_p(w_p)G_k(w_p)dx + \frac{C(p-1)}{p}|\Omega|$$
$$\le C\int_\Omega \frac{1}{|x|^b}(1+|w_p|^{\bar{q}-1})|G_k(w_p)|dx + \frac{C(p-1)}{p}|\Omega|$$
$$\le C\left(\int_{A_{kp}} \frac{1}{|x|^b}(1+|w_p|^{\bar{q}-1})^{\frac{N}{1+a-b}}\right)^{\frac{1+a-b}{N}} \left(\int_\Omega \frac{1}{|x|^b}|G_k(w_p)|^{1_a^*}dx\right)^{\frac{1}{1_a^*}}$$
$$+ \frac{C(p-1)}{p}|\Omega| \qquad (5.65)$$

On the other hand, by Lemma 5.4 there exists $k_0 \in \mathbb{N}$ such that

$$\int_{A_{k,p}} \frac{1}{|x|^b}(1+|w_p|^{\bar{q}-1})^{\frac{N}{1+a-b}}dx < \frac{1}{(2C)^{\frac{N}{1+a-b}}} \qquad \text{for all } k \ge k_0. \qquad (5.66)$$

Using (5.66) in (5.65) we get

$$\left(\int_\Omega \frac{1}{|x|^b}|G_k(w_p)|^{1_a^*}dx\right)^{\frac{1}{1_a^*}} \le \frac{2C(p-1)}{p}|\Omega|. \qquad (5.67)$$

Since $w_p(x) \to w(x)$ a. e. in Ω when $p \to 1^+$, Fatou's Lemma implies

$$\int_\Omega \frac{1}{|x|^b}|G_k(w)|^{1_a^*}dx = 0 \qquad \text{for all } k \ge k_0.$$

Therefore $\|w\|_\infty \le k_0$.

5.5. Existence of the vector field

We begin by using the notation of Remark 2.1 and observing that (5.55) yields

$$m_a^p \int_\Omega |\nabla w_p|^p\, dx \le \tilde{C} \qquad \forall p \in (1,\bar{p})$$

and then

$$\int_\Omega |\nabla w_p|^p\, dx \le \tilde{C}\left(1+\frac{1}{m_a}\right)^{\bar{p}} \qquad \forall p \in (1,\bar{p})\,.$$

So, we may apply the same argument than that in [30, Theorem 3.5.] and obtain a subsequence (not relabeled) and $z \in L^\infty(\Omega;\mathbb{R}^N)$ satisfying $\|z\|_\infty \le 1$

and

$$|\nabla w_p|^{p-2}\nabla w_p \rightharpoonup z \quad \text{weakly in } L^s(\Omega;\mathbb{R}^N)\,, \quad \text{for all } 1 \le s < \infty\,. \tag{5.68}$$

In order to pass to the limit in the following stages, these weak convergences must slightly be improved. Fix $1 < s < \infty$ such that $1 < s' < \frac{N}{a}$, and take $\bar{p}$ small enough to have $1 < s' < \frac{N}{a\bar{p}}$, so that $\int_\Omega \frac{1}{|x|^{a\bar{p}s'}}dx < \infty$. Since

$$\frac{1}{|x|^{aps'}} \le \max\left\{\frac{1}{|x|^{a\bar{p}s'}}, 1\right\}$$

for all $1 < p < \bar{p}$, Lebesgue Convergence Dominated Theorem implies

$$\int_\Omega \left|\frac{1}{|x|^{ap}} - \frac{1}{|x|^a}\right|^{s'} dx \to 0 \quad \text{ as } p \to 1^+. \tag{5.69}$$

Thus, the convergences $\frac{1}{|x|^{ap}} \to \frac{1}{|x|^a}$ strongly in $L^{s'}(\Omega)$ and $|\nabla w_p|^{p-2}\nabla w_p \rightharpoonup z$ weakly in $L^s(\Omega;\mathbb{R}^N)$ lead to

$$\frac{1}{|x|^{ap}}|\nabla w_p|^{p-2}\nabla w_p \rightharpoonup \frac{1}{|x|^a}z \quad \text{weakly in } L^1(\Omega;\mathbb{R}^N). \tag{5.70}$$

5.6. w satisfies condition (1) of Definition 4.9

Let $\varphi \in C_c^\infty(\Omega)$ and take it as test function in (5.42) to obtain

$$\int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p|^{p-2}\nabla w_p \cdot \nabla\varphi\, dx = \int_\Omega \frac{1}{|x|^b} f_p(w_p)\varphi\, dx. \tag{5.71}$$

Our aim is to let $p \to 1^+$ in (5.71). On the left hand side it is enough to apply (5.70), while in the right hand side, just observe that

$$f_p(w_p(x)) \to f(w(x)) \qquad \text{a. e. } x \in \Omega$$

due to (5.59). Moreover, by (f_{3p}) and Young's inequality, we get

$$\begin{aligned} |f_p(w_p(x))| &\le c_1 + c_2|w_p(x)|^{\bar{q}-1} \\ &\le c_1 + c_2 g(x)^{\bar{q}-1} \\ &\le c_1 + \frac{1}{\bar{q}}c_2^{\bar{q}} + \frac{\bar{q}-1}{\bar{q}}g(x)^{\bar{q}} \end{aligned}$$

and $g \in L_b^{\bar{q}}(\Omega)$. Hence, the Lebesgue Dominated Convergence Theorem implies

$$\lim_{p\to 1^+}\int_\Omega \frac{1}{|x|^b} f_p(w_p)\varphi dx = \int_\Omega \frac{1}{|x|^b} f(w)\varphi dx. \tag{5.72}$$

Therefore, letting $p \to 1^+$ in (5.71), we obtain

$$\int_\Omega \frac{1}{|x|^a} z \cdot \nabla\varphi\, dx = \int_\Omega \frac{1}{|x|^b} f(w)\varphi\, dx \tag{5.73}$$

and thus item (1) of Definition 4.9 is verified.

5.7. w satisfies condition (2) of Definition 4.9

In this subsection, we show that the identity

$$\left(\frac{1}{|x|^a}z, Dw\right) = \frac{1}{|x|^a}|Dw|,$$

holds as Radon measures.

Firstly note that we may apply Corollary 4.6 (since $\|z\|_\infty \le 1$) getting

$$\int_\Omega \left(\frac{1}{|x|^a}z, Dw\right) \le \int_\Omega \left|\left(\frac{1}{|x|^a}z, Dw\right)\right| \le \int_\Omega \frac{1}{|x|^a}|Dw|.$$

Now let us check the opposite inequality, i. e.,

$$\left\langle \left(\frac{1}{|x|^a}z, Dw\right), \varphi \right\rangle \ge \left\langle \frac{1}{|x|^a}|Dw|, \varphi \right\rangle, \tag{5.74}$$

for all $\varphi \in C_c^1(\Omega)$ such that $\varphi \ge 0$.

Fix $0 \le \varphi \in C_c^1(\Omega)$ and choose $k > \|w\|_\infty$. Taking $T_k(w_p)\varphi \in \mathcal{D}^{1,p}_{0,a}(\Omega)$ as test function in (5.42), we get

$$\int_\Omega \frac{1}{|x|^{ap}}\varphi|\nabla T_k(w_p)|^p\,dx + \int_\Omega \frac{1}{|x|^{ap}}T_k(w_p)|\nabla w_p|^{p-2}\nabla w_p \cdot \nabla\varphi\,dx$$
$$= \int_\Omega \frac{1}{|x|^b}f_p(w_p)T_k(w_p)\varphi\,dx \tag{5.75}$$

Moreover, applying Young's inequality, one deduces

$$\int_\Omega \frac{1}{|x|^a}\varphi|\nabla T_k(w_p)|dx \le \frac{1}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla T_k(w_p)|^p\varphi dx + \frac{p-1}{p}\int_\Omega \varphi dx$$
$$\le -\frac{1}{p}\int_\Omega \frac{1}{|x|^{ap}}T_k(w_p)|\nabla w_p|^{p-2}\nabla w_p\cdot\nabla\varphi\,dx + \frac{1}{p}\int_\Omega \frac{1}{|x|^b}f_p(w_p)T_k(w_p)\varphi\,dx$$
$$+ \frac{p-1}{p}\int_\Omega \varphi dx\,. \tag{5.76}$$

Our next objective is to let $p \to 1^+$. On the left hand side, since $T_k(w_p) \to T_k(w)$ in $L^1(\Omega)$, the lower semicontinuity of (2.11) may be applied:

$$\int_\Omega \frac{1}{|x|^a}\varphi|DT_k(w)| \le \liminf_{p\to 1}\int_\Omega \frac{1}{|x|^a}\varphi|\nabla T_k(w_p)|dx\,. \tag{5.77}$$

We turn to analyze the right hand side of (5.76). The convergence of the first integral is a consequence of (5.59) and (5.70). Thus,

$$\int_\Omega \frac{1}{|x|^a}T_k(w)z\cdot\nabla\varphi\,dx = \lim_{p\to 1}\int_\Omega \frac{1}{|x|^{ap}}T_k(w_p)|\nabla w_p|^{p-2}\nabla w_p\cdot\nabla\varphi\,dx\,. \tag{5.78}$$

We deal with the second integral applying the Lebesgue Dominated Convergence Theorem as in the previous subsection. So, we obtain

$$\int_\Omega \frac{1}{|x|^b} f(w) T_k(w)\varphi\, dx = \lim_{p\to 1} \int_\Omega \frac{1}{|x|^b} f_p(w_p) T_k(w_p)\varphi\, dx\,. \tag{5.79}$$

The last term on the right hand side, obviously, tends to 0.

Therefore, from (5.77), (5.78) and (5.79), inequality (5.76) becomes

$$\int_\Omega \frac{1}{|x|^a}\varphi |DT_k(w)| + \int_\Omega \frac{1}{|x|^a} T_k(w) z\cdot\nabla\varphi\, dx \le \int_\Omega \frac{1}{|x|^b} f(w) T_k(w)\varphi\, dx.$$

Our choice of k leads to

$$\int_\Omega \frac{1}{|x|^a}\varphi |Dw| + \int_\Omega \frac{1}{|x|^a} wz\cdot\nabla\varphi\, dx \le \int_\Omega \frac{1}{|x|^b} f(w) w\varphi\, dx\,,$$

so that (5.73) implies

$$\begin{aligned}\int_\Omega \frac{1}{|x|^a}\varphi|Dw| &\le -\int_\Omega w\varphi \operatorname{div}\left(\frac{1}{|x|^a} z\right) - \int_\Omega \frac{1}{|x|^a} wz\cdot\nabla\varphi dx \\ &= \left\langle \left(\frac{1}{|x|^a} z, Dw\right), \varphi\right\rangle\,.\end{aligned}$$

Thus (5.74) holds.

5.8. w satisfies condition (3) of Definition 4.9

It only remains to check

$$[z,\nu] \in \operatorname{sign}(-w) \qquad \text{on } \partial\Omega. \tag{5.80}$$

It is equivalent to show that

$$\int_{\partial\Omega} \left(\frac{1}{|x|^a}|w| + w\frac{1}{|x|^a}\,[z,\nu]\right) d\mathcal{H}^{N-1} = 0. \tag{5.81}$$

Indeed, $\|z\|_\infty \le 1$ yields

$$-w\left[\frac{1}{|x|^a} z, \nu\right] \le \frac{1}{|x|^a}\|z\|_\infty |w| \qquad \mathcal{H}^{N-1}-\ \text{a. e. on } \partial\Omega \tag{5.82}$$

and so the integrand is nonnegative. Then (5.81) implies $\frac{1}{|x|^a}|w| + w\frac{1}{|x|^a}\,[z,\nu] = 0$ and it follows from (5.82) that (5.80) holds. Actually, due to the nonnegativeness of the integrand, it is enough to check

$$\int_{\partial\Omega} \left(\frac{1}{|x|^a}|w| + w\frac{1}{|x|^a}\,[z,\nu]\right) d\mathcal{H}^{N-1} \le 0. \tag{5.83}$$

In order to do so, we take w_p as a test function in (5.42) obtaining

$$\int_\Omega \frac{1}{|x|^{ap}} |\nabla w_p|^p\, dx = \int_\Omega \frac{1}{|x|^b} f_p(w_p) w_p\, dx\,.$$

Using Young's inequality and the boundary condition $w_p\big|_{\partial\Omega} = 0$, we get

$$p\int_\Omega \frac{1}{|x|^a}|\nabla w_p|\,dx + p\int_{\partial\Omega} \frac{1}{|x|^a}|w_p|\,d\mathcal{H}^{N-1} \le \int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p|^p\,dx + (p-1)|\Omega|$$
$$= \int_\Omega \frac{1}{|x|^b} f_p(w_p)w_p\,dx + (p-1)|\Omega|. \quad (5.84)$$

Our aim is to let $p \to 1^+$ again. The lower semicontinuity of the functional in (2.10) gives

$$\int_\Omega \frac{1}{|x|^a}|Dw| + \int_{\partial\Omega} \frac{1}{|x|^a}|w|\,d\mathcal{H}^{N-1}$$
$$\le \liminf_{p\to 1^+}\left(\int_\Omega \frac{1}{|x|^a}|\nabla w_p|\,dx + \int_{\partial\Omega} \frac{1}{|x|^a}|w_p|\,d\mathcal{H}^{N-1}\right). \quad (5.85)$$

On the other hand, we may apply the Lebesgue Dominated Convergence Theorem on the right hand side of (5.84), owing to

$$f_p(w_p(x))w_p(x) \to f(w(x))w(x) \qquad \text{a. e. } x \in \Omega$$

and the following consequence of condition (f_{3p}):

$$\begin{aligned} |f_p(w_p(x))w_p(x)| &\le c_1|w_p(x)| + c_2|w_p(x)|^{\bar q-1}|w_p(x)| \\ &\le c_3 + c_4 g(x)^{\bar q}\,. \end{aligned}$$

Thus,

$$\int_\Omega \frac{1}{|x|^b} f(w)w\,dx = \lim_{p\to 1}\int_\Omega \frac{1}{|x|^b} f_p(w_p)w_p\,dx \quad (5.86)$$

and the remainder term tends to 0.

Consequently, using (5.85) and (5.86) in (5.84) we get

$$\int_\Omega \frac{1}{|x|^a}|Dw| + \int_{\partial\Omega} \frac{1}{|x|^a}|w|d\mathcal{H}^{N-1} \le \int_\Omega \frac{1}{|x|^b} f(w)w\,dx. \quad (5.87)$$

Applying (5.73) and Green's formula (Theorem 4.7), we arrive at

$$\begin{aligned} \int_\Omega \frac{1}{|x|^b} f(w)w dx &= -\int_\Omega w \operatorname{div}\left(\frac{1}{|x|^a}z\right)dx \quad (5.88) \\ &= -\int_{\partial\Omega} w\left[\frac{1}{|x|^a}z,\nu\right]d\mathcal{H}^{N-1} + \int_\Omega \frac{1}{|x|^a}(z,Dw) \\ &= -\int_{\partial\Omega} w\left[\frac{1}{|x|^a}z,\nu\right]d\mathcal{H}^{N-1} + \int_\Omega \frac{1}{|x|^a}|Dw|. \end{aligned}$$

Gathering together (5.87) and (5.88), we obtain

$$\int_{\partial\Omega} w\left[\frac{1}{|x|^a}z,\nu\right]d\mathcal{H}^{N-1} + \int_\Omega \frac{1}{|x|^a}|w|d\mathcal{H}^{N-1} \le 0, \quad (5.89)$$

and we are done.

Therefore, since w satisfies conditions (1), (2) and (3) of Definition 4.9, we conclude that w is a solution to problem (1.2).

5.9. w is a nontrivial solution of (1.2)

Now, what is left to do is to show that $w \neq 0$. In order to do so, we should introduce the energy functional $\Phi : BV_a(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) = \int_\Omega \frac{1}{|x|^a}|Du| + \int_{\partial\Omega} \frac{1}{|x|^a}|u|d\mathcal{H}^{N-1} - \int_\Omega \frac{1}{|x|^b}F(u)dx.$$

First of all, let us prove that

$$\lim_{p\to 1^+}\left(I_p(w_p) - \int_\Omega \frac{1}{|x|^b}F_p(w_p)dx\right) = \Phi(w). \tag{5.90}$$

Indeed, since w satisfies (1), (2) and (3) in Definition 4.9 and w_p satisfies (5.42), it follows from Remark 4.8, (5.57), (f_{3p}) and the Lebesgue Dominated Convergence Theorem that, as $p \to 1^+$

$$\begin{aligned}
\|w\|_{BV_a(\Omega),1} &= \int_\Omega \frac{1}{|x|^a}|Dw| + \int_{\partial\Omega} \frac{1}{|x|^a}|w|d\mathcal{H}^{N-1} \\
&= \int_\Omega \left(\frac{1}{|x|^a}z, Dw\right) - \int_{\partial\Omega} \frac{1}{|x|^a}w\,[z,\nu]\,d\mathcal{H}^{N-1} \\
&= -\int_\Omega w \operatorname{div}\left(\frac{1}{|x|^a}z\right)\,dx \\
&= \int_\Omega \frac{1}{|x|^b}f(w)wdx \\
&= \frac{1}{p}\int_\Omega f_p(w_p)w_pdx + o_p(1) \\
&= \frac{1}{p}\int_\Omega \frac{1}{|x|^{ap}}|\nabla w_p|^p dx + o_p(1).
\end{aligned} \tag{5.91}$$

Moreover, again by (f_{3p}), (5.57) and the Lebesgue Dominated Convergence Theorem, as $p \to 1^+$, we have that

$$\int_\Omega \frac{1}{|x|^b}F(w)dx = \int_\Omega \frac{1}{|x|^b}F_p(w_p)dx + o_p(1). \tag{5.92}$$

Then, (5.91) and (5.92) imply in (5.90).

We remark that, by (f_1) and (f_2), given $\epsilon > 0$, we may find $\delta > 0$ satisfying

$$|f(s)| < \epsilon \qquad \forall |s| < \delta$$

so that (f_3) implies that there exists a positive constant $\tilde{C}_\epsilon > 0$ such that

$$|f(s)| < \epsilon + \tilde{C}_\epsilon|s|^{q-1} \qquad \forall s \in \mathbb{R}\,.$$

Integrating this inequality, we deduce

$$|F(s)| \leq \epsilon|s| + C_\epsilon |s|^q \qquad \forall s \in \mathbb{R}\,, \tag{5.93}$$

for certain constant $C_\epsilon > 0$. Thus, by Theorem 3.2,

$$\begin{aligned}
\Phi(u) &= \|u\|_{BV_a(\Omega),1} - \int_\Omega \frac{1}{|x|^b} F(u)\,dx \\
&\geq \|u\|_{BV_a(\Omega),1} - \epsilon \int_\Omega \frac{1}{|x|^b}|u|\,dx - C_\epsilon \int_\Omega \frac{1}{|x|^b}|u|^q\,dx \\
&\geq (1-\epsilon C_1)\|u\|_{BV_a(\Omega),1} - C_\epsilon C_q \|u\|^q_{BV_a(\Omega),1}.
\end{aligned}$$

Let us consider $\epsilon > 0$ small enough such that $1 - \epsilon C_1 > 1/2$. So, if $\|u\|_{BV_a(\Omega),1} \leq \rho$, where $0 < \rho < \left(\dfrac{(1-\epsilon C_1) - 1/2}{C_\epsilon C_q}\right)^{\frac{1}{q-1}}$, then

$$\Phi(u) \geq \frac{\|u\|_{BV_a(\Omega),1}}{2}. \tag{5.94}$$

On the other hand, for all $1 < p < \bar{p}$, Young's inequality implies that $I_p(u) - \int_\Omega \frac{1}{|x|^b} F_p(u)dx \geq \Phi(u) + o_p(1)$. Then, for all $\gamma \in \Gamma_p$, from the continuity of $t \mapsto I_p(\gamma(t)) - \int_\Omega \frac{1}{|x|^b} F_p(\gamma(t))dx$ and from the fact that $I_p(e) - \int_\Omega \frac{1}{|x|^b} F_p(e)dx < 0$, it follows that there exists $t_0 \in [0,1]$ such that $\|\gamma(t_0)\|_{BV_a(\Omega),1} = \rho$. Then,

$$I_p(w_p) - \int_\Omega \frac{1}{|x|^b} F_p(w_p) = \inf_{\gamma\in\Gamma_p} \max_{t\in[0,1]} \left(I_p(\gamma(t)) - \int_\Omega \frac{1}{|x|^b|} F_p(\gamma(t))dx \right) \geq \frac{\rho}{2}.$$

Hence, from the last inequality and (5.90), it follows that

$$\Phi(w) > 0$$

and then w is a nontrivial solution of (1.2). It remains to prove that w is a nonnegative solution of (1.2), but Corollary 4.12 does the job. This finishes the proof of Theorem 1.4.

As a consequence of Proposition 4.13, we deduce the following result.

Corollary 5.5. *If $u \in BV_a(\Omega) \cap L^\infty(\Omega)$ and $\frac{1}{|x|^b} f(w) \in \partial\|u\|_{BV(\Omega),1}$, then u is a solution to problem* (1.2)*.*

6. Existence by variational methods

First of all, let us consider the energy functional $\Phi : BV_a(\Omega) \to \mathbb{R}$, given by

$$\begin{aligned}
\Phi(u) &= \int_\Omega \frac{1}{|x|^a}|Du| + \int_{\partial\Omega} \frac{1}{|x|^a}|u| d\mathcal{H}^{N-1} - \int_\Omega \frac{1}{|x|^b} F(u)dx \\
&= \mathcal{J}_a(u) - \mathcal{F}_b(u),
\end{aligned}$$

where

$$\mathcal{J}_a(u) = \|u\|_{BV_a(\Omega),1}$$

and

$$\mathcal{F}_b(u) = \int_\Omega \frac{1}{|x|^b} F(u)dx.$$

It is straightforward to see that $\mathcal{F}_b$ is a smooth functional. Moreover, by the same arguments of [8], it is possible to show that the functional $\mathcal{J}_a$ admits some directional derivatives. More specifically, given $u \in BV_a(\Omega)$, for all $v \in BV_a(\Omega)$ such that $(Dv)^s$ is absolutely continuous with respect to $(Du)^s$, $(Dv)^a$ vanishes a.e. on the set $\{x \in \Omega : (Du)^a(x) = 0\}$ and $v \equiv 0$, $\mathcal{H}^{N-1}-$a.e. on $\{x \in \partial\Omega : u(x) = 0\}$, it follows that

$$\begin{aligned}\mathcal{J}_a'(u)v = &\int_\Omega \frac{1}{|x|^a} \frac{(Du)^a(Dv)^a}{|(Du)^a|} dx + \\ &+ \int_\Omega \frac{1}{|x|^a} \frac{Du}{|Du|}(x) \frac{Dv}{|Dv|}(x)|(Dv)|^s + \int_{\partial\Omega} \frac{1}{|x|^a} \operatorname{sgn}(u) v d\mathcal{H}^{N-1}. \quad (6.95)\end{aligned}$$

In particular, note that, for all $u \in BV_a(\Omega)$,

$$\mathcal{J}_a'(u)u = \mathcal{J}_a(u). \tag{6.96}$$

Then, the directional derivatives $\Phi'(u)u$ exist and

$$\Phi'(u)u = \|u\|_{BV_a(\Omega),1} - \int_\Omega \frac{1}{|x|^b} f(u)u dx. \tag{6.97}$$

Note that Φ can we written as the difference between a Lipschitz and a smooth functional in $BV_a(\Omega)$. Taking into account the theory of subdifferentials of Clarke (see [16, 17]) , we say that $w \in BV_a(\Omega)$ is a critical point of Φ if $0 \in \partial\Phi(w)$, where $\partial\Phi(w)$ denotes the generalized gradient of Φ in w. It follows that this is equivalent to $\mathcal{F}'(w) \in \partial\mathcal{J}_a(w)$ and, since $\mathcal{J}_a$ is convex, this can be written as

$$\mathcal{J}_a(v) - \mathcal{J}_a(w) \geq \mathcal{F}'(w)(v - w), \quad \forall v \in BV_a(\Omega). \tag{6.98}$$

Henceforth, every $w \in BV_a(\Omega)$ such that (6.98) holds is going to be called a critical point of Φ.

Let us prove that Φ satisfies the first geometric condition of the Mountain Pass Theorem (see [22]). Note again (see inequality (5.93)) that, by (f_1), (f_2) and (f_3), it follows that for all $\epsilon > 0$, there exists $A_\epsilon > 0$ such that

$$|F(s)| \leq \epsilon|s| + A_\epsilon|s|^q, \quad \forall s \in \mathbb{R}. \tag{6.99}$$

Note also that, by (6.99) and the embeddings of $BV_a(\Omega)$ (see Theorem 3.2), it follows that

$$\begin{aligned}\Phi(u) &= \|u\|_{BV_a(\Omega),1} - \int_\Omega \frac{1}{|x|^b} F(u)dx \\ &\geq \|u\|_{BV_a(\Omega),1} - \epsilon \|u\|_{L^1_b(\Omega)} - A_\epsilon \|u\|^q_{L^q_b(\Omega)} \\ &= \|u\|_{BV_a(\Omega),1}\left(1 - \epsilon C - c_3 \|u\|^{q-1}_{BV_a(\Omega),1}\right) \\ &\geq \alpha,\end{aligned}$$

for all $u \in BV_a(\Omega)$, such that $\|u\|_{BV_a(\Omega),1} = \rho$, where $0 < \epsilon < 1$ is fixed, $0 < \rho < \left(\frac{1-\epsilon C}{c_3}\right)^{\frac{1}{p-1}}$ and $\alpha = \rho(1 - \epsilon C - c_3\rho^{p-1})$.

Now let us check that Φ satisfies the second geometric condition of the Mountain Pass Theorem. Recall (see Remark 1.2) that condition (f_4) implies that there exists constants $d_1, d_2 > 0$ such that

$$F(s) \geq d_1|s|^\mu - d_2, \quad \forall s \in \mathbb{R}. \tag{6.100}$$

Let $\phi \in C_c^\infty(\Omega)$ be nontrivial and nonnegative and let $t > 0$. Since $\mu > 1$, it follows that

$$\Phi(t\phi) \leq t\|\phi\|_{BV_a(\Omega),1} - d_1 t^\mu \|\phi\|^\mu_{L^\mu\Omega} + d_2|\mathrm{supp}(\phi)| \to -\infty,$$

as $t \to +\infty$, and so we can choose $e \in BV_a(\Omega)$ such that $\Phi(e) < 0$.

Then, the Mountain Pass Theorem (see [22, Theorem 4.1]) implies that there exist sequences $\tau_n \to 0$ and $(w_n) \subset BV_a(\Omega)$ satisfying the following conditions

(1)

$$\lim_{n\to\infty} \Phi(w_n) = c \tag{6.101}$$

where c is given by

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t))$$

and $\Gamma = \{\gamma \in C^0([0,1], BV_a(\Omega));\ \gamma(0) = 0 \text{ and } \gamma(1) = \phi\}$.

(2)

$$\begin{aligned}&\|v\|_{BV_a(\Omega),1} - \|w_n\|_{BV_a(\Omega),1} \\ &\qquad \geq \int_{\mathbb{R}^N} \frac{1}{|x|^b} f(w_n)(v - w_n)dx - \tau_n\|v - w_n\|_{BV_a(\Omega),1},\end{aligned} \tag{6.102}$$

for all $v \in BV_a(\Omega)$.

Let us prove that the sequence (w_n) is bounded in $BV_a(\Omega)$. First of all, note that by taking $v = w_n + tw_n$ in (6.102), dividing by t and letting $t \to 0^{\pm}$, we have that

$$\int_\Omega \frac{1}{|x|^b} f(w_n) w_n dx - \tau_n \|w_n\|_{BV_a(\Omega),1} \leq \|w_n\|_{BV_a(\Omega),1} \\ \leq \int_\Omega \frac{1}{|x|^b} f(w_n) w_n dx + \tau_n \|w_n\|_{BV_a(\Omega),1}. \quad (6.103)$$

Then, by (f_4) and (6.103), note that

$$\begin{aligned} c + o_n(1) &\geq \Phi(w_n) \\ &= \|w_n\|_{BV_a(\Omega),1} - \int_{\Omega\cap[w_n\leq s_0]} \frac{1}{|x|^b} F(w_n) dx - \int_{\Omega\cap[w_n> s_0]} \frac{1}{|x|^b} F(w_n) dx \\ &\geq \|w_n\|_{BV_a(\Omega),1} - C - \frac{1}{\mu} \int_{\Omega\cap[w_n> s_0]} \frac{1}{|x|^b} f(w_n) w_n dx \\ &\geq \|w_n\|_{BV_a(\Omega),1} - C - \frac{1}{\mu} \int_\Omega \frac{1}{|x|^b} f(w_n) w_n dx \\ &\geq \left(1 - \frac{1}{\mu} - \frac{\tau_n}{\mu}\right) \|w_n\|_{BV_a(\Omega),1} - C \\ &\geq C\|w_n\|_{BV_a(\Omega),1} - C + o_n(1), \end{aligned}$$

for some $C > 0$ uniform in $n \in \mathbb{N}$. Then it follows that (w_n) is bounded in $BV_a(\Omega)$.

By the boundedness of $(w_n) \subset BV_a(\Omega)$ and Theorem 3.3, we find $w \in BV_a(\Omega)$ such that

$$w_n \to w \quad \text{in } L^r(\Omega) \text{ for all } r \in \left[1, \tfrac{N}{N-(1+a-b)}\right). \quad (6.104)$$

Then, by (6.104) and the lower semicontinuity of $\mathcal{J}_a$ with respect to the $L^1(\Omega)$ convergence, calculating the $\limsup$ on both sides of (6.102), it yields that w satisfies (6.98). Moreover, by taking $v = w + tw$ in (6.98) and considering the sign of t, we obtain

$$\|w\|_{BV_a(\Omega),1} = \int_\Omega \frac{1}{|x|^b} f(w) w dx. \quad (6.105)$$

On the other hand, taking the limit as $n \to +\infty$ in (6.103), it follows that

$$\|w_n\|_{BV_a(\Omega),1} = \int_\Omega \frac{1}{|x|^b} f(w_n) w_n dx + o_n(1). \quad (6.106)$$

Hence, from (6.104), (6.105), (6.106) and the Lebesgue Dominated Convergence Theorem, it follows that

$$c = \Phi(w)$$

and then w is a nontrivial critical point of Φ.

Our next concern is to check that $w \in L^\infty(\Omega)$. To this end, consider $k > 0$ and the function $G_k(s)$ defined in (5.64). Taking $v = w \pm G_k(w)$ in (6.98), it yields

$$\pm \int_\Omega \frac{1}{|x|^b} f(w) G_k(w)\, dx \le \|w \pm G_k(w)\|_{BV_a(\Omega),1} - \|w\|_{BV_a(\Omega),1} \le \|G_k(w)\|_{BV_a(\Omega),1}$$

and we infer that

$$\|G_k(w)\|_{BV_a(\Omega),1} = \int_\Omega \frac{1}{|x|^b} f(w) G_k(w) dx.$$

Setting $1_a^* = \frac{N}{N-(1+a-b)}$ again and reasoning as in Subsection 5.4, we obtain

$$\left(\int_\Omega \frac{1}{|x|^b} |G_k(w)|^{1_a^*} dx \right)^{\frac{1}{1_a^*}}$$
$$\le C \left(\int_{\{|w| \ge k\}} \frac{1}{|x|^b} (1 + |w|^{q-1})^{\frac{N}{1+a-b}}\, dx \right)^{\frac{1+a-b}{N}} \left(\int_\Omega \frac{1}{|x|^b} |G_k(w)|^{1_a^*} dx \right)^{\frac{1}{1_a^*}}.$$

Since

$$\lim_{k \to \infty} \int_{\{|w| \ge k\}} \frac{1}{|x|^b} (1 + |w|^{q-1})\, dx = 0,$$

we may find $k_0 > 0$ such that

$$C \left(\int_{\{|w| \ge k_0\}} \frac{1}{|x|^b} (1 + |w|^{q-1}\, dx)^{\frac{N}{1+a-b}} \right)^{\frac{1+a-b}{N}} < 1$$

and then

$$\left(\int_\Omega \frac{1}{|x|^b} |G_k(w)|^{1_a^*} dx \right)^{\frac{1}{1_a^*}} = 0$$

holds. Therefore, $G_{k_0}(w) = 0$ and so $|w| \le k_0$.

As a consequence of Corollary 5.5, since $w \in BV_a(\Omega) \cap L^\infty(\Omega)$ satisfies (6.98), it also satisfies all the conditions of Definition 4.9 and, moreover, it is nonnegative thanks to Corollary 4.12.

It just remains to justify that w is a ground-state solution, i.e., that w has the lowest energy level among all nontrivial bounded variation solutions. In order to prove it, we have to recall [23], where it is proved that we can define the Nehari set associated to Φ, given by

$$\mathcal{N} = \left\{ u \in BV_a(\Omega) \backslash \{0\} \; : \; \|u\|_{BV_a(\Omega),1} = \int_\Omega \frac{1}{|x|^b} f(u) u dx \right\}.$$

It can be proven as in [23] that $\mathcal{N}$ is a set which contains all nontrivial bounded variation solutions of (1.2). Then, if we manage to prove that the solution w is such that $\Phi(w) = \inf_{\mathcal{N}} \Phi$, then w would have the lowest energy level among the nontrivial solutions.

By using the same kind of arguments that Rabinowitz in [33], which consists in study the map $t \mapsto \Phi(tv)$ and verify that it has a unique maximum point $t_v > 0$, which is such that $t_v v \in \mathcal{N}$ ((f_5) is mandatory to prove the uniqueness), in the light of $(f_1) - (f_5)$, one can see that $\mathcal{N}$ is radially homeomorphic to the unit sphere in $BV_a(\Omega)$ and also that the minimax level c satisfies

$$c = \inf_{v \in BV_a(\Omega)\setminus\{0\}} \max_{t>0} \Phi(tv) = \inf_{v \in \mathcal{N}} \Phi(v).$$

Since w is such that $\Phi(w) = c$, it follows that w is a solution which has the lowest energy among all the nontrivial ones.

References

[1] Abdellaoui, B., Colorado, E., Peral, I.: *Some remarks on elliptic equations with singular potentials and mixed boundary conditions.* Adv. Nonlinear Stud. 4, No. 4, 503-533 (2004).

[2] Abdellaoui, B., Peral, I.: *On quasilinear elliptic equations related to some Caffarelli-Kohn-Nirenberg inequalities.* Commun. Pure Appl. Anal. 2, No. 4, 539-566 (2003).

[3] Ambrosetti, A., Rabinowitz, P.H.: *Dual variational methods in critical point theory and applications.* J. Funct. Anal. 14, 349-381 (1973).

[4] Ambrosio, L., Fusco, N., Pallara, D.: *Functions of bounded variation and free discontinuity problems*, Oxford University Press, Oxford (2000).

[5] Andreu, F., Ballester, C., Caselles, V., Mazón, J.M.: *Minimizing total variation flow*, C. R. Acad. Sci. Paris Sér. I Math., 331, no. 11, 867 - 872 (2000).

[6] Andreu, F., Ballester, C., Caselles, V., Mazón, J.M.: *The Dirichlet problem for the total variation flow*, J. Funct. Anal. 180, No. 2, 347-403 (2001).

[7] Anzellotti, G.: *Pairings Between Measures and Bounded Functions and Compensated Compactness*, Ann. Mat. Pura Appl., 135, no. 1, 293 - 318 (1983).

[8] Anzellotti, G.: *The Euler equation for functionals with linear growth*, Trans. Amer. Math. Soc., 290, No. 2, 483 - 501 (1985).

[9] Assunção, R.B., Carrião, P.C., Miyagaki, O.H.: *Subcritical perturbations of a singular quasilinear elliptic equation involving the critical Hardy-Sobolev exponent.* Nonlinear Anal., 66, 1351-1364 (2007).

[10] Baldi, A.: *Weighted BV functions*, Houston J. Math, 27, no. 3, 683 - 705 (2001).

[11] Brezis, H.: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer (2010).

[12] Brock, F., Iturriaga, L., Sánchez, J., Ubilla, P.: *Existence of positive solutions for p-Laplacian problems with weights.* Communications on Pure and Applied Analysis, 5, no. 4, 941(2006).

[13] Caffarelli, L., Kohn, R., Nirenberg, L.: *First order interpolation inequalities with weights.* Compositio Mathematica, 53, 259 - 275 (1984).

[14] Carrião, P.C., de Figueiredo, D.G., Miyagaki, O.H.: *Quasilinear elliptic equations of the Henon-type: existence of non-radial solutions.* Commun. Contemp. Math., 11, no. 5, 783 - 798 (2009).

[15] Caselles, V.: *On the entropy conditions for some flux limited diffusion equations*, J. Differential Equation, 250, 3311 - 3348 (2011).

[16] Chang, K.: *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl., 80, 102 - 129 (1981).

[17] Clarke, F.: *Generalized gradients and applications*, Trans. Amer. Math. Soc., 205, 247 - 262 (1975).

[18] Chen, G.-Q., Frid, H.: *Divergence–measure fields and hyperbolic conservation laws*, Arch. Ration. Mech. Anal. 147, no. 2, 89 - 118 (1999).

[19] Demengel, F.: *Functions locally almost* $1-$*harmonic*, Appl. Anal., 83, no. 9, 865 - 896 (2004).

[20] Demengel, F.: *On some nonlinear partial differential equations involving the* $1-$*Laplacian and critical Sobolev exponent*, ESAIM Control Optim. Calc. Var., 4, 667 - 686 (1999).

[21] Evans, L.C., Gariepy, R.F.: *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Florida (1992).

[22] Figueiredo, G.M., Pimenta, M.T.O.: *Existence of bounded variation solution for a* $1-$*Laplacian problem with vanishing potentials*, J. Math. Anal. Appl., 459, 861 - 878 (2018).

[23] Figueiredo, G.M., Pimenta, M.T.O.: *Nehari method for locally Lipschitz functionals with examples in problems in the space of bounded variation functions*, Nonlinear Differential Equations Appl., Vol. 25, no. 5, 18pp (2018).

[24] Figueiredo, G.M., Pimenta, M.T.O.: *Strauss' and Lions' type results in* $BV_a(\Omega)$ *with an application to* $1-$*Laplacian problem* , Milan J. Math, 86, 15 - 30 (2018).

[25] García-Cuerva, J., Rubio de Francia, J.L.: *Weighted norm inequalities and related topics*, North-Holland, Amsterdam - New York - Oxford (1985).

[26] Heinonen, J., Kilpeläinen, T., Martio, O.: *Nonlinear potential theory of degenerate elliptic equations*, Courier Dover Publications (2018).

[27] Kilpeläinen, T.: *Weighted Sobolev spaces and capacity*, Ann. Acad. Sci. Fenn. Ser. AI Math, 19, no. 1, 95 - 113 (1994).

[28] Kufner, A.: *Weighted Sobolev spaces*, Teubner (1980).

[29] Mazón, J.M.: *The Euler-Lagrange equation for the anisotropic least gradient problem*, Nonlinear Anal. Real World Appl., 31, 452 - 472 (2016).

[30] Mercaldo, A., Rossi, J.D., Segura de León, S., Trombetti, C.: *Behaviour of p–Laplacian problems with Neumann boundary conditions when p goes to 1*, Commun. Pure Appl. Anal. 12, no. 1, 253 - 267 (2013).

[31] Molino, A., Segura de León, S.: *Elliptic equations involving the* $1-$*Laplacian and a subcritical source term*, Nonlinear Anal., 168, 50 - 66 (2018).

[32] Moll, J.S.: *The anisotropic total variation flow.*, Math. Ann. 332, no. 1, 177 - 218 (2005).

[33] Rabinowitz, P.: *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys., 43, 270 - 291 (1992).

[34] Stein, E.M.: *Singular integrals and differentiability properties of functions.* Princeton university press (1970).

[35] Willem, M.: *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24. Birkäuser, Boston (1996).

[36] Xuan, B.: *The solvability of quasilinear Brezis-Nirenberg-type problems with singular weights.* Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 62, No. 4, 703-725 (2005).

[37] Zúñiga, A.: *Continuity of minimizers to weighted least gradient problems*, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 178, 86–109 (2019).