Existence of solutions to a 1−Laplacian problem with a concave-convex nonlinearity

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Abstract

In this paper, we analyze a "concave-convex" type problem involving the 1-Laplacian operator in a general Lipschitz–continuous domain and prove the existence of two positive solutions. Owing to 1-Laplacian is 0-homogeneous, the "concave" term must be singular. Hence, we should deal with an energy functional having two non–differentiable terms: the total variation and that one coming from the singular term. Due to these difficulties, we do not get solutions as critical points of the energy functional defined in the $BV(\Omega)$ space. Instead, we study problems involving the p-Laplacian operator and let p goes to 1.

Keywords: 1-Laplacian operator, singular term, concave-convex nonlinearities

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1 Introduction

This paper is devoted to study multiplicity of positive solutions for an elliptic equation driven by the 1-Laplace operator which combines a singular term and a supercritical [1]one. More precisely, we search for positive solutions to problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = \dfrac{\lambda}{u^\gamma} + u^{q-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set having Lipschitz–continuous boundary, $N \geq 2$, $\lambda > 0$ and $0 < \gamma < 1 < q < 1^* = N/(N-1)$. Our aim is to find two positive solutions to (1.1) for λ small enough.

[1] Here and in what follows, supercritical term means that its growth is bigger than the growth of the term that governs the equation. For instance, if the equation is driven by the p-Laplacian, whose growth is $p-1$, then u^{q-1} is a supercritical term when $p < q$.

In recent years problems involving the 1-Laplacian operator have been extensively studied. One of the main interests for studying this kind of equations comes from the variational approach to image restoration after the pioneering paper [34]. The suitable energy space to handle this type of equations is $BV(\Omega)$, the space of functions of bounded variation. The introduction of the proper concept of solution is due to [19, 3, 4]; it consists of considering a vector field $\mathbf{z}$ that plays the role of $\frac{Du}{|Du|}$ by means of the identity $(\mathbf{z}, Du) = |Du|$ (the definition of $(\mathbf{z}, Du)$ as a Radon measure is due to [5]). As to the boundary condition, it does not hold in the sense of traces, but in a very weak sense that involves the weak trace of the normal component of $\mathbf{z}$. For a precise statement of what we understand as a solution to problem (1.1), we refer to Definition 2.1 below.

The study of stationary configurations of reaction-diffusion problems forms a relevant research area in nonlinear partial differential equations. Within this field, a very active subject is that of problems of concave–convex type. The model problem in a bounded domain $\Omega \subset \mathbb{R}^N$ consists of finding a positive solution to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f(t) = \lambda t^r + t^s$ satisfies $0 < r < 1 < s < \frac{N+2}{N-2}$, that is, the reaction is made up of a concave term and a convex one. The combined effects of these two terms leads to an interval $(0, \Lambda)$ where if $\lambda \in (0, \Lambda)$, two positive solutions can be obtained, while only one positive solution exists for $\lambda = \Lambda$ and there is no such a solution for $\lambda > \Lambda$. Already in 1982 the bifurcation diagram for this problem is shown in [25, Remark 1.7, Case 4]. A milestone to understand this diagram was a paper by Ambrosetti, Brezis and Cerami [1], in which the authors get the second positive solution applying an interesting result by Brezis and Nirenberg (see [13]). This paper [1] has been extended and generalized by a large number of authors. Let us cite [21] in which the equation is driven by the p–Laplacian operator instead of the Laplacian, and the source is made up of two terms: one subcritical and the other supercritical (that is, exponents satisfy $0 < r < p - 1 < s < p^* - 1$). It is in this paper that the existence of two positive solutions in the whole interval $(0, \Lambda)$ is called global multiplicity of positive solutions. We point out that we do not reach a global multiplicity result but a local one.

With respect to singular elliptic equations, they have been widely studied. Let us cite the pioneering work [16], which was followed by the analysis in [26] of a more specific problem:

$$\begin{cases} -\Delta u = \dfrac{h(x)}{u^\gamma} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where h is a positive function and $\gamma > 0$. The existence of a positive solution is established and it is proved that regularity of the solution fails when γ is greater than a certain threshold. This problem was further studied by Boccardo and Orsina [12] for non continuous data, and extended to problems governed by the p-Laplacian by Mohammed [29] and by De Cave [17] (see also [18] for the limiting case $p = 1$). Since these singular terms have a subcritical growth, it was natural to analyze the features of combining a singular term and a supercritical one (so that the reaction becomes $f(t) = \lambda t^{-\gamma} + t^s$ with $p < s + 1 < p^*$). Indeed, this kind of problems has been studied by many authors. For problems driven by the Laplacian, it was considered in [35, 23, 24, 11, 6, 7]. For the p-Laplacian case, we refer to [10, 22, 31] (for a related problem, see [32, 33]).

Typically, to get multiplicity of positive solutions of concave–convex problems, the associated energy functional I must be analyzed and solutions are critical points of it: a solution is a local minimum of I, while the other is typically a saddle point obtained by applying the Mountain Pass Theorem. The energy functional associated to (1.1) is $I : BV(\Omega) \to \mathbb{R}$, given by

$$I(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} - \int_\Omega F(u)\, dx \tag{1.2}$$

where F is the primitive of the real function defined by

$$f(s) = \begin{cases} \dfrac{\lambda}{s^\gamma} + s^{q-1} & \text{if } s > 0 \\ 0 & \text{if } s \le 0. \end{cases}$$

It is worth noting that this functional includes the total variation $\int_\Omega |Du|$ and a singular term, and both terms are not differentiable. We point out that the critical point theory for non–smooth functionals will not be applied. Instead we study singular problems of concave–convex type involving the p–Laplacian and let p go to 1. Solutions to these problems are essentially known, but we have to obtain them from the very beginning to have estimates that do not depend on p and thus be able to arrive at the limit.

The main result of the present paper is stated as follows.

Theorem 1.1 *There exist at least two positive solutions to* (1.1) *for each* λ *small enough.*

The plan of this paper is the following. In Section 2 we introduce our notation and present some basic results on the space of functions of bounded variation $BV(\Omega)$. In Section 3 we obtain two families of approximate solutions by approximating problem (1.1) through p–Laplacian problems (see (3.1) below). In Section 4 we show the convergence of the two families and check that their limits are different positive solutions to problem (1.1).

2 Preliminaries

2.1 Notation

Hereafter, we deal with a bounded open set $\Omega \subset \mathbb{R}^N$, with $N \ge 2$ whose boundary is written by $\partial\Omega$. For a set $E \subset \mathbb{R}^N$, we denote by $|E|$ its Lebesgue measure, while $\mathcal{H}^{N-1}(E)$ stands for its $(N-1)$–dimensional Hausdorff measure. We will always consider domains having Lipschitz–continuous boundary. Thus, for $\mathcal{H}^{N-1}$–almost all $x \in \partial\Omega$ there is an outward unit normal vector $\nu(x)$.

We say that a function $u : \Omega \to \mathbb{R}$ is nonnegative if $u(x) \ge 0$ a.e. in Ω and we write $u \ge 0$, and it is positive if $u(x) > 0$ a.e. in Ω and then we write $u > 0$.

We will make use of the usual Lebesgue and Sobolev spaces, denoted by $L^q(\Omega)$ (with norm $\|\cdot\|_q$) and $W_0^{1,p}(\Omega)$ (whose norm is given by $\|u\|_{W_0^{1,p}} = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$), respectively. Recall that by Sobolev's embedding, there exists a constant $S_p > 0$ satisfying $S_p\|u\|_{p^*}^p \le \|u\|_{W_0^{1,p}}^p$ for all $u \in W_0^{1,p}(\Omega)$.

2.2 The energy space $BV(\Omega)$

We say that u is a function of bounded variation, if $u \in L^1(\Omega)$, and its distributional derivative Du is a vector Radon measure. We then write $u \in BV(\Omega)$. It can be proved that $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and its total variation is finite, that is,

$$\int_\Omega |Du| := \sup\left\{\int_\Omega u\,\mathrm{div}\,\phi dx;\ \phi \in C_c^1(\Omega, \mathbb{R}^N),\ \|\phi\|_\infty \le 1\right\} < +\infty.$$

The space $BV(\Omega)$ is a Banach space when endowed with the norm

$$\|u\| := \int_\Omega |Du| + \int_\Omega |u| dx.$$

It can also be seen that a trace operator $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$ is well defined. Using this trace, we define the norm

$$\|u\|_{BV} := \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1},$$

which is equivalent to $\|\cdot\|$. The space $BV(\Omega)$ is continuously embedded into $L^r(\Omega)$ for all $r \in [1, 1^*]$, where $1^* = N/(N-1)$. We denote as $S_1 > 0$ a constant satisfying $S_1\|u\|_{1^*} \le \|u\|_{BV}$ for all $u \in BV(\Omega)$. We may assume that $S_1 = \lim_{p\to 1} S_p$.

A compactness result in $BV(\Omega)$ will be used in what follows. It states that every bounded sequence in $BV(\Omega)$ has a subsequence which strongly converges in $L^r(\Omega)$ for all $r \in [1, 1^*)$ to a certain $u \in BV(\Omega)$. To pass to the limit we will often apply that some functionals defined on $BV(\Omega)$ are lower semicontinuous with respect to the convergence in $L^1(\Omega)$. The most important are the total variation $u \mapsto \int_\Omega |Du|$ and the norm $u \mapsto \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}$.

For a given $u \in BV(\Omega)$, in several arguments it is important to handle with its precise representative, denoted by u^*. It is defined $\mathcal{H}^{N-1}$–a.e. in Ω and satisfies the following property: if $(\rho_\epsilon)_{\epsilon>0}$ is a mollifier family, then

$$u * \rho_\epsilon \to u^* \text{ (as } \epsilon \to 0^+) \quad \mathcal{H}^{N-1}\text{–a.e. in } \Omega\,. \tag{2.1}$$

We point out that, as a consequence, if $u > 0$ a.e., then $u^* > 0$ $\mathcal{H}^{N-1}$–a.e.

For further information on functions of bounded variation, we refer to [2, 9].

2.3 L^∞–divergence–measure vector fields

The theory of L^∞–divergence–measure vector fields provides, under some conditions, a "dot product" of a vector field $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ and Du the gradient of a function of bounded variation, together with a generalized Green's formula. It was introduced in [5] and extended in [15] and it will be essential in our notion of solution to (1.1).

Let us denote by $\mathcal{DM}^\infty(\Omega)$ the space of all vector fields $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distribution is a Radon measure with finite total variation. It should be remarked that, for every $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ can be defined (see [5]). It is denoted $[\mathbf{z}, \nu]$ and satisfies $\|[\mathbf{z}, \nu]\|_\infty \le \|\mathbf{z}\|_\infty$. The symbol $X(\Omega)_1$ stands for the subset of those $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ such that $\operatorname{div}\mathbf{z} \in L^1(\Omega)$. On the other hand, we denote by $\mathcal{DM}^\infty_{loc}(\Omega)$ the set of vector fields belonging to $\mathcal{DM}^\infty(\omega)$ for all open set $\omega \subset\subset \Omega$.

The Anzellotti theory states that the pairing $(\mathbf{z}, Du)$ is a Radon measure if $z \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$. Moreover, the above pairing is a Radon measure when $\mathbf{z} \in X(\Omega)_1$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. It was extended in [15] for $z \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$ (see also [14, Section 5] and [28, Apendix A]). A further extension $z \in \mathcal{DM}^\infty_{loc}(\Omega)$ and $u \in BV_{loc}(\Omega) \cap L^\infty(\Omega)$ can be found in [18]

Let $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$. Note that, by (2.1) and the fact that $|\operatorname{div}\mathbf{z}| \ll \mathcal{H}^{N-1}$, we have that

$$u * \rho_\epsilon \to u^* \quad |\operatorname{div} z|\text{–a.e. in } \Omega \tag{2.2}$$

and so, the precise representative u^* is $|\operatorname{div}\mathbf{z}|$–summable. Let $u \in BV_{loc}(\Omega) \cap L^\infty(\Omega)$ and $z \in \mathcal{DM}^\infty_{loc}(\Omega)$. Define the distribution $(\mathbf{z}, Du) : C_c^\infty(\Omega) \to \mathbb{R}$ by

$$\langle(\mathbf{z}, Du), \varphi\rangle = -\int_\Omega u^*\varphi \operatorname{div}\mathbf{z} - \int_\Omega u\mathbf{z}\cdot\nabla\varphi\, dx \quad \forall\, \varphi \in C_c^\infty(\Omega)\,. \tag{2.3}$$

Since all the terms have sense, $(\mathbf{z}, Du)$ is well–defined. Notice, however, that this definition depends on the precise representative of u; if we choose another representative, the above functional can be different. Nevertheless, it is independent of the representative when $\mathbf{z} \in X(\Omega)_1$.

Proposition 2.1 *Let $u \in BV_{loc}(\Omega) \cap L^\infty(\Omega)$ and let $z \in \mathcal{DM}^\infty_{loc}(\Omega)$. Then the functional $(\mathbf{z}, Du)$ satisfy*

$$|\langle(\mathbf{z}, Du), \varphi\rangle| \leq \|\varphi\|_\infty \|\mathbf{z}\|_{L^\infty(A)} \int_A |Du| \,, \tag{2.4}$$

for all open set $A \subset\subset \Omega$ and for all $\varphi \in C^\infty_c(A)$. Hence $(\mathbf{z}, Du)$ is a Radon measure in Ω.

Corollary 2.1 *The measures $(\mathbf{z}, Du)$ and $|(\mathbf{z}, Du)|$ are absolutely continuous with respect to the measure $|Du|$ in Ω and one has*

$$|(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_\infty |Du| \,,$$

as measures in Ω.

We remark that if $u \in BV(\Omega)$, then the measure $(\mathbf{z}, Du)$ has finite total variation.

Proposition 2.2 *Let $\mathbf{z} \in X(\Omega)_1$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then the following Green's Formula holds*

$$\int_\Omega u \, div \, \mathbf{z} + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u[\mathbf{z}, \nu] \, d\mathcal{H}^{N-1} \,. \tag{2.5}$$

2.4 Definition of solution to problem (1.1)

The above subsections allow us to define what we mean by a solution of (1.1).

Definition 2.1 *We say that $u \in BV(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.1) if*

a) $u > 0$ *a.e. in* Ω,

b) $\frac{1}{u^\gamma} \in L^1(\Omega)$,

and there exists $\mathbf{z} \in X(\Omega)_1$ such that $\|\mathbf{z}\|_\infty \leq 1$,

c) $-div \, \mathbf{z} = \frac{\lambda}{u^\gamma} + u^{q-1}$ *in* $\mathcal{D}'(\Omega)$,

d) $(\mathbf{z}, Du) = |Du|$ *as mesures in* Ω,

e) $[\mathbf{z}, \nu] \in sign(-u)$ $\mathcal{H}^{N-1}$*–a.e. on* $\partial\Omega$.

3 An approximate singular problem driven by the p–Laplacian

In this section, we consider the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u &= \dfrac{\lambda}{u^\gamma} + u^{q-1} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

where $\lambda > 0$ and $0 < \gamma < 1 < q < 1^*$.

To find positive solutions of (3.1), we must consider the energy functional associated to (3.1), defined in $W^{1,p}_0(\Omega)$ by

$$I^p(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \int_\Omega F(u) \, dx. \tag{3.2}$$

where

$$F(s) = \begin{cases} \frac{\lambda}{1-\gamma} s^{1-\gamma} + \frac{1}{q} s^q & \text{if } s > 0 \\ 0 & \text{if } s \le 0 \end{cases},$$

which is the same function considered in the definition of functional I. The objective of this section is to find two positive solutions u_p and v_p to problem (3.1), where u_p is a local minimum of I^p, while v_p is a critical point of I^p.

In the next section we will study the convergence of families (u_p) and (v_p) as p goes to 1. We have mentioned that the study of problems and functionals related to (3.1) and I^p is an active area of research. Hence, these solutions have already been obtained. Nevertheless, we have to get again the above solutions since the convergence of both families lies in deducing estimates not depending on p (at least for p close to 1).

3.1 Connecting the functionals I^p and I

This subsection is devoted to analyze the connection between the geometry of I^p and that of I. We prove that we can pass continuously from the parameters associated to I^p to those associated to I.

Lemma 3.1 *For each*

$$0 < \rho < \left(q S_1^q |\Omega|^{\frac{q-1^*}{1^*}} \right)^{\frac{1}{q-1}}$$

there exist $\lambda^*(\rho), \beta(\rho) > 0$ such that

$$I(u) \ge \beta(\rho),$$

for all $0 < \lambda < \lambda^*(\rho)$ and noonnegative $u \in BV(\Omega)$ with $\|u\|_{BV} = \rho$.

Proof. From Hölder's inequality and Sobolev's embedding for $BV(\Omega)$, it follows that

$$I(u) \ge \|u\|_{BV} \left(1 - \frac{\lambda}{1-\gamma} S_1^{\gamma-1} |\Omega|^{\frac{1^*+\gamma-1}{1^*}} \|u\|_{BV}^{-\gamma} - \frac{S_1^{-q}}{q} |\Omega|^{\frac{1^*-q}{1^*}} \|u\|_{BV}^{q-1} \right).$$

Note that if $0 < \rho < \left(q S_1^q |\Omega|^{\frac{q-1^*}{1^*}} \right)^{\frac{1}{q-1}}$, then

$$1 - \frac{S_1^{-q}}{q} |\Omega|^{\frac{1^*-q}{1^*}} \rho^{q-1} > 0.$$

Now choose $\lambda^*(\rho) > 0$ such that

$$1 - \frac{\lambda^*(\rho)}{1-\gamma} S_1^{\gamma-1} |\Omega|^{\frac{1^*+\gamma-1}{1^*}} \rho^{-\gamma} - \frac{S_1^{-q}}{q} |\Omega|^{\frac{1^*-q}{1^*}} \rho^{q-1} = 0$$

and take $0 < \lambda < \lambda^*(\rho)$. Then it is easy to see that

$$I(u) \ge \rho \left(1 - \frac{\lambda}{1-\gamma} S_1^{\gamma-1} |\Omega|^{\frac{1^*+\gamma-1}{1^*}} \rho^{-\gamma} - \frac{S_1^{-q}}{q} |\Omega|^{\frac{1^*-q}{1^*}} \rho^{q-1} \right) = \beta(\rho) > 0,$$

for all nonnegative $u \in BV(\Omega)$ such that $\|u\|_{BV} = \rho$. □

Lemma 3.2 *For each*

$$0 < \rho_p < \left(\frac{q}{p} S_p^{\frac{q}{p}} |\Omega|^{\frac{q-p^*}{p^*}} \right)^{\frac{1}{q-p}}$$

there exist $\lambda_p^*(\rho_p), \beta_p(\rho_p) > 0$ such that

$$I^p(u) \geq \beta_p(\rho_p).$$

Moreover, we may assume that the families ρ_p, $\beta_p(\rho_p)$ and $\lambda_p^*(\rho_p)$ satisfy

$$\lim_{p\to 1^+} \rho_p = \rho, \quad \lim_{p\to 1^+} \beta_p(\rho_p) = \beta(\rho) \quad \text{and} \quad \lim_{p\to 1^+} \lambda_p^*(\rho_p) = \lambda^*(\rho).$$

Proof. From Hölder's inequality and Sobolev's embedding for $W_0^{1,p}(\Omega)$, it follows that

$$I^p(u) \geq \|u\|^p_{W_0^{1,p}} \left(\frac{1}{p} - \frac{\lambda}{1-\gamma} S_p^{\frac{\gamma-1}{p}} |\Omega|^{\frac{p^*+\gamma-1}{p^*}} \|u\|^{1-\gamma-p}_{W_0^{1,p}} - \frac{S_p^{\frac{-q}{p}}}{q} |\Omega|^{\frac{p^*-q}{p^*}} \|u\|^{q-p}_{W_0^{1,p}} \right).$$

Standard calculations imply that if $0 < \rho_p < \left(\frac{q}{p} S_p^{\frac{q}{p}} |\Omega|^{\frac{q-p^*}{p^*}} \right)^{\frac{1}{q-p}}$, then

$$\frac{1}{p} - \frac{S_p^{\frac{-q}{p}}}{q} |\Omega|^{\frac{p^*-q}{p^*}} \rho_p^{q-p} > 0.$$

Observe that, owing to $\lim_{p\to 1^+} S_p = S_1$, we get

$$\lim_{p\to 1^+} \left(\frac{q}{p} S_p^{\frac{q}{p}} |\Omega|^{\frac{q-p^*}{p^*}} \right)^{\frac{1}{q-p}} = \left(q S_1^q |\Omega|^{\frac{q-1^*}{1^*}} \right)^{\frac{1}{q-1}},$$

so that ρ_p can be taken in such a way that $\lim_{p\to 1^+} \rho_p = \rho$.

Then it is enough to choose $\lambda_p^*(\rho_p) > 0$ such that

$$\frac{1}{p} - \frac{\lambda_p^*(\rho_p)}{1-\gamma} S_p^{\frac{\gamma-1}{p}} |\Omega|^{\frac{p^*+\gamma-1}{p^*}} \rho_p^{1-p-\gamma} - \frac{S_p^{\frac{-q}{p}}}{q} |\Omega|^{\frac{p^*-q}{p^*}} \rho_p^{q-p} = 0.$$

Hence, if $0 < \lambda < \lambda_p^*(\rho_p)$, then

$$I^p(u) \geq \rho_p \left(\frac{1}{p} - \frac{\lambda}{1-\gamma} S_p^{\frac{\gamma-1}{p}} |\Omega|^{\frac{p^*+\gamma-1}{p^*}} \rho_p^{1-p-\gamma} - \frac{S_p^{\frac{-q}{p}}}{q} |\Omega|^{\frac{p^*-q}{p^*}} \rho_p^{q-p} \right) = \beta_p(\rho_p) > 0,$$

for all $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_{W_0^{1,p}} = \rho_p$. We point out that $0 < \lambda < \lambda^*(\rho_p)$ can be chosen to satisfy $0 < \lambda < \lambda_p^*(\rho_p)$ for all p close enough to 1 and so $\lim_{p\to 1^+} \beta_p(\rho_p) = \beta(\rho)$ holds. $\square$

From now on, we fix $0 < \lambda < \lambda^*(\rho)$ and assume that p_0 is such that, for every $1 < p < p_0$, we have that $0 < \lambda < \lambda_p^*(\rho_p)$. Later on, we will require further restrictions on the parameter λ and p_0 and we will consider $\lambda^* = \lambda^*(\rho), \beta = \beta(\rho), \lambda_p^* = \lambda_p^*(\rho_p)$ and $\beta_p = \beta_p(\rho_p)$ if there is no confusion.

Lemma 3.3 *Let $\phi \in C_c^\infty(\Omega)$ be such that $\phi \geq 0$ and $\phi \neq 0$. If $t > 0$ is small enough, then $I^p(t\phi) < 0$ and also $I(t\phi) < 0$.*

Proof. If $t > 0$ is small enough, then we may write

$$I^p(t\phi) = t^{1-\gamma} \left(\frac{t^{p-1+\gamma}}{p} \|\phi\|^p_{W_0^{1,p}} - \frac{\lambda}{(1-\gamma)} \int_\Omega \phi^{1-\gamma} dx - \frac{t^{q-1+\gamma}}{q} \int_\Omega \phi^q dx \right) < 0$$

and the same arguments apply to I. The result is just a consequence of $\lim_{p\to 1} I^p(t\phi) = I(t\phi)$. $\square$

3.2 Critical points of (3.2)

In this subsection, we find two different nonnegative critical points of I^p (named u_p and v_p) satisfying

$$I^p(u_p) < 0 < \beta_p \le I^p(v_p)\,. \tag{3.3}$$

First of all, let us define the well known *Nehari manifold* (even though it is not a manifold properly) associated to (3.1)

$$\mathcal{N}_p = \left\{ u \in W_0^{1,p}(\Omega)\backslash\{0\}\,;\ \int_\Omega |\nabla u|^p\,dx - \int_\Omega |u|^q - \lambda \int_\Omega |u|^{1-\gamma}\,dx = 0 \right\},$$

as well as some subsets of it

$$\mathcal{N}_p^+ = \left\{ u \in \mathcal{N}_p\,;\ (p-1+\gamma)\int_\Omega |\nabla u|^p\,dx - (q-1+\gamma)\int_\Omega |u|^q > 0 \right\},$$

$$\mathcal{N}_p^- = \left\{ u \in \mathcal{N}_p\,;\ (p-1+\gamma)\int_\Omega |\nabla u|^p\,dx - (q-1+\gamma)\int_\Omega |u|^q < 0 \right\}$$

and

$$\mathcal{N}_p^0 = \left\{ u \in \mathcal{N}_p\,;\ (p-1+\gamma)\int_\Omega |\nabla u|^p\,dx - (q-1+\gamma)\int_\Omega |u|^q = 0 \right\}.$$

For a given $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, let us denote

$$\varphi_u(t) := I^p(tu). \tag{3.4}$$

It is straightforward to see that

$$\mathcal{N}_p = \left\{ u \in W_0^{1,p}(\Omega)\backslash\{0\}\,;\ \varphi_u'(1) = 0 \right\},$$

and also that

$$\mathcal{N}_p^+ = \{u \in \mathcal{N}_p\,;\ \varphi_u''(1) > 0\}\,,$$
$$\mathcal{N}_p^- = \{u \in \mathcal{N}_p\,;\ \varphi_u''(1) < 0\}\,,$$

and

$$\mathcal{N}_p^0 = \{u \in \mathcal{N}_p\,;\ \varphi_u''(1) = 0\}\,.$$

Now, we are going to analyze functions φ_u and deduce some crucial geometric and compactness properties about these subsets of the Nehari manifold. In Figure 1 a typical profile of φ_u is shown.

Lemma 3.4 *Let $0 < \lambda < \overline{\lambda_p}$, where*

$$\overline{\lambda_p} := S_p^{\frac{q+\gamma-1}{q-p}} |\Omega|^{-\sigma_p} C_{p,q,\gamma},$$

$$\sigma_p = \frac{p(p^*-q)}{p^*q}\frac{(q+\gamma-1)}{(q-p)} + \frac{q+\gamma-1}{q}$$

and

$$C_{p,q,\gamma} = \left(\frac{p+\gamma-1}{q+\gamma-1}\right)^{\frac{p+\gamma-1}{q-p}} - \left(\frac{p+\gamma-1}{q+\gamma-1}\right)^{\frac{q+\gamma-1}{q-p}} = \left(\frac{q-p}{q+\gamma-1}\right)\left(\frac{p+\gamma-1}{q+\gamma-1}\right)^{\frac{p+\gamma-1}{q-p}}.$$

Then, for each nonnegative $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, there exist unique real numbers $0 < t^+ < t^-$, such that $t^+u \in \mathcal{N}_p^+$ and $t^-u \in \mathcal{N}_p^-$. Moreover,

$$I^p(t^-u) = \max_{t>0} I^p(tu)$$

and, for $u \in W_0^{1,p}(\Omega)$, $t^- = 1$ if, and only if, $u \in \mathcal{N}_p^-$.

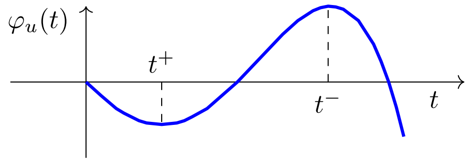

Figure 1: Typical profile of φ_u

Proof. Fix one of those functions $u \in W_0^{1,p}(\Omega)$ and let φ_u be as in (3.4). Note that $\varphi_u'(t) = t^{-\gamma}\eta(t)$, being

$$\eta(t) = \|u\|^p_{W_0^{1,p}} t^{p+\gamma-1} - \|u\|_q^q t^{q+\gamma-1} - \lambda \int_\Omega |u|^{1-\gamma} dx.$$

It is straightforward that φ_u' and η have the same roots. Main features of η are $\eta(0) < 0$ and $\lim_{t\to+\infty} \eta(t) = -\infty$ for all $p < p_0 < q$. Moreover, differentiating η, we deduce it admits a unique maximum point $\bar{t}$, which is given by

$$\bar{t} = \left(\frac{(p+\gamma-1)\|u\|^p_{W_0^{1,p}}}{(q+\gamma-1)\|u\|_q^q} \right)^{\frac{1}{q-p}}.$$

Simple calculations show that

$$\eta(\bar{t}) = C_{p,q,\gamma} \frac{\|u\|_{W_0^{1,p}}^{p\frac{q+\gamma-1}{q-p}}}{\|u\|_q^{q\frac{p+\gamma-1}{q-p}}} - \lambda \int_\Omega |u|^{1-\gamma} dx.$$

Now the Sobolev and Hölder inequalities imply

$$\eta(\bar{t}) \geq \left(C_{p,q,\gamma} S_p^{\frac{q+\gamma-1}{q-p}} |\Omega|^{-p\frac{p^*-q}{qp^*}\frac{q+\gamma-1}{q-p}} - \lambda |\Omega|^{\frac{q+\gamma-1}{q}} \right) \|u\|_q^{1-\gamma}$$

and so this maximum point is positive, as far as $0 < \lambda < \overline{\lambda_p}$. Then, taking our hypothesis into account, we have that there exist unique real numbers $0 < t^+ < \bar{t} < t^-$, such that $\eta(t^+) = \eta(t^-) = 0$, so that $\varphi_u'(t^+) = \varphi_u'(t^-) = 0$. We also deduce that $\varphi_u''(t^+) > 0$ and $\varphi_u''(t^-) < 0$. Hence, since $\varphi_{tu}''(1) = t^2\varphi_u''(t)$ for all $t \in \mathbb{R}$, it follows that

$$t^+u \in \mathcal{N}_p^+ \quad \text{and} \quad t^-u \in \mathcal{N}_p^-.$$

To end up with the proof, it follows straightforwardly from the definition of $\mathcal{N}_p^-$ and φ_u, that $t^- = 1$ if and only if $u \in \mathcal{N}_p^-$. $\square$

Lemma 3.5 *Let*

$$\widetilde{\lambda_p} = \left(\frac{q-p+2\gamma-2}{q-1+\gamma} \right) \left(\frac{p-1+\gamma}{q-1+\gamma} \right)^{\frac{1-\gamma-p}{p-q}} C_p^{-p-p\left(\frac{1-\gamma-p}{p-q}\right)} |\Omega|^{\frac{1-q-\gamma}{q}},$$

where C_p is the best constant of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Then for all $\lambda \in (0, \widetilde{\lambda_p})$ it holds that:

i) $\mathcal{N}_p^0 = \emptyset$;

ii) $\mathcal{N}_p^-$ *is a closed set in* $W_0^{1,p}(\Omega)$.

Proof. The proof just follows as in [8, Lemma 2.2]. $\square$

Remark 3.1 *Note that, as* $p \to 1^+$, $\overline{\lambda_p} \to \overline{\lambda_1} =: \overline{\lambda}$ *and* $\widetilde{\lambda_p} \to \widetilde{\lambda_1} =: \widetilde{\lambda}$.

Lemma 3.6 *Given* $u \in \mathcal{N}_p^-$, *there exist* $\epsilon > 0$ *and a continuous function* $h > 0$ *defined in* $B_\epsilon(0) \subset W_0^{1,p}(\Omega)$, *such that*

$$h(0) = 1, \quad h(w)(u + w) \in \mathcal{N}_p^-, \quad \forall\, w \in B_\epsilon(0).$$

Proof. Just follow the proof in [8, Lemma 2.4]. □

Lemma 3.7 *Let* $0 < \lambda < \min\{\lambda_p^*, \overline{\lambda_p}\}$, *where* λ_p^* *and* $\overline{\lambda_p}$ *are given as in Lemmas 3.2 and 3.4, respectively. Then*

$$\inf_{u \in \mathcal{N}_p^-} I^p(u) \geq \beta_p > 0,$$

where β_p *is given in Lemma 3.2.*

Proof. Given $u \in \mathcal{N}_p^-$, let $t^- > 0$ be such that $t^- u \in \mathcal{N}_p^-$ and $I^p(t^-u) = \max_{t>0} I^p(tu)$, which exists by Lemma 3.4. Let β_p and ρ_p be as in Lemma 3.2. Then

$$I^p(t^-u) \geq I^p\left(\rho_p \frac{u}{\|u\|_{W_0^{1,p}}}\right) \geq \beta_p.$$

□

Lemma 3.8 *Assume that there exists* $u_0 \in \mathcal{N}_p$, $u_0 > 0$, *such that*

$$0 \leq \int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi dx - \lambda \int_\Omega u_0^{-\gamma}\phi\, dx - \int_\Omega u_0^{q-1}\phi\, dx, \quad \forall \phi \in W_0^{1,p}(\Omega),\ \phi \geq 0. \tag{3.5}$$

Then u_0 *is a weak solution of* (3.1).

Proof. Let $\phi \in W_0^{1,p}(\Omega)$ and $\epsilon > 0$, and define $w \in W_0^{1,p}(\Omega)$ by

$$w \equiv (u_0 + \epsilon\phi)^+.$$

Using w as test function in (3.5) and taking into account that $(I^p)'(u_0)u_0 = 0$, we obtain

$$\begin{aligned}
0 \;&\leq\; \int_\Omega \Big(|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla w - u_0^{q-1} w - \lambda u_0^{-\gamma} w\Big)\, dx \\
&=\; \int_{[u_0+\epsilon\phi \geq 0]} \Big(|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla(u_0+\epsilon\phi) - u_0^{q-1}(u_0+\epsilon\phi) - \lambda u_0^{-\gamma}(u_0+\epsilon\phi)\Big)\, dx \\
&=\; \left(\int_\Omega - \int_{[u_0+\epsilon\phi<0]}\right) \Big(|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla(u_0+\epsilon\phi) - u_0^{q-1}(u_0+\epsilon\phi) - \lambda u_0^{-\gamma}(u_0+\epsilon\phi)\Big)\, dx \\
&=\; \int_\Omega |\nabla u_0|^p\, dx - \int_\Omega u_0^q\, dx - \lambda \int_\Omega u_0^{1-\gamma}\, dx \\
&\quad + \epsilon\left(\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi - \int_\Omega u_0^{q-1}\phi\, dx - \lambda\int_\Omega u_0^{-\gamma}\phi\, dx\right) \\
&\quad - \int_{[u_0+\epsilon\phi<0]} \Big(|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla(u_0+\epsilon\phi) - u_0^{q-1}(u_0+\epsilon\phi) - \lambda u_0^{-\gamma}(u_0+\epsilon\phi)\Big)\, dx \\
&\leq\; \epsilon\left(\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi - \int_\Omega u_0^{q-1}\phi\, dx - \lambda\int_\Omega u_0^{-\gamma}\phi\, dx\right) \\
&\quad - \epsilon\int_{[u_0+\epsilon\phi<0]} |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi\, dx.
\end{aligned} \tag{3.6}$$

Since $\lim_{\epsilon\to 0^+} |[u_0 + \epsilon\phi < 0]| = 0$ we have

$$\lim_{\epsilon\to 0} \int_{[u_0+\epsilon\phi<0]} |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi\, dx = 0.$$

Thus, dividing (3.6) by ϵ and letting $\epsilon \to 0$, we have that

$$0 \le \int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi - \int_\Omega u_0^{q-1}\phi\, dx - \lambda \int_\Omega u_0^{-\gamma}\phi\, dx. \tag{3.7}$$

Since this inequality holds for every $\phi \in W_0^{1,p}(\Omega)$, by taking $-\phi$ as test function, we get the opposite inequality and, thus, the equality itself. Therefore, u_0 is a weak positive solution of (3.1). □

Now let us show the existence of two positive solutions for (3.1), one with a positive energy level and another with a negative one.

Lemma 3.9 *The functional I^p is bounded from below in $\overline{B_{\rho_p}(0)}$, where $B_{\rho_p}(0) = \{u \in W_0^{1,p}(\Omega);\ \|u\|_{W_0^{1,p}} < \rho_p\}$.*

Proof. Note that, for nonnegative $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_{W_0^{1,p}} \le \rho_p$, the Hölder and Sobolev inequalities imply

$$\begin{aligned}
|I^p(u)| &\le \frac{1}{p}\|u\|^p_{W_0^{1,p}} + \frac{\lambda}{1-\gamma}\|u\|_q^{\frac{1-\gamma}{q}}|\Omega|^{\frac{q-1+\gamma}{q}} + \frac{1}{q}\|u\|_q^q \\
&\le \frac{1}{p}\|u\|^p_{W_0^{1,p}} + C\frac{\lambda}{1-\gamma}\|u\|_{W_0^{1,p}}^{\frac{1-\gamma}{q}}|\Omega|^{\frac{q-1+\gamma}{q}} + \frac{C}{q}\|u\|^q_{W_0^{1,p}} \\
&\le \frac{1}{p}\rho_p^p + C\frac{\lambda}{1-\gamma}\rho_p^{\frac{1-\gamma}{q}}|\Omega|^{\frac{q-1+\gamma}{q}} + \frac{C\rho_p^q}{q}.
\end{aligned}$$

□

From the last lemma, it is straightforward to see that I^p attains a local minimum in $B_{\rho_p}(0)$. Hence, there exists $u_p \in B_{\rho_p}(0)$ such that

$$I^p(u_p) = \alpha_p := \inf_{v\in\overline{B_{\rho_p}(0)}} I^p(v). \tag{3.8}$$

By Lemma 3.3, we have $\alpha^p < 0$. Moreover, fixing $\phi \in C_c^\infty(0)$ and choosing $t > 0$ in such a way that $\|t\phi\|_{W_0^{1,p}} < \rho_p$ and $\|t\phi\|_{BV} < \rho$, since $I^p(t\phi) \to I(t\phi)$, as $p \to 1^+$, it follows that

$$I^p(u_p) = \alpha_p < \frac{\alpha}{2} < 0, \tag{3.9}$$

for p sufficiently close to 1, where $\alpha := I(t\phi)$.

Lemma 3.10 *u_p defined as in* (3.8) *is a weak solution of* (3.1)

Proof. As in [35], one can show that

$$-\Delta_p u_p \ge 0 \quad \text{in } \Omega,$$

which, by the Strong Maximum Principle for this operator (see [36]), implies that $u_p > 0$ a.e. in Ω. Again, as in [35], one can show that $u_p \in \mathcal{N}_p$ and also that

$$0 \le \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\phi dx - \lambda\int_\Omega u_p^{-\gamma}\phi\, dx - \int_\Omega u_p^{q-1}\phi\, dx, \quad \forall \phi \in W_0^{1,p}(\Omega),\ \phi \ge 0. \tag{3.10}$$

Hence, from (3.10) and Lemma 3.8, it follows that u_p is a weak solution of (3.1). □

Lemma 3.11 *There exists a weak solution v_p of* (3.1) *satisfying $\beta_p < I^p(v_p)$.*

Proof. Let us consider $0 < \lambda < \Lambda$, where $\Lambda = \frac{1}{2}\min\{\lambda^*, \overline{\lambda}, \widetilde{\lambda}\}$. From Lemma 3.2 and Remark 3.1, there exists p_0 such that $\lambda < \min\{\lambda_p^*, \overline{\lambda_p}, \widetilde{\lambda_p}\}$, for all $1 < p < p_0$. First of all note that I^p is coercive on $\mathcal{N}_p$. Indeed, if $u \in \mathcal{N}_p$, then Hölder's inequality implies that

$$\begin{aligned} I^p(u) &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^p_{W_0^{1,p}} - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{q}\right)\int_\Omega |u|^{1-\gamma}\,dx \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^p_{W_0^{1,p}} - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{q}\right)C\|u\|^{1-\gamma}_{W_0^{1,p}}. \end{aligned}$$

Hence, since $1-\gamma < 1 < p$, it follows that I^p is coercive on $\mathcal{N}_p$.

Now, from Lemma 3.5, $\mathcal{N}_p^-$ is a closed set in $W_0^{1,p}(\Omega)$. Moreover, from the compact embeddings, the Lebesgue Dominated Convergence Theorem and since $p < q < q^*$, it follows that I^p is lower semicontinuous. Then, by Ekeland's Variational Principle, there exists a minimizing sequence $(v_n) \subset \mathcal{N}_p^-$ such that

$$I^p(v_n) < \inf_{\mathcal{N}_p^-} I^p + \frac{1}{n}\,, \tag{3.11}$$

and

$$I^p(u) \geq I^p(v_n) - \frac{1}{n}\|u - v_n\|_{W_0^{1,p}}\,, \quad \forall u \in \mathcal{N}_p^-\,. \tag{3.12}$$

Moreover, we can choose $v_n \geq 0$ in Ω for all $n \in \mathbb{N}$, since $I^p(|u|) = I^p(u)$, for all $u \in W_0^{1,p}(\Omega)$. Since I^p is coercive on $\mathcal{N}_p^-$, it follows that (v_n) is bounded in $W_0^{1,p}(\Omega)$ and so, there exists $v_p \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, as $n \to \infty$,

$$\begin{array}{ll} v_n \rightharpoonup v_p & \text{in } W_0^{1,p}(\Omega) \\ v_n \to v_p & \text{in } L^r(\Omega), \text{ for all } 1 \leq r < p^*, \end{array} \tag{3.13}$$

where we have used the compactness of the embeddings of $W_0^{1,p}(\Omega)$.

A remark is in order. Our aim is to apply Lemma 3.8, so that we have to check $v_p \in \mathcal{N}_p$, $v_p > 0$ and that inequality (3.5) holds. We cannot infer that v_p belongs to $\mathcal{N}_p^-$, since it is just the weak limit of a sequence in $\mathcal{N}_p^-$ and $\mathcal{N}_p^-$ is only closed with respect to the strong convergence.

To begin with, we are seeing that $v_p \neq 0$. Note that the Sobolev and Hölder inequalities imply that

$$\|v_n\|^p_{W_0^{1,p}} \geq \frac{S_p}{|\Omega|^\alpha}\|v_n\|_q^p\,,$$

where $\alpha = p(p^* - q)/p^*q$. On the other hand, since $v_n \in \mathcal{N}_p^-$, it yields $\|v_n\|_q^q \geq \frac{p+\gamma-1}{q+\gamma-1}\|v_n\|^p_{W_0^{1,p}}$. Then we obtain

$$\|v_n\|_q \geq \left[\frac{S_p(p-1+\gamma)}{|\Omega|^\alpha(q-1+\gamma)}\right]^{\frac{1}{q-p}}\,. \tag{3.14}$$

Using (3.13) and passing to the limit as $n \to \infty$, we get $v_p \neq 0$.

From Lemma 3.6 with $u = v_n$ and $w = t\phi$, $\phi \in W_0^{1,p}(\Omega)$, $\phi \geq 0$ and $t > 0$ small enough, we find $h_n(t) := h_n(t\phi)$ such that

$$h_n(0) = 1 \quad \text{and} \quad h_n(t)(v_n + t\phi) \in \mathcal{N}_p^-\,.$$

As in [35], one can prove that the right derivative $h_n'^{+}(0)$ is well defined and

$$|h_n'^{+}(0)| \leq C, \quad \forall n \in \mathbb{N},$$

where $C > 0$ does not depend on n.

Now we show that $v_p \in \mathcal{N}_p^-$ and it is a weak positive solution of (3.1). From (3.12), we obtain

$$
\begin{aligned}
\frac{1}{n}\left[|h_n(t)-1|\,\|v_n\|_{W_0^{1,p}} + t h_n(t)\|\phi\|_{W_0^{1,p}}\right] &\geq I^p(v_n) - I^p(h_n(t)(v_n+t\phi)) \\
&= -\frac{1}{p}(h_n^p(t)-1)\|v_n\|^p_{W_0^{1,p}} + \frac{1}{p}h_n^p(t)\left(\|v_n\|^p_{W_0^{1,p}} - \|v_n+t\phi\|^p_{W_0^{1,p}}\right) \\
&\quad + \frac{1}{q}(h_n^q(t)-1)\|v_n+t\phi\|_q^q + \frac{1}{q}\left(\|v_n+t\phi\|_q^q - \|v_n\|_q^q\right) \\
&\quad + \frac{\lambda}{1-\gamma}(h_n^{1-\gamma}(t)-1)\int_\Omega |v_n+t\phi|^{1-\gamma}\,dx \\
&\quad + \frac{\lambda}{1-\gamma}\int_\Omega \left(|v_n+t\phi|^{1-\gamma} - |v_n|^{1-\gamma}\right)\,dx.
\end{aligned}
$$

Dividing by $t > 0$, calculating the lim inf as $t \to 0^+$, recalling that $h_n(0) = 1$ and taking into account that $v_n \in \mathcal{N}_p$, it yields

$$
\begin{aligned}
&\frac{1}{n}(|h_n'^{+}(0)|\|v_n\|_{W_0^{1,p}} + \|\phi\|_{W_0^{1,p}}) \qquad (3.15)\\
&\geq -h_n'^{+}(0)\|v_n\|^p_{W_0^{1,p}} - \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\cdot\nabla\phi\,dx + h_n'^{+}(0)\|v_n\|_q^q + \int_\Omega v_n^{q-1}\phi\,dx \\
&\qquad + \lambda h_n'^{+}(0)\int_\Omega v_n^{1-\gamma}\,dx + \liminf_{t\to 0^+}\frac{\lambda}{1-\gamma}\int_\Omega \frac{(v_n+t\phi)^{1-\gamma} - v_n^{1-\gamma}}{t}\,dx \\
&= -h_n'^{+}(0)\left(\|v_n\|^p_{W_0^{1,p}} - \|v_n\|_q^q - \lambda\int_\Omega v_n^{1-\gamma}\,dx\right) - \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\cdot\nabla\phi\,dx \\
&\qquad + \int_\Omega v_n^{q-1}\phi\,dx + \liminf_{t\to 0^+}\frac{\lambda}{1-\gamma}\int_\Omega \frac{(v_n+t\phi)^{1-\gamma} - v_n^{1-\gamma}}{t} \\
&= -\int_\Omega |\nabla v_n|^{p-2}\nabla v_n\cdot\phi\,dx + \int_\Omega v_n^{q-1}\phi\,dx \\
&\qquad + \liminf_{t\to 0^+}\frac{\lambda}{1-\gamma}\int_\Omega \frac{(v_n+t\phi)^{1-\gamma} - v_n^{1-\gamma}}{t}\,dx\,.
\end{aligned}
$$

Note that, by Fatou's Lemma, we get

$$
\lambda\int_\Omega v_n^{-\gamma}\phi\,dx \leq \liminf_{t\to 0^+}\frac{\lambda}{1-\gamma}\int_\Omega \frac{(v_n+t\phi)^{1-\gamma} - v_n^{1-\gamma}}{t}\,dx\,. \tag{3.16}
$$

Using (3.16) in (3.15) we obtain

$$
\begin{aligned}
\lambda\int_\Omega v_n^{-\gamma}\phi\,dx &\leq \frac{1}{n}(|h_n'^{+}(0)|\|v_n\|_{W_0^{1,p}} + \|\phi\|_{W_0^{1,p}}) + \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\cdot\nabla\phi\,dx - \int_\Omega v_n^{q-1}\phi\,dx \qquad (3.17)\\
&\leq \frac{1}{n}(CC_1 + \|\phi\|_{W_0^{1,p}}) + \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\cdot\nabla\phi\,dx - \int_\Omega v_n^{q-1}\phi\,dx. \qquad (3.18)
\end{aligned}
$$

Now, calculating the limit as $n \to +\infty$ and using once more Fatou's Lemma, we have

$$
\lambda\int_\Omega v_p^{-\gamma}\phi\,dx \leq \int_\Omega |\nabla v_p|^{p-2}\nabla v_p\cdot\nabla\phi\,dx - \int_\Omega v_p^{q-1}\phi\,dx\,.
$$

Hence

$$
\int_\Omega |\nabla v_p|^{p-2}\nabla v\cdot\nabla\phi\,dx - \int_\Omega v_p^{q-1}\phi\,dx - \lambda\int_\Omega v_p^{-\gamma}\phi\,dx \geq 0\,, \quad \forall\phi\in W_0^{1,p}(\Omega)\,, \phi\geq 0\,, \tag{3.19}
$$

which implies that v_p satisfies in the weak sense

$$-\Delta_p v_p \geq 0 \quad \text{in } \Omega\,,$$

since $v_p \geq 0$ and $v_p \neq 0$ in Ω. Hence, from the Strong Maximum Principle (see [36]), it holds

$$v_p > 0 \quad \text{in } \Omega\,.$$

To apply Lemma 3.8 we still have to check that $v_p \in \mathcal{N}_p$. Note that, for $\phi = v_p$ in (3.19), we have

$$\|v_p\|^p_{W_0^{1,p}} \geq \|v_p\|^q_q + \lambda \int_\Omega v_p^{1-\gamma}\, dx. \tag{3.20}$$

On the other hand, by the weakly lower semi-continuity of the norm,

$$\begin{aligned} \|v_p\|^p_{W_0^{1,p}} &\leq \liminf_{n\to\infty} \|v_n\|^p_{W_0^{1,p}} \\ &\leq \limsup_{n\to\infty} \|v_n\|^p_{W_0^{1,p}} \\ &= \lim_{n\to\infty} \left[\|v_n\|^q_q + \lambda \int_\Omega v_n^{1-\gamma}\, dx\right] \\ &= \|v_p\|^q_q + \lambda \int_\Omega v_p^{1-\gamma}\, dx. \end{aligned} \tag{3.21}$$

Thus, from (3.20) and (3.21), it follows that

$$\|v_p\|^p_{W_0^{1,p}} = \lim_{n\to\infty} \|v_n\|^p_{W_0^{1,p}} = \|v_p\|^q_q + \lambda \int_\Omega v_p^{1-\gamma}\, dx\,. \tag{3.22}$$

Then, since $\nabla v_n \rightharpoonup \nabla v_p$ in $L^p(\Omega;\mathbb{R}^N)$ and $\|\nabla v_n\|_p \to \|\nabla v_p\|_p$, the Radon-Riesz theorem implies that in fact

$$v_n \to v_p \quad \text{in } W_0^{1,p}(\Omega), \text{ as } n \to +\infty. \tag{3.23}$$

This, in turn, thanks to Lemma 3.5, implies that $v_p \in \mathcal{N}_p^-$ and also that

$$I^p(v_p) = \inf_{\mathcal{N}_p^-} I^p. \tag{3.24}$$

Therefore, from Lemma 3.8, (3.19) and (3.24), it follows that v_p is a weak solution of (3.1). $\square$

Hence, we have proved that there exist two weak solutions of (3.1), u_p and v_p, such that, by Lemmas 3.2 and 3.7 and by (3.9), it holds that

$$I^p(u_p) < \frac{\alpha}{2} < 0 < \frac{\beta}{2} < \beta_p < I^p(v_p), \tag{3.25}$$

for all $p > 1$ sufficiently close to 1.

3.3 BV-estimates

In this subsection, we are checking that both families of critical points of I^p are bounded in $BV(\Omega)$.

Lemma 3.12 *The family $(u_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$.*

Proof. From its definition, $\|u_p\|_{W_0^{1,p}} < \rho_p$. On the other hand, since $\rho_p \to \rho$ as $p \to 1^+$, then there exists $M > 0$ such that

$$\int_\Omega |\nabla u_p|^p dx \le M, \quad \text{for all } 1 < p < p_0. \tag{3.26}$$

Then, by using Young's inequality and (3.26),

$$\begin{aligned}
\|u_p\| &= \int_\Omega |\nabla u_p| dx \\
&\le \frac{1}{p}\int_\Omega |\nabla u_p|^p dx + \frac{p-1}{p}|\Omega| \\
&\le M + |\Omega| = M_1
\end{aligned} \tag{3.27}$$

and this proves that $(u_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$. ☐

In order to show that the family $(v_p)_{1<p<p_0}$ is also bounded in $BV(\Omega)$, let us follow the arguments of [20], which was inspired by those in [30]. First of all, let us define

$$J^p(u) = I^p(u) + \frac{p-1}{p}|\Omega|. \tag{3.28}$$

Observe that I^p and J^p have the same critical points. Moreover, by their definition,

$$J^p(v_p) = \inf_{\mathcal{N}_p^-} J^p. \tag{3.29}$$

By Young's inequality, the following monotone property holds,

$$p_1 \le p_2 \Rightarrow J^{p_1}(u) \le J^{p_2}(u) \quad \forall u \in W_0^{1,p_1}(\Omega). \tag{3.30}$$

Moreover, the statement of Lemma 3.4 also holds for J^p.

Lemma 3.13 *The family $(v_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$.*

Proof. Let $1 < p_1 \le p_2 < N$ and consider $v_{p_1} \in W_0^{1,p_1}(\Omega)$ and $v_{p_2} \in W_0^{1,p_2}(\Omega)$ such that (3.29) holds for J^{p_1} and J^{p_2}, respectively. Since $p_2 \ge p_1$, then $W_0^{1,p_2}(\Omega) \subset W_0^{1,p_1}(\Omega)$. From Lemma 3.4, there exists $t > 0$ such that

$$tv_{p_2} \in \mathcal{N}_{p_1}^-. \tag{3.31}$$

Then, from Lemma 3.4 and (3.30), it follows that

$$\begin{aligned}
J^{p_2}(v_{p_2}) &\ge J^{p_2}(tv_{p_2}) \\
&\ge J^{p_1}(tv_{p_2}) \\
&\ge J^{p_1}(v_{p_1}).
\end{aligned}$$

Then, for $1 < p < p_0$, it follows that

$$J^p(v_p) < J^{p_0}(v_{p_0}) =: C. \tag{3.32}$$

By (3.32), Hölder's inequality and the definition of S_p leads to

$$\begin{aligned}
C &\ge J^p(v_p) \\
&= J^p(v_p) - \frac{1}{q}(J^p)'(v_p)v_p \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)\|v_p\|^p_{W_0^{1,p}} - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{q}\right)\int_\Omega |v_p|^{1-\gamma}dx \\
&\ge \left(\frac{1}{p} - \frac{1}{q}\right)\|v_p\|^p_{W_0^{1,p}} - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{q}\right) S_p^{\frac{\gamma-1}{p}}|\Omega|^{\frac{p^*(p+\gamma-1)+(1-\gamma)(p^*-p)}{pp^*}}\|v_p\|^{1-\gamma}_{W_0^{1,p}}.
\end{aligned} \tag{3.33}$$

Then, since $1-\gamma < p$, it follows that $\|v_p\|_{W_0^{1,p}}$ is bounded and there exists $M > 0$ (which does not depend on p), such that

$$\int_\Omega |\nabla v_p|^p dx \le M, \quad \text{for all } 1 < p < p_0. \tag{3.34}$$

Then, as in (3.27), it follows that $(v_p)_{1<p<p_0}$ is bounded in $BV(\Omega)$. □

4 Convergence to positive solutions

In this section, we prove that the families of solutions $(u_p)_{1<p<p_0}$ and $(v_p)_{1<p<p_0}$ to (3.1) converge to functions which are positive solutions to (1.1). Actually, in order to calculate their limits, we just need that they are bounded in $BV(\Omega)$. Hence, we just detail the convergence of $(u_p)_{1<p<p_0}$, since the other convergence can be obtained by repeating the proof verbatim.

First of all, let us recall that the BV–estimate implies that there exists u_0 such that, up to subsequences,

$$u_p \to u_0 \quad \text{in } L^r(\Omega)\,,\ \forall\, r \in [1, 1^*). \tag{4.1}$$

as $p \to 1^+$. Moreover,

$$u_p \rightharpoonup u_0 \quad \text{in } L^{1^*}(\Omega) \quad \text{as} \quad p \to 1^+\,. \tag{4.2}$$

4.1 Comparison with a simpler singular problem

In this subsection, we connect critical points of functional I^p with the positive solution of a simpler problem, namely,

$$\begin{cases} -\Delta_p w &= \dfrac{\lambda}{w^\gamma} \quad \text{in } \Omega, \\ \quad\ \ w &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Existence and uniqueness to this problem has been studied in [17] (see also [18]); we next state its main properties:
There exists a unique solution $w_p \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to problem (4.3) in the sense that $\frac{1}{w_p^\gamma} \in L^1(\Omega)$ and

$$\int_\Omega |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \varphi\, dx = \int_\Omega \frac{\lambda}{w_p^\gamma}\, \varphi\, dx\,, \tag{4.4}$$

for every $\varphi \in W_0^{1,p}(\Omega)$. Moreover, this solution satisfies the following positiveness feature: for every open set $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $w_p \ge c_\omega$ a.e. in ω.

Proposition 4.1 *The inequality $u_p \ge w_p$ holds for every $p > 1$.*

Proof. Take $(w_p - u_p)^+$ as test function in both formulations and subtract them, then we deduce

$$\int_\Omega (|\nabla w_p|^{p-2}\nabla w_p - |\nabla u_p|^{p-2}\nabla u_p) \cdot \nabla (w_p - u_p)^+$$
$$= \lambda \int_\Omega \left(\frac{1}{w_p^\gamma} - \frac{1}{u_p^\gamma} \right) (w_p - u_p)^+ - \int_\Omega u_p^{q-1} (w_p - u_p)^+.$$

Now it is enough to realize that the left hand side is nonnegative while the right hand side is nonpositive. So, both sides vanish and we conclude that $(w_p - u_p)^+ = 0$. □

In [18] the family $(w_p)_{1<p<p_0}$ is considered. It is proved that there exists $w_0 \in BV(\Omega)$ satisfying $w_0 > 0$, $\frac{1}{w_0^\gamma} \in L^1(\Omega)$ and

$$w_p \to w_0 \qquad \text{in } L^r(\Omega) \quad \text{for all } 1 \le r < N/(N-1).$$

Corollary 4.1 *Inequality $u_0 \geq w_0$ holds. As a consequence, $u_0 > 0$ and $\frac{1}{u_0^\gamma} \in L^1(\Omega)$.*

4.2 L^∞-estimate

This subsection is devoted to check that $u_0 \in L^\infty(\Omega)$.

Lemma 4.1 *Let u_0 be as in (4.1). Then*

$$u_0 \in L^\infty(\Omega)\,.$$

Proof. Taking $G_k(u_p) = (u_p - k)^+$ as test function in (3.1), we get

$$\int_\Omega |\nabla G_k(u_p)|^p\,dx = \lambda \int_\Omega \frac{1}{u_p^\gamma} G_k(u_p)\,dx + \int_\Omega u_p^{q-1} G_k(u_p)\,dx\,.$$

Since Hölder's inequality implies that

$$\int_\Omega \frac{1}{u_p^\gamma} G_k(u_p)\,dx \leq \frac{1}{k^\gamma} \|G_k(u_p)\|_{\frac{N}{N-1}} |\Omega|^{\frac{1}{N}}$$

and

$$\int_\Omega u_p^{q-1} G_k(u_p)\,dx \leq \left(\int_{A_{k,p}} u_p^{(q-1)N}\,dx\right)^{\frac{1}{N}} \|G_k(u_p)\|_{\frac{N}{N-1}}\,,$$

where $A_{k,p} := \{x \in \Omega\,;\, u_p(x) > k\}$, we have

$$\int_\Omega |\nabla G_k(u_p)|^p\,dx \leq \left[\frac{\lambda}{k^\gamma}|\Omega|^{\frac{1}{N}} + \left(\int_{A_{k,p}} u_p^{(q-1)N}\,dx\right)^{\frac{1}{N}}\right] \|G_k(u_p)\|_{\frac{N}{N-1}}\,. \tag{4.5}$$

Thus, by the definition of S_1 (see Lemma 3.1), Young's inequality and (4.5), we have

$$\begin{aligned}
\left(\int_\Omega |G_k(u_p)|^{\frac{N}{N-1}}\,dx\right)^{\frac{N-1}{N}} &\leq S_1^{-1} \int_\Omega |\nabla G_k(u_p)|\,dx \\
&\leq S_1^{-1}\left(\frac{1}{p}\int_\Omega |\nabla G_k(u_p)|^p\,dx + \frac{p-1}{p}|\Omega|\right) \\
&\leq \frac{S_1^{-1}}{p}\left[\frac{\lambda}{k^\gamma}|\Omega|^{\frac{1}{N}} + \left(\int_{A_{k,p}} u_p^{(q-1)N}\,dx\right)^{\frac{1}{N}}\right] \|G_k(u_p)\|_{\frac{N}{N-1}} \\
&\quad + \frac{S_1^{-1}(p-1)}{p}|\Omega|\,,
\end{aligned} \tag{4.6}$$

or equivalently,

$$\left(1 - \frac{S_1^{-1}}{p}\left(\frac{\lambda}{k^\gamma}|\Omega|^{\frac{1}{N}} + \left(\int_{A_{k,p}} u_p^{(q-1)N}\,dx\right)^{\frac{1}{N}}\right)\right) \|G_k(u_p)\|_{\frac{N}{N-1}} \leq \frac{S_1^{-1}(p-1)}{p}|\Omega|\,. \tag{4.7}$$

We claim that, there exists $k_0 > 0$ and $p_0 > 1$ such that

$$\int_{A_{k,p}} u_p^{(q-1)N}\,dx < \epsilon \quad \forall\, k \geq k_0\,, \forall\, p \in (1, p_0)\,. \tag{4.8}$$

Indeed, since
$$|A_{k,p}| \le \frac{1}{k^{\frac{N}{N-1}}} \int_{A_{k,p}} u_p^{\frac{N}{N-1}}\,dx\,,$$
we have, by (3.27) and Hölder's inequality, that
$$\begin{aligned}
\int_{A_{k,p}} u_p^{(q-1)N}\,dx &\le \left(\int_{A_{k,p}} u_p^{\frac{N}{N-1}}\,dx\right)^{(q-1)(N-1)} |A_{k,p}|^{1-(q-1)(N-1)} \\
&\le \left(\int_{A_{k,p}} u_p^{\frac{N}{N-1}}\,dx\right)^{(q-1)(N-1)} \frac{1}{k^{\frac{N[1-(q-1)(N-1)]}{N-1}}} \left(\int_{A_{k,p}} u_p^{\frac{N}{N-1}}\,dx\right)^{1-(q-1)(N-1)} \\
&\le \frac{1}{k^{\frac{N[1-(q-1)(N-1)]}{N-1}}} \int_{A_{k,p}} u_p^{\frac{N}{N-1}}\,dx \\
&\le \frac{1}{k^{\frac{N[1-(q-1)(N-1)]}{N-1}}} S_1^{\frac{-N}{N-1}} \left(\int_\Omega |\nabla u_p|\,dx\right)^{\frac{N}{N-1}} \\
&\le \frac{1}{k^{\frac{N[1-(q-1)(N-1)]}{N-1}}} S_1^{\frac{-N}{N-1}} M_1^{\frac{N}{N-1}}\,,
\end{aligned}$$
for all $p \in (1, p_0)$. Since $q < N/(N-1)$ implies $1-(q-1)(N-1) > 0$, it follows that
$$\lim_{k\to\infty} \int_{A_{k,p}} u_p^{(q-1)N}\,dx = 0$$
uniformly on p and then (4.8) holds. Consequently, we can choose $k_1 > 0$ such that
$$\int_{A_{k,p}} u_p^{(q-1)N}\,dx \le \left(\frac{pS_1}{2}\right)^N \qquad \forall\, k \ge k_1, p \in (1, p_0)\,. \tag{4.9}$$
Thus, by using (4.9) in (4.7) we get
$$\left(\frac{1}{2} - \frac{S_1^{-1}\lambda}{pk^\gamma}|\Omega|^{\frac{1}{N}}\right) \|G_k(u_p)\|_{\frac{N}{N-1}} \le \frac{S_1^{-1}(p-1)}{p}|\Omega| \qquad \forall\, k \ge k_1, p \in (1, p_0)\,. \tag{4.10}$$
Now, if we take a larger $k_2 > k_1$ such that
$$\frac{1}{2} - \frac{S_1^{-1}\lambda}{pk^\gamma}|\Omega|^{\frac{1}{N}} > 0 \quad \forall\, k \ge k_2 \tag{4.11}$$
then, by (4.10) and (4.11), we have that
$$0 \le \left(\frac{1}{2} - \frac{S_1^{-1}\lambda}{pk^\gamma}|\Omega|^{\frac{1}{N}}\right) \|G_k(u_p)\|_{\frac{N}{N-1}} \le \frac{S_1^{-1}(p-1)}{p}|\Omega| \quad \forall\, k \ge k_2\ , \forall\, p \in (1, p_0)\,. \tag{4.12}$$

Therefore, letting $p \to 1^+$ in (4.12), by Fatou's Lemma, (4.2) and the definition of G_k, we obtain
$$u_0 \le k \quad \text{ a.e. in } \Omega.$$
We conclude that $u_0 \in L^\infty(\Omega)$. □

4.3 Existence of the vector field

In this subsection we obtain the vector field appearing in the definition of solution and we check that the equation holds in the sense of distributions.

Lemma 4.2 *Let u_0 be as in* (4.1). *Then, there exists $\mathbf{z} \in \mathcal{DM}^\infty_{loc}(\Omega)$ with $\|\mathbf{z}\|_\infty \le 1$ such that*

$$-\mathrm{div}\,\mathbf{z} \ge \frac{\lambda}{u_0^\gamma} + u_0^{q-1} \qquad \text{in } \mathcal{D}'(\Omega). \tag{4.13}$$

Proof. Let us fix $1 \le s < \infty$ and consider $1 < p < s'$. By Hölder's inequality and (3.26), we have

$$\int_\Omega ||\nabla u_p|^{p-2}\nabla u_p|^s\,dx = \int_\Omega |\nabla u_p|^{s(p-1)}\,dx \tag{4.14}$$

$$\le \left(\int_\Omega |\nabla u_p|^p\,dx\right)^{\frac{s}{p'}} |\Omega|^{1-\frac{s}{p'}} \tag{4.15}$$

$$\le M^{\frac{s}{p'}} |\Omega|^{1-\frac{s}{p'}}\,, \tag{4.16}$$

for all $p \in (1, p_0)$. Then we may follow the proof of [27, Theorem 3.5] and show that there exists $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \le 1$. It also holds, up to a subsequence,

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z} \quad \text{in } L^s(\Omega, \mathbb{R}^N)\,, \quad \forall\, s \in [1, +\infty)\,, \tag{4.17}$$

as $p \to 1^+$.

Using $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$ as a test function in (3.1), we get

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi\,dx = \lambda \int_\Omega \frac{1}{u_p^\gamma}\varphi\,dx + \int_\Omega u_p^{q-1}\varphi\,dx\,. \tag{4.18}$$

Calculating the limit inferior, as $p \to 1^+$, in both sides of the last expression, by (4.1), (4.17) and Fatou's Lemma, we obtain

$$\int_\Omega \mathbf{z}\cdot\nabla\varphi\,dx \ge \lambda \int_\Omega \frac{1}{u_0^\gamma}\varphi\,dx + \int_\Omega u_0^{q-1}\varphi\,dx \ge 0 \quad \forall\,\varphi \in C_c^\infty(\Omega)\,,\ \varphi \ge 0\,. \tag{4.19}$$

Thus, by the Riesz representation theorem, the functional $\varphi \mapsto \int_\Omega \mathbf{z}\cdot\nabla\varphi\,dx$ defines a nonnegative Radon measure in Ω. On the other hand, this functional coincides with $-\mathrm{div}\,\mathbf{z}$ in the sense of distributions. Therefore, inequality (4.19) is just (4.13) in the sense of distributions. As a consequence $-\mathrm{div}\,\mathbf{z}$ is a Radon measure.

Moreover, by (3.26), (4.17) and Young's inequality, we have that

$$0 \le -\int_\Omega \varphi\,\mathrm{div}\,\mathbf{z} = \int_\Omega \mathbf{z}\cdot\nabla\varphi\,dx = \lim_{p\to 1}\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx \le M + \int_\Omega (|\nabla\varphi| + 1)^{p_0}\,,$$

for all $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$. Observe that, at this point, one can only deduce that the total variation of $-\mathrm{div}\,\mathbf{z}$ is locally finite and so $\mathbf{z} \in \mathcal{DM}^\infty_{loc}(\Omega)$. □

The next step is to verify that the equation holds in the sense of distributions. On this issue we will adapt and simplify the arguments in [18].

Lemma 4.3 *Let u_0 be as in* (4.1) *and let $\mathbf{z}$ be as in Lemma 4.2. Then, the following identity holds*

$$-u_0^*\,\mathrm{div}\,\mathbf{z} = \lambda u_0^{1-\gamma} + u_0^q \qquad \text{in } \mathcal{D}'(\Omega)\,, \tag{4.20}$$

Proof. Since $u_0 > 0$, it follows from (4.13) that

$$-u_0^*\mathrm{div}\,\mathbf{z} \ge \lambda u_0^{1-\gamma} + u_0^q \qquad \text{in } \mathcal{D}'(\Omega)\,. \tag{4.21}$$

Let us prove the opposite inequality. Let $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$. Taking $u_p\varphi$ as test function in (3.1) we get

$$\int_\Omega \varphi|\nabla u_p|^p\,dx + \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx = \lambda\int_\Omega u_p^{1-\gamma}\varphi\,dx + \int_\Omega u_p^q\varphi\,dx\,.$$

Thus, using Young's inequality we have that

$$\begin{aligned}&\int_\Omega \varphi|\nabla u_p|\,dx + \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx\\ &\leq \frac{1}{p}\int_\Omega \varphi|\nabla u_p|^p\,dx + \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx + \frac{p-1}{p}\int_\Omega\varphi\,dx\\ &\leq \lambda\int_\Omega u_p^{1-\gamma}\varphi\,dx + \int_\Omega u_p^q\varphi\,dx + \frac{p-1}{p}\int_\Omega\varphi\,dx\,.\end{aligned}\tag{4.22}$$

Now, we are going to pass to the limit above, as $p\to 1^+$. On the left-hand side, by (4.1) and (4.17), we have

$$\liminf_{p\to1^+}\left(\int_\Omega \varphi|\nabla u_p|\,dx + \int_\Omega u_p|\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi\,dx\right) \geq \int_\Omega\varphi|Du_0| + \int_\Omega u_0\mathbf{z}\cdot\nabla\varphi\,dx\,.\tag{4.23}$$

On the right-hand side, by (4.1) and the Dominated Convergence Theorem, it follows that

$$\lim_{p\to1^+}\left(\lambda\int_\Omega u_p^{1-\gamma}\varphi\,dx + \int_\Omega u_p^q\varphi\,dx\right) = \lambda\int_\Omega u_0^{1-\gamma}\varphi\,dx + \int_\Omega u_0^q\varphi\,dx\,.\tag{4.24}$$

Thus, by (4.23) and (4.24), (4.22) becomes

$$\int_\Omega\varphi|Du_0| + \int_\Omega u_0\mathbf{z}\cdot\nabla\varphi\,dx \leq \lambda\int_\Omega u_0^{1-\gamma}\varphi\,dx + \int_\Omega u_0^q\varphi\,dx\,.\tag{4.25}$$

But this implies, by (2.3), Corollary 2.1 and (4.25), that

$$-\int_\Omega u_0^*\varphi\operatorname{div}\mathbf{z} = \int_\Omega\varphi(\mathbf{z},Du_0) + \int_\Omega u_0\mathbf{z}\cdot\nabla\varphi\,dx\tag{4.26}$$

$$\leq \int_\Omega\varphi|Du_0| + \int_\Omega u\mathbf{z}\cdot\nabla\varphi\,dx\tag{4.27}$$

$$\leq \lambda\int_\Omega u_0^{1-\gamma}\varphi\,dx + \int_\Omega u_0^q\varphi\,dx\,,\tag{4.28}$$

or equivalently,

$$-u_0^*\operatorname{div}\mathbf{z} \leq \lambda u_0^{1-\gamma} + u_0^q \quad \text{ in } \mathcal{D}'(\Omega)\,.\tag{4.29}$$

Consequently, by (4.21) and (4.29), we infer (4.20). □

Finally, taking into account that $u_0 > 0$ a.e. (and so $u_0^* > 0$ $\mathcal{H}^{N-1}$–a.e.), the division by u_0 is allowed and we may arrive at the expected identity.

Corollary 4.2 *Let u_0 be as in* (4.1) *and let $\mathbf{z}$ be as in Lemma 4.2. Then, the following identity holds*

$$-div\,\mathbf{z} = \lambda u_0^{-\gamma} + u_0^{q-1} \qquad in\ \mathcal{D}'(\Omega)\,.\tag{4.30}$$

As a consequence, $div\,\mathbf{z}\in L^1(\Omega)$, so that $\mathbf{z}\in X(\Omega)_1$.

4.4 End of the proof of Theorem 1.1

In order to finish the proof of this theorem, let us show that u_0 satisfies the conditions **(a)**–**(e)** of the Definition 2.1.

Conditions **(a)** and **(b)** were already proved in Corollary 4.1.

Condition **(c)** is just Corollary 4.2.

Conditions **(d)** and **(e)** can be proved as in [4]: take $u_p\varphi$ (with $\varphi \in C_c^\infty(\Omega)$ a nonnegative function) and u_p as test functions in (3.1), respectively; then in both **(d)** and **(e)** let p go to 1 and apply Anzellotti's theory.

Therefore, u_0 satisfies the conditions **(a)**–**(e)** of the Definition 2.1 and thus it is a solution of (1.1).

As mentioned, it can be proved in the same way that v_0 also satisfies these conditions and so v_0 is also a solution of (1.1).

Now, it remains to prove that $u_0 \neq v_0$. Indeed, by (3.25) and the fact that $u_p, v_p \in \mathcal{N}_p$, we have

$$\begin{aligned}\lambda\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\int_\Omega u_p^{1-\gamma}\,dx+\left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega u_p^q\,dx&\\ <\frac{\alpha}{2}<0<\frac{\beta}{2}&\\ <\lambda\left(\frac{1}{p}-\frac{1}{1-\gamma}\right)\int_\Omega v_p^{1-\gamma}\,dx+\left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega v_p^q\,dx\,,&\end{aligned}$$

for all $p \in (1, p_0)$. Thus, calculating the limit as $p \to 1^+$ in the last expression, by (4.1), it holds that

$$\lambda\left(1-\frac{1}{1-\gamma}\right)\int_\Omega u_0^{1-\gamma}\,dx+\left(1-\frac{1}{q}\right)\int_\Omega u_0^q\,dx<\lambda\left(1-\frac{1}{1-\gamma}\right)\int_\Omega v_0^{1-\gamma}\,dx+\left(1-\frac{1}{q}\right)\int_\Omega v_0^q\,dx\,.$$

Then $u_0 \neq v_0$ and Theorem 1.1 is completely proved. □

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