Existence and uniqueness for L^1 data of some elliptic equations with natural growth

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Abstract - We deal with the following nonlinear elliptic problem:

$$\begin{cases} -\text{div }\mathbf{a}(x,u,\nabla u) + b(x,u,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

where Ω is a bounded open in $\mathbb{R}^N$, $f \in L^1(\Omega)$, $-\text{div }\mathbf{a}(x,u,\nabla u)$ defines an operator satisfying Leray-Lions type conditions, and the lower order term satisfies natural growth conditions and some other properties; we point out that these properties do not include a sign assumption (see in (2) below our model example). We prove existence of an entropy solution for this problem (see definition 2.2 below) and we show that, under a natural monotonicity hypothesis, there exists a smallest entropy solution.

1 Introduction

This paper is concerned with existence and uniqueness of solutions of some nonlinear second-order elliptic equations posed in a bounded set $\Omega \subset \mathbb{R}^N$. These equations have a lower-order term which depends on the gradient with natural growth and L^1 functions as data. We point out that the results proved in this paper are new, even for regular data (this is so except for some simple cases like Euler's equations of functionals, which can be studied using methods from the Calculus of Variations: see, for instance, [12]).

In order to deal with L^1 data, we consider the notion of entropy solution (see definition 2.2 below). Entropy solutions were defined in [2] in order to get an L^1 theory of existence and uniqueness for elliptic equations such as

$$\begin{cases} -\text{div }\mathbf{a}(x,\nabla u) + b(x,u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is an open set in $\mathbb{R}^N$, $1 < p < N$, $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a function satisfying the classical Leray-Lions conditions such that $-\text{div }\mathbf{a}(x,\nabla u)$ defines a strictly monotone operator from $W_0^{1,p}(\Omega)$ onto $W^{-1,p'}(\Omega)$, and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, nondecreasing in its second variable and such that $\sup_{|s|\le k}|b(x,s)| \in L^1_{loc}(\Omega)$. (Note that the restriction on p is a consequence of the

Sobolev imbedding $L^1(\Omega) \subset W^{-1,p'}(\Omega)$ for $p > N$, since then problem (1) can be solved in the variational setting.) Afterwards, entropy solutions have been used to prove existence and/or uniqueness of similar type of equations in [1, 3, 5, 9, 15, 18]. Another approach to define a suitable generalized solution is that of renormalized solution which was introduced in [17] and then used, for instance, in [14, 19, 20]. Yet another work on existence and uniqueness of solutions of second-order quasilinear equations with L^1 or measure data is [13]. Let us also mention [22] which deals with a system where a gradient term appears, for such system existence of a solution is proved but it does not provide uniqueness of that solution. For a survey about the search of a definition of generalized solution which will make problem (1) well-posed see [21].

Our purpose is to investigate the effect of a lower-order term, depending on the gradient with natural growth, on the suitable notion of generalized solution. Our model equation is the simplest case: Euler's equation for a functional on $H_0^1(\Omega)$ defined by $J(u) = \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{\sigma(u)} - \int_\Omega fu$ (with σ being smooth enough and positive, and satisfying appropriate bound conditions); that is,

$$\begin{cases} -\operatorname{div}\Big(\frac{\nabla u}{\sigma(u)}\Big) - \frac{1}{2}\frac{|\nabla u|^2\sigma'(u)}{\sigma(u)^2} = f, & \text{in } \Omega; \\ \qquad\qquad\qquad\qquad\qquad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We emphasize that the lower-order term does not satisfy a sign condition. Thus, we cannot obtain a priori estimates on $T_k u$, a truncature of the solution, by taking it as test function in a weak formulation of problem (2). This is a setback since we need some a priori estimates on the truncatures to prove existence of a solution for an integrable datum. Observe that we can overcome this difficulty by considering $\sqrt{\sigma(u)}\, T_k u$, instead $T_k u$, as test function and so getting

$$\int_\Omega \frac{|\nabla T_k u|^2}{\sqrt{\sigma(u)}} = \int_\Omega f\sqrt{\sigma(u)}\, T_k u \quad \Longrightarrow \quad \frac{1}{\sqrt{M}}\int_\Omega |\nabla T_k u|^2 \le k\sqrt{M}\int_\Omega |f|,$$

where M is an upper bound of σ. Hence, we shall obtain the desired estimates in our general equation by developing the above idea.

If in (2) we choose σ strictly concave on $[0, +\infty[$, then it yields

$$\Big(\frac{\xi}{\sigma(r)} - \frac{\eta}{\sigma(s)}\Big)\cdot(\xi - \eta) - \frac{1}{2}\Big(\frac{|\xi|^2\sigma'(r)}{\sigma(r)^2} - \frac{|\eta|^2\sigma'(s)}{\sigma(s)^2}\Big)(r - s) > 0 \tag{3}$$

for $r, s \ge 0$ and $\eta \neq \xi$ (see Corollary 4.1 at the end of this paper). It follows from (3) that uniqueness should be almost automatic for positive solutions. This is so for regular data (when the solution u belongs to the right Sobolev space $H_0^1(\Omega)$), but does not apply for every L^1 data. Nevertheless, we can obtain an entropy solution as limit of solutions of approximating problems and, on the other hand, these solutions belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$ by an L^∞-estimate. Both facts together allow us to prove that there exists the smallest entropy solution, showing in this way a kind of uniqueness. In the same way, we may deal with a general equation which satisfies a monotone hypothesis similar to (3).

A remark concerning (3) is in order. That inequality is only satisfied for $r, s \ge 0$. The claim that (3) holds for all $r, s \in \mathbb{R}$ is not true, and this fact would contradict [12, Proposition 2.4].

Let us mention that our model problem (2) is also the one considered in [8, Section 5] to study existence results for a class of nonlinear elliptic equations. Nevertheless, their development does not coincide with ours: the classes of problems investigated are very different and general questions on uniqueness of solutions are not considered in [8] (in [8, Section 3] they also deal with uniqueness, but only for the model problem).

This paper is organized into three sections. The next one is on preliminaries: we include the precise hypotheses on our problem and the definition of entropy solution, and we also state the results. Section 2 is devoted to existence of a solution, while section 3 is on uniqueness. At the end of section 3, we give some examples to show general hypotheses on uniqueness in particular cases.

2 Assumptions and Statement of Results

Let $N \geq 3$, $1 < p < N$ and $p' = p/(p-1)$. Throughout this paper $\Omega \subset \mathbb{R}^N$ will denote an open bounded set, μ Lebesgue measure on Ω and c (possible different) positive constants which only depend on the parameters of our problem. Let us consider three functions $\mathbf{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $B : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

$(H1)$ These functions satisfy the Carathéodory conditions: i.e., for almost all $x \in \Omega$, the functions $(s,\xi) \to \mathbf{a}(x,s,\xi)$, $(s,\xi) \to b(x,s,\xi)$ and $s \to B(x,s)$ are continuous, and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, the functions $x \to \mathbf{a}(x,s,\xi)$, $x \to b(x,s,\xi)$ and $x \to B(x,s)$ are measurable.

$$(H2) \qquad \begin{cases} |\mathbf{a}(x,s,\xi)| \leq \Lambda\big(|s|^{p-1} + |\xi|^{p-1} + g_a(x)\big) \\[2ex] \big(\mathbf{a}(x,s,\xi) - \mathbf{a}(x,s,\eta)\big) \cdot (\xi - \eta) > 0 \quad \text{for all} \quad \xi \neq \eta \\[2ex] \mathbf{a}(x,s,\xi) \cdot \xi \geq \alpha(s)|\xi|^p, \end{cases}$$

where $\Lambda > 0$, $g_a \in L^{p'}(\Omega)$ and α is a continuous real function such that $\alpha(s) \geq \lambda$ for some $\lambda > 0$. Observe that $\alpha(s) \leq \Lambda$, so that α is a bounded positive function.

$$(H3) \qquad |b(x,s,\xi) - B(x,s)| \leq \beta(s)|\xi|^p,$$

β being a nonnegative and continuous function which is integrable on $\mathbb{R}$.

$$(H4) \qquad \begin{cases} \sup_{|s|\leq k} |B(x,s)| \in L^1(\Omega) \quad \text{for every} \quad k > 0 \\[2ex] B(x,s)s \geq 0. \end{cases}$$

Observe that $(H2)$ and $(H3)$ imply that the quotient β/α is also integrable on $\mathbb{R}$. We will denote $\gamma(s) = \int_0^s \beta/\alpha$, which obviously is a bounded function.

Remark 2.1 *In* $(H2)$ *we have to impose the condition* $\alpha(s) \geq \lambda > 0$ *to guarantee the coerciveness of the operator defined by* $-div\, \mathbf{a}(x,u,\nabla u)$*. However, noncoercive* *operators are considered in [3], where existence results are proved. Since their methods can be adapted to our situation, the above assumption on α can be removed. Actually, the basic requirement on functions α and β needed in most of what follows is β/α integrable on $I\!R$ (see also [10]). Thus, even though our hypotheses on both functions are independent, we just need a compatibility condition between them.*

Consider the following nonlinear elliptic problem: Given $f \in L^1(\Omega)$ find a measurable function u such that $\mathbf{a}(x,u,\nabla u)$ and $b(x,u,\nabla u)$ belong to $L^1(\Omega)$ and satisfying

$$\begin{cases} -\text{div}\ \mathbf{a}(x,u,\nabla u) + b(x,u,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

in a generalized sense to be determined.

This problem was studied, for instance, in [4] under two fundamental hypotheses on b: a sign condition (i.e., $b(x,s,\xi)s \geq 0$) and a growth hypothesis on the second variable (that is, $|b(x,s,\xi)| \geq |s|^p$). These assumptions allow to find a solution in the Sobolev space $W^{1,p}(\Omega)$.

We shall prove existence of a solution of (4) assuming our hypotheses $(H1)$, $(H2)$, and $(H3)$. As in [2], it is impossible to solve problem (4) in $W^{1,p}(\Omega)$. Our first step is to prove existence of solutions when functions B and f are good enough to find a weak solution.

Proposition 2.1 *Assume that $|B(x,s)| \leq |s|^{p-1} + g_B(x)$, where $g_B \in L^{p'}(\Omega)$, and $f \in L^{\frac{Np}{Np-N+p}}(\Omega)$. Then there exists $u \in W_0^{1,p}(\Omega)$ such that $b(x,u,\nabla u) \in L^1(\Omega)$, which is a weak solution of (4) in the sense that*

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla v + \int_\Omega b(x,u,\nabla u)v = \int_\Omega fv \tag{5}$$

hold for every $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Moreover, $f \geq 0$ implies $u \geq 0$.

If we also assume that $\beta(s)|s| \leq \alpha(s)$ holds for all $s \in I\!R$, then $b(x,u,\nabla u)u \in L^1(\Omega)$ and u may be taken as test function obtaining the following energy type identity:

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla u + \int_\Omega b(x,u,\nabla u)u = \int_\Omega fu \tag{6}$$

When f is an arbitrary integrable function some difficulties appear (they are very well explained in [21] for equations without lower-order terms depending on the gradient). The main obstacle is that then the solution does not belong to the correct Sobolev space $W_0^{1,p}(\Omega)$ and, for small p, does not even belong to $W_{loc}^{1,1}(\Omega)$.

On the one hand, since ∇u may no longer be in $L^1(\Omega)$, there appears the problem of interpreting what ∇u means, that is, we have to define the gradient of u in this situation. To do so we need some preliminaries. For $k > 0$ we define the truncature at level $\pm k$ as $T_k(r) = (-k)\vee[k\wedge r]$.

Definition 2.1 *Following [2], we introduce $\mathcal{T}_0^{1,p}(\Omega)$ as the set of all measurable functions $u : \Omega \to I\!R$ such that $T_k u \in W_0^{1,p}(\Omega)$ for all $k > 0$.*

For a measurable function u belonging to $\mathcal{T}_0^{1,p}(\Omega)$, a gradient can be defined: it is a measurable function which is also denoted by ∇u and satisfies $\nabla T_k u = (\nabla u)\chi_{\{|u|<k\}}$ for all $k > 0$ (see [2]).

On the other hand, it follows from $u \notin W_0^{1,p}(\Omega)$ that the function $\mathbf{a}(x,u,\nabla u)\cdot\nabla v$ is not integrable for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; thus, weak solutions, as functions satisfying (5), have no sense. However, we can prove that a solution exists in the sense of distributions and, moreover, as an entropy solution; let us next define it.

Definition 2.2 *Let $f \in L^1(\Omega)$. We will say that $u \in \mathcal{T}_0^{1,p}(\Omega)$ is an entropy solution of (4) if $b(x,u,\nabla u) \in L^1(\Omega)$ and the identity*

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(u-v) + \int_\Omega b(x,u,\nabla u)T_k(u-v) = \int_\Omega fT_k(u-v) \tag{7}$$

holds for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$.

We point out that every term in (7) is well defined: since $f, b(x,u,\nabla u) \in L^1(\Omega)$, the only term which offers some difficulty is the first one. Observe that

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(u-v) = \int_{\{|u-v|<k\}} \mathbf{a}(x,T_K u,\nabla T_K u)\cdot\nabla T_k(u-v),$$

where $K = k + \|v\|_\infty$. Now, it follows from $T_K u \in W_0^{1,p}(\Omega)$ and $T_k(u-v) \in W_0^{1,p}(\Omega)$ that $\mathbf{a}(x,T_K u,\nabla T_K u)\cdot\nabla T_k(u-v)$ is integrable on the set $\{|u-v|<k\}$ and so the first term in (7) is well defined.

Recall que a measurable function $u : \Omega \to \mathrm{I\!R}$ belongs to the Marcinkiewicz (or weak-Lebesgue) space $\mathcal{M}^q(\Omega)$, with $0 < q < \infty$, if there exists $c > 0$ satisfying

$$\mu\{x \in \Omega : |u(x)| > k\} \le ck^{-q}$$

for every $k > 0$. It is straightforward from the definition that $L^q(\Omega) \subset \mathcal{M}^q(\Omega) \subset L^{q-\epsilon}(\Omega)$ for all $0 < \epsilon < q$.

The main result on existence of a solution of (4) is the following.

Theorem 2.1 *For each $f \in L^1(\Omega)$ there exists u such that it is a solution of (4) in the sense of distributions, $b(x,u,\nabla u) \in L^1(\Omega)$ and it is also an entropy solution. Moreover, $u \in \mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$ and $|\nabla u| \in \mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$, so that if $p > 2 - \frac{1}{N}$, then $u \in W_0^{1,q}(\Omega)$ for all $1 \le q < \frac{N(p-1)}{N-1}$.*

We also have that if $f \ge 0$, then $u \ge 0$.

It is now time to discuss uniqueness for nonnegative data, and to do it some extra conditions on functions $\mathbf{a}$ and b are needed. In the remaining results we will suppose that the following condition is also satisfied.

$(H5)$ The inequality

$$\Big(\mathbf{a}(x,s,\xi) - \mathbf{a}(x,r,\eta)\Big)\cdot(\xi-\eta) + \Big(b(x,s,\xi) - b(x,r,\eta)\Big)(s-r) > 0$$

holds for almost all $x \in \Omega$, for $r, s \ge 0$ and for $\xi \ne \eta$.

It is straightforward that this inequality is satisfied when, for almost all $x \in \Omega$, the functions $(s,\xi) \to \mathbf{a}(x,s,\xi)$, $(s,\xi) \to b(x,s,\xi)$ are of class C^1 on $[0,+\infty[\times \mathbb{R}^N$ and the matrix

$$\begin{pmatrix} \frac{\partial a_1}{\partial \xi_1} & \cdots & \frac{\partial a_1}{\partial \xi_N} & \frac{\partial a_1}{\partial s} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial a_N}{\partial \xi_1} & \cdots & \frac{\partial a_N}{\partial \xi_N} & \frac{\partial a_N}{\partial s} \\ \\ \frac{\partial b}{\partial \xi_1} & \cdots & \frac{\partial b}{\partial \xi_N} & \frac{\partial b}{\partial s} \end{pmatrix}$$

generates a quadratic form which is positive-definite. Other sufficient conditions, in a particular case, will be shown at the end of Section 3.

Concerning entropy solutions we are only able to prove the following result.

Proposition 2.2 *Let u and v be two entropy solutions of (4), where $f \in L^1(\Omega)$ and $f \geq 0$. Then one has*

$$\limsup_{k\to\infty} k \int_{\{|u-v|\geq k\}} \big(b(x,u,\nabla u) - b(x,v,\nabla v)\big)\, sign(u-v) \leq 0,$$

and the condition

$$\limsup_{k\to\infty} k \int_{\{|u-v|\geq k\}} \big(b(x,u,\nabla u) - b(x,v,\nabla v)\big)\, sign(u-v) \geq 0$$

implies $u = v$.

In particular, if $u - v \in L^\infty(\Omega)$, then $u = v$.

Notice that these conditions resemble those of [14]. As a consequence, in the particular case that the assumptions of Proposition 2.1 hold true, we may get uniqueness.

Corollary 2.1 *If u and v are two entropy solutions of (4) belonging to $W^{1,p}(\Omega)$, then $u = v$.*

Unfortunately, the argument used to prove Corollary 2.1 cannot be applied to data f which are just summable. The results we can prove for these data are the following.

Theorem 2.2 *For each $f \in L^1(\Omega)$, with $f \geq 0$, there exists an entropy solution of (4) which is the smallest one. This smallest entropy solution is obtained as limit of solutions of approximating problems.*

Proposition 2.3 *Let $f, g \in L^1(\Omega)$ be such that $0 \leq g \leq f$. Denote by u and v the smallest entropy solutions of (4) with data f and g, respectively. Then $0 \leq v \leq u$.*

We finally point out that a stability result can be obtained reasoning as in the proof of Theorem 2.1.

3 Existence

This section is devoted to prove Proposition 2.1 and Theorem 2.1. The main idea in our proofs is to consider test functions of exponential type; these resemble, in some sense, those used in the proofs of [6].

Proof of Proposition 2.1: Let us consider the approximating problems

$$\begin{cases} -\text{div } \mathbf{a}(x,u_n,\nabla u_n) + \frac{nb(x,u_n,\nabla u_n)}{n+|b(x,u_n,\nabla u_n)|} = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

By classical results (see [16]), we know that there exists $u_n \in W_0^{1,p}(\Omega)$, which is a weak solution of (8). Hence,

$$\int_\Omega \mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla v + \int_\Omega \frac{nb(x,u_n,\nabla u_n)}{n+|b(x,u_n,\nabla u_n)|}v = \int_\Omega fv \tag{9}$$

holds for every $v \in W_0^{1,p}(\Omega)$.

First step: a priori estimates.

We claim that if $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $v \geq 0$, then $e^{\gamma(u_n)}v$ may be taken as test function in (9) and as a consequence the inequality

$$\int_\Omega e^{\gamma(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla v + \int_\Omega \frac{nB(x,u_n)}{n+|b(x,u_n,\nabla u_n)|}e^{\gamma(u_n)}v \leq \int_\Omega fe^{\gamma(u_n)}v \tag{10}$$

holds for every nonnegative $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Let $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. First observe that we have $e^{\gamma(T_ku_n)} \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, for every $k>0$. Thus, $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ implies $e^{\gamma(T_ku_n)}v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Taking it as test function in (9), it yields

$$\begin{aligned} &\textstyle\int_\Omega \frac{\beta(T_ku_n)}{\alpha(T_ku_n)}e^{\gamma(T_ku_n)}v\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla T_ku_n + \int_\Omega e^{\gamma(T_ku_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla v + \\ &+ \textstyle\int_\Omega \frac{n}{n+|b(x,u_n,\nabla u_n)|}b(x,u_n,\nabla u_n)e^{\gamma(T_ku_n)}v = \int_\Omega fe^{\gamma(T_ku_n)}v. \end{aligned} \tag{11}$$

Now we are going to study these integrals. Note that the first integral is equal to

$$\int_{\{|u_n|<k\}} \frac{\beta(u_n)}{\alpha(u_n)}e^{\gamma(u_n)}v\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla u_n;$$

so that, by the positivity of α and β, and by $(H2)$, the function is nonnegative. Hence, applying the Monotone Convergence theorem, we have

$$\lim_{k\to\infty}\int_\Omega \frac{\beta(T_ku_n)}{\alpha(T_ku_n)}e^{\gamma(T_ku_n)}v\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla T_ku_n =$$

$$= \int_\Omega \frac{\beta(u_n)}{\alpha(u_n)}e^{\gamma(u_n)}v\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla u_n.$$

On the other hand, the functions $\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla v$, $\frac{nb(x,u_n,\nabla u_n)}{n+|b(x,u_n,\nabla u_n)|}v$ and fv are integrable, and the functions $e^{\gamma(T_ku_n)}$ are bounded in $L^\infty(\Omega)$; so Lebesgue's Dominated Convergence theorem may be applied in the remaining integrals. Thus, letting k tend to ∞ in (11), we obtain

$$\int_\Omega \frac{\beta(u_n)}{\alpha(u_n)}e^{\gamma(u_n)}v\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla u_n + \int_\Omega e^{\gamma(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla v +$$

$$+\int_\Omega \frac{n}{n+|b(x,u_n,\nabla u_n)|} b(x,u_n,\nabla u_n) e^{\gamma(u_n)} v = \int_\Omega f e^{\gamma(u_n)} v;$$

Hence, $e^{\gamma(u_n)} v$ may be taken as test function in (9).

Finally, since $(H2)$ and $(H3)$ imply

$$\int_\Omega \frac{\beta(u_n)}{\alpha(u_n)} e^{\gamma(u_n)} v \mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla u_n +$$

$$+\int_\Omega \frac{n}{n+|b(x,u_n,\nabla u_n)|} b(x,u_n,\nabla u_n) e^{\gamma(u_n)} v \geq$$

$$\geq \int_\Omega \frac{n}{n+|b(x,u_n,\nabla u_n)|} B(x,u_n) e^{\gamma(u_n)} v,$$

it follows that (10) holds true.

Similarly, for each $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $v \leq 0$, one obtains that $e^{-\gamma(u_n)} v$ can be taken as test function in (9) and deduces that the inequality

$$\int_\Omega e^{-\gamma(u_n)} \mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla v + \int_\Omega \frac{nB(x,u_n)}{n+|b(x,u_n,\nabla u_n)|} e^{-\gamma(u_n)} v \leq \int_\Omega f e^{-\gamma(u_n)} v \tag{12}$$

holds for every nonpositive $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

From (10) and (12) we are going to obtain our a priori estimates. Taking $v = T_k u_n^+$ in (10), we get

$$\int_\Omega e^{\gamma(u_n)} \mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla T_k u_n^+ + \int_\Omega \frac{nB(x,u_n)}{n+|b(x,u_n,\nabla u_n)|} e^{\gamma(u_n)} T_k u_n^+ \leq$$

$$\leq \int_\Omega f e^{\gamma(u_n)} T_k u_n^+$$

so that, by $(H2)$,

$$\int_{\{0\leq u_n<k\}} e^{\gamma(u_n)} \alpha(u_n)|\nabla u_n|^p + \int_\Omega \frac{nB(x,u_n)}{n+|b(x,u_n,\nabla u_n)|} e^{\gamma(u_n)} T_k u_n^+ \leq$$

$$\leq \int_\Omega |f| e^{\gamma(u_n)} T_k u_n^+.$$

Disregarding first a nonnegative term, it follows from the boundedness of γ that

$$\int_{\{0\leq u_n<k\}} \alpha(u_n)|\nabla u_n|^p \leq c \int_\Omega |f| T_k u_n^+,$$

for some positive constant c. Analogously, taking $v = -T_k u_n^-$ in (12), we obtain

$$\int_{\{-k<u_n\leq 0\}} \alpha(u_n)|\nabla u_n|^p \leq c \int_\Omega |f| T_k u_n^-.$$

Adding up both results it yields

$$\int_{\{|u_n|<k\}} \alpha(u_n)|\nabla u_n|^p \leq c \int_\Omega |f|\cdot|T_k u_n| \leq c \int_\Omega |f|\cdot|u_n|$$

and, since this holds for every $k>0$, we obtain from Fatou's lemma that

$$\int_\Omega \alpha(u_n)|\nabla u_n|^p \le c\int_\Omega |f|\cdot|u_n|.$$

As a consequence,

$$\lambda\int_\Omega |\nabla u_n|^p \le \int_\Omega \alpha(u_n)|\nabla u_n|^p \le c\int_\Omega |f|\cdot|u_n|$$

and then Hölder's and Sobolev's inequalities imply

$$\|\nabla u_n\|_p^p \le c\|f\|_{\frac{Np}{Np-N+p}}\|u_n\|_{p^*} \le c\|f\|_{\frac{Np}{Np-N+p}}\|\nabla u_n\|_p.$$

Hence, the sequence $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$, so that we can extract a subsequence (still denoted by $(u_n)_n$), such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(\Omega), \quad (\text{and in } L^{p^*}(\Omega)). \tag{13}$$

By Rellich-Kondrachov's theorem, we may also assume

$$u_n \to u \quad \text{in} \quad L^p(\Omega) \quad \text{and a.e.} \tag{14}$$

Moreover, having in mind $(H2)$ and $|B(x,u_n)| \le |u_n|^{p-1}+g_B(x)$, we also have that

$$\mathbf{a}(x,u_n,\nabla u) \to \mathbf{a}(x,u,\nabla u) \quad \text{and} \quad B(x,u_n) \to B(x,u) \quad \text{in} \quad L^{p'}(\Omega). \tag{15}$$

Second step: convergence.

Our aim in this step is to prove that

$$u_n \to u \quad \text{in} \quad W^{1,p}(\Omega) \tag{16}$$

and

$$b(x,u_n,\nabla u_n) \to b(x,u,\nabla u) \quad \text{in} \quad L^1(\Omega). \tag{17}$$

In order to see (16), we will prove

$$\lim_{n\to\infty}\int_\Omega \big(\mathbf{a}(x,u_n,\nabla u_n)-\mathbf{a}(x,u_n,\nabla u)\big)\cdot\nabla(u_n-u)=0. \tag{18}$$

To begin with the proof of it, let $\delta = e^{-\sup|\gamma|}$. Then

$$\delta\int_\Omega \Big(\mathbf{a}(x,u_n,\nabla u_n)-\mathbf{a}(x,u_n,\nabla u)\Big)\cdot\nabla(u_n-u) \le$$

$$\int_{\{u_n-u\ge 0\}} e^{\gamma(u_n)}\Big(\mathbf{a}(x,u_n,\nabla u_n)-\mathbf{a}(x,u_n,\nabla u)\Big)\cdot\nabla(u_n-u)+$$

$$+\int_{\{u_n-u\le 0\}} e^{-\gamma(u_n)}\Big(\mathbf{a}(x,u_n,\nabla u_n)-\mathbf{a}(x,u_n,\nabla u)\Big)\cdot\nabla(u_n-u)=$$

$$=I_n^1-I_n^2+I_n^3-I_n^4,$$

where

$$I_n^1 = \int_{\{u_n - u \geq 0\}} e^{\gamma(u_n)} \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u)$$

$$I_n^2 = \int_{\{u_n - u \geq 0\}} e^{\gamma(u_n)} \mathbf{a}(x, u_n, \nabla u) \cdot \nabla(u_n - u)$$

$$I_n^3 = \int_{\{u_n - u \leq 0\}} e^{-\gamma(u_n)} \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u)$$

$$I_n^4 = \int_{\{u_n - u \leq 0\}} e^{-\gamma(u_n)} \mathbf{a}(x, u_n, \nabla u) \cdot \nabla(u_n - u).$$

Thus, we only have to prove that $\limsup_{n\to\infty} I_n^1 - I_n^2 + I_n^3 - I_n^4 \leq 0$. From (15), it follows that $e^{\gamma(u_n)}\mathbf{a}(x, u_n, \nabla u) \to e^{\gamma(u)}\mathbf{a}(x, u, \nabla u)$ in $L^{p'}(\Omega)$ and then (13) implies $\lim_{n\to\infty} I_n^2 = 0$. In the same way, $\lim_{n\to\infty} I_n^4 = 0$.

To estimate I_n^1, take $v = T_k(u_n - u)^+$ in (10) and get

$$\int_\Omega e^{\gamma(u_n)} \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - u)^+ + \int_\Omega \frac{nB(x, u_n)}{n + |b(x, u_n, \nabla u_n)|} e^{\gamma(u_n)} T_k(u_n - u)^+$$

$$\leq c \int_\Omega |f| \cdot T_k(u_n - u)^+ \leq c \int_\Omega |f| \cdot |u_n - u|.$$

From $e^{\gamma(u_n)}\mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - u)^+ \geq -e^{\gamma(u_n)}|\mathbf{a}(x, u_n, \nabla u_n)| \cdot |\nabla(u_n - u)| \in L^1(\Omega)$ for all $k > 0$ and Fatou's lemma, we deduce

$$\begin{aligned} &\textstyle\int_\Omega e^{\gamma(u_n)} \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u)^+ + \int_\Omega \frac{nB(x,u_n)e^{\gamma(u_n)}(u_n-u)^+}{n+|b(x,u_n,\nabla u_n)|} \leq \\ &\leq c \textstyle\int_\Omega |f| \cdot |u_n - u|. \end{aligned} \tag{19}$$

Now, from (13) and (14), it follows that $|u_n - u| \rightharpoonup 0$ weakly in $L^{p^*}(\Omega)$ and so

$$\lim_{n\to\infty} \int_\Omega |f| \cdot |u_n - u| = 0. \tag{20}$$

On the other hand, $(u_n - u)^+ \to 0$ in $L^p(\Omega)$, and the sequence $\left(\frac{nB(x,u_n)e^{\gamma(u_n)}}{n+|b(x,u_n,\nabla u_n)|}\right)_n$ is bounded in $L^{p'}(\Omega)$, so that

$$\lim_{n\to\infty} \int_\Omega \frac{nB(x, u_n)e^{\gamma(u_n)}(u_n - u)^+}{n + |b(x, u_n, \nabla u_n)|} = 0. \tag{21}$$

Thus, from (19), (20) and (21), we conclude that $\limsup_{n\to\infty} I_n^1 \leq 0$.

Finally, take $v = -T_k(u_n - u)^-$ in (12) and obtain

$$-\int_\Omega e^{-\gamma(u_n)} \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u)^- - \int_\Omega \frac{nB(x, u_n)e^{\gamma(u_n)}(u_n - u)^-}{n + |b(x, u_n, \nabla u_n)|} \leq$$

$$\leq c \int_\Omega |f| \cdot |u_n - u|.$$

Reasoning as above, one deduces that

$$\lim_{n\to\infty} \int_\Omega |f| \cdot |u_n - u| = \lim_{n\to\infty} \int_\Omega \frac{nB(x, u_n)e^{-\gamma(u_n)}(u_n - u)^-}{n + |b(x, u_n, \nabla u_n)|} = 0,$$

and consequently

$$\limsup_{n\to\infty} I_n^3 = \limsup_{n\to\infty} -\int_\Omega e^{-\gamma(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla(u_n-u)^- \le 0.$$

Therefore, (18) holds true. Now, according to a result of [7] or [11], which is a variation of a classical result by Leray-Lions [16], it yields $u_n \to u$ in $W^{1,p}(\Omega)$. After passing to a suitable subsequence, if necessary, we have

$$\nabla u_n \to \nabla u \quad \text{a.e.} \tag{22}$$

An immediate consequence of (16) and $(H2)$ is

$$\mathbf{a}(x,u_n,\nabla u_n) \to \mathbf{a}(x,u,\nabla u) \quad \text{in} \quad L^{p'}(\Omega). \tag{23}$$

Next, we will see (17); first note that $b(x,u_n,\nabla u_n) \to b(x,u,\nabla u)$ a.e. and so we only need the equi-integrability of the sequence $\big(b_n(x,u_n,\nabla u_n)\big)_n$ and Vitali's Convergence theorem. We now claim that the sequence $\big(\beta(u_n)|\nabla u_n|^p\big)_n$ is equi-integrable. Indeed, take $e^{\gamma(u_n)}[\gamma(u_n - T_k u_n + k) - \gamma(k)]^+$ and $-e^{-\gamma(u_n)}[\gamma(u_n - T_k u_n - k) - \gamma(-k)]^-$ as test functions in (9) to get

$$\int_{\{u_n>k\}} \beta(u_n)|\nabla u_n|^p \le c\int_{\{u_n>k\}} |f|$$

and

$$\int_{\{u_n<-k\}} \beta(u_n)|\nabla u_n|^p \le c\int_{\{u_n<-k\}} |f|,$$

respectively; thus, we obtain

$$\lim_{k\to\infty}\int_{\{|u_n|>k\}} \beta(u_n)|\nabla u_n|^p = 0 \quad \text{uniformly on } n.$$

The desired equi-integrability is now consequence of the following standard argument. Given $\epsilon > 0$ find $k > 0$ such that $\int_{\{|u_n|>k\}} \beta(u_n)|\nabla u_n|^p < \frac{\epsilon}{2}$ for all $n \in I\!N$. Fixed $k > 0$, denote $\beta_k = \max\{\beta(s) : |s| \le k\}$ and observe that

$$\int_A \beta(u_n)|\nabla u_n|^p \le \int_{A\cap\{|u_n|\le k\}} \beta(u_n)|\nabla u_n|^p + \int_{\{|u_n|>k\}} \beta(u_n)|\nabla u_n|^p \le$$

$$\le \int_A \beta_k|\nabla u_n|^p + \frac{\epsilon}{2}$$

holds for every measurable set $A \subset \Omega$. Since $\nabla u_n \to \nabla u$ in $L^p(\Omega)^N$, it yields that there is $\delta > 0$ satisfying that $\mu(A) < \delta$ implies $\int_A |\nabla u_n|^p < \frac{\epsilon}{2\beta_k}$ for every $n \in I\!N$. Hence, it follows from $\mu(A) < \delta$ that $\int_A \beta(u_n)|\nabla u_n|^p < \epsilon$ for every $n \in I\!N$ and so the sequence $\big(\beta(u_n)|\nabla u_n|^p\big)_n$ is equi-integrable. On the other hand, it follows from (15) that the sequence $\big(\frac{nB(x,u_n)}{n+|b(x,u_n,\nabla u_n)|}\big)_n$ is also equi-integrable. We then deduce from $(H3)$ that the sequence $\big(b(x,u_n,\nabla u_n)\big)_n$ is equi-integrable and, by Vitali's theorem, that (17) holds true.

Finally, let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; holding (9) for this v, it follows from (23) and (17) that (5) is satisfied.

Third step: f nonnegative implies u nonnegative.

Assume that $f \geq 0$. If we take $-e^{-\gamma(u)}T_k u^-$ as test function in (5) and reason as in (12), then

$$-\int_\Omega e^{-\gamma(u)}\mathbf{a}(x,u,\nabla u)\cdot\nabla T_k u^- - \int_\Omega B(x,u)e^{-\gamma(u)}T_k u^- \leq -\int_\Omega f e^{-\gamma(u)}T_k u^- \leq 0$$

and, dropping a nonnegative term, there exists $c>0$ such that

$$c\int_{\{-k<u\leq 0\}}|\nabla u|^p \leq \int_{\{-k<u\leq 0\}} e^{-\gamma(u)}\mathbf{a}(x,u,\nabla u)\cdot\nabla u \leq 0$$

This holds for every $k>0$ and so Fatou's lemma implies $\int_{\{u\leq 0\}}|\nabla u|^p \leq 0$. Then $\|\nabla u^-\|_p = 0$ and, by Poincaré inequality, $u^- = 0$.

Last step: assuming $\beta(s)|s| \leq \alpha(s)$, u can be taken as test function.

First of all, recall that by (13) and $(H2)$, the function $\alpha(u)|\nabla u|^p$ is integrable on Ω. Thus, the assumption $\beta(s)|s| \leq \alpha(s)$ for all $s\in\mathbb{R}$ implies that the function $\beta(u)|\nabla u|^p|u|$ belongs to $L^1(\Omega)$. On the other hand, the function $B(x,u)u$ also belongs to $L^1(\Omega)$, since $|B(x,s)| \leq |s|^{p-1}+g_B(x)$, where $g_B\in L^{p'}(\Omega)$, and $u\in L^p(\Omega)$. It follows from $(H3)$ that

$$|b(x,u,\nabla u)u| \leq \beta(u)|\nabla u|^p|u| + |B(x,u)u|$$

Hence, $b(x,u,\nabla u)u\in L^1(\Omega)$.

Now taking $T_k u$ as test function in (5), we deduce that

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k u + \int_\Omega b(x,u,\nabla u)T_k u = \int_\Omega f T_k u \tag{24}$$

holds for all $k>0$. Since functions $\mathbf{a}(x,u,\nabla u)\cdot\nabla u$, $b(x,u,\nabla u)u$, and fu are integrable, applying Lebesgue's theorem on Dominated Convergence to take limits in (24), we get (6).

Proof of Theorem 2.1: The proof will also be divided into several steps.

1.- Approximating problems and a priori estimates

We have to approximate the function b by functions $b_n : \Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$, with $n\in\mathbb{N}$, satisfying the assumptions of Proposition 2.1. For instance, consider the functions defined by

$$b_n(x,s,\xi) = \max\{-\beta(s)|\xi|^p + T_n B(x,s), \min\{b(x,s,\xi), \beta(s)|\xi|^p + T_n B(x,s)\}\}.$$

Then, given the approximating problem

$$\begin{cases} -\text{div }\mathbf{a}(x,u_n,\nabla u_n) + b_n(x,u_n,\nabla u_n) = T_n f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}$$

and applying Proposition 2.1, we may find u_n which is a weak solution of (25) in the sense of (5).

Arguing as in (10) and (12), we deduce that

$$\int_\Omega e^{\gamma(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla v+\int_\Omega T_nB(x,u_n)e^{\gamma(u_n)}v\le\int_\Omega T_nfe^{\gamma(u_n)}v \tag{26}$$

holds for every nonnegative $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, and

$$\int_\Omega e^{-\gamma(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla v+\int_\Omega T_nB(x,u_n)e^{-\gamma(u_n)}v\le\int_\Omega fe^{-\gamma(u_n)}v \tag{27}$$

holds for every nonpositive $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Taking $v=T_h(u_n-T_ku_n)^+$ in (26), and applying $(H2)$ and $(H3)$, it yields that there is $c>0$ such that

$$\int_\Omega \alpha(u_n)|\nabla T_h(u_n-T_ku_n)^+|^p+\int_\Omega T_nB(x,u_n)T_h(u_n-T_ku_n)^+\le$$

$$\le c\int_\Omega T_nfT_h(u_n-T_ku_n)^+.$$

In the same way, if we take $v=T_h(u_n-T_ku_n)^-$ in (27), we will obtain

$$\int_\Omega \alpha(u_n)|\nabla T_h(u_n-T_ku_n)^-|^p+\int_\Omega T_nB(x,u_n)T_h(u_n-T_ku_n)^-\le$$

$$\le c\int_\Omega T_nfT_h(u_n-T_ku_n)^-.$$

Adding up both inequalities, we have

$$\int_\Omega \alpha(u_n)|\nabla T_h(u_n-T_ku_n)|^p+\int_\Omega T_nB(x,u_n)T_h(u_n-T_ku_n)\le$$

$$\le c\int_\Omega |T_nf\ T_h(u_n-T_ku_n)|.$$

Thus, on the one hand,

$$\lambda\int_\Omega|\nabla T_h(u_n-T_ku_n)|^p\le\int_\Omega\alpha(u_n)|\nabla T_h(u_n-T_ku_n)|^p\le ch\int_\Omega|f|,$$

so that

$$\int_{\{k<|u_n|<k+h\}}|\nabla u_n|^p\le ch\int_{\{|u_n|>k\}}|f| \tag{28}$$

and, on the other hand, $\int_\Omega T_nB(x,u_n)T_h(u_n-T_ku_n)\le ch\int_{\{|u_n|>k\}}|f|$. From here, we have that $\int_{\{|u_n|>k\}}T_nB(x,u_n)\frac{T_h(u_n-k)}{h}\le c\int_{\{|u_n|>k\}}|f|$ for all $h>0$; taking limits as $h\to0^+$ and applying the sign condition $B(x,s)s\ge0$ we deduce that

$$\int_{\{|u_n|>k\}}|T_nB(x,u_n)|\le c\int_{\{|u_n|>k\}}|f|. \tag{29}$$

In particular, as k tends to 0^+ in (28) and (29), we obtain

$$\int_{\{|u_n|<h\}}|\nabla u_n|^p\le ch\int_\Omega|f| \tag{30}$$

and
$$\int_\Omega |T_n B(x,u_n)| \le c\int_\Omega |f|. \tag{31}$$

Finally, we shall obtain a priori bounds on $(u_n)_n$ and $(\nabla u_n)_n$. First we show that the sequence $(u_n)_n$ is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$. Indeed, given $k>0$, Sobolev's imbedding theorem and (30) imply that

$$\mu\{|u_n|\ge k\} \le \int_\Omega \frac{|T_k u_n|^{p^*}}{k^{p^*}} \le \frac{c}{k^{p^*}}\|\nabla T_k u_n\|_p^{p^*} \le \frac{c}{k^{p^*}} k^{\frac{p^*}{p}} \le ck^{\frac{-N(p-1)}{N-p}}. \tag{32}$$

On the other hand, it follows from $\{|\nabla u_n|\ge h\}\subset\{|u_n|\ge k\}\cup\{|\nabla T_k u_n|\ge h\}$, (30) and (32), that

$$\mu\{|\nabla u_n|\ge h\} \le \frac{c}{k^{\frac{N(p-1)}{N-p}}} + \int_\Omega \frac{|\nabla T_k u_n|^p}{h^p} \le \frac{c}{k^{\frac{N(p-1)}{N-p}}} + \frac{ck}{h^p}$$

so that, taking $k=h^{\frac{N-p}{N-1}}$, we deduce that

$$\mu\{|\nabla u_n|\ge h\} \le ch^{\frac{-N(p-1)}{N-1}}. \tag{33}$$

Hence, the sequence $(\nabla u_n)_n$ is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$.

2.- Convergence

To begin with, it will be proved that

$$u_n \to u \quad \text{a.e.} \tag{34}$$

holds for some measurable function u. Indeed, consider $\Phi(s)=s/(1+|s|)$, which defines a bounded and increasing function. Note also that $\left|\int_0^{u_n}(\Phi')^p\right| \le \int_0^{|u_n|}\Phi' = \Phi(|u_n|)\le 1$. So taking $v=\int_0^{u_n^+}(\Phi')^p$ in (26) and $v=-\int_{-u_n^-}^0(\Phi')^p$ in (27), we can argue as above and find $c>0$ such that

$$\int_{\{u_n\ge 0\}} \alpha(u_n)|\nabla\Phi(u_n)|^p + \int_\Omega T_n B(x,u_n)\Big(\int_0^{u_n^+}(\Phi')^p\Big) \le$$
$$\le c\int_\Omega (T_n f)\Big(\int_0^{u_n^+}(\Phi')^p\Big)$$

and

$$\int_{\{u_n\le 0\}} \alpha(u_n)|\nabla\Phi(u_n)|^p - \int_\Omega T_n B(x,u_n)\Big(\int_{-u_n^-}^0(\Phi')^p\Big) \le$$
$$\le -c\int_\Omega (T_n f)\Big(\int_{-u_n^-}^0(\Phi')^p\Big).$$

Dropping nonnegative terms, we obtain by $(H2)$ that $\int_\Omega |\nabla\Phi(u_n)|^p \le c\int_\Omega |f|$. Thus, the sequence $\big(\Phi(u_n)\big)_n$ is bounded in $W_0^{1,p}(\Omega)$ and then a subsequence, still denoted by $\big(\Phi(u_n)\big)_n$, converges weakly in $W_0^{1,p}(\Omega)$. As a consequence of Rellich-Kondrachov's theorem, it also converges in measure. We may assume (taking another subsequence, if necessary) that $\big(\Phi(u_n)\big)_n$ converges almost everywhere. Since Φ is increasing, it follows that the sequence $(u_n)_n$ also converges a.e. and so (34) holds. Moreover, (34) and (32) imply that $u \in \mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$.

Obviously, (34) also implies that $T_k u_n \to T_k u$ a.e. for all $k > 0$. Since, by (30), each sequence $(T_k u_n)_n$ is also bounded in $W_0^{1,p}(\Omega)$, we deduce from the pointwise convergence that

$$T_k u_n \rightharpoonup T_k u \quad \text{weakly in} \quad W_0^{1,p}(\Omega). \tag{35}$$

Furthermore, $u \in \mathcal{T}_0(\Omega)$, since $T_k u \in W_0^{1,p}(\Omega)$ for all $k > 0$.

We shall show, in the next step, that

$$T_n B(x, u_n) \to B(x, u) \quad \text{in} \quad L^1(\Omega). \tag{36}$$

From $(H1)$ and (34), we first deduce that $T_n B(x, u_n) \to B(x, u)$ a.e. and, by $(H4)$ and Lebesgue's Dominated Convergence theorem, $T_n B(x, T_k u_n) \to B(x, T_k u)$ in $L^1(\Omega)$. Thus, each sequence $\big(T_n B(x, T_k u_n)\big)_n$ is equi-integrable. In order to obtain that the sequence $\big(T_n B(x, u_n)\big)_n$ is also equi-integrable, let $\epsilon > 0$ and consider $A \subset \Omega$. Then

$$\int_A |T_n B(x, u_n)| \le \int_A |T_n B(x, T_k u_n)| + \int_{\{|u_n|>k\}} |T_n B(x, u_n)|. \tag{37}$$

Since (29) implies $\lim_{n\to\infty} \int_{\{|u_n|>k\}} |T_n B(x, u_n)| = 0$ uniformly with respect to n, it follows from (37) that we can fix $k > 0$ big enough to have

$$\int_A |T_n B(x, u_n)| \le \int_A |T_n B(x, T_k u_n)| + \frac{\epsilon}{2}.$$

Now the sequence $\big(T_n B(x, T_k u_n)\big)_n$ is equi-integrable and so there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\int_A |T_n B(x, T_k u_n)| < \epsilon/2$. Hence, $\int_A |T_n B(x, u_n)| < \epsilon$: that is, (36) is proved by Vitali's theorem.

As in Proposition 2.1, we now prove that

$$\nabla T_k u_n \to \nabla T_k u \quad \text{in} \quad L^p(\Omega), \tag{38}$$

by seeing that

$$\lim_{n\to\infty} \int_\Omega \big(\mathbf{a}(x, T_k u_n, \nabla T_k u_n) - \mathbf{a}(x, T_k u_n, \nabla T_k u)\big) \cdot \nabla(T_k u_n - T_k u) = 0 \tag{39}$$

and applying the same result of [7] or [11].

We shall prove (39) following the technique introduced in [15]. Let $\delta = e^{-\sup|\gamma|}$ and decompose

$$\delta \int_\Omega \big(\mathbf{a}(x, T_k u_n, \nabla T_k u_n) - \mathbf{a}(x, T_k u_n, \nabla T_k u)\big) \cdot \nabla(T_k u_n - T_k u) \le I_n^1 - I_n^2 + I_n^3 - I_n^4,$$

where

$$I_n^1 = \int_{\{u_n - T_h u_n + T_k u_n - T_k u \ge 0\}} e^{\gamma(u_n)} \mathbf{a}(x, T_k u_n, \nabla T_k u_n) \cdot \nabla(T_k u_n - T_k u)$$

$$I_n^2=\int_{\{u_n-T_hu_n+T_ku_n-T_ku\geq 0\}} e^{\gamma(u_n)}\mathbf{a}(x,T_ku_n,\nabla T_ku)\cdot\nabla(T_ku_n-T_ku)$$

$$I_n^3=\int_{\{u_n-T_hu_n+T_ku_n-T_ku\leq 0\}} e^{-\gamma(u_n)}\mathbf{a}(x,T_ku_n,\nabla T_ku_n)\cdot\nabla(T_ku_n-T_ku)$$

$$I_n^4=\int_{\{u_n-T_hu_n+T_ku_n-T_ku\leq 0\}} e^{-\gamma(u_n)}\mathbf{a}(x,T_ku_n,\nabla T_ku)\cdot\nabla(T_ku_n-T_ku)$$

and $h>0$ to be fixed. It follows from (35) that $T_ku_n\to T_ku$ in $L^p(\Omega)$ and so $\mathbf{a}(x,T_ku_n,\nabla T_ku)\to\mathbf{a}(x,T_ku,\nabla T_ku)$ in $L^{p'}(\Omega)$; thus, (35) implies that $\lim_{n\to\infty} I_n^2+I_n^4=0$. Then we just need to see $\limsup_{n\to\infty} I_n^1+I_n^3\leq 0$.

To prove $\limsup_{n\to\infty} I_n^1\leq 0$, let $\epsilon>0$ and fix $h>k$ so big to have

$$\int_\Omega |f|\cdot T_{2k}(u-T_hu)^+ + \int_\Omega |B(x,u)|\cdot T_{2k}(u-T_hu)^+ < \epsilon.$$

Taking $v=T_{2k}(u_n-T_hu_n+T_ku_n-T_ku)^+$ in (26) we deduce that

$$\int_\Omega e^{\gamma(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla T_{2k}(u_n-T_hu_n+T_ku_n-T_ku)^+\leq$$

$$\leq c\int_\Omega |f|\cdot T_{2k}(u_n-T_hu_n+T_ku_n-T_ku)^+ +$$

$$+c\int_\Omega |T_nB(x,u_n)|\cdot T_{2k}(u_n-T_hu_n+T_ku_n-T_ku)^+$$

for some $c>0$. Since, by the Dominated Convergence theorem,

$$\lim_{n\to\infty}\Big[\textstyle\int_\Omega |f|\cdot T_{2k}(u_n-T_hu_n+T_ku_n-T_ku)^+ +$$

$$+\textstyle\int_\Omega |T_nB(x,u_n)|\cdot T_{2k}(u_n-T_hu_n+T_ku_n-T_ku)^+\Big]=$$

$$=\textstyle\int_\Omega |f|\cdot T_{2k}(u-T_hu)^+ + \int_\Omega |B(x,u)|\cdot T_{2k}(u-T_hu)^+,$$

we obtain

$$\limsup_{n\to\infty}\int_\Omega e^{\gamma(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla T_{2k}(u_n-T_hu_n+T_ku_n-T_ku)^+ < c\epsilon. \tag{40}$$

On the other hand, if $A_n=\{x\in\Omega: u_n(x)-T_hu_n(x)+T_ku_n(x)-T_ku(x)\geq 0\}$, then

$$\int_\Omega e^{\gamma(u_n)}\mathbf{a}(x,u_n,\nabla u_n)\cdot\nabla T_{2k}(u_n-T_hu_n+T_ku_n-T_ku)^+=$$

$$=\int_{\{|u_n|\leq k\}\cap A_n} e^{\gamma(u_n)}\mathbf{a}(x,T_ku_n,\nabla T_ku_n)\cdot\nabla(T_ku_n-T_ku)+$$

$$+\int_{\{k\leq|u_n|\leq h\}\cap A_n} e^{\gamma(u_n)}\mathbf{a}(x,u_n,\nabla T_hu_n)\cdot\nabla(T_ku_n-T_ku)+$$

$$+\int_{\{|u_n|\geq h\}\cap\{-h-2k\leq|u_n|\geq h+4k\}} e^{\gamma(u_n)}\mathbf{a}(x,T_{h+4k}u_n,\nabla T_{h+4k}u_n)\cdot\nabla(u_n-T_ku)$$

$$\geq I_n^1 - 2\int_{\{|u_n|\geq k\}\cap A_n} e^{\gamma(u_n)}\mathbf{a}(x, T_{h+4k}u_n, \nabla T_{h+4k}u_n)\cdot\nabla T_k u,$$

so that, by (40),

$$\limsup_{n\to\infty} I_n^1 \leq c\epsilon + 2\limsup_{n\to\infty}\int_{\{|u_n|\geq k\}\cap A_n} e^{\gamma(u_n)}\mathbf{a}(x, T_{h+4k}u_n, \nabla T_{h+4k}u_n)\cdot\nabla T_k u. \qquad (41)$$

The limit in this last integral is easy to evaluate by standard arguments; indeed, the sequence $\big(\chi_{A_n}e^{\gamma(u_n)}\mathbf{a}(x, T_{h+4k}u_n, \nabla T_{h+4k}u_n)\big)_n$ is bounded in $L^{p'}(\Omega)$ and $\chi_{\{|u_n|\geq k\}}\nabla T_k u \to 0$ in $L^p(\Omega)$. Hence,

$$\lim_{n\to\infty}\int_{\{|u_n|\geq k\}\cap A_n} e^{\gamma(u_n)}\mathbf{a}(x, T_{h+4k}u_n, \nabla T_{h+4k}u_n)\cdot\nabla T_k u = 0.$$

This fact and (41) imply $\limsup_{n\to\infty} I_n^1 \leq c\epsilon$ and the arbitrariness of $\epsilon > 0$ shows that $\limsup_{n\to\infty} I_n^1 \leq 0$, as desired.

Finally, by considering $v = -T_{2k}(u_n - T_h u_n + T_k u_n - T_k u)^-$ in (27) and by arguing in the same way as above, it yields $\limsup_{n\to\infty} I_n^3 \leq 0$. Therefore, (38) is proved and, using now a diagonal argument, we also obtain that, up to subsequences,

$$\nabla u_n \to \nabla u \quad \text{in measure and a.e..} \qquad (42)$$

Moreover, by (33), $|\nabla u| \in \mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$.

It is straighforward, from (34), (42) and $(H1)$, that $\mathbf{a}(x, u_n, \nabla u_n) \to \mathbf{a}(x, u, \nabla u)$ a.e. We next see that

$$\mathbf{a}(x, u_n, \nabla u_n) \to \mathbf{a}(x, u, \nabla u) \quad \text{in} \quad L^1(\Omega). \qquad (43)$$

According to (32) and (33), the sequences $(u_n)_n$ and $(\nabla u_n)_n$ are bounded in the spaces $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega)$ and $\mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)^N$, respectively. Since $\mathcal{M}^{\frac{N(p-1)}{N-p}}(\Omega) \subset \mathcal{M}^{\frac{N(p-1)}{N-1}}(\Omega)$, it follows from $(H2)$ that the sequence $\big(\mathbf{a}(x, u_n, \nabla u_n)\big)_n$ is bounded in $\mathcal{M}^{\frac{N}{N-1}}(\Omega)$ and, consequently, it is bounded in $L^q(\Omega)$ for all $1 \leq q < N/(N-1)$. It follows from the pointwise convergence of this sequence that

$$\mathbf{a}(x, u_n, \nabla u_n) \rightharpoonup \mathbf{a}(x, u, \nabla u) \quad \text{weakly in} \quad L^q(\Omega)$$

for all $1 \leq q < N/(N-1)$. Hence, (43) holds true and $\mathbf{a}(x, u, \nabla u) \in L^1(\Omega)$.

Another easy consequence of (34), (42) and $(H1)$ is $b_n(x, u_n, \nabla u_n) \to b(x, u, \nabla u)$ a.e. On the other hand, to show that the sequence $\big(b_n(x, u_n, \nabla u_n)\big)_n$ is equi-integrable, we may follow the same argument used to prove (17), having in mind (36) instead (15). By Vitali's theorem,

$$b_n(x, u_n, \nabla u_n) \to b(x, u, \nabla u) \quad \text{in} \quad L^1(\Omega), \qquad (44)$$

and so $b(x, u, \nabla u) \in L^1(\Omega)$.

3.- u is solution of (4)

We begin this step by proving that u is a distributional solution of (4). Let $v \in \mathcal{D}(\Omega)$. Then

$$\int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla v + \int_\Omega b_n(x, u_n, \nabla u_n) v = \int_\Omega T_n f v$$

holds for all $n \in I\!N$. Now, applying Lebesgue's Dominated Convergence theorem in the right-hand side, and taking into account (43) and (44), we may take limits and conclude that u is a solution of (4) in the sense of distributions.

It remains to see that u is also an entropy solution. If $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and take $T_k(u_n - v)$ as test function in the weak formulation of (25), then

$$\int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - v) + \int_\Omega b_n(x, u_n, \nabla u_n) T_k(u_n - v) = \int_\Omega T_n f T_k(u_n - v) \quad (45)$$

holds for all $n \in I\!N$. As above, we may take limits in the right hand side and in the second term of the left hand side. To take limits in the first term, let $K = k + \|v\|_\infty$. Then

$$\mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - v) = \mathbf{a}(x, T_K u_n, \nabla T_K u_n) \cdot \nabla T_k(T_K(u_n) - v).$$

On the other hand, (38) implies $\nabla T_k(T_K(u_n) - v) \to \nabla T_k(T_K(u) - v)$ in $L^p(\Omega)$ and $\mathbf{a}(x, T_K u_n, \nabla T_K u_n) \to \mathbf{a}(x, T_K u, \nabla T_K u)$ in $L^{p'}(\Omega)$, so that

$$\mathbf{a}(x, T_K u_n, \nabla T_K u_n) \cdot \nabla T_k(T_K(u_n) - v) \to \mathbf{a}(x, T_K u, \nabla T_K u) \cdot \nabla T_k(T_K(u) - v)$$

in $L^1(\Omega)$. Therefore, $\lim_{n\to\infty} \int_\Omega \mathbf{a}(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - v) = \int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla T_k(u - v)$ and, taking limits in (45), we conclude that u is an entropy solution of (4).

4.- f nonnegative implies u nonnegative

Let $f \in L^1(\Omega)$ be a nonnegative function and let u be an entropy solution of (4). For $h > k > j > 0$, take $v = T_h u + e^{-\gamma(u)} T_j u^-$ in (7). Then we have

$$\int_{\{u \ge 0\}} \mathbf{a}(x, u, \nabla u) \cdot \nabla T_k(u - T_h u) + \int_{\{u<0\}} \mathbf{a}(x, u, \nabla u) \cdot \nabla T_k(u - T_h u - e^{-\gamma(u)} T_j u^-) +$$

$$+ \int_\Omega b(x, u, \nabla u) T_k(u - T_h u - e^{-\gamma(u)} T_j u^-) = \int_\Omega f T_k(u - T_h u - e^{-\gamma(u)} T_j u^-).$$

The first two integrals are nonnegative: so that we may drop the first one and apply Fatou's lemma to the other as h tend to infinity. These facts and Lebesgue's Dominated Convergence theorem yield

$$-\int_{\{u<0\}} \mathbf{a}(x, u, \nabla u) \cdot \nabla T_k(e^{-\gamma(u)} T_j u^-) - \int_\Omega b(x, u, \nabla u) T_k(e^{-\gamma(u)} T_j u^-) \le$$

$$\le -\int_\Omega f T_k(e^{-\gamma(u)} T_j u^-) \le 0.$$

So, denoting $A_k = \{x \in \Omega : e^{-\gamma(u)} T_j u^- < k\}$, we deduce, reasoning as in (10) or (12), that

$$0 \ge -\int_{\{u<0\} \cap A_k} \mathbf{a}(x, u, \nabla u) \cdot \nabla (e^{-\gamma(u)} T_j u^-) -$$

$$-\int_{A_k} b(x,u,\nabla u)e^{-\gamma(u)}T_j u^- - \int_{\Omega\setminus A_k} b(x,u,\nabla u)k \geq$$

$$\geq -\int_{\{u<0\}\cap A_k} e^{-\gamma(u)}\mathbf{a}(x,u,\nabla u)\cdot\nabla T_j u^- - \int_{\{u<0\}\cap A_k} B(x,u)e^{-\gamma(u)}T_j u^- - I_k,$$

where $I_k = \int_{\Omega\setminus A_k} b(x,u,\nabla u)k$. Thus, since $\int_{\{u<0\}\cap A_k} B(x,u)e^{-\gamma(u)}T_j u^- \leq 0$, it follows that

$$0 \geq -\int_{\{u<0\}\cap A_k} e^{-\gamma(u)}\mathbf{a}(x,u,\nabla u)\cdot\nabla T_j u^- - I_k. \tag{46}$$

Now, having in mind $b(x,u,\nabla u)\in L^1(\Omega)$, $e^{-\gamma(u)}T_j u^- \in L^\infty(\Omega)$ and

$$|I_k| \leq \int_{\Omega\setminus A_k} |b(x,u,\nabla u)|\cdot e^{-\gamma(u)}T_j u^-,$$

we obtain $\lim_{k\to\infty} I_k = 0$. On the other hand, we have $\mathbf{a}(x,u,\nabla u)\cdot\nabla T_j u^- \in L^1(\Omega)$, and so, by Lebesgue's theorem,

$$\lim_{k\to\infty}\int_{\{u<0\}\cap A_k} e^{-\gamma(u)}\mathbf{a}(x,u,\nabla u)\cdot\nabla T_j u^- = \int_{\{u<0\}} e^{-\gamma(u)}\mathbf{a}(x,u,\nabla u)\cdot\nabla T_j u^-.$$

Hence, letting k tend to infinity in (46), we get

$$-\int_{\{u<0\}} e^{-\gamma(u)}\mathbf{a}(x,u,\nabla u)\cdot\nabla T_j u^- \leq 0.$$

Now, it follows from $(H2)$ and the boundedness of γ, that $\int_{\{-j<u<0\}} |\nabla u|^p \leq 0$. Finally, a further limit on j yields $\|\nabla u^-\|_p = 0$ and, by Poincare's inequality, $u^- = 0$. This finishes the proof of step 4 and, consequently, of Theorem 2.1.

4 Uniqueness

This section is divided into three parts. In the first one we will prove Proposition 2.2 and Corollary 2.1, which shown what we can prove with the concept of entropy solution; the second part is devoted to the proof of Theorem 2.2 and Proposition 2.3, and the in last part we give simple examples to show our general hypotheses on uniqueness in particular cases.

4.1 *3.1.- Results on uniqueness of entropy solutions*

Proof of Proposition 2.2: Consider the formulation (see Definition 2.2) of u and v as entropy solutions of (4) and let $h>0$. Taking $T_h v$ as test function in the formulation of u and $T_h u$ as test function in that of v, and adding up both identities we deduce

$$\begin{aligned}&\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(u-T_h v) + \int_\Omega \mathbf{a}(x,v,\nabla v)\cdot\nabla T_k(v-T_h u) +\\ &\quad \textstyle\int_\Omega b(x,u,\nabla u)T_k(u-T_h v) + \int_\Omega b(x,v,\nabla v)T_k(v-T_h u) =\\ &\quad = \textstyle\int_\Omega f\big[T_k(u-T_h v) + T_k(v-T_h u)\big].\end{aligned} \tag{47}$$

The two integrals where function $\mathbf{a}$ occurs, can be worked out by the same method of [2, Theorem 5.1] obtaining

$$\liminf_{h\to\infty} \int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(u-T_h v) + \int_\Omega \mathbf{a}(x,v,\nabla v)\cdot\nabla T_k(v-T_h u) \geq$$

$$\geq \int_\Omega \Big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v)\Big)\cdot\nabla T_k(u-v).$$

In the remaining integrals, applying Lebesgue's Dominated Convergence theorem, it yields

$$\lim_{h\to\infty} \int_\Omega b(x,u,\nabla u)T_k(u-T_h v) + \int_\Omega b(x,v,\nabla v)T_k(v-T_h u) =$$

$$= \int_\Omega \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)T_k(u-v)$$

and

$$\lim_{h\to\infty} \int_\Omega f\big[T_k(u-T_h v)+T_k(v-T_h u)\big] = 0.$$

Hence, it follows from (47) that

$$\int_\Omega \Big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v)\Big)\cdot\nabla T_k(u-v)+$$

$$+\int_\Omega \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)T_k(u-v) \leq 0,$$

in other words,

$$\begin{aligned}&\textstyle\int_{\{|u-v|<k\}} \Big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v)\Big)\cdot\nabla T_k(u-v)+\\ &+\textstyle\int_{\{|u-v|<k\}} \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)T_k(u-v) \leq\\ &\leq -k\textstyle\int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)\mathrm{sign}(u-v).\end{aligned} \tag{48}$$

Hypothesis $(H5)$ implies that integrands in the left-hand side of (48) are positive. Thus, on the one hand, $-k\int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)\mathrm{sign}(u-v) \geq 0$ for all $k>0$ and so

$$\limsup_{k\to\infty} k\int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)\mathrm{sign}(u-v) \leq 0.$$

On the other hand, if

$$\limsup_{k\to\infty} k\int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)\mathrm{sign}(u-v) \geq 0,$$

then $\liminf_{k\to\infty} -k\int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)\mathrm{sign}(u-v) \leq 0$ and, by Fatou's lemma, one obtains from (48) that

$$\int_\Omega \Big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v)\Big)\cdot\nabla(u-v) + \int_\Omega \Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)(u-v) \leq 0.$$

Applying $(H5)$ again, we conclude that $u = v$.

Proof of Corollary 2.1: By Proposition 2.2, it is enough to see that

$$\limsup_{k\to\infty} k \int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big)\mathrm{sign}(u-v) \geq 0 \tag{49}$$

Actually, we can see that $\liminf_{k\to\infty} k \int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big)\mathrm{sign}(u-v) \geq 0$ and so, by Proposition 2.2, we obtain that limit exists and is equal to 0.

To prove (49) we only need to control the sign of the integrand on different integration sets, by applying $(H5)$, and pass to the limit using our assumptions on functions u and v. Let us consider the sets $A = \{x \in \Omega : b(x,u(x),\nabla u(x)) - b(x,v(x),\nabla v(x)) > 0\}$ and $B = \{x \in \Omega : b(x,v(x),\nabla v(x)) - b(x,u(x),\nabla u(x)) > 0\}$. Then

$$k \int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big)\mathrm{sign}(u-v) =$$

$$= k \int_{\{u-v\geq k\}\cap A} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big) -$$

$$-k \int_{\{u-v\leq -k\}\cap A} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big) -$$

$$-k \int_{\{u-v\geq k\}\cap B} \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big) +$$

$$+k \int_{\{u-v\leq -k\}\cap B} \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big),$$

and consequently

$$\begin{aligned} &-k \textstyle\int_{\{u-v\leq -k\}\cap A} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big) - \\ &-k \textstyle\int_{\{v-u\leq -k\}\cap B} \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big) \leq \\ &\leq k \textstyle\int_{\{|u-v|\geq k\}} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big)\mathrm{sign}(u-v). \end{aligned} \tag{50}$$

Now, computing in the first member of (50), it yields

$$-k \textstyle\int_{\{u-v\leq -k\}\cap A} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big) -$$

$$-k \textstyle\int_{\{v-u\leq -k\}\cap B} \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big) \geq$$

$$\geq \textstyle\int_{\{u-v\leq -k\}\cap A} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big)(u-v) + + \int_{\{v-u\leq -k\}\cap B} \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big)(v -$$

$$= \textstyle\int_{(\{u-v\leq -k\}\cap A)\cup(\{v-u\leq -k\}\cap B)} \Big(b(x,u,\nabla u) - b(x,v,\nabla v)\Big)(u-v) \geq$$

$$\geq -\textstyle\int_{(\{u-v\leq -k\}\cap A)\cup(\{v-u\leq -k\}\cap B)} \Big(\mathbf{a}(x,u,\nabla u) - \mathbf{a}(x,v,\nabla v)\Big)\cdot\nabla(u-v),$$

by $(H5)$. Thus, (50) becomes

$$\begin{aligned}&\int_{(\{u-v\leq -k\}\cap A)\cup(\{v-u\leq -k\}\cap B)}\Big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v)\Big)\cdot\nabla(u-v)\leq\\&\leq k\int_{\{|u-v|\geq k\}}\Big(b(x,u,\nabla u)-b(x,v,\nabla v)\Big)\mathrm{sign}(u-v).\end{aligned}\tag{51}$$

Since $u,v\in W^{1,p}(\Omega)$, it follows from $(H2)$ that $\Big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,v,\nabla v)\Big)\cdot\nabla(u-v)$ is integrable. On the other hand, the measure of the integration sets in the first member of (51) tends to 0 as k goes to $+\infty$. Therefore, taking limits in (51), it yields (49).

4.2 *3.2.- Uniqueness through the smallest entropy solution*

In order to prove Theorem 2.2 and Proposition 2.3 two lemmata are needed. The first one extends the class of test functions which can be taken.

Lemma 4.1 *Let $f\in L^1(\Omega)$ and let u be an entropy solutions of (4). Then every $w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ satisfying $\nabla w=0$ on the set $\{x\in\Omega : |u|\geq M\}$ for some $M>0$, can be taken as test function: that is,*

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla w+\int_\Omega b(x,u,\nabla u)w=\int_\Omega fw.$$

In particular, if $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, then every $w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ is an admissible test function.

Proof: Let $h>0$ and take $v=T_hu-w$ in the formulation of entropy solution. Then we have

$$\begin{aligned}&\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(u-T_hu+w)+\int_\Omega b(x,u,\nabla u)\ T_k(u-T_hu+w)=\\&=\int_\Omega fT_k(u-T_hu+w).\end{aligned}\tag{52}$$

Now, on the one hand, $\mathbf{a}(x,u,\nabla u)\cdot\nabla w=\mathbf{a}(x,T_Mu,\nabla T_Mu)\cdot\nabla w$, thus $\mathbf{a}(x,u,\nabla u)\cdot\nabla w$ is an integrable function. On the other hand, it follows from $\mathbf{a}(x,u,\nabla u)\cdot\nabla(u-T_hu)\geq 0$ that

$$\begin{aligned}\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(u-T_hu+w)&=\int_{\{|u-T_hu+w|\leq k\}}\mathbf{a}(x,u,\nabla u)\cdot\nabla(u-T_hu+w)\geq\\&\geq\int_{\{|u-T_hu+w|\leq k\}}\mathbf{a}(x,u,\nabla u)\cdot\nabla w.\end{aligned}$$

Therefore,

$$\liminf_{h\to+\infty}\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_k(u-T_hu+w)\geq\int_{\{|w|\leq k\}}\mathbf{a}(x,u,\nabla u)\cdot\nabla w.\tag{53}$$

Let h tend to infinity in (52) by using inequality (53) in the first term and applying Lebesgue's Dominated Convergence theorem in the others; it yields

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla T_kw+\int_\Omega b(x,u,\nabla u)\ T_kw\leq\int_\Omega fT_kw.$$

Since $k > \|w\|_\infty$ implies $T_k w = w$, one deduces

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot \nabla w + \int_\Omega b(x,u,\nabla u)\; w \le \int_\Omega f w.$$

Finally, considering also $-w$ we get the desired equality.

Lemma 4.2 *Let u and v entropy solutions of (4) with data f and g, respectively. Assume that $0 \le g \le f$ and $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Then $0 \le v \le u$.*

Proof: Consider the function $w = -T_k(u-v)^-$ and observe that $\nabla w = 0$ on the set $\{x \in \Omega : |u| \ge M\}$, where $M = k + \|v\|_\infty$. Applying Lemma 4.1, we have

$$-\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot \nabla T_k(u-v)^- - \int_\Omega b(x,u,\nabla u)\; T_k(u-v)^- = -\int_\Omega f T_k(u-v)^-$$

as well as

$$\int_\Omega \mathbf{a}(x,v,\nabla v)\cdot \nabla T_k(u-v)^- + \int_\Omega b(x,v,\nabla v)\; T_k(u-v)^- = \int_\Omega g T_k(u-v)^-.$$

Adding up both equalities, we deduce

$$\begin{aligned}
&\textstyle\int_\Omega \Big(\mathbf{a}(x,v,\nabla v) - \mathbf{a}(x,u,\nabla u)\Big)\cdot \nabla T_k(u-v)^- + \\
&+\textstyle\int_\Omega \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big)\; T_k(u-v)^- = \\
&= \textstyle\int_\Omega (g-f) T_k(u-v)^- \le 0,
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_{\{0\le v-u<k\}} \Big(\mathbf{a}(x,v,\nabla v) - \mathbf{a}(x,u,\nabla u)\Big)\cdot \nabla (u-v) + \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big)(u-v) + \\
&\qquad + k\int_{\{v-u\ge k\}} \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big) \le 0. \qquad (54)
\end{aligned}$$

Note that $k > \|v\|_\infty$ implies $\{v-u \ge k\} = \emptyset$, since $u \ge 0$; so that

$$\int_{\{v-u\ge k\}} \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big) = 0$$

for those k. Thus, by (54),

$$\int_{\{0\le v-u<k\}} \Big(\mathbf{a}(x,v,\nabla v) - \mathbf{a}(x,u,\nabla u)\Big)\cdot \nabla (u-v) + \Big(b(x,v,\nabla v) - b(x,u,\nabla u)\Big)(u-v)$$

is nonpositive and we deduce from $(H5)$ that $u = v$ on the sets $\{0 \le v-u < k\}$, for k big enough. Therefore, $v \le u$.

Proof of Theorem 2.2: Let u be any entropy solution of (4). Consider now the problem (4) with $T_n f$ as datum and denote by v_n its solution. By the L^∞-estimates proved in [10], we obtain $v_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. By Lemma 4.2, it follows from $T_n f \leq f$ that $v_n \leq u$.

On the other hand, reasoning as in the proof of Theorem 2.1, we can find a subsequence $(v_{n_k})_k$ which converges a.e. to an entropy solution of (4), say v. Hence, $v \leq u$ and v is the smallest entropy solution of (4). Note that this smallest entropy solution is obtained as limit of solutions of approximating problems, so that Theorem 2.2 is proved.

Proof of Proposition 2.3: Let u_n and v_n be entropy solutions of (4) with data $T_n f$ and $T_n g$, respectively. Applying again a result of [10], it yields $u_n, v_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Since $0 \leq T_n g \leq T_n f$, it follows from Lemma 4.2 that $0 \leq v_n \leq u_n$. Now, up to subsequences, $(u_n)_n$ and $(v_n)_n$ converge a.e. to u and v, respectively (we only have to follow the same argument that that of the proof of Theorem 2.1). Hence, $0 \leq v \leq u$.

4.3 *3.3.- Examples*

We finish this paper by giving a sufficient condition to obtain $(H5)$ in some cases which include our model equation (see Corollary 4.1).

Proposition 4.1 *Let α and β be real positive functions of class C^1 on $[0,+\infty[$. Suppose that α is bounded by some positive constants, i.e., $0 < \lambda \leq \alpha(s) \leq \Lambda$, and that β is nonincreasing on $[0,+\infty[$. If either*

(1) $\big(\alpha'(s) - p\beta(s)\big)^2 \leq -4(p-1)\alpha(s)\beta'(s)$ for all $s \geq 0$ and $B(x,.)$ is increasing, or

(2) $\big(\alpha'(s) - p\beta(s)\big)^2 < -4(p-1)\alpha(s)\beta'(s)$ for all $s \geq 0$ and $B(x,.)$ is nondecreasing,

then $(H5)$ holds true for $\mathbf{a}(x,s,\xi) = \alpha(s)|\xi|^{p-2}\xi$ and $b(x,s,\xi) = -\beta(s)|\xi|^p + B(x,s)$.

Proof: Let $w = s - r$ and $\rho = \xi - \eta$ (we may and will assume that $w \geq 0$, if not we would define $w = r - s$ and $\rho = \eta - \xi$ without lost of generality) and define $f : [0,1] \to \mathbb{R}$ by

$$f(t) = \alpha(r+tw)|\eta + t\rho|^{p-2}(\eta + t\rho)\cdot\rho - \beta(r+tw)|\eta + t\rho|^p w$$

Observe that f is well defined, and differentiable at every point, except perhaps at a such that $\eta + a\rho = 0$, when $p < 2$. For the sake of simplicity let us write

$$A(t) = \sqrt{-(p-1)\alpha(r+tw)\beta'(r+tw)} - \frac{1}{2}|\alpha'(r+tw) - p\beta(r+tw)|,$$

which is nonnegative when assumption (1) holds true and positive if one assumes condition (2).

We have to see that $f(1) - f(0) \geq 0$ under condition (1), respectively $f(1) - f(0) > 0$ under condition (2). Since both proofs are similar, we will only prove $f(1) - f(0) \geq 0$ assuming (1). To do this derive and apply Cauchy-Schwarz's inequality to get

$$\begin{aligned} f'(t) &\geq (p-1)\alpha(r+tw)|\eta + t\rho|^{p-4}((\eta + t\rho)\cdot\rho)^2 + \\ &+ \big[\alpha'(r+tw) - p\beta(r+tw)\big]|\eta + t\rho|^{p-2}((\eta + t\rho)\cdot\rho)w - \beta'(r+tw)|\eta + t\rho|^p w^2. \end{aligned} \tag{55}$$

Under assumption (1), it is possible that $\beta'(r+tw)=0$ for some $t\in]0,1[$, but then $\alpha'(r+tw)-p\beta(r+tw)=0$ and so

$$f'(t)\geq (p-1)\alpha(r+tw)|\eta+t\rho|^{p-4}((\eta+t\rho)\cdot\rho)^2\geq 0.$$

On the other hand, if $\beta'(r+tw)\neq 0$, then to get $f'(t)\geq 0$, just complete a square and simplify disregarding some nonnegative terms. Indeed, it follows from (55) that

$$f'(t)\geq |\eta+t\rho|^{p-4}\Big[(p-1)\alpha(r+tw)((\eta+t\rho)\cdot\rho)^2+$$

$$+\big(\alpha'(r+tw)-p\beta(r+tw)\big)|\eta+t\rho|^2((\eta+t\rho)\cdot\rho)w-\beta'(r+tw)|\eta+t\rho|^4w^2\Big]=$$

$$=|\eta+t\rho|^{p-4}\Bigg[\Bigg((p-1)\alpha(r+tw)-$$

$$\tfrac{-1}{2}\sqrt{\tfrac{(p-1)\alpha(r+tw)}{-\beta'(r+tw)}}\big|\alpha'(r+tw)-p\beta(r+tw)\big|\Bigg)((\eta+t\rho)\cdot\rho)^2+$$

$$+\Bigg(-\beta'(r+tw)-\tfrac{1}{2}\sqrt{\tfrac{-\beta'(r+tw)}{(p-1)\alpha(r+tw)}}\big|\alpha'(r+tw)-p\beta(r+tw)\big|\Bigg)|\eta+t\rho|^4w^2+$$

$$+\Bigg(\sqrt[4]{\tfrac{(p-1)\alpha(r+tw)}{-4\beta'(r+tw)}\big(\alpha'(r+tw)-p\beta(r+tw)\big)^2}\ |(\eta+t\rho)\cdot\rho|-$$

$$-\sqrt[4]{\tfrac{-\beta'(r+tw)}{4(p-1)\alpha(r+tw)}\big(\alpha'(r+tw)-p\beta(r+tw)\big)^2}\ |\eta+t\rho|^2w\Bigg)^2\ \Bigg]\geq$$

$$\geq|\eta+t\rho|^{p-4}\Bigg[\sqrt{\tfrac{(p-1)\alpha(r+tw)}{-\beta'(r+tw)}}A(t)((\eta+t\rho)\cdot\rho)^2+\sqrt{\tfrac{-\beta'(r+tw)}{(p-1)\alpha(r+tw)}}A(t)|\eta+t\rho|^4w^2\Bigg].$$

Since $A(t)\geq 0$, it follows that $f'(t)\geq 0$, for all $t\in]0,1[$, except perhaps at a such that $\eta+a\rho=0$, so that one concludes that $f(1)-f(0)\geq 0$.

As a consequence of Proposition 4.1, we have the following result whose proof is elementary.

Corollary 4.1 *Let σ be a real function of class C^2 on $[0,+\infty[$ which is bounded by some positive constants. Taking $\alpha(s)=\frac{1}{\sigma(s)^{p-1}}$ and $\beta(s)=\frac{p-1}{p}\frac{\sigma'(s)}{\sigma(s)^p}$, then condition (2) in Proposition 4.1 holds if and only if $-\sigma''(s)\sigma(s)>0$ for all $s\geq 0$; in other words, when σ is strictly concave on $[0,+\infty[$.*

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