Nonlinear elliptic equations having a gradient term with natural growth

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Abstract.

In this paper, we study a class of nonlinear elliptic Dirichlet problems whose simplest model example is

$$\begin{cases} -\Delta_p u = g(u)|\nabla u|^p + f, & \text{in } \Omega; \\ \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here Ω is a bounded open set in $I\!R^N$ $(N \geq 2)$, Δ_p denotes the so-called p-Laplace operator $(p > 1)$ and g is a continuous real function. Given $f \in L^m(\Omega)$ $(m > 1)$, we study under which growth conditions on g problem (1) admits a solution. If $m \geq N/p$, we prove that there exists a solution under assumption (3) (see below), and that it is bounded when $m > \frac{N}{p}$; while if $1 < m < N/p$ and g satisfies the condition (4) below, we prove the existence of an unbounded generalized solution. Note that no smallness condition is asked on f. Our methods rely on a priori estimates and compactness arguments and are applied to a large class of equations involving operators of Leray–Lions type. We also make several examples and remarks which give evidence of the optimality of our results.

Résumé.

Dans cet article, nous traitons le problème de Dirichlet pour une classe d'équations elliptiques non linéaires, dont l'exemple modèle est (1) ci-dessus, où Ω est un ensemble ouvert bornée dans $I\!R^N$ $(N \geq 2)$, Δ_p dénote l'opérateur p-Laplacien $(p > 1)$ et g est une fonction continue à valeurs réelles.

Étant donnée $f \in L^m(\Omega)$ $(m > 1)$, nous étudions les conditions de croissance sur g qui assurent l'existence de solutions. Si $m \geq N/p$, nous démontrons qu'il existe une solution sous l'hypothèse (3) (voir ci-dessous), solution que est bornée quand $m > \frac{N}{p}$; tandis que si $1 < m < N/p$ et g satisfait la condition (4) (voir ci-dessous), nous prouvons l'existence de solutions généralisées non bornées. Nous soulignons qu'aucune condition de petitesse n'est imposée sur f. Les méthodes reposent sur estimations a priori et arguments de compacité, et sont appliqués sur une large classe d'equations qui incluent les opérateurs de type Leray–Lions. Nous presentons aussi plusieurs exemples et remarques qui montrent l'optimalité des nos résultats.

1 Introduction.

This article is devoted to study the Dirichlet problem for some nonlinear elliptic equations whose simplest model is (1). This kind of problems has been widely studied. The classical references are [18] and [22]; since then, many authors have proved results for second order elliptic problems with lower order terms depending on the gradient: these works include, for instance, [4], [5], [6], [8], [13], [14], [15], [16], [20], [25], [27], or [29]. After the classical example by Kazdan and Kramer (see [17]), which shows that (1) can not always have solutions, two different kind of questions have been considered. On the one hand, in some papers it is proved existence of solutions when the source f is small in a suitable norm. On the other hand, conditions on the function g have been considered in order to get a solution for all f in a given Lebesgue space. This is the way chosen in [8], [27] and [25] under the hypothesis

$$g \in L^1(\mathbb{R}). \tag{2}$$

In [8] it is considered the problem (1) for $p = 2$ and datum $f \in L^m(\Omega)$, with $m \geq 2N/(N+2)$, and it is proved first an L^∞-estimate, when $m > N/2$, and then the existence of a generalized solution (so–called entropy solution, see Definition 2.3 below) when $2N/(N+2) \leq m < N/2$. Assuming also this condition (2), the problem (1) has been studied in [27] for datum $f \in L^1(\Omega)$ and in [25] for measure datum. In this last paper an example is given which shows the optimality of (2) in order to have solutions for **any** measure on the right hand side. A natural question is whether (2) is still necessary for different classes of data f. In this work we prove that this is not the case and that the results in [8] can be extended relaxing assumption (2) in dependence on the regularity of the data. We look then for optimal conditions on the growth of g at infinity to ensure that, given f with a certain summability, problem (1) admits a solution. It turns out that, as f varies in the class of Lebesgue's spaces $L^m(\Omega)$, a different growth assumption at infinity (depending on m and p) is required for (1) to have a solution. For the model problem (1), our results read as follows.

Theorem 1.1 *Let us set $G(s) = \int_0^s |g(t)|dt$, and $\Phi(s) = \int_0^s e^{|G(t)|}dt$. Then we have:*

(i) Let $f \in L^m(\Omega)$, with $m > \frac{N}{p}$; if

$$\lim_{s\to\pm\infty} \frac{e^{|G(s)|}}{(1+|\Phi(s)|)^{p-1}} = 0\,, \tag{3}$$

then (1) has a solution, which belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

(ii) Let $f \in L^{\frac{N}{p}}(\Omega)$*; if (3) holds true, then (1) has a solution, which belongs to* $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ *for all* $q \in [1, +\infty[$.

(iii) Let $f \in L^m(\Omega)$, *with* $1 < m < \frac{N}{p}$*; if there exist* θ, $0 < \theta < \frac{p^*}{pm'}$ *and positive constants* M_1, M_2 *such that*

$$M_1 \le \frac{e^{|G(s)|}}{(1 + |\Phi(s)|)^{\theta(p-1)}} \le M_2 \qquad \text{for all } s \in \mathbb{R}, \tag{4}$$

then (1) has a solution, which belongs to $W_0^{1,p}(\Omega) \cap L^{\frac{Nm(p-1)}{N-pm}}(\Omega)$ *when* $m \ge (p^*)'$, *and to* $W_0^{1,m^*(p-1)}(\Omega)$ *when* $1 < m < (p^*)'$.

As an example, (i) implies that, given any $f \in L^m(\Omega)$ with $m > \frac{N}{p}$, (1) has a (bounded) solution if $g(s)$ tends to zero at infinity. This assumption should then be compared with the counterexample to existence given in [17], where g behaves like a constant; in that situation existence fails unless f has a suitable small norm (see also [13]). Thus (3) seems to be optimal to find solutions to (1) for **any** $f \in L^m(\Omega)$, $m > \frac{N}{p}$. Moreover, we show with an example that when a sequence of admissible functions g (i.e. satisfying (3)) approximates the constant of Kazdan–Kramer's counterexample, then the corresponding solutions blow–up everywhere in Ω, giving reason for such failure of existence.

As far as (4) is concerned, as a main example it is satisfied if $g(s) \le \frac{\lambda}{s}$, with $\lambda < \lambda_m := \frac{N(m-1)}{N-pm}$. Again, this value λ_m (as well as the limiting $\theta = \frac{p^*}{pm'}$ in (4)) seems to be optimal in order to have solutions to (1) for **any** $f \in L^m(\Omega)$. We give examples showing that if $g(s) \ge \frac{\lambda_m}{s}$ no a priori estimates depending on $\|f\|_{L^m(\Omega)}$ can be expected and complete blow–up phenomena may occur; in particular, there exist functions in $L^m(\Omega)$ such that no solution is expected to exist.

It is not surprising that our assumptions (3) and (4) involve the primitive function $G(s) = \int_0^s g(t)dt$ and its exponential $\exp(G(t))$, which usually play a crucial role in this kind of problems, making a link with a semilinear structure underlying problem (1). Indeed, our main estimates are obtained by considering test functions of exponential type which, through a sort of cancellation lemma (see Lemma 2.1 below), transform the equation in a problem of semilinear type. Our results are then also related to possibly singular sublinear problems, which were studied in [7]. We also point out that the parabolic problem associated to (1), with $p = 2$, is studied in [11].

The plan of the paper is the following. Next section is devoted to a precise description of our assumptions (which are set in a general framework including possibly non uniformly coercive problems) and of our results. In Section 3 we prove the L^∞-estimate and deduce the existence of bounded solutions, while Section 4 deals with the existence of unbounded solutions (and their regularity); in this context we use the more general framework of entropy solutions, which can be applied to the full range of $f \in L^m(\Omega)$, $m > 1$. Last but not least, in Section 5 we give several examples concerning the optimality of our results in the range of these assumptions, showing that no a priori estimates can be expected if these are violated and proving some results of nonexistence of solutions or of complete blow–up for approximating problems.

2 Assumptions, Statement of Results and Comments.

Let Ω be a bounded open set in $I\!R^N$, with $N \geq 2$, and let $p > 1$. Throughout this article, c will denote a positive constant which only depends on the parameters of our problem, its value may vary from line to line. We will also denote by $|E|$ the Lebesgue measure of $E \subset \Omega$. Moreover, for $1 < q < +\infty$, we denote $q' = \dfrac{q}{q-1}$, and if $1 \leq q < N$, $q^* = \dfrac{Nq}{N-q}$.

We are going to investigate the existence of a solution of the following nonlinear elliptic problem:

$$\begin{cases} -\mathrm{div}\, \mathbf{a}(x,u,\nabla u) + b(x,u,\nabla u) = f, & \text{in } \Omega; \\ \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\mathbf{a} : \Omega \times I\!R \times I\!R^N \to I\!R^N$ and $b : \Omega \times I\!R \times I\!R^N \to I\!R$ are two Carathéodory functions satisfying:

(A1) for every $k > 0$ there is a positive constant C_k such that

$$|\mathbf{a}(x,s,\xi)| \leq C_k(1 + |\xi|^{p-1})$$

holds for all $(s,\xi) \in [-k,k] \times I\!R^N$ and almost all $x \in \Omega$.

(A2) there exists a continuous positive function $\alpha : I\!R \to I\!R$ satisfying

$$\mathbf{a}(x,s,\xi) \cdot \xi \geq \alpha(s)|\xi|^p$$

for all $(s,\xi) \in I\!R \times I\!R^N$ and almost all $x \in \Omega$. Moreover, this function α satisfies

$$\alpha^{\frac{1}{p-1}} \notin L^1(-\infty,0) \cup L^1(0,+\infty). \tag{6}$$

(A3) if $\xi, \eta \in I\!R^N$, with $\xi \neq \eta$, then

$$[\mathbf{a}(x,s,\xi) - \mathbf{a}(x,s,\eta)] \cdot (\xi - \eta) > 0$$

holds for all $s \in I\!R$ and almost all $x \in \Omega$.

(B) there exist $b_0 \in L^m(\Omega)$, for some $m > 1$, and a continuous non negative function $g : I\!R \to I\!R$ satisfying

$$|b(x,s,\xi)| \leq b_0(x) + g(s)|\xi|^p$$

for all $(s,\xi) \in I\!R \times I\!R^N$ and almost all $x \in \Omega$.

Let us remark that condition (B) implies that the growth rate of function b with respect to ξ may have any order $q \leq p$ and so we are not restricted to the limit case $q = p$, which nevertheless remains our main interest.

Finally, we assume that, for the same $m > 1$ as above,

(F) $f \in L^m(\Omega)$.

(The parameter m occurring in (B) and (F) will be precised in the results below.)

Remark 2.1 Note that assumption (A2) includes the case of a non uniformly coercive operator, as considered for instance in [3] or [1]. Actually, we remark that no great loss of generality would result by assuming that the function $\mathbf{a}(x,s,\xi)$ is uniformly coercive, that is $\alpha = 1$ in (A2). Indeed, problem (5) is (at least formally) equivalent to

$$\begin{cases} -\text{div } \tilde{\mathbf{a}}(x,v,\nabla v) + \tilde{b}(x,v,\nabla v) = f, & \text{in } \Omega; \\ \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $v = A(u) = \int_0^u \alpha(s)^{\frac{1}{p-1}} ds$ and

$$\tilde{\mathbf{a}}(x,s,\xi) = \mathbf{a}\Big(x, A^{-1}(s), \frac{\xi}{\alpha(A^{-1}(s))}\Big), \qquad \tilde{b}(x,s,\xi) = b\Big(x, A^{-1}(s), \frac{\xi}{\alpha(A^{-1}(s))}\Big).$$

The function $\tilde{\mathbf{a}}(x,s,\xi)$ would then satisfy (A2) with $\alpha = 1$ (and $\tilde{b}$ would satisfy (B) with a different function $\tilde{g}$). The results obtained on the coercive problem for v imply the results for the possibly non uniformly coercive equation on u provided the equivalence between (5) and (7) is rigorously proved (in the desired formulation). This is not difficult to do but still needs a few technicalities when dealing with the entropy formulation (see later), so that it seemed to us easier to work directly from the beginning with the generalized assumption (A2). Note however that for bounded solutions u, v, the equivalence between the two problems is immediate.

It is well known that (A1)–(A3) and (B) are not enough in general to ensure the existence of a solution, so that some supplementary condition on the lower order term b is needed. Before stating our additional assumptions, some notation is in order. We denote

$$G(s) = \int_0^s \frac{g(\sigma)}{\alpha(\sigma)}\, d\sigma \tag{8}$$

$$A(s) = \int_0^s \alpha^{1/(p-1)}(\sigma)\, d\sigma \tag{9}$$

$$\Phi(s) = \int_0^s \alpha(\sigma)^{1/(p-1)} e^{|G(\sigma)|/(p-1)}\, d\sigma. \tag{10}$$

Note that, with this notation, (6) is transformed into

$$\lim_{s\to+\infty} A(s) = +\infty \qquad \text{and} \qquad \lim_{s\to-\infty} A(s) = -\infty. \tag{11}$$

We deal first with the problem of finding bounded weak solutions of (5) and assume

$$(\mathbf{C1}) \qquad \lim_{s\to\pm\infty} \frac{e^{|G(s)|}}{(1+|\Phi(s)|)^{p-1}} = 0.$$

Let us briefly analyze this hypothesis.

Remark 2.2 Let us observe that, on account of (C1),

$$\lim_{s\to\pm\infty} A(s) = \pm\infty \Longleftrightarrow \lim_{s\to\pm\infty} \Phi(s) = \pm\infty.$$

Indeed, on the one hand, clearly $|A(s)| \le |\Phi(s)|$, so that (11) implies $\lim_{s\to\pm\infty} \Phi(s) = \pm\infty$. On the other hand, it follows from (C1) that

$$\frac{\Phi'(s)}{1+|\Phi(s)|} = \frac{\alpha^{1/(p-1)}(s)e^{|G(s)|/(p-1)}}{1+|\Phi(s)|} \le c\alpha^{1/(p-1)}(s)$$

for some positive constant c. Consequently,

$$\log\left(1+|\Phi(s)|\right) \le c|A(s)| \Longrightarrow |\Phi(s)| \le e^{c|A(s)|}.$$

Hence, $\lim_{s\to\pm\infty} \Phi(s) = \pm\infty$ implies $\lim_{s\to\pm\infty} A(s) = \pm\infty$ and both conditions are equivalent.

Remark 2.3 If $\alpha = 1$, a simple way to obtain many functions g satisfying (C1) is to take $g = g_1 + g_2$, where $g_1 \in L^1(I\!R)$ and $\lim_{s\to\pm\infty} g_2(s) = 0$: writing $G_i = \int_0^s g_i(\sigma)\,d\sigma$, $i = 1, 2$, we have that G_1 is bounded and so $e^{|G_1|}$ is bounded from above and from below (by a positive constant); hence,

$$0 \le \left(\frac{e^{|G(s)|}}{(1+|\Phi(s)|)^{p-1}}\right)^{\frac{1}{p-1}} = \frac{e^{\frac{|G(s)|}{p-1}}}{1+\left|\int_0^s e^{\frac{|G(\sigma)|}{p-1}}\,d\sigma\right|} \le \frac{ce^{\frac{|G_2(s)|}{p-1}}}{1+c\left|\int_0^s e^{\frac{|G_2(\sigma)|}{p-1}}\,d\sigma\right|}$$

and now the right hand side tends to 0 by l'Hôpital's rule.

Nevertheless, our condition is strictly more general as the following example shows.

Example 2.1 Let $p = 2$ and $\alpha(s) = 1$ for all $s \in I\!R$. Consider

$$g(s) = \begin{cases} \pi\cos(\pi s/2), & \text{if } |s| \in [0,1]; \\ \frac{(-1)^n}{2} n\pi \sin(n^2\pi s), & \text{if } |s| \in [n, n+\frac{1}{n^2}]; \\ 0, & \text{if } |s| \in [n+\frac{1}{n^2}, n+1]. \end{cases}$$

Notice that $\int_n^{n+1} g(\sigma)\,d\sigma = \frac{1}{n}$ for all $n \in I\!N$. Thus, for s big enough, we have $G(s) \approx \operatorname{sign}(s)\log|s|$ and $e^{|G(s)|} \approx |s|$, so that $\left|\int_0^s e^{|G(\sigma)|}\,d\sigma\right| \approx |s|^2/2$. Hence,

$$\frac{e^{|G(s)|}}{1+\left|\int_0^s e^{|G(\sigma)|}\,d\sigma\right|} \approx \frac{2}{|s|} \to 0 \qquad \text{as} \quad s \to \pm\infty.$$

On the other hand, if $g = g_1 + g_2$, where $g_1 \in L^1(I\!R)$ and $\lim_{s\to\pm\infty} g_2(s) = 0$, then g_2 is bounded and we may find a constant $C > 0$ such that $g = (g-C)^+ + g\wedge C$ with $(g-C)^+$ a summable function. However, it follows that $\displaystyle\int_0^s \big(g(\sigma)-C\big)^+\,d\sigma \approx \operatorname{sign}(s)\log|s|$ for every $C > 0$ and so g cannot be decomposed as above.

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Let us point out that, in this example,

$$\lim_{s\to\pm\infty} \frac{e^{|G(s)|}}{\left(1+\left|\int_0^s e^{|G(\sigma)|}\,d\sigma\right|\right)^{1/2}} = \sqrt{2}. \tag{12}$$

We define weak solutions of problem (5) in the sense of finite energy solutions.

Definition 2.1 *We will say that a function* u *is a weak solution of problem (5) if* $\mathbf{a}(x,u,\nabla u) \in L^{p'}(\Omega)$, $b(x,u,\nabla u) \in L^1(\Omega)$ *and*

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla\varphi + \int_\Omega b(x,u,\nabla u)\varphi = \int_\Omega f\varphi$$

holds for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Concerning bounded solutions of (5), the main result we will prove in this paper is the following one.

Theorem 2.1 *Assume (A1), (A2), (A3), (B), (C1) and (F), with* $m > \max\{\frac{N}{p},1\}$. *Then there exists a function* $u \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, *which is a weak solution of (5).*

Consider now the case in which the datum f is less regular, that is, assume $1 < m \le \frac{N}{p}$ (consequently $1 < p < N$). In this situation solutions are expected to be unbounded, and even if $u \in W_0^{1,p}(\Omega)$ we may have that $\mathbf{a}(x,u,\nabla u)$ does not belong to $L^1(\Omega)^N$, since the growth assumption (A1) does not imply that $\mathbf{a}(x,u,\nabla u) \in L^1(\Omega)^N$ for all $u \in W_0^{1,p}(\Omega)$. In particular, under this general growth condition we cannot work in the framework of distributional solutions as soon as u is unbounded. Moreover, if $f \in L^m(\Omega)$ with m close to 1, then solutions are also not expected to belong to the energy space $W_0^{1,p}(\Omega)$ and, for small p, not even to $W_{loc}^{1,1}(\Omega)$. Consequently, in this situation the notion of ∇u has to be precised, since ∇u may no longer be in $L^1(\Omega)$. To overcome the above mentioned problems it will be helpful (as it is done in [8]) to use the generalized framework of so–called "entropy solutions", introduced in [2], which allows us a unified presentation for the whole range $1 < m \le \frac{N}{p}$, including the case of both finite and infinite energy solutions. To do so we need some preliminaries. For $k > 0$ we define the truncature at level $\pm k$ as $T_k(r) = \max\big\{-k, \min\{k,r\}\big\}$.

Definition 2.2 *Following [2], we introduce* $\mathcal{T}_0^{1,p}(\Omega)$ *as the set of all measurable functions* $u:\Omega\to I\!\!R$ *almost everywhere finite and such that* $T_k(u)\in W_0^{1,p}(\Omega)$ *for all* $k>0$.

For a measurable function u belonging to $\mathcal{T}_0^{1,p}(\Omega)$, a gradient can be defined: it is a measurable function, also denoted by ∇u, which satisfies $\nabla T_k(u) = (\nabla u)\chi_{\{|u|<k\}}$ for all $k>0$ (see [2]). Let us next define entropy solutions.

Definition 2.3 *Let* $f \in L^1(\Omega)$. *We will say that* $u \in \mathcal{T}_0^{1,p}(\Omega)$ *is an entropy solution of (5) if* $b(x,u,\nabla u)\in L^1(\Omega)$ *and the identity*

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(u-v) + \int_\Omega b(x,u,\nabla u)T_k(u-v) = \int_\Omega fT_k(u-v) \qquad (13)$$

holds for every $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ *and* $k>0$.

We point out that every term in (13) is well defined, see [2], where the reader can find an introduction to this concept. We also remark that in dealing with our problem the equivalent notion of renormalized solution (see [19], [21] and [12]) also works.

Let then $f \in L^m(\Omega)$ with $1 < m \leq \frac{N}{p}$. First of all we remark that also in the limit case $mp = N$ assumption (C1) is still sufficient to have a solution, though unbounded.

Theorem 2.2 *Assume (A1), (A2), (A3), (B), (C1) and (F), with $m = \frac{N}{p}$. Then there exists an entropy solution u of (5), which is such that $\Phi(u) \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ for all $q \in [1, +\infty[$.*

Remark 2.4 We point out that the inequalities $|A(u)| \leq |\Phi(u)|$ and

$$|\nabla A(u)|^p \leq e^{\frac{p\,|G(u)|}{p-1}} |\nabla A(u)|^p = |\nabla \Phi(u)|^p$$

imply that $A(u) \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ for all $q \in [1, +\infty[$.

Remark 2.5 We will not enter into possible detailed estimates for this case; nevertheless, we will briefly compare our situation with a borderline one. Let us turn our attention to (1), when the function g is constant, say γ. This problem was studied by V. Ferone and F. Murat (see [13] and [14]). They proved, under smallness assumptions on $\|f\|_{N/p}$, that there exists a solution u for each of these data and, moreover, the common regularity shared for all these solutions is

$$\left(e^{\frac{\gamma\,|u|}{p-1}} - 1\right) \operatorname{sign}(u) \in W_0^{1,p}(\Omega).$$

(Actually, solutions have more regularity, but depending on how small $\|f\|_{N/p}$ is.) Since in this case our function Φ would be defined by

$$\Phi(s) = \frac{p-1}{\gamma}\left(e^{\frac{\gamma\,|s|}{p-1}} - 1\right) \operatorname{sign}(s),$$

it follows that the regularity stated in the above theorem is the same as that proved in [14].

Next consider that $f \in L^m(\Omega)$ and $1 < m < \frac{N}{p}$. An additional hypothesis, stronger than (C1), will now be needed for having an existence result; we assume here that

(C2) there exist θ, with $0 < \theta < \frac{p^*}{pm'}$, and a constant $M > 0$ such that

$$e^{|G(s)|} \leq M(1 + |\Phi(s)|)^{\theta(p-1)} \qquad \text{for all } s \in I\!R.$$

Observe that since $\theta < 1$ this condition implies (C1). Moreover as $m \to 1$, then $\theta \to 0$ and (C2) reduces to ask $g \in L^1(I\!R)$, the assumption already used in [27] and [25]; since, when $m = 1$, the previous results can not be improved (see also Proposition 5.1), this is the reason to limiting ourselves to consider $m > 1$. As it is shown in Theorem 2.3 below, condition (C2) implies existence of an entropy solution. Nevertheless, to obtain the regularity stated in Theorem 1.1, we need a stronger hypothesis. Indeed, the desired regularity is a consequence of the following assumption.

(C3) there exist constants $0 < \theta < \frac{p^*}{pm'}$ and $0 < M_1 \le M_2$ satisfying

$$M_1(1+|\Phi(s)|)^{\theta(p-1)} \le e^{|G(s)|} \le M_2(1+|\Phi(s)|)^{\theta(p-1)} \qquad \text{for all } s \in I\!R.$$

Note that, because of (12), Example 2.1 satisfies (C3) with $\theta = 1/2$.

Theorem 2.3 *Assume (A1), (A2), (A3), (B), (C2) and (F), with* $1 < m < \frac{N}{p}$. *Then there exists an entropy solution* u *of (5). Furthermore, this entropy solution satisfies*

$$\int_\Omega |\Phi(u)|^{\frac{rp^*}{p}}\,dx + \int_\Omega |\nabla\Phi(u)|^{\frac{Nr}{N-p+r}} \le c\,, \qquad \textit{where} \quad r = \frac{(p-1)\,(1-\theta)m'}{m' - \frac{N}{N-p}}. \tag{14}$$

In particular, when $\frac{Nr}{N-p+r} \ge 1$, *we have* $\Phi(u) \in W_0^{1,\frac{Nr}{N-p+r}}(\Omega)$.

Moreover, assuming (C3) instead of (C2), we also have:

(i) If $(p^*)' \le m < N/p$, *then* $A(u) \in W_0^{1,p}(\Omega) \cap L^{\frac{Nm(p-1)}{N-pm}}(\Omega)$.

(ii) If $1 < m < (p^*)'$, *then* $A(u) \in W_0^{1,m^*(p-1)}(\Omega)$.

Remark 2.6 The main example of functions g satisfying (C2) is given if g satisfies $|g(s)| \le \frac{\lambda}{|s|}$ for $|s|$ large, with $\lambda < \frac{N(m-1)}{N-pm}$. This value actually plays the role of a borderline: in Corollary 5.3 we show that if $\lambda = \frac{N(m-1)}{N-pm}$, there are no general a priori estimates depending on $\|f\|_{L^m(\Omega)}$ and existence may fail for some f in this class.

Remark 2.7 Observe that if $\mathbf{a}(x,u,\nabla u) \in L^1_{loc}(\Omega)$, the entropy solution obtained above is a solution in the sense of distributions and if $\mathbf{a}(x,u,\nabla u) \in L^{p'}(\Omega)$, it is a weak solution as defined in (2.1). We also stress that, if $\alpha = 1$, the previous theorem states that under assumption (C3) we recover the regularity directly on u.

The main tool for proving Theorems 2.1 and 2.3 are a priori estimates together with compactness arguments applied to sequences of bounded approximating solutions. Eventually, we will often use the following cancellation lemma (see [8]), which underlines the variational structure of the problem once the equation in (5) is multiplied by $\exp(G(u))$.

Lemma 2.1 *Assume (A1)–(A3) and (B). Let* $u \in W_0^{1,p}(\Omega)$ *be a weak solution of problem (5).*

(1) If $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and* $e^{\operatorname{sign}(v)G(u)}v$ *can be taken as test function in the weak formulation of (5), we have:*

$$\int_\Omega e^{\operatorname{sign}(v)G(u)}\mathbf{a}(x,u,\nabla u)\cdot\nabla v \le \int_\Omega e^{\operatorname{sign}(v)G(u)}v(f+b_0).$$

(2) If Ψ *is a locally Lipschitz continuous and increasing function such that* $\Psi(0) = 0$ *and* $\Psi(u)e^{|G(u)|}$ *can be taken as test function in the weak formulation of (5), then*

$$\int_\Omega e^{|G(u)|}\Psi'(u)\alpha(u)|\nabla u|^p \le \int_\Omega e^{|G(u)|}\Psi(u)(f+b_0).$$

Remark 2.8 Actually, all main estimates will be deduced from (1) of Lemma 2.1, so that they essentially will hold as soon as the equation can be multiplied by $\exp(|G(u)|)$. Although we will apply Lemma 2.1 to sequences of bounded approximating solutions, it remains also true for entropy solutions. In particular, if

$$g(u)|\nabla u|^p e^{|G(u)|} + (b_0 + |f|)\, e^{|G(u)|} \in L^1(\Omega)\,,$$

then any entropy solution satisfies the estimates of Theorem 3.1 and Corollary 4.1 and the regularity stated in Theorems 2.1 and 2.3. A sketch of the proof of this fact will be given in Remark 4.2.

3 An L^∞ estimate and proof of Theorem 2.1

In this section, we will prove the existence of a bounded solution to problem (5). We begin by proving a Stampacchia's type result (see [28] and [23]). It follows from it and the cancellation lemma (2.1) that an priori L^∞ estimate holds.

L^∞-estimates for second order elliptic equations have been widely studied: the 1960's works by G. Stampachia [28] and Ladyzenskaja-Ural'ceva [22] are the classical references; others include, for instance, [3], [4], [6], [8], [15], [16], [26] or [29]. In every case, once the L^∞-estimate is proved, a solution is obtained. This also works in our situation: as a straightforward consequence, Theorem 2.1 is proved.

Proposition 3.1 *Let $a > 0$ and let $\varphi : [a, +\infty[\to [0, +\infty[$ be a non increasing function satisfying*

$$\varphi(h) \le \frac{\omega(k)^\rho}{(h-k)^\rho} \varphi(k)^{1+\nu} \qquad \forall h > k \ge a, \tag{15}$$

where $\lim_{k\to\infty} \omega(k)/k = 0$ and $\rho, \nu > 0$. Then there exists $k^* > a$ such that $\varphi(k^*) = 0$.

Proof: First of all, denote $C_0 = \frac{1}{2}[\varphi(a)]^{-\nu/\rho} 2^{-(1+\nu)/\nu}$. Since $\lim_{k\to\infty} \omega(k)/k = 0$, there exists $k_0 > a$ such that

$$\omega(k) < C_0 k \qquad \forall k \ge k_0. \tag{16}$$

Let $d > 0$ satisfy

$$d^\rho = \Lambda[\varphi(k_0)]^\nu 2^{(1+\nu)\mu},$$

where we denote $\mu = \rho/\nu$ and Λ is a positive number to be chosen later; consider also the increasing sequence

$$k_r = k_0 + d - \frac{d}{2^r}, \qquad r \in I\!N.$$

Next, we claim that

$$\varphi(k_r) \le \varphi(k_0)\, 2^{-r\mu}, \qquad \forall r \in I\!N. \tag{17}$$

We argue by induction. This inequality holds trivially for $r = 0$; if it is satisfied for some $r \ge 0$, applying (15) with $h = k_{r+1}$ and $k = k_r$ we have

$$\varphi(k_{r+1}) \le \frac{\omega(k_r)^\rho\, 2^{(1+r)\rho}}{d^\rho}\, \varphi(k_r)^{1+\nu} \le \frac{\omega(k_r)^\rho\, 2^{r\rho-\mu}}{\Lambda[\varphi(k_0)]^\nu} \left(\frac{\varphi(k_0)}{2^{r\mu}}\right)^{1+\nu} = \frac{\omega(k_r)^\rho}{\Lambda}\, \frac{\varphi(k_0)}{2^{(1+r)\mu}}.$$

So, our claim (17) holds for $r+1$ provided $\Lambda > 0$ is such that $\omega(k_r) \le \Lambda^{1/\rho}$. Recalling that by (16), $\omega(k_r) < C_0 k_r \le C_0(k_0+d)$, we look for $\Lambda > 0$ satisfying $C_0(k_0+d) \le \Lambda^{1/\rho}$ (note that d depends on Λ). Since

$$\frac{\Lambda^{1/\rho}}{k_0+d} = \frac{\Lambda^{1/\rho}}{k_0 + \Lambda^{1/\rho}[\varphi(k_0)]^{\nu/\rho}2^{(1+\nu)/\nu}} \ge \frac{\Lambda^{1/\rho}}{k_0 + \Lambda^{1/\rho}[\varphi(a)]^{\nu/\rho}2^{(1+\nu)/\nu}},$$

we deduce that, taking Λ big enough,

$$\frac{\Lambda^{1/\rho}}{k_0+d} \ge \frac{1}{2[\varphi(a)]^{\nu/\rho}2^{(1+\nu)/\nu}} = C_0.$$

Thus, we have seen that there exists $\Lambda > 0$ such that $C_0(k_0+d) \le \Lambda^{1/\rho}$, and so claim (17) is proved.

Since φ is non increasing, (17) implies as r goes to infinity:

$$\varphi(k_0+d) \le \lim_{r\to\infty} \varphi(k_r) \le \lim_{r\to\infty} \varphi(k_0)\, 2^{-r\mu} = 0.$$

Therefore, we may take $k^* = k_0 + d$ and Proposition 3.1 is proved. ∎

From now on in this paper, we use the following notation: for every $k > 0$ and $s \in \mathbb{R}$, $G_k(s) = s - T_k(s) = (|s|-k)^+ \operatorname{sign}(s)$. Recall also the function Φ defined in (10).

Theorem 3.1 *Let* $m > \max\{\frac{N}{p}, 1\}$. *If* u *is a weak solution of (5) such that* $e^{|G(u)|}\Phi(u)$ *may be taken as test function, then* $\|\Phi(u)\|_\infty \le c$, *where* $c > 0$ *is a constant that only depends on the parameters* p, m, $\|f+b_0\|_m$, N, *and* $|\Omega|$; *so that,* $\|u\|_\infty \le \max\{-\Phi^{-1}(-c), \Phi^{-1}(c)\}$.

Proof: Take $\Psi(u) = G_k\big(\Phi(u)\big)$ in Lemma 2.1 to get

$$\int_\Omega |\nabla G_k\big(\Phi(u)\big)|^p = \int_\Omega e^{|G(u)|}\Psi'(u)\alpha(u)|\nabla u|^p \le \int_\Omega e^{|G(u)|} G_k\big(\Phi(u)\big)(f+b_0), \qquad (18)$$

and then, by Hölder's inequality,

$$\textstyle\int_\Omega |\nabla G_k\big(\Phi(u)\big)|^p \le \|f+b_0\|_m \left(\int_\Omega e^{|G(u)|\,m'}\, |G_k\big(\Phi(u)\big)|^{m'}\right)^{1/m'}. \qquad (19)$$

Let us set

$$\eta(k) = \sup_{\{|\Phi(s)|>k\}} \frac{e^{|G(s)|}}{(1+|\Phi(s)|)^{(p-1)}}.$$

Since $\lim_{s\to\pm\infty} \Phi(s) = \pm\infty$ (see Remark 2.2) and due to (C1) we have that $\eta(k)$ tends to zero as k goes to infinity. Moreover

$$e^{|G(u)|\,m'} \le \frac{e^{|G(u)|\,m'}}{(1+|\Phi(u)|)^{m'(p-1)}}(1+k+|G_k(\Phi(u))|)^{m'(p-1)}$$

so that

$$\left(\int_\Omega e^{|G(u)|\,m'}\, |G_k\big(\Phi(u)\big)|^{m'}\right)^{1/m'} \le c\eta(k)\left(\int_\Omega k^{m'(p-1)}\, |G_k(\Phi(u))|^{m'} + |G_k(\Phi(u))|^{pm'}\right)^{\frac{1}{m'}}$$

$$\le c\eta(k)k^{p-1}\left(\int_\Omega |G_k(\Phi(u))|^{m'}\right)^{\frac{1}{m'}} + c\eta(k)\left(\int_\Omega |G_k(\Phi(u))|^{pm'}\right)^{\frac{1}{m'}}.$$

Let for the moment $p < N$, then $p^* = \frac{Np}{N-p} > m' p$ since $m > \frac{N}{p}$. We deduce

$$\begin{aligned} \left(\int_\Omega e^{|G(u)|\, m'}\ |G_k\big(\Phi(u)\big)|^{m'}\right)^{1/m'} &\le c\eta(k)k^{p-1}\left(\int_\Omega |G_k(\Phi(u))|^{m'}\right)^{\frac{1}{m'}} \\ &+ c\eta(k)\left(\int_\Omega |G_k(\Phi(u))|^{p^*}\right)^{\frac{p}{p^*}} |\Omega|^{\frac{1}{m'}-\frac{p}{p^*}}. \end{aligned} \tag{20}$$

Thus putting together (20) and (19) and using Sobolev's embedding we obtain

$$\begin{aligned} \left(\int_\Omega |G_k\big(\Phi(u)\big)|^{p^*}\right)^{\frac{p}{p^*}} &\le c\|f+b_0\|_m\, \eta(k)k^{p-1}\left(\int_\Omega |G_k(\Phi(u))|^{m'}\right)^{\frac{1}{m'}} + \\ &+c\|f+b_0\|_m\, \eta(k)\left(\int_\Omega |G_k(\Phi(u))|^{p^*}\right)^{\frac{p}{p^*}}. \end{aligned}$$

Since $\eta(k)$ tends to zero as k goes to infinity, there exists k_0 such that for any $k > k_0$ we have the inequality:

$$\left(\int_\Omega |G_k(\Phi(u))|^{p^*}\right)^{\frac{p}{p^*}} \le c\eta(k)k^{p-1}\left(\int_\Omega |G_k(\Phi(u))|^{m'}\right)^{\frac{1}{m'}},$$

which, denoting $A_k = \{|\Phi(u)| \ge k\}$, yields

$$\left(\int_\Omega |G_k(\Phi(u))|^{p^*}\right)^{\frac{p}{p^*}} \le c\eta(k)k^{p-1}|A_k|^{\frac{1}{m'}-\frac{1}{p^*}}\left(\int_\Omega |G_k(\Phi(u))|^{p^*}\right)^{\frac{1}{p^*}}.$$

Thus

$$\left(\int_\Omega |G_k(\Phi(u))|^{p^*}\right)^{\frac{p-1}{p^*}} \le c\eta(k)k^{p-1}|A_k|^{\frac{1}{m'}-\frac{1}{p^*}}.$$

From here, denoting $\varphi(k) = |A_k|$ and taking into account Stampacchia's procedure (see [28]), we obtain

$$\varphi(h) \le \frac{(\eta(k)^{\frac{1}{p-1}}k)^{p^*}}{(h-k)^{p^*}}\varphi(k)^{(\frac{p^*}{m'}-1)\frac{1}{p-1}},$$

for $h > k > k_0$. Hence, since $(\frac{p^*}{m'}-1)\frac{1}{p-1} > 1$ and $\eta(k)$ tends to zero, Lemma 3.1 implies there is k^* such that $\big|\{|\Phi(u)| \ge k^*\}\big| = 0$, that is, $\Phi(u)$ is bounded as desired. If we have $p \ge N$, one has to use that $W_0^{1,p}(\Omega)$ is embedded into $L^r(\Omega)$ for any $r < \infty$; then the same calculation applies provided p^* is replaced by any number r such that $r > p\, m'$. ∎

Proof of Theorem 2.1: Let us consider the following sequence of approximating problems:

$$\begin{cases} -\mathrm{div}[\mathbf{a}(x,u_n,\nabla u_n)] + T_n[b(x,u_n,\nabla u_n)] = f, & \text{in } \Omega; \\ u_n = 0, & \text{on } \partial\Omega; \end{cases} \tag{21}$$

Applying standard results (see [18]), one can easily see that problem (21) has a bounded solution u_n (indeed the results of [18] are directly applied to the sequence of functions $v_n = A(u_n)$, see also Remark 2.1). We can apply Theorem 3.1 to (21) and deduce that u_n is uniformly bounded in $L^\infty(\Omega)$. Once we have the uniform bound on u_n, we are in the position to use the classical arguments in [4] (note that $\mathbf{a}(x,s,\xi)$ may lack of coerciveness only if $|s|$ is unbounded). Then u_n is bounded in $L^\infty(\Omega)$ and relatively compact in $W_0^{1,p}(\Omega)$, and then (using (B)) $T_n(b(x,u_n,\nabla u_n))$ is also compact in $L^1(\Omega)$. We conclude that there exists $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which is a weak solution of problem (5). ■

Remark 3.1 Concerning the optimality of our test functions, one can wonder what happens if, instead of hypothesis (C1), the borderline condition $e^{|G(s)|} \le c(1 + |\Phi(s)|)^{p-1}$ holds. In this case, our proof still applies but requires f to have a suitable small norm. For instance, in the model case

$$-\alpha\Delta u = \beta|\nabla u|^2 + f$$

applying our procedure one has (see (18))

$$\int_\Omega |\nabla G_k(\Phi(u))|^2 \le \int_\Omega f e^{\beta|u|/\alpha} G_k(\Phi(u)) \le \frac{\beta}{\alpha^2}\int_\Omega |f\Phi(u)G_k(\Phi(u))|\,.$$

Thus, we may obtain an L^∞-estimate if $\|f\|_m$ is small enough (e.g. as in [14]).

On the other hand, we may also consider a equation such as

$$u - \alpha\Delta u = \beta|\nabla u|^2 + f$$

with $f \in L^\infty(\Omega)$. In this case, (18) becomes

$$\int_\Omega |\nabla G_k(\Phi(u))|^2 \le \frac{\beta}{\alpha^2}\int_\Omega (f - M)\Phi(u)G_k(\Phi(u))\,,$$

where $M = (\alpha/\beta)\log(1 + (\beta/\alpha^2)k)$. Taking k so big to get $M > \|f\|_\infty$, we deduce

$$\Phi(u)G_k(\Phi(u)) = 0 \qquad \text{on} \qquad \{|\Phi(u)| > k\}$$

and so an L^∞-estimate is proved. It is worth noting that the function $\Psi(u) = e^{\beta|s|/\alpha}\Phi(s)$ satisfies

$$\alpha\Psi'(s) - \beta|\Psi(s)| \ge \alpha^2 \qquad \text{for all} \quad s \in I\!R,$$

which is the same basic property used in [4] to deal with the above equation.

4 Unbounded solutions

We study here the possibility to have unbounded solutions of (5) if the datum f is less regular. Under assumption (C1), we can still handle the case $m = \frac{N}{p}$.

Theorem 4.1 *Assume that (A1), (A2), (B), (C1) and (F) hold true, with $m = \frac{N}{p}$. If u is a weak solution of (5) such that $e^{|G(u)|}\Phi(u)$ may be taken as test function, then there exists a constant $c > 0$, that only depends on the parameters p, N, $\|f + b_0\|_{\frac{N}{p}}$, and $|\Omega|$, such that $\|\Phi(u)\|_{W_0^{1,p}(\Omega)} + \|\Phi(u)\|_{L^q(\Omega)} \le c$ for all $q \in [1, +\infty[$.*

Proof. Given $q \geq p^*$ let us consider $\gamma = q\frac{N-p}{N} - (p-1) \geq 1$ and the function $\Psi(s) = (1+|\Phi(s)|)^{\gamma-1}\Phi(s)$, which satisfies $\Psi'(s) \geq (1+|\Phi(s)|)^{\gamma-1}\Phi'(s)$. Using Lemma 2.1 with $\Psi(u)$ we get

$$\int_\Omega (1+|\Phi(u)|)^{\gamma-1}|\nabla\Phi(u)|^p \leq \int_\Omega e^{|G(u)|}\Psi'(u)\alpha(u)|\nabla u|^p \leq$$

$$\leq \int_\Omega e^{|G(u)|}(1+|\Phi(u)|)^\gamma\,|f+b_0|,$$

and then, by Sobolev and Hölder's inequalities,

$$\begin{aligned}
&\|(1+|\Phi(u)|)^{\frac{\gamma-1}{p}+1} - 1\|^p_{L^{p^*}} \leq c\textstyle\int_\Omega \left|\nabla(1+|\Phi(u)|)^{\frac{\gamma-1}{p}+1}\right|^p = \\
&= c\textstyle\int_\Omega (1+|\Phi(u)|)^{\gamma-1}|\nabla\Phi(u)|^p \leq c\int_\Omega e^{|G(u)|}(1+|\Phi(u)|)^\gamma\,|f+b_0| \leq \\
&\leq c\eta(k)\|f+b_0\|_{\frac{N}{p}}\|(1+|\Phi(u)|)^{\frac{\gamma-1}{p}+1}\|^p_{L^{p^*}} + M(k)\textstyle\int_\Omega |f+b_0|,
\end{aligned} \tag{22}$$

where

$$\eta(k) = \sup_{\{|\Phi(s)|>k\}} \frac{e^{|G(s)|}}{(1+|\Phi(s)|)^{(p-1)}} \qquad \text{and} \qquad M(k) = \sup_{\{|\Phi(s)|\leq k\}} e^{|G(s)|}|\Phi(s)|.$$

Since (C1) implies that $\lim\limits_{k\to+\infty} \eta(k) = 0$, we can choose in (22) a level k so that we get $\|(1+|\Phi(u)|)^{\frac{\gamma-1}{p}+1}\|^p_{L^{p^*}} \leq c$, and then also $\int_\Omega \left|\nabla(1+|\Phi(u)|)^{\frac{\gamma-1}{p}+1}\right|^p \leq c$. It follows from $\frac{\gamma-1}{p}+1 = \frac{q}{p^*}$, that $\|\Phi(u)\|_{L^q} \leq c$. Finally if we set $q = p^*$, i.e. $\gamma = 1$, then we also deduce that $\|\Phi(u)\|_{W_0^{1,p}} \leq c$. ■

Remark 4.1 As it was remarked to us by L. Boccardo, the above proof for having an estimate on $\Phi(u)$ in $W_0^{1,p}(\Omega)$ still works under possibly weaker assumptions on f, provided (C1) always holds true. A significant example is given by $f(x) = \lambda\,|x|^{-p}$, which only belongs to the Marcinkiewicz space $M^{\frac{N}{p}}(\Omega)$. Indeed, with the same notations as before one has (take $b_0 = 0$ for simplicity)

$$\int_\Omega |\nabla\Phi(u)|^p \leq \int_\Omega e^{|G(u)|}|\Phi(u)f| \leq \lambda\,\eta(k)\int_\Omega \frac{(1+|\Phi(u)|)^p}{|x|^p} + M(k)\int_\Omega |f|$$

and one concludes using Hardy–Sobolev's inequality and the fact that $\eta(k)$ is arbitrarily small for large k. Once the estimate is obtained, the compactness arguments developed below allow to prove the existence of a solution even for this weaker case.

Let now consider $m < \frac{N}{p}$. We begin by proving the basic a priori estimates in this case.

Theorem 4.2 *Assume that (A1), (A2), (B), (C2) and (F) hold true, with* $1 < m < \frac{N}{p}$. *If* u *is a bounded weak solution of (5), then there exists a constant* c *which only depends on* N, p, θ, m *and* $\|f+b_0\|_{L^m(\Omega)}$ *such that*

$$\begin{gathered}
\int_\Omega \frac{|\nabla\Phi(u)|^p}{(1+|\Phi(u)|)^{p-r}} \leq c \qquad \textit{and} \\
\int_\Omega |\Phi(u)|^{\frac{rp^*}{p}} + \int_\Omega |\nabla\Phi(u)|^{\frac{Nr}{N-p+r}} \leq c\,,
\end{gathered} \tag{23}$$

where $r = \dfrac{(p-1)\,(1-\theta)m'}{m' - \frac{N}{N-p}}$. *In particular, if* $Nr/(N-p+r) \geq 1$, *we have an estimate on* $\Phi(u)$ *in* $W_0^{1,\frac{Nr}{N-p+r}}(\Omega)$.

Proof: We take $\Psi(s) = (1+|\Phi(s)|)^{r-p}\,\Phi(s)$, with $r > p-1$, in Lemma 2.1. Since $\Psi'(s) \geq \min\{r+1-p, 1\}\,(1+|\Phi(s)|)^{r-p}\,\Phi'(s)$, we get

$$\int_\Omega (1+|\Phi(u)|)^{r-p}\,|\nabla\Phi(u)|^p \leq c\int_\Omega e^{|G(u)|}(1+|\Phi(u)|)^{r-p+1}\,|f+b_0|$$

and then, using (C2),

$$\int_\Omega (1+|\Phi(u)|)^{r-p}\,|\nabla\Phi(u)|^p \leq c\,M\int_\Omega (1+|\Phi(u)|)^{r-(p-1)\,(1-\theta)}\,|f+b_0|.$$

Using Hölder's inequality on the right hand side and Sobolev's inequality on the left one, we obtain

$$\begin{aligned}\left(\int_\Omega |(1+|\Phi(u)|)^{\frac{r}{p}}-1|^{p^*}\right)^{\frac{p}{p^*}} &\leq c\int_\Omega (1+|\Phi(u)|)^{r-p}\,|\nabla\Phi(u)|^p \\ &\leq c\,\|f+b_0\|_{L^m(\Omega)}\left(\int_\Omega (1+|\Phi(u)|)^{(r-(p-1)\,(1-\theta))m'}\right)^{\frac{1}{m'}}\end{aligned} \tag{24}$$

If we choose $r = \frac{(p-1)\,(1-\theta)m'}{m'-\frac{N}{N-p}}$ we have

$$(r-(p-1)\,(1-\theta))m' = \frac{rp^*}{p} = \frac{Nm(p-1)\,(1-\theta)}{N-pm}\,.$$

Note that $r > p-1$ if and only if $\theta < \frac{p^*}{pm'}$, as given by (C2). Having in mind that $\frac{p}{p^*} > \frac{1}{m'}$ (since $m < \frac{N}{p}$), (24) implies two estimates in (23). To see the third one, note that it also follows from (24) that

$$\int_\Omega (1+|\Phi(u)|)^{\frac{Nm(p-1)\,(1-\theta)}{N-pm}} \leq c.$$

Now denote $q = Nr/(N-p+r)$ and take into account $\frac{p-r}{p-q}q = \frac{Nm(p-1)\,(1-\theta)}{N-pm}$, and so it follows from Hölder's inequality that

$$\begin{aligned}\int_\Omega |\nabla\Phi(u)|^q &= \int_\Omega (1+|\Phi(u)|)^{(p-r)q/p}\frac{|\nabla\Phi(u)|^q}{(1+|\Phi(u)|)^{(p-r)q/p}} \\ &\leq \left(\int_\Omega (1+|\Phi(u)|)^{(p-r)q/(p-q)}\right)^{(p-q)/p}\left(\int_\Omega \frac{|\nabla\Phi(u)|^p}{(1+|\Phi(u)|)^{p-r}}\right)^{q/p}.\end{aligned}$$

Therefore, we also have $\int_\Omega |\nabla\Phi(u)|^q \leq c$ and (23) is completed. Clearly, in case $\frac{Nr}{N-p+r} \geq 1$, we obtained an estimate for $\Phi(u)$ in $W_0^{1,Nr/(N-p+r)}(\Omega)$. ■

Assuming now that the stronger hypothesis (C3) holds instead of (C2), we are able to prove the following estimates, which generalize those proved in [8].

Corollary 4.1 *Assume (A1), (A2), (B), (C3) and (F), with $1 < m < \frac{N}{p}$, and let u be a bounded weak solution of (5). There exists a constant c, only depending on $\|f + b_0\|_{L^m(\Omega)}$ (and N, p, θ, m) such that:*

$$\text{if } m \geq \tfrac{Np}{Np-N+p}, \text{ then} \qquad \|A(u)\|_{L^{\frac{Nm(p-1)}{N-pm}}(\Omega)} + \|A(u)\|_{W_0^{1,p}(\Omega)} \leq c\,, \tag{25}$$

and

$$\text{if } 1 < m < \tfrac{Np}{Np-N+p}, \text{ then} \qquad \|A(u)\|_{W_0^{1,\frac{Nm(p-1)}{N-m}}(\Omega)} \leq c\,.$$

Proof: By (C3), we have for any $s \in I\!R$,

$$\frac{\Phi'(s)}{(1+|\Phi(s)|)^\theta} = \frac{e^{|G(s)|/(p-1)}\alpha^{1/(p-1)}(s)}{(1+|\Phi(s)|)^\theta} \geq M_1\, \alpha^{1/(p-1)}(s)$$

which implies that $M_1|A(s)| \leq c\,(1+|\Phi(s)|)^{(1-\theta)}$. Since we have an estimate of $\Phi(u)$ in $L^{\frac{Nm(p-1)\,(1-\theta)}{N-pm}}(\Omega)$ we deduce an estimate on $A(u)$ in $L^{\frac{Nm(p-1)}{N-pm}}(\Omega)$. Moreover we have, again thanks to (C3),

$$M_1^{p/(p-1)} \int_\Omega (1+|\Phi(u)|)^{r-p(1-\theta)}\, |\nabla A(u)|^p \leq \int_\Omega (1+|\Phi(u)|)^{r-p}\, |\nabla \Phi(u)|^p\,.$$

Since $r - p(1-\theta) = (1-\theta)\frac{p^*-m'}{m'-\frac{N}{N-p}} \geq 0$ when $\frac{Np}{Np-N+p} \leq m < \frac{N}{p}$, we deduce that $A(u)$ is estimated in $W_0^{1,p}(\Omega)$ in this case. If, instead, we consider $1 < m < \frac{Np}{Np-N+p}$, then, by Hölder's inequality,

$$\int_\Omega |\nabla A(u)|^{m^*(p-1)} =$$

$$= \int_\Omega (1+|\Phi(u)|)^{(r-p(1-\theta))\frac{m^*(p-1)}{p}}\, |\nabla A(u)|^{m^*(p-1)} \cdot (1+|\Phi(u)|)^{(p(1-\theta)-r)\frac{m^*(p-1)}{p}} \leq$$

$$\leq c\left(\int_\Omega (1+|\Phi(u)|)^{r-p(1-\theta)}\, |\nabla A(u)|^p\right)^{\frac{m^*(p-1)}{p}} \left(\int_\Omega (1+|\Phi(u)|)^{\frac{(p(1-\theta)-r)m^*(p-1)}{p-m^*(p-1)}}\right)^{1-\frac{m^*(p-1)}{p}}.$$

It follows from

$$\tfrac{(p(1-\theta)-r)m^*(p-1)}{p-m^*(p-1)} = (p-1)(1-\theta)\left(\tfrac{pm^*}{p-m^*(p-1)} - \tfrac{r}{(p-1)\,(1-\theta)}\tfrac{m^*(p-1)}{p-m^*(p-1)}\right) = \tfrac{Nm(p-1)\,(1-\theta)}{N-pm},$$

that the estimates on $\Phi(u)$ yield the estimate for $A(u)$ in $W_0^{1,m^*(p-1)}(\Omega)$.

■

Remark 4.2 The estimates found in Theorem 4.2 and Corollary 4.1 apply not only to bounded weak solutions of (5), but actually to any entropy solution such that

$$g(u)|\nabla u|^p\, e^{|G(u)|} + (|f| + b_0)\, e^{|G(u)|} \in L^1(\Omega)\,. \tag{26}$$

This can be proved by means of a bootstrap regularity argument, which we only sketch here. To begin with fix $1 < m < N/p$. We remark that the entropy formulation implies, for any $k,\ h > 0$:

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(e^{|G(T_h(u))|}\,\psi(\Phi(u))) \leq \int_\Omega (b(x,u,\nabla u)+f)T_k(e^{|G(T_h(u))|}\,\psi(\Phi(u)))\,,$$

where ψ is any nondecreasing function such that $\psi(0) = 0$ and ψ' has compact support. By Fatou's lemma and thanks to (26) it is possible to take the limit on h and k obtaining:

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla(e^{|G(u)|}\,\psi(\Phi(u))) \le \int_\Omega (|b(x,u,\nabla u)| + |f|)e^{|G(u)|}\,\psi(\Phi(u)))\,,$$

and then, applying (A2) and (B),

$$\int_\Omega (\alpha(u)e^{|G(u)|})^{p'}|\nabla u|^p\psi'(\Phi(u)) \le \int_\Omega (|f| + b_0)e^{|G(u)|}\,\psi(\Phi(u))\,.$$

Setting $v = \Phi(u)$, the previous inequality can be rewritten as

$$\int_\Omega |\nabla v|^p\psi'(v) \le \int_\Omega (|f| + b_0)e^{|G(u)|}\;\psi(v) \tag{27}$$

for any nondecreasing ψ such that $\psi(0) = 0$ and ψ' has compact support. From (27) one can deduce the desired estimates using assumption (C2) and bootstrap arguments. Indeed, as a first step, choosing $\psi(s) = T_k(s)$ and since $(|f|+b_0)e^{|G(u)|} \in L^1(\Omega)$, one has that $v \in L^q(\Omega)$ for any $q < \frac{N(p-1)}{N-p}$ (see [2]). Then, using $m < \frac{N}{p}$ and $e^{|G(u)|} \le c\,|v|^{\theta(p-1)}$ with $\theta < \frac{N}{(N-p)m'}$, one deduces that there exists $\delta > 0$ such that $e^{|G(u)|}\,|v|^\delta \in L^{m'}(\Omega)$. Taking $\psi(s) = |T_k(s)|^{\delta-1}T_k(s)$ in (27), it follows from $(|f| + b_0)e^{|G(u)|}\,|v|^\delta \in L^1(\Omega)$ that

$$\left(\int_\Omega |T_k(v)|^{\frac{\delta+p-1}{p}p^*}\right)^{p/p^*} \le c\int_\Omega \left|\nabla|T_k(v)|^{\frac{\delta+p-1}{p}}\right|^p \le c\int_\Omega (|f| + b_0)e^{|G(u)|}\,|T_k(v)|^\delta$$

and so Fatou's lemma implies $v \in L^{\frac{\delta+p-1}{p}p^*}(\Omega)$. In particular, $v \in L^{\frac{N(p-1)}{N-p}}(\Omega)$ and so a first step of the bootstrap process has been attained. Subsequently, one can take $\psi(s) = |T_k(s)|^{\gamma-1}T_k(s)$ and perform a power type iteration process, since (27) implies:

$$\begin{aligned}\left(\int_\Omega |T_k(v)|^{(\frac{\gamma-1}{p}+1)p^*}\right)^{\frac{p}{p^*}} &\le c\int_\Omega |\nabla T_k(v)|^p\,|T_k(v)|^{\gamma-1}\\ &\le \left(\int_\Omega ((|f| + b_0)e^{|G(u)|})^{r_k}\right)^{\frac{1}{r_k}}\;\left(\int_\Omega |T_k(v)|^{\gamma r_k'}\right)^{\frac{1}{r_k'}}\,.\end{aligned} \tag{28}$$

If one chooses γ such that $(\frac{\gamma-1}{p}+1)p^* = \gamma\,r_k' = \frac{Nr_k(p-1)}{N-pr_k}$, clearly one obtains an estimate on v in $L^{\frac{Nr_k(p-1)}{N-pr_k}}(\Omega)$ provided

$$\frac{1}{r_k'} < \frac{p}{p^*}\,, \qquad (|f| + b_0)e^{|G(u)|} \in L^{r_k}(\Omega)\,. \tag{29}$$

By (C2), we have $e^{|G(u)|} \le c\,|v|^{\theta(p-1)}$. Then defining r_k such that

$$r_1 = 1\,, \qquad \frac{1}{m} + \frac{\theta(N - pr_k)}{Nr_k} = \frac{1}{r_{k+1}}\,,$$

one can prove by induction, using $\theta < \frac{p^*}{pm'}$, that r_k is an increasing sequence such that (29) is always satisfied. Therefore using that $r_k \to \frac{Nm(1-\theta)}{N-\theta pm}$ one deduces that

$v \in L^{\frac{Nm(p-1)(1-\theta)}{N-pm}}(\Omega)$, which is the regularity found for $\Phi(u)$ in (23). In particular, $\Phi(u)^{r-p+1}$, where $r = \frac{(p-1)\,(1-\theta)m'}{m'-\frac{N}{N-p}}$, can then be taken as test function in the weak formulation and the full conclusions of Theorem 4.2 and Corollary 4.1 (under (C3)) hold true.

In case $m > \frac{N}{p}$ one can use a similar argument (with $\theta = 1$) to obtain that $v \in L^q(\Omega)$ for any $q < \infty$; then one can take $G_k(\Phi(u))$ as test function as in (18) and obtain the L^∞ bound on u.

Once we have obtained a priori estimates, we consider the following approximating problems

$$\begin{cases} -\mathrm{div}\,(\mathbf{a}(x, u_n, \nabla u_n)) + b_n(x, u_n, \nabla u_n) = f_n, & \text{in } \Omega; \\ u_n = 0, & \text{on } \partial\Omega, \end{cases} \tag{30}$$

where f_n is a sequence of bounded functions converging to f in $L^m(\Omega)$ and $b_n(x, s, \xi) = T_n(b(x, s, \xi))$. In order to study the compactness properties of u_n, we will need the following preliminary result whose proof, essentially, can be found in [2].

Lemma 4.1 *Let v_n denote a sequence of measurable functions such that*

$$\|T_k(v_n)\|^p_{W_0^{1,p}(\Omega)} \le c(k+1) \qquad \textit{for all } \ k > 0.$$

Then there is a function $v \in \mathcal{T}_0^{1,p}(\Omega)$ and a subsequence, still denoted by v_n, satisfying

$$v_n \to v \qquad \text{a.e. in } \ \Omega$$
$$T_k(v_n) \rightharpoonup T_k(v) \qquad \text{weakly in } \ W_0^{1,p}(\Omega), \quad \forall k > 0.$$

Our main compactness result is then the following.

Theorem 4.3 *Assume (A1), (A2), (A3), (B), (C2) and (F), with $1 < m < \frac{N}{p}$. Let u_n be a sequence of solutions of (30), then there exists a function u such that, up to subsequences,*

$$T_k(u_n) \to T_k(u) \qquad \text{strongly in } \ W_0^{1,p}(\Omega), \quad \forall k > 0,$$
$$b_n(x, u_n, \nabla u_n) \to b(x, u, \nabla u) \qquad \text{strongly in } \ L^1(\Omega).$$

Moreover, the function u satisfies the estimates stated in Theorem 4.2 and, if (C3) holds true, those in Corollary 4.1.

Proof: First observe that we can apply Theorem 4.2 to deduce that $\Phi(u_n)$ is uniformly bounded in $L^{\frac{Nm(p-1)(1-\theta)}{N-pm}}(\Omega)$. Another observation is that $\theta < \frac{p^*}{pm'}$ implies

$$\theta m'(p-1) < \frac{Nm(p-1)\,(1-\theta)}{N-pm}\,. \tag{31}$$

Since by (C2) we have that

$$(e^{|G(u_n)|})^{m'} \le M^{m'}(1 + |\Phi(u_n)|)^{\theta\, m'(p-1)}\,, \tag{32}$$

we deduce from (31) that $e^{|G(u_n)|}$ is bounded in $L^{m'}(\Omega)$. As a consequence, we have

$$\int_\Omega |f_n + b_0|\, e^{|G(u_n)|} \le \|f_n + b_0\|_{L^m(\Omega)}\, \|e^{|G(u_n)|}\|_{L^{m'}(\Omega)} \le c\,.$$

Choosing then $\Psi(s) = T_k(A(s))$ in (2) of Lemma 2.1 we get

$$\int\limits_{\{|A(u_n)|<k\}} \alpha(u_n)^{\frac{p}{p-1}} |\nabla u_n|^p \le kc\,,$$

that is

$$\|T_k(A(u_n))\|^p_{W_0^{1,p}(\Omega)} \le ck \qquad \forall k > 0\,.$$

We can apply then Lemma 4.1 with $v_n = A(u_n)$ so that v_n almost everywhere converges, up to subsequences, to a function v such that $T_k(v) \in W_0^{1,p}(\Omega) \quad \forall k > 0$. Clearly we deduce, since $A'(s) > 0$, that there exists a function u such that $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$ and

$$\begin{aligned} &u_n \to u \qquad \text{a.e. in} \quad \Omega \\ &T_k(u_n) \rightharpoonup T_k(u) \qquad \text{weakly in} \quad W_0^{1,p}(\Omega), \quad \forall k > 0. \end{aligned} \tag{33}$$

Using again that, thanks to (32) and the estimate on $\Phi(u_n)$, we have $e^{|G(u_n)|}$ bounded in some $L^r(\Omega)$ with $r > m'$, we deduce that $e^{|G(u_n)|}$ strongly converges to $e^{|G(u)|}$ in $L^{m'}(\Omega)$, and then

$$(|f_n| + b_0)e^{|G(u_n)|} \to (|f| + b_0)e^{|G(u)|} \qquad \text{strongly in } L^1(\Omega). \tag{34}$$

We are going to prove now the strong convergence of $T_k(u_n)$ in $W_0^{1,p}(\Omega)$. The argument can be carried on in a very similar way as in [25], so we will not insist on some details. Let us consider the function

$$w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)), \quad \text{with} \quad h > k\,.$$

Set $M_h = h+4k$. Two main properties of w_n will be used, namely: that $\nabla w_n = 0$ if $|u_n| > M_h$ and that

$$w_n \overset{n\to+\infty}{\to} T_{2k}(u - T_h(u)) \overset{h\to+\infty}{\to} 0\,, \tag{35}$$

and such convergences take place weakly–$*$ in $L^\infty(\Omega)$ and in $L^q(\Omega)$ with any $q < +\infty$, due to Lebesgue's theorem.

We take w_n in Lemma 2.1 to obtain

$$\int_\Omega e^{\operatorname{sign}(w_n)G(u_n)} \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla w_n \le \int_\Omega (|f_n| + b_0)\, e^{\operatorname{sign}(w_n)G(u_n)}\, w_n\,. \tag{36}$$

Thanks to (34) and (35), we have

$$\lim_{n\to+\infty} \int_\Omega (|f_n| + b_0)\, e^{|G(u_n)|}\, |w_n| = \int_\Omega (|f| + b_0|)\, e^{|G(u)|}\, |T_{2k}(u - T_h(u))|,$$

and then

$$\lim_{h\to+\infty} \lim_{n\to+\infty} \int_\Omega (|f_n| + b_0)\, e^{|G(u_n)|}\, |w_n| = 0\,.$$

Hence,

$$\lim_{h\to+\infty}\ \lim_{n\to+\infty}\ \int_\Omega (|f_n|+b_0)\, e^{\operatorname{sign}(w_n)G(u_n)}\, w_n = 0\,. \tag{37}$$

We also have

$$\begin{aligned}
&\int_\Omega e^{\operatorname{sign}(w_n)G(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\nabla w_n \geq \\
&\geq \int_\Omega e^{\operatorname{sign}(w_n)G(T_k(u_n))}\mathbf{a}(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u)) \\
&- \int\limits_{\{k\leq|u_n|\leq M_h\}} e^{|G(u_n)|}|\mathbf{a}(x,u_n,\nabla u_n)|\,|\nabla T_k(u)|\,.
\end{aligned} \tag{38}$$

Applying now that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for any $k>0$, and since $|\nabla T_k(u)|\chi_{\{|u_n|>k\}}$ strongly converges to zero in $L^p(\Omega)$, we get (using also (A1))

$$\lim_{n\to+\infty}\ \int\limits_{\{k\leq|u_n|\leq M_h\}} e^{|G(u_n)|}|\mathbf{a}(x,u_n,\nabla u_n)|\,|\nabla T_k(u)| = 0\,. \tag{39}$$

Similarly we observe that

$$\lim_{n\to+\infty}\int_\Omega e^{\operatorname{sign}(w_n)G(T_k(u_n))}\mathbf{a}(x,T_k(u_n),\nabla T_k(u))\nabla(T_k(u_n)-T_k(u)) = 0\,. \tag{40}$$

Therefore from (36) and (37),(38),(39),(40) we obtain, letting first n then h tend to infinity,

$$\begin{aligned}
&\limsup_{h\to+\infty}\ \limsup_{n\to+\infty}\int_\Omega e^{\operatorname{sign}(w_n)G(T_k(u_n))} \\
&\qquad [\mathbf{a}(x,T_k(u_n),\nabla T_k(u_n))-\mathbf{a}(x,T_k(u_n),\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u)) \leq 0\,,
\end{aligned}$$

which yields, by (A3) and since $e^{\operatorname{sign}(w_n)G(T_k(u_n))}\geq c_k>0$,

$$\lim_{n\to+\infty}\int_\Omega [\mathbf{a}(x,T_k(u_n),\nabla T_k(u_n))-\mathbf{a}(x,T_k(u_n),\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u)) = 0\,.$$

It is well known (see e.g. [5]) that due to (A1)–(A3), this implies the strong convergence of $T_k(u_n)$ to $T_k(u)$ in $W_0^{1,p}(\Omega)$. Moreover, by a diagonal argument, it yields (up to subsequences) that ∇u_n converges to ∇u in measure, and then (extracting another subsequence, if necessary) that

$$\nabla u_n \to \nabla u \qquad \text{a.e. in} \quad \Omega \tag{41}$$

We now point out that it follows from the almost everywhere convergence of u_n and ∇u_n jointly with Fatou's Lemma that the estimates (23) and (in case (C3) is also satisfied) (25) hold for u.

Finally, observe that it also follows from (41) that

$$b_n(x,u_n,\nabla u_n)\to b(x,u,\nabla u) \qquad \text{a.e. in} \quad \Omega,$$

so that to apply Vitali's Theorem, we only have to check the equi-integrability of the sequence. To this end, take $\Psi(s)=\int_0^s \frac{g(t)}{\alpha(t)}\chi_{\{|t|>k\}}dt$ in Lemma 2.1 written for u_n, then we obtain

$$\int\limits_{\{|u_n|>k\}} e^{|G(u_n)|}g(u_n)|\nabla u_n|^p \le \int_\Omega (|f_n|+b_0)\,\Psi(u_n)e^{|G(u_n)|}\,,$$

which yields, using that $\Psi(s)\le |G(s)|\chi_{\{|s|>k\}}$,

$$\int\limits_{\{|u_n|>k\}} e^{|G(u_n)|}g(u_n)|\nabla u_n|^p\,dx \le \int\limits_{\{|u_n|>k\}} (|f_n|+b_0)\,|G(u_n)|e^{|G(u_n)|}\,dx\,. \tag{42}$$

Now, by (31), there exists $\delta>0$ such that $\theta(1+\delta)m'(p-1)\le \frac{Nm(p-1)\,(1-\theta)}{N-pm}$, so that $\Phi(u_n)$ is bounded in $L^{\theta(p-1)(1+\delta)m'}(\Omega)$ thanks to the estimate (23). Since $|G(s)|\le \frac{1}{\delta}e^{\delta|G(s)|}$ we have, using (C2) and Hölder's inequality,

$$\begin{aligned}
\int\limits_{\{|u_n|>k\}} (|f_n|+b_0)\,|G(u_n)|e^{|G(u_n)|} &\le \frac{1}{\delta}\int\limits_{\{|u_n|>k\}} (|f_n|+b_0)\,e^{(1+\delta)|G(u_n)|}\\
&\le \frac{M}{\delta}\int\limits_{\{|u_n|>k\}} (|f_n|+b_0)\,(1+|\Phi(u_n)|)^{\theta(p-1)(1+\delta)}\\
&\le \frac{M}{\delta}\left(\int\limits_{\{|u_n|>k\}} (|f_n|+b_0)^m\right)^{\frac{1}{m}}\left(\int_\Omega (1+|\Phi(u_n)|)^{\theta(p-1)(1+\delta)m'}\right)^{\frac{1}{m'}}\\
&\le c\left(\int\limits_{\{|u_n|>k\}} (|f_n|+b_0)^m\right)^{\frac{1}{m}}.
\end{aligned}$$

Then we deduce from (42)

$$\sup_{n\in I\!N}\int\limits_{\{|u_n|>k\}} e^{|G(u_n)|}g(u_n)|\nabla u_n|^p\,dx \le c\sup_{n\in I\!N}\left(\int\limits_{\{|u_n|>k\}} |f_n+b_0|^m\,dx\right)^{\frac{1}{m}},$$

which in particular implies, since $(|f_n|+b_0)$ strongly converges in $L^m(\Omega)$,

$$\lim_{k\to+\infty}\sup_{n\in I\!N}\int\limits_{\{|u_n|>k\}} g(u_n)|\nabla u_n|^p = 0\,. \tag{43}$$

Next, being $E\subset\Omega$, it yields:

$$\begin{aligned}
\int_E g(u_n)|\nabla u_n|^p &\le \int_{E\cap\{|u_n|<k\}} g(u_n)|\nabla u_n|^p + \int_{E\cap\{|u_n|>k\}} g(u_n)|\nabla u_n|^p\\
&\le \int_E g(u_n)|\nabla T_k(u_n)|^p + \int_{\{|u_n|>k\}} g(u_n)|\nabla u_n|^p.
\end{aligned}$$

From here, by the strong convergence of $T_k(u_n)$ to $T_k(u)$ in $W_0^{1,p}(\Omega)$ and (43), we obtain that $g(u_n)|\nabla u_n|^p$ strongly converges in $L^1(\Omega)$, so that by means of (B) we deduce that

$$b_n(x,u_n,\nabla u_n)\to b(x,u,\nabla u) \qquad \text{strongly in } L^1(\Omega).$$

■

Let us see that the previous proof also applies to the limiting case $m=\frac{N}{p}$ under the weaker assumption (C1).

Theorem 4.4 *Assume (A1), (A2), (A3), (B), (C1) and (F), with* $m=\frac{N}{p}$. *Let* u_n *be a sequence of solutions of (30), then there exists a function* u *such that, up to subsequences,*

$$\begin{aligned} &T_k(u_n)\to T_k(u) \qquad \text{strongly in} \quad W_0^{1,p}(\Omega), \quad \forall k>0,\\ &b_n(x,u_n,\nabla u_n)\to b(x,u,\nabla u) \qquad \text{strongly in} \quad L^1(\Omega). \end{aligned}$$

Moreover, the function u *is such that* $\Phi(u)\in W_0^{1,p}(\Omega)\cap L^q(\Omega)$ *for all* $q\in[1,+\infty[$.

Proof. Thanks to Theorem 4.1, we have that $\Phi(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, hence, up to subsequences, we have that it strongly converges in $L^p(\Omega)$ and almost everywhere. Since Φ is bijective, this immediately implies (33). Moreover, using (C1) and the estimate on $\Phi(u_n)$ in Theorem 4.1, we also have that $(f_n+b_0)e^{|G(u_n)|}$ is bounded in $L^\gamma(\Omega)$ for some $\gamma>1$, which together with the almost everywhere convergence of u_n implies (34). We have then both ingredients used in the previous proof to obtain that (always up to subsequences)

$$T_k(u_n)\to T_k(u) \qquad \text{strongly in } W_0^{1,p}(\Omega) \text{ for any } k>0.$$

Finally, as in Theorem 4.3 we obtain (42) which implies, for any $\delta>0$:

$$\int_{\{|u_n|>k\}} e^{|G(u_n)|}g(u_n)|\nabla u_n|^p\,dx \le$$

$$\le \frac{c}{\delta}\left(\int_{\{|u_n|>k\}}(|f_n|+b_0)^{\frac{N}{p}}\right)^{\frac{p}{N}}\left(\int_\Omega(1+|\Phi(u_n)|)^{(p-1)(1+\delta)\frac{N}{N-p}}\right)^{1-\frac{p}{N}}.$$

Hence, the estimate on $\Phi(u_n)$ in $L^q(\Omega)$, for all $q<+\infty$, gives

$$\int_{\{|u_n|>k\}} e^{|G(u_n)|}g(u_n)|\nabla u_n|^p\,dx \le C\left(\int_{\{|u_n|>k\}}(|f_n|+b_0)^{\frac{N}{p}}\right)^{\frac{p}{N}}.$$

As in Theorem 4.3, we get (43) which in turns implies that $b_n(x,u_n,\nabla u_n)$ strongly converges in $L^1(\Omega)$. ■

Proof of Theorems 2.2 and 2.3: Let $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and take $T_k(u_n-v)$ as test function in the weak formulation of (30), then

$$\begin{aligned} &\int_\Omega \mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n-v)+\int_\Omega b_n(x,u_n,\nabla u_n)T_k(u_n-v)=\\ &\qquad =\int_\Omega f_nT_k(u_n-v) \end{aligned} \tag{44}$$

holds for all $n \in I\!N$. We may take limits in the right hand side and in the second term of the left hand side. To take limits in the first term, let $K = k + \|v\|_\infty$. Then

$$\mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - v) = \mathbf{a}(x, T_K(u_n), \nabla T_K(u_n)) \cdot \nabla T_k(T_K(u_n) - v).$$

On the other hand, Theorems 4.3, when $1 < m < N/p$, and 4.4, when $m = N/p$, imply $\nabla T_k(T_K(u_n) - v) \to \nabla T_k(T_K(u) - v)$ in $L^p(\Omega)^N$ and $\mathbf{a}(x, T_K(u_n), \nabla T_K(u_n)) \to \mathbf{a}(x, T_K(u), \nabla T_K(u))$ in $L^{p'}(\Omega)^N$, so that

$$\mathbf{a}(x, T_K(u_n), \nabla T_K(u_n)) \cdot \nabla T_k(T_K(u_n) - v) \to \mathbf{a}(x, T_K(u), \nabla T_K(u)) \cdot \nabla T_k(T_K(u) - v)$$

in $L^1(\Omega)$. Therefore, taking limits in (44), we conclude that u is an entropy solution of (5). Moreover, it clearly satisfies the desired estimates. ■

5 Examples and Remarks

In this last Section we give examples concerning the optimality of our assumptions. We restrict ourselves to the case $p = 2$, and assume $N > 2$.

Example 5.1 The optimality of assumption (C1) for having bounded solutions is somehow showed by the classical example of Kazdan and Kramer ([17], see also [13]), as discussed in Remark 3.1 as well. In view of Theorem 2.1, one could wonder what happens to the bounded solutions of

$$\begin{cases} -\Delta u_\epsilon = \frac{1}{(1+|u_\epsilon|)^\epsilon} |\nabla u_\epsilon|^2 + f, & \text{in } \Omega \\ u_\epsilon = 0, & \text{on } \partial\Omega \end{cases} \tag{45}$$

as ϵ tends to zero. Let f be bounded and sufficiently large, for instance $f > \lambda_1$ (as in the counterexample by Kazdan and Kramer [17]), then the sequence u_ϵ blows up everywhere in Ω. Indeed, note that u_ϵ is non negative and let $v_\epsilon = \int_0^{u_\epsilon} e^{\frac{(1+s)^{1-\epsilon}-1}{1-\epsilon}} ds$, which solves

$$-\Delta v_\epsilon = \psi_\epsilon(u_\epsilon) \, f \,, \qquad \text{with} \quad \psi_\epsilon(t) = e^{\frac{(1+t)^{1-\epsilon}-1}{1-\epsilon}} \,.$$

It is easy to see that (any subsequence of) v_ϵ cannot be bounded in $L^\infty(\Omega)$, otherwise standard compactness arguments would imply that there exists a limit function v which solves $-\Delta v = f\,(v+1) \geq \lambda_1 \, v + f$, and this is impossible since f is positive. On the other hand, since $\psi_\epsilon(u_\epsilon) \leq (v_\epsilon + 1)$, a bootstrap regularity argument implies that if $\psi_\epsilon(u_\epsilon)$ is bounded in $L^1(\Omega)$ (or even in $L^1(\Omega, \delta(x))$, $\delta(x) = \text{dist }(x, \partial\Omega)$) then v_ϵ is bounded in $L^\infty(\Omega)$. The conclusion is that $\psi_\epsilon(u_\epsilon)$ is not bounded in $L^1(\Omega, \delta(x))$, and since we have (using the representation of v_ϵ through convolution, see e.g. [9])

$$v_\epsilon(x) \geq \delta(x) \int_\Omega f \, \psi_\epsilon(u_\epsilon(y)) \delta(y) dy \,, \qquad \delta(x) = \text{dist }(x, \partial\Omega) \,,$$

we deduce that $v_\epsilon(x)$ tends to infinity for every $x \in \Omega$, hence the same holds for u_ϵ. ■

The value of θ in assumption (C2) also plays a crucial role. Actually, if (C2) is satisfied with $\theta \geq \frac{2^*}{2m'}$, complete blow up of approximating solutions may occur for data $f \in L^m(\Omega)$. To point out this feature, we restrict to a model example with $b(x,s,\xi) = \frac{\lambda}{1+|s|}|\xi|^2$, which satisfies (C2) (even (C3)) with $\theta = \frac{\lambda}{\lambda+1}$. In this case, observe that formally u is a positive solution of

$$\begin{cases} -\Delta u = \frac{\lambda}{1+|u|}|\nabla u|^2 + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{46}$$

if and only if the function $v = \frac{(1+u)^{1+\lambda}-1}{1+\lambda}$ is a solution of the semilinear problem

$$\begin{cases} -\Delta v = f\,((\lambda+1)v+1)^\theta & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{47}$$

In case of unbounded data f, problems of the type (47) have been considered in [7] where the value $\theta = \frac{2^*}{2m'}$ also appeared as the borderline for having solutions with $f \in L^m(\Omega)$. We proved in Theorem 2.3 that problem (46) has a solution if $\lambda < \lambda_m := \frac{N(m-1)}{N-2m}$ (note that $\frac{\lambda_m}{\lambda_m+1} = \frac{2^*}{2m'}$). In virtue of Theorem 2.1, we can always consider a sequence of bounded solutions u_n of

$$\begin{cases} -\Delta u_n = \frac{\lambda}{1+|u_n|}|\nabla u_n|^2 + T_n(f) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{48}$$

In the following example, we deal with the case that $\lambda > \frac{N(m-1)}{N-2m}$ in (48), which corresponds to $\theta > \frac{2^*}{2m'}$ in assumption (C2). We prove that if $f \in L^m(\Omega)$ but not to $L^{m+\delta}(\Omega)$ for any $\delta > 0$, then the solutions u_n blow up everywhere in Ω; this can be essentially deduced from a counterexample given by L. Orsina ([24]) for problem (47).

Example 5.2 In order to simplify, we assume that Ω is a ball and f is a positive radial nonincreasing function. Let u_n be a solution of problem (48); observe that $u_n \geq 0$, since $f \geq 0$. Setting $v_n = \frac{(1+u_n)^{1+\lambda}-1}{1+\lambda}$, we have that v_n satisfies

$$-\Delta v_n \geq T_n(f)\,(1+u_n)^\lambda \geq T_n(f)\,v_n^\theta\,,$$

with $\theta = \frac{\lambda}{\lambda+1}$. Now, let $\epsilon_n = f^{-1}(n)$; since f is decreasing, it follows that $T_n(f) = n$ in the ball $B_{\epsilon_n}(0)$ and so

$$-\Delta v_n \geq n\,v_n^\theta \qquad \text{in } B_{\epsilon_n}(0).$$

If φ_1 is the first eigenfunction of the Laplacian (with Dirichlet boundary conditions) in the unit ball, we have that there exists a constant γ such that the function $z_n = \gamma\,(\epsilon_n^2\,n)^{\frac{1}{1-\theta}}\varphi_1(\frac{x}{\epsilon_n})$ satisfies

$$-\Delta z_n \leq n\,z_n^\theta \qquad \text{in } B_{\epsilon_n}(0),$$

so that by comparison (see e.g. [10]) we get $v_n \geq z_n$ in $B_{\epsilon_n}(0)$, and in particular $v_n \geq \gamma'\,(\epsilon_n^2\,n)^{\frac{1}{1-\theta}}$ in $B_{\frac{\epsilon_n}{2}}(0)$ for a possibly different constant γ'. Finally, since we have

$$v_n(x) \geq \int_\Omega T_n(f(y))v_n^\theta(y)\,G(x,y)dy\,,$$

where $G(x,y)$ is the Green function of the Laplacian, we deduce (for a positive constant c_0)

$$v_n(x) \geq \int_{B_{\frac{\epsilon_n}{2}}(0)} T_n(f(y))v_n^\theta(y)\, G(x,y)dy \geq c_0(\epsilon_n^2\, n)^{\frac{\theta}{1-\theta}} n\, \epsilon_n^N = c_0(\epsilon_n^N\, n^\beta)^{1+\frac{2\theta}{(1-\theta)N}}\,, \tag{49}$$

where $\beta = \frac{1+\frac{\theta}{1-\theta}}{1+\frac{2\theta}{(1-\theta)N}}$. Now, if f is a function which belongs to $L^m(\Omega)$ but not to $L^{m+\delta}(\Omega)$ for any $\delta > 0$, we have

$$n^\beta \quad \big|\{x\, :\, |f(x)| > n\}\big| \to +\infty \qquad \text{for any } \beta > m.$$

Thus, using the definition of ϵ_n (recall that f is radial and decreasing), we have $\epsilon_n^N\, n^\beta$ tends to infinity as soon as $\beta > m$. Since $\beta > m$ if and only if $\theta > \frac{2^*}{2m'}$, we deduce that, for this range of values of θ, $v_n(x)$ goes to infinity for every $x \in \Omega$, which implies, for the solutions u_n of (48),

$$u_n(x) \to \infty \qquad \text{for every } x \in \Omega.$$

Therefore, if $\lambda > \frac{N(m-1)}{N-2m}$, and if f is a positive nonincreasing radial function which belongs to $L^m(\Omega)$ but not to $L^{m+\delta}(\Omega)$ for any $\delta > 0$, every sequence of solutions of (48) blows up everywhere in Ω. ■

The case $\lambda = \frac{N(m-1)}{N-2m}$ in (46), which corresponds to asking $\theta = \frac{2^*}{2m'}$ in assumption (C2) or in (47), is more delicate and the previous argument does not apply. Indeed, we point out that in this borderline case existence can still be possible for some functions in $L^m(\Omega)$ which are not in $L^{m+\delta}(\Omega)$ for any $\delta > 0$, for instance functions belonging to some Orlicz space $L^m \log L^\alpha(\Omega)$. However, a general solvability result for f in $L^m(\Omega)$ cannot hold: next examples show that there exist functions in $L^m(\Omega)$ for which approximate solutions blow–up everywhere and no solution of (46) is expected with $\lambda = \frac{N(m-1)}{N-2m}$, showing the optimality of our results in the class of Lebesgue spaces.

We give first a general nonexistence result, in the radial case, for sublinear equations as (47).

Lemma 5.1 *Let* $\theta = \frac{2^*}{2m'}$ *and assume that*

$$\exists\, M > 0 \;: \quad f(x) \geq M\, |x|^{-\frac{N}{m}} \,(-\log|x|)^{-1}\,, \quad \textit{in a neighborhood of } x = 0. \tag{50}$$

Then there is no positive radial function $w(|x|) \in C^1(0,R)$ *which solves*

$$-\Delta w = f\, w^\theta \quad \textit{in } B_R(0) \setminus \{0\}. \tag{51}$$

Proof. Let $\rho = |x|$ and assume that $w = w(\rho) \in C^1(0,R)$ is a positive solution of (51), hence $-(w'\,\rho^{N-1})' = \rho^{N-1}\, f\, w^\theta$. Since $w'\,\rho^{N-1}$ is decreasing, it admits a limit at $\rho = 0$, say ℓ. This limit cannot be positive, otherwise (for ρ close to 0)

$$w(\rho) \geq \int_0^\rho w'(\xi)\, d\xi \geq \int_0^\rho \frac{\ell}{2\xi^{N-1}}\, d\xi = +\infty.$$

Thus we deduce that $\ell \leq 0$; on the one hand, it implies that the limit is finite and so $f\, w^\theta$ is integrable in a neighborhood of $x = 0$. On the other, $w' \leq 0$ and then w is decreasing and satisfies the key inequality (for shortness we take $M = 1$ in (50)):

$$-w' \geq \frac{1}{\rho^{N-1}} \int_0^\rho \xi^{N-1-\frac{N}{m}} (-\log \xi)^{-1}\, w(\xi)^\theta \, d\xi \,. \tag{52}$$

Without loss of generality we can assume that $w \geq 1$ on $(0, \frac{1}{4})$. Then, using that

$$\text{if } \gamma > 0,\ \beta > -1, \text{ then } \int_0^\rho \xi^\beta (-\log \xi)^{-\gamma} d\xi \geq \frac{\rho^{\beta+1}}{\beta+1+\gamma} (-\log \rho)^{-\gamma} \quad \forall \rho \leq \tfrac{1}{e}, \tag{53}$$

we get

$$-w' \geq \frac{\rho^{1-\frac{N}{m}} (-\log \rho)^{-1}}{\frac{N}{m'} + 1} \qquad \forall \rho \in (0, \frac{1}{4}),$$

which yields, integrating and applying that $(-\log \rho)^{-1}$ is an increasing function,

$$w(\rho) \geq -\int_\rho^{\frac{1}{4}} w' d\xi \geq (-\log \rho)^{-1} \int_\rho^{\frac{1}{4}} \frac{\xi^{1-\frac{N}{m}}}{(\frac{N}{m'} + 1)} d\xi$$

We deduce that

$$w(\rho) \geq \frac{(-\log \rho)^{-1} \rho^{-(\frac{N}{m} - 2)}}{2(\frac{N}{m} - 2)(\frac{N}{m'} + 1)} \quad \forall \rho \in \left(0, \frac{1}{4} 2^{-\frac{1}{\frac{N}{m} - 2}}\right) . \tag{54}$$

Set $\lambda = \frac{N}{m} - 2$ (note that $\lambda > 0$ since $m < \frac{N}{2}$): we can use again (54) into (52) and get, with same arguments, a refined estimate on u from below. By induction, we get in fact the following estimate:

$$w(\rho) \geq \frac{\rho^{-\lambda \sum\limits_{k=0}^{n} \theta^k} (-\log \rho)^{-\sum\limits_{k=0}^{n} \theta^k}}{(2\lambda)^{\sum\limits_{k=0}^{n} \theta^k} \left(\frac{N}{m'} + 1\right)^{\theta^n} \prod\limits_{k=1}^{n} \left[\left(\sum\limits_{j=0}^{k} \theta^j \right)^{\theta^{n-k}} \left(\frac{N}{m'} + \sum\limits_{j=0}^{k} \theta^j - \lambda \sum\limits_{j=1}^{k} \theta^j \right)^{\theta^{n-k}} \right]}$$

$$\text{if} \qquad \rho < \frac{1}{4}\, 2^{-\frac{1}{\lambda} \sigma_n} \qquad \sigma_n = \sum_{k=0}^{n} \frac{1}{\sum\limits_{j=0}^{k} \theta^j} \,.$$

Since $\theta < 1$, we have

$$\log \left(\tfrac{N}{m'} + 1\right)^{\theta^n} + \sum_{k=1}^{n} \log \left[\left(\sum_{j=0}^{k} \theta^j \right)^{\theta^{n-k}} \left(\frac{N}{m'} + \sum_{j=0}^{k} \theta^j - \lambda \sum_{j=1}^{k} \theta^j \right)^{\theta^{n-k}} \right] \leq$$
$$\leq \sum_{k=0}^{n} \theta^{n-k} \left[\log(\tfrac{1}{1-\theta}) + \log(\tfrac{N}{m'} + 1 + (1-\lambda)^+ \tfrac{\theta}{1-\theta} \right] \leq C$$

and so it follows that

$$w(\rho) \geq C\, \rho^{-\lambda \sum\limits_{k=0}^{n} \theta^k} (-\log \rho)^{-\sum\limits_{k=0}^{n} \theta^k}$$

$$\text{if} \qquad \rho < \frac{1}{4}\, 2^{-\frac{1}{\lambda}\,\sigma_n} \qquad\qquad \sigma_n = \sum_{k=0}^{n} \frac{1}{\sum\limits_{j=0}^{k} \theta^j}\,.$$

Note that $\sigma_n \le n+1$, so that we deduce

$$w(\rho)\rho^{\frac{\lambda}{1-\theta}}\,(-\log\rho)^{\frac{1}{1-\theta}} \ge C\rho^{\frac{\lambda\theta^{n+1}}{1-\theta}}(-\log\rho)^{\frac{\theta^{n+1}}{1-\theta}}\,, \qquad \text{if } \rho\in(0,\tfrac14\,2^{-\frac{n+1}{\lambda}}),$$

and in particular, if $\rho \in (\frac14\,2^{-\frac{n+2}{\lambda}}, \frac14\,2^{-\frac{n+1}{\lambda}})$, then

$$w(\rho)\rho^{\frac{\lambda}{1-\theta}}\,(-\log\rho)^{\frac{1}{1-\theta}} \ge C(\frac{1}{4}\,2^{-\frac{n+2}{\lambda}})^{\frac{\lambda\theta^{n+1}}{1-\theta}}(-\log(\frac{1}{4}\,2^{-\frac{n+1}{\lambda}}))^{\frac{\theta^{n+1}}{1-\theta}}\,. \tag{55}$$

Since $(0,\frac14\,2^{-\frac{1}{\lambda}}) = \bigcup\limits_{n=0}^{\infty}(\frac14\,2^{-\frac{n+2}{\lambda}},\frac14\,2^{-\frac{n+1}{\lambda}})$, and the right hand side in (55) has a positive limit as n goes to infinity, we conclude that there exist $\rho_0>0$ and a constant c_0 such that

$$w(\rho) \ge c_0\,\rho^{-\frac{\lambda}{1-\theta}}\,(-\log\rho)^{-\frac{1}{1-\theta}} = c_0\,\rho^{2-N}\,(-\log\rho)^{-\frac{1}{1-\theta}} \qquad \forall\rho\in(0,\rho_0)\,. \tag{56}$$

Again we can use this information in (52) which now implies

$$-w' \ge \frac{c_0^\theta}{\rho^{N-1}}\int_0^\rho \frac{1}{\xi}\,(-\log\xi)^{-\frac{1}{1-\theta}}d\xi = \frac{c_0^\theta(-\log\rho)^{1-\frac{1}{1-\theta}}}{\rho^{N-1}(\frac{1}{1-\theta}-1)}.$$

Taking into account that $(-\log\rho)^{1-\frac{1}{1-\theta}}$ is increasing, it yields

$$\begin{aligned} w(\rho) &\ge \tfrac{c_0^\theta(-\log\rho)^{1-\frac{1}{1-\theta}}}{(\frac{1}{1-\theta}-1)}\int_\rho^{\rho_0}\tfrac{1}{\xi^{N-1}}d\xi \\ &\ge \tfrac{1}{2(N-2)}\,\tfrac{c_0^\theta(-\log\rho)^{1-\frac{1}{1-\theta}}}{\rho^{N-2}(\frac{1}{1-\theta}-1)} \qquad \text{if } \rho\in(0,\rho_0\,2^{\frac{1}{2-N}}). \end{aligned}$$

Iterating this estimate by using (52) we can again deduce by induction that

$$w(\rho) \ge \frac{c_0^{\theta^{n+1}}\,\rho^{2-N}\,(-\log\rho)^{-\frac{1}{1-\theta}+\sum\limits_{k=0}^{n}\theta^k}}{(2(N-2))^{\sum\limits_{k=0}^{n}\theta^k}\prod\limits_{k=0}^{n}\left(\frac{1}{1-\theta}-\sum\limits_{j=0}^{k}\theta^j\right)^{\theta^{n-k}}}$$

if $\rho\in(0,2^{-\frac{n+1}{N-2}}\,\rho_0)$. Note that we have $\frac{1}{1-\theta} > \sum\limits_{j=0}^{k}\theta^j$ for every k. In particular we get, if $\rho\in(2^{-\frac{n+2}{N-2}}\,\rho_0, 2^{-\frac{n+1}{N-2}}\,\rho_0)$

$$w(\rho)\,\rho^{N-2} \ge \frac{c_0^{\theta^{n+1}}\,(-\log\rho_0 + \frac{n+2}{N-2}\log 2)^{-\frac{\theta^{n+1}}{1-\theta}}}{(2(N-2))^{\sum\limits_{k=0}^{n}\theta^k}\prod\limits_{k=0}^{n}\left(\frac{\theta^{k+1}}{1-\theta}\right)^{\theta^{n-k}}} \ge c_1\,.$$

Thus, we obtain that $w \ge c_1\,\rho^{2-N}$ in $(0,2^{-\frac{1}{N-2}}\rho_0)$, which due to (52) implies

$$-w'(\rho) \ge \frac{c_1^\theta}{\rho^{N-1}}\int_0^\rho \frac{(-\log\xi)^{-1}}{\xi}d\xi \equiv +\infty\,.$$

Therefore, a positive solution w cannot exist. ■

Note that, in the previous statement, w is just taken to be a solution of the equation in $B_R(0) \setminus \{0\}$, and that no requirement is made *a priori* on its behaviour near $x = 0$. Simply, the singularity of f is not allowed in the equation. By comparing with radial solutions, the above proof implies the following nonexistence result for distributional solutions, and, in particular, gives evidence of the optimality of the existence results in [7].

Corollary 5.1 *Let Ω be a bounded open set containing $x = 0$. Let $\theta = \frac{2^*}{2m'}$ and assume that f satisfies (50). Then there is no positive function $w \in L^1(\Omega)$ such that $fw^\theta \in L^1(\Omega)$ and*

$$-\Delta w = f\, w^\theta \quad \text{in } \mathcal{D}'(\Omega).$$

Proof. Since $w > 0$, up to rescaling we can assume that $w \geq 1$ in a ball $B_R(0)$ and satisfies $-\Delta w \geq f$. By comparison principle for the Laplace equation we deduce $w \geq G(f)$ in $B_R(0)$, where we have denoted $G(f)$ the Green operator: $G = (-\Delta)^{-1}$ in $B_R(0)$ with Dirichlet boundary conditions. It follows again that

$$-\Delta w \geq fG(f)^\theta \qquad \text{in } B_R(0),$$

hence $w \geq G(fG(f)^\theta)$. Thus, defining the operator $T(z) = fG(z)^\theta$, we simply have by induction that $w \geq G(T^n(f))$ for any $n \geq 0$ (we only use the comparison principle for distributional solutions of Laplace operator with L^1–data). Now, it should be clear that this is precisely what we did in the proof of Lemma 5.1 for the radial case; in particular, using (50) and the fact that both G and T are monotone operators, we can estimate $G(T^n(f))$ in terms of radial solutions; for instance, (54) gives an estimate for $G(f)$ which can be used to obtain an estimate on $G(T(f)) = G(f\, G(f)^\theta))$ and so on. First we obtain (56), i.e.

$$w(x) \geq \psi(x) \, := c_0 \, |x|^{2-N} \, (-\log|x|)^{-\frac{1}{1-\theta}} \chi_{B_{\rho_0}(0)} \, ,$$

then iterating again we get $w \geq G(T^n(f\, \psi^\theta))$, so that

$$\int_\Omega f\, w^\theta dx \geq \int_\Omega T^{n+1}(f\, \psi^\theta)\, dx \, .$$

Estimating the right hand side with radial solutions we obtain as in Lemma 5.1 that the right hand side integral goes to infinity, which gives a contradiction with the assumption $f\, w^\theta \in L^1(\Omega)$. ■

We immediately deduce the counterpart for problem (46).

Corollary 5.2 *Let $\lambda = \frac{N(m-1)}{N-2m}$ and assume that (50) holds true. Then there is no positive radial function $u(|x|) \in C^1(0, R)$ which solves*

$$-\Delta u = \lambda \frac{|\nabla u|^2}{1 + |u|} + f \quad \text{in } B_R(0) \setminus \{0\}.$$

Moreover if Ω is a bounded open set containing 0, then (46) has no solution u such that $|\nabla u|^2 (1 + u)^{\lambda - 1}$ and fu^λ belong to $L^1(\Omega)$.

In virtue of the previous result, we can easily deduce a complete blow–up result for solutions of (48) if $f(x)$ satisfies (50).

Corollary 5.3 *If f satisfies (50) and $\lambda = \frac{N(m-1)}{N-2m}$, then the solutions of (48) blow–up everywhere in Ω.*

Proof. Setting, as in Example 5.2, $v_n = \frac{(1+u_n)^{1+\lambda}-1}{1+\lambda}$, v_n solves

$$-\Delta v_n \geq T_n(f)\, v_n^\theta \,.$$

Let $g(x) = |x|^{-\frac{N}{m}}(-\log|x|)^{-1}$ and let w_n be the unique (and therefore radial) positive solution of

$$-\Delta w_n = T_n(g)\, w_n^\theta \,, \qquad \text{in } B_R(0),\ w_n = 0 \text{ on } \partial B_R(0). \tag{57}$$

By comparison, if f satisfies (50) (without lost of generality take $M = 1$ and assume that (50) holds in $B_R(0)$), we have $v_n \geq w_n$. Since the sequence w_n is increasing, there exists a (possibly infinite valued) function w such that $w_n(x)$ pointwise converges to $w(x)$, so that by monotone convergence's theorem, we have

$$\lim_{n\to+\infty} \int_{B_{\frac{R}{2}}(0)} T_n(g) w_n^\theta \, dx = \int_{B_{\frac{R}{2}}(0)} g\, w^\theta \, dx \,. \tag{58}$$

Assume by contradiction that the right hand side is finite. Since w_n satisfies

$$-w_n' = \frac{1}{\rho^{N-1}} \int_0^\rho T_n(g)(\xi)\, w_n(\xi)^\theta\, \xi^{N-1}\, d\xi \,, \tag{59}$$

in particular it is possible to pass to the limit in (59), so that w is a radial function belonging to $C^1(0, R)$ which solves

$$-\Delta w = |x|^{-\frac{N}{m}}(-\log|x|)^{-1}\, w^\theta \,, \quad |x| \in (0, R).$$

By Lemma 5.1, this is not possible, then we conclude that

$$\int_{B_{\frac{R}{2}}(0)} T_n(g)\, w_n^\theta \, dx \to +\infty \,.$$

Since again we have the integral representation

$$w_n(x) \geq \delta(x) \int_\Omega T_n(g)\, w_n^\theta\, \delta(y)\, dy \qquad \delta(x) = \text{dist }(x, \partial\Omega)\,,$$

we deduce that $w_n(x)$ converges to $+\infty$ for every x in Ω. By comparison we also have that $v_n(x)$ blows-up completely, and therefore the sequence of solutions u_n of (48) as well. ■

Note that if $f(x) = |x|^{-\frac{N}{m}}(-\log|x|)^{-\alpha}$, then f belongs to $L^m(\Omega)$ if $\alpha > \frac{1}{m}$, and that the nonexistence result for (46) (or complete blow–up for approximating solutions of (48)) holds for $\alpha \leq 1$. In view of Lemma 5.1, it seems that the value $\alpha = 1$ represents again a borderline, and we believe that solutions exist if $\alpha > 1$.

Finally, in the same spirit we provide a generic counterexample in case $m = 1$, i.e. $f \in L^1(\Omega)$. This case was treated in [27] assuming that the function g in (1) is integrable. Again, we show that this condition cannot be improved in order to have solutions for any L^1–datum.

Proposition 5.1 *Let $g : I\!R \to I\!R^+$ be a positive continuous function, and suppose that it is nonincreasing on $I\!R^+$ and such that $\int_0^{+\infty} g(s)ds = +\infty$. Then there exists a nonnegative radial function $f \in L^1(B_R(0)) \cap C^1(B_R(0) \setminus \{0\})$ such that the problem*

$$-\Delta u \geq g(u)|\nabla u|^2 + f \quad \text{in } B_R(0) \setminus \{0\}$$

does not have any nonnegative radial solution $u \in C^1(B_R(0) \setminus \{0\})$.

The proof of Proposition 5.1 will follow from a simple result on semilinear equations.

Proposition 5.2 *Let $h : I\!R \to I\!R^+$ be a positive function. Assume that it is nondecreasing, concave, and such that $\lim_{s\to+\infty} h(s) = +\infty$. Then*

(i) there exists a nonnegative radial function $f \in L^1(B_R(0)) \cap C^1(B_R(0) \setminus \{0\})$ such that problem

$$-\Delta w \geq h(w)\, f \quad \text{in } B_R(0) \setminus \{0\} \tag{60}$$

does not have any nonnegative radial solution $w \in C^1(B_R(0) \setminus \{0\})$.

(ii) there exists a nonnegative $f \in L^1(\Omega)$ such that the problem

$$-\Delta w = h(w)\, f \qquad \text{in } \mathcal{D}'(\Omega) \tag{61}$$

does not have any nonnegative solution $w \in L^1(\Omega)$ satisfying $fh(w) \in L^1(\Omega)$.

Proof. By contradiction, let w be a nonnegative radial solution of (60). As in the proof of Lemma 5.1, we have that w is nonincreasing and satisfies

$$-w' \geq \frac{1}{\rho^{N-1}} \int_0^\rho \xi^{N-1} f(\xi)\, h(w(\xi))\, d\xi\,. \tag{62}$$

Without loss of generality, let $w \geq 1$ in $(0, \rho_0)$, so that

$$-w' \geq \frac{h(1)}{\rho^{N-1}} \int_0^\rho \xi^{N-1} f(\xi)\; d\xi.$$

Let $\phi(t)$ define a C^1, bounded and increasing function; in particular, we may take $\phi \leq 0$ and $\lim_{t\to+\infty} \phi(t) = 0$. Now set $f(\xi) = \xi^{-N}\phi'(-\log \xi)$, and so

$$\int_0^R \xi^{N-1} f(\xi)\, d\xi = \int_0^R \xi^{-1}\phi'(-\log \xi)\, d\xi = -\phi(-\log R);$$

hence, $f \in L^1(B_R(0))$. With this f, we get

$$-w' \geq \frac{h(1)}{\rho^{N-1}} (-\phi(-\log \rho))\,,$$

which gives, since ϕ is increasing,

$$w(\rho) \geq \int_\rho^{\rho_0} \frac{h(1)}{\xi^{N-1}} (-\phi(-\log \xi)) d\xi \geq h(1)\,(-\phi(-\log \rho)) \int_\rho^{\rho_0} \xi^{1-N} d\xi\,.$$

We deduce that there exists a constant c_1 and a neighborhood $(0,\rho_1)$ such that

$$w(\rho) \geq c_1\,(-\phi(-\log\rho))\,\rho^{2-N}$$

so that (62) yields

$$-w' \geq \frac{1}{\rho^{N-1}}\int_0^\rho \phi'(-\log\xi)\xi^{-1}\,h(c_1(-\phi(-\log\xi))\xi^{2-N})\,d\xi\,. \tag{63}$$

Now observe that, since h is positive and concave on $I\!R^+$, we deduce $h(s)\geq h'(s)s$ for all $s\geq 0$. Moreover, it follows that $\log h$ is also concave on $I\!R^+$; thus, for any $0\leq\lambda<1$, we obtain

$$\log h(s)\leq \log h(\lambda s)+(1-\lambda)s\frac{h'(\lambda s)}{h(\lambda s)}\leq \log h(\lambda s)+\frac{1-\lambda}{\lambda}\qquad \forall s\geq 0.$$

As a consequence, for every $0\leq\lambda<1$ and $s\geq 0$, we have

$$\frac{h(\lambda s)}{h(s)}\geq \exp\left(\frac{\lambda-1}{\lambda}\right).$$

Applying these inequalities to (63) and having in mind that ϕ is bounded, it yields

$$\begin{aligned}-\,w' &\geq \frac{c_2}{\rho^{N-1}}\int_0^\rho \phi'(-\log\xi)\xi^{-1}\,\exp(\frac{1}{\phi(-\log\xi)})\,h(\xi^{2-N})\,d\xi = \\ &= \frac{c_2}{\rho^{N-1}}\int_{-\log\rho}^{+\infty}\phi'(t)e^{\frac{1}{\phi(t)}}h(e^{(N-2)t})dt\,.\end{aligned}$$

Choose for instance ϕ such that

$$\int_{\phi(t)}^0 e^{\frac{1}{\xi}}d\xi=\frac{1}{h(e^{(N-2)t})}\,.$$

Then we have that ϕ is a bounded increasing function and $\lim_{t\to+\infty}\phi(t)=0$ since h is unbounded; moreover

$$\int^{+\infty}\phi'(t)e^{\frac{1}{\phi(t)}}h(e^{(N-2)t})dt=\int^{+\infty}\frac{h'(s)}{h(s)}ds=+\infty\,,$$

so that we get $-w'\geq+\infty$; this proves that no solution w exists with $f=\xi^{-N}\phi'(-\log\xi)$, and (i) is proved.

In order to prove (ii), it is enough to observe that any solution (possibly non radial) of (61) is positive in a ball $B\subset\Omega$, so it satisfies $w\geq c_0G(f)$ for a positive constant c_0, where $G(f)$ is the Green operator as in Corollary 5.1. In particular there exists a ball $B\subset\Omega$ such that

$$-\Delta w\geq fh(c_0G(f))\qquad\text{in } B,$$

with $fh(c_0G(f))\leq fh(w)\in L^1(\Omega)$. Thus we can use the above contruction in B: if f is (greater than) the radial function constructed above, we have $G(fh(c_0G(f)))\equiv+\infty$, and since $w\geq G(fh(c_0G(f)))$ we deduce that such a w cannot exist.

∎

Proof of Proposition 5.1. As in (8) and (10), consider $G(s) = \int_0^s g(t)dt$ and $\Phi(s) = \int_0^t \exp\big(G(s)\big)\,ds$. Set now $w(\rho) = \Phi\big(u(\rho)\big)$, where $\rho = |x|$. Then $w \in C^1(0,R)$ and solves

$$-\Delta w \geq fh(w) \quad \text{in} \quad B_R(0)\setminus\{0\}, \qquad \text{with} \quad h(t) = \exp(G(\Phi^{-1}(t))).$$

Note that Φ is an increasing unbounded function and that, since g is not integrable, h is also unbounded (and increasing). Moreover, the assumption g nonincreasing implies that h is concave. We conclude applying (i) of Proposition 5.2. ∎

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