Multiplicity of solutions to a concave–convex problem

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Abstract

This paper studies two related problems. A first one,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{r-2}u & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p–Laplacian operator, Ω is a *ball* of $\mathbb{R}^N$, ν stands for the outer unit normal and $\lambda > 0$ is a parameter. Exponents are supposed to satisfy $1 < q < p < r \le p^*$, $p^* = Np/(N-p)$ if $1 < p < N$, $p^* = \infty$ otherwise. The existence of $\Lambda > 0$ is shown so that (P) does not admit positive solutions if $\lambda > \Lambda$, a minimal positive solution exists when $0 < \lambda \le \Lambda$ and most importantly, a further second positive solution arises if $0 < \lambda \le \Lambda$. Hence, extensions of results in [2], [3], [12] and [13] to the framework of (P) are provided. Second problem considered is the variant of (P):

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{r-2}u & x \in \partial\Omega, \end{cases} \tag{Q}$$

where Ω is a ball and $1 < q < p < r$. Features described above are shown to be also exhibited by (Q) and more importantly, it is proved that minimal solutions to (Q) develop flat patterns in the degenerate regime $p > 2$. Finally, it should be stressed that some of the properties satisfied by (P) and (Q) hold true when Ω is a general smooth domain.

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1 Introduction and statement of the main results

Reaction–diffusion problems has been an active research field in nonlinear partial differential equations since the late sixties. Analysis of existence or multiplicity of their equilibrium states (semilinear or quasilinear elliptic boundary value problems), together with the variation of their number as a response to parameters perturbation have been trending subjects from the beginning. Of course, there is a huge amount of available literature on the field and we just refer to some few general texts or reviews to catch some general traits on the topic. Namely, [22], [28], [24], [23]. For the purposes of the present work, a relevant recent one is [25], specifically concerned with nonlinear boundary conditions.

A particular interest area in nonlinear diffusion theory deals with the interaction between a concave nonlinearity with a convex one. These are the so–called "concave–convex" problems and its study can be traced back to the works [10], [11],

[4] (involving the p–Laplacian operator) and [2]. The latter analyzes the problem

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + |u|^{r-2}u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain and $\lambda > 0$ is a parameter. More importantly,

$$1 < q < 2 < r \leq 2^*,$$

where $2^* = \dfrac{2N}{N-2}$. Among other main features, it was shown there the existence of a critical value Λ of λ so that no positive solutions are possible for $\lambda > \Lambda$ while a minimal positive solution u_λ exists for all $0 < \lambda \leq \Lambda$. Moreover, the existence of a second positive solution was also stated for such range of the parameter λ.

Results in [2] were later achieved in [3] for the radially symmetric p–Laplacian version of (1.1),

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{r-2}u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

with

$$1 < q < p < r < p^*, \tag{1.3}$$

$p^* = \dfrac{Np}{N-p}$ for $p > N$, $p^* = \infty$ otherwise. In [3] Ω is a ball, solutions are radial and a value Λ is found so that no positive solutions exist for $\lambda > \Lambda$ while two positive solutions exist if $0 < \lambda < \Lambda$.

Extension of results in [3] to a general domain Ω was a quite delicate task that was addressed in [12]. The crux of the matter was the difficulty in getting L^∞ uniform estimates for bounded values of λ. Obstacle was circumvented by extending the variational approach in [6] to the context of (1.2). Nevertheless, this also required an elaborated analysis of uniform $C^{1,\alpha}$ estimates of the solutions of an associated auxiliary problem.

By the way, it should be observed that in problems (1.1) and (1.2), both concave and convex nonlinearities are source terms. For a model where such interaction is switched to "absorption terms" readers are referred to [16]. In [7] concave and convex terms appear in the equation with opposite signs. On the other hand, possible combinations of the concave–convex effect do not need to be necessarily located together in the volumetric reaction term. In fact, analysis in [2] was carried out in [13] for a reaction–diffusion problem subject to a nonlinear flux boundary condition. Namely,

$$\begin{cases} -\Delta u + u = |u|^{r-2}u & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & x \in \partial\Omega, \end{cases} \tag{1.4}$$

exponents q, r under the same restrictions as in [2]. Again, range of existence of positive solutions is shown to be an interval $0 < \lambda \leq \Lambda$ while existence of two positive solutions is attained in such interval. This fact is what is termed in [12] as "global multiplicity" of solutions.

In the present paper we are focusing our interest in the p–Laplacian version of the problem treated in [13]. Specifically,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{r-2}u & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where exponents q, r satisfies,

$$1 < q < p < r \leq p^*. \tag{1.6}$$

It should be stressed that a new extra difficulty arises when dealing with (1.5). Namely, the lack of a strong comparison principle to strictly separate two minimal solutions at different values of λ. While a version of such result is available for (1.2) (see [12] and [8], [9], [17], [30] for restricted versions of the strong comparison principle for the case of Dirichlet conditions) and is, of course, well known for (1.4), no similar general result works for our problem (1.5) (see further comments in [27]). When studying (1.5) such kind of comparison must be proved "ad hoc". Moreover, in the variant (1.16) below of that problem, strong comparison only holds if certain conditions are met. In fact, the presence of "flat patterns" under certain regime for (1.16) entails that strong comparison fails (see Remark 1.1 below).

A preliminary study of (1.5) was presented in [27], where existence of positive solutions was shown to occur only for $0 < \lambda \leq \Lambda$, for certain positive Λ, and the existence of a minimal positive solution was attained in that range. However, the validity of a global multiplicity result similar to the ones mentioned before remained open there.

Here we are showing such multiplicity result by confining ourselves to the case where Ω is a ball of $\mathbb{R}^N$ and solutions are radially symmetric. We are delaying the study of the general case to a forthcoming work. A first main result is the following.

Theorem 1.1 *Assume $\Omega = B$, $B = \{x \in \mathbb{R}^N : |x| < 1\}$, while exponents q, r satisfies* (1.6)*. Then problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{r-2}u & x \in B \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & x \in \partial B, \end{cases} \tag{1.7}$$

exhibits the following properties.

i) There exists $\Lambda > 0$ such that positive solutions to (1.7) *are only possible if*

$$0 < \lambda \leq \Lambda. \tag{1.8}$$

ii) For all λ satisfying (1.8) *there exists a minimal positive radial solution $u_\lambda \in C^{1,\alpha}(\overline{B})$ for a certain $0 < \alpha < 1$. Moreover, as a function $\lambda \mapsto u_\lambda \in C^{1,\alpha}(\overline{B})$, u_λ is continuous from the left and*

$$\lim_{\lambda \to 0} u_\lambda = 0. \tag{1.9}$$

Furthermore, u_λ is smooth in λ for λ small and:

$$u_\lambda(x) = I^{-\frac{1}{p-q}} v_0(x) \lambda^{\frac{1}{p-q}} + o(\lambda^{\frac{1}{p-q}}) \qquad \text{as } \lambda \to 0, \tag{1.10}$$

in $C^1(\overline{B})$, where $v_0(x)$ is the unique solution to:

$$\begin{cases} -\Delta_p v + |v|^{p-2} v = 0 & x \in \Omega \\ v = 1 & x \in \partial\Omega, \end{cases} \tag{1.11}$$

and $I = \int_0^1 t^{N-1} v_0(t)^{p-1}\, dt$.

iii) Assume w_λ is a family of positive radial solutions to (1.7) *satisfying $\|w_\lambda\|_\infty \to 0$ as $\lambda \to 0+$. Then*

$$w_\lambda = u_\lambda, \tag{1.12}$$

for λ sufficiently small.

iv) Suppose $r < p^*$. Then, for every $0 < \lambda < \Lambda$, problem (1.7) *posses a second positive solution $v_\lambda \in C^{1,\alpha}(\overline{B})$. Moreover, a positive constant M exists such that every possible positive radial solution u to* (1.7) *with $0 < \lambda \le \Lambda$ satisfies,*

$$\|u\|_{1,\alpha} \le M. \tag{1.13}$$

Remark 1.1

a) Relation (1.9) says that minimal solution u_λ bifurcates from zero at $\lambda = 0$. On the contrary, (1.13) implies that no bifurcation from infinity occurs at $\lambda = 0$.

b) Existence of a minimal solution to (1.4) for $0 < \lambda < \Lambda$ was obtained in [27] for a general domain Ω.

c) Restriction $r \le p^*$ in (1.6) can be relaxed in statements i) to iii) (see Section 2 and Remark 6.1).

A related associated problem to (1.4) that was studied in [13] is

$$\begin{cases} -\Delta u + u = |u|^{q-2} u & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda |u|^{r-2} u & x \in \partial\Omega, \end{cases} \tag{1.14}$$

with exponents q, r now satisfying,

$$1 < q < 2 < r < 2_*, \tag{1.15}$$

where $2_* = \dfrac{2(N-1)}{N-2}$. In other words, concave and convex terms interchange in (1.4) their rôles in Ω and $\partial\Omega$. In [27] some preliminary facts on the p-Laplacian version of (1.14) were analyzed. Such problem is,

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = |u|^{q-2} u & x \in \Omega \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda |u|^{r-2} u & x \in \partial\Omega, \end{cases} \tag{1.16}$$

with q, r verifying

$$1 < q < r \leq p_* , \tag{1.17}$$

where $p_* = \dfrac{p(N-1)}{N-p}$ if $p > N$, $p_* = \infty$ otherwise.

We now state a existence result of a minimal solution to (1.16) which extends the corresponding one for the linear diffusion problem (1.14) contained in [13].

Theorem 1.2 *Let Ω be a bounded smooth domain and assume that exponents fall in the range*

$$1 < q < p < r \leq p_*. \tag{1.18}$$

Then problem (1.16) *exhibits the following properties.*

i) There exists $\Lambda > 0$ so that positive solutions to (1.16) *are only possible when:*

$$0 < \lambda \leq \Lambda.$$

ii) All possible positive solutions u to (1.16) *satisfies*

$$u(x) \geq 1 \qquad x \in \overline{\Omega}. \tag{1.19}$$

iii) For $0 < \lambda \leq \Lambda$ there exists a minimal solution $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. In addition

$$u_\lambda \to 1 \tag{1.20}$$

in $C^{1,\alpha}(\overline{\Omega})$ as $\lambda \to 0+$. Moreover, any possible family w_λ of positive solutions to (1.16) *verifying*

$$\|w_\lambda\|_\infty = O(1), \tag{1.21}$$

as $\lambda \to 0$ necessarily satisfies (1.20).

iv) In the degenerate regime $p > 2$, there exists $\lambda_0 > 0$ such that for every $0 < \lambda < \lambda_0$ the region

$$\mathcal{F}_\lambda = \{x \in \Omega : u_\lambda(x) = 1\},$$

becomes nonempty. Furthermore,

$$\mathcal{F}_\lambda \supset \{x \in \Omega : d(x) \geq d(\lambda)\},$$

with $d(x) = \text{dist }(x, \partial\Omega)$ and certain $d(\lambda) \to 0$ as $\lambda \to 0+$. Such features are also exhibited by any family w_λ of positive solutions to (1.16) *fulfilling* (1.21) *as $\lambda \to 0+$.*

Remark 1.1 Relevant pieces of information in Theorem 1.2 are the estimate (1.19) and, more importantly, the presence of "flat patterns" $\mathcal{F}_\lambda$ in the minimal solution when $\lambda \to 0+$. Contrary to what happens with the problem under convex volumetric reaction (1.5), for which strong comparison between minimal solutions hold (see Section 4), such solutions u_λ to (1.16) clearly violates such principle when $p > 2$. These features were not noticed in [27].

Next statement sharpens the results in Section 9 in [13] concerning problem (1.14). As mentioned above, problem (1.14) posses a minimal solution u_λ for every $0 < \lambda < \Lambda$. It was further shown in [13] the existence of a second positive solution $\hat{u}_\lambda$ for λ in that range. Our next result completes in some sense the global picture of the set of positive solutions to (1.14) as $\lambda \to 0+$.

Theorem 1.3 *Let Ω be a bounded smooth domain and assume that q, r satisfy* (1.15) *and for $0 < \lambda \leq \Lambda$ let u_λ be the minimal solution to* (1.14). *The following features hold true.*

i) Assume that $\tilde{u}_\lambda$ is any family of positive solutions to (1.14) *that satisfy*

$$\|\tilde{u}_\lambda\|_\infty = O(1),$$

as $\lambda \to 0+$. Then, there exists $\lambda_1 > 0$ such that

$$\tilde{u}_\lambda = u_\lambda$$

for $0 < \lambda < \lambda_1$.

ii) Let $\hat{u}_\lambda$ be any family of positive solutions to (1.14) *distinct from the family of minimal solutions u_λ. Then,*

$$\|\hat{u}_\lambda\|_\infty \to \infty,$$

as $\lambda \to 0+$.

Remark 1.2 Theorem 1.3 states that the unique family of positive solutions to (1.14) that remains bounded as $\lambda \to 0+$ is just u_λ. On the contrary, any other possible family of "extra" positive solutions to (1.14) must bifurcate from infinity as $\lambda \to 0+$. This fact is in strong contrast with both problems (1.4) and (1.5) where all possible families of positive solutions remain uniformly bounded as $\lambda \to 0+$ (see [27]).

Our next result furnishes a global multiplicity result for the alternative version (1.16) of problem (1.5), in the radially symmetric case.

Theorem 1.4 *Assume that*

$$1 < q < p < r, \tag{1.22}$$

while $B = \{x \in \mathbb{R}^N : |x| < 1\}$. Then problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & x \in B \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{r-2}u & x \in \partial B, \end{cases} \tag{1.23}$$

exhibits the following properties.

i) There exists $\Lambda > 0$ such that problem (1.23) *only admits positive radial solutions when λ satisfies $0 < \lambda \leq \Lambda$.*

ii) For every $0 < \lambda \le \Lambda$ *there exists a minimal radial positive solution* $u_\lambda(|x|)$ *such that*

$$u_\lambda \to 1$$

as $\lambda \to 0+$ *in* $C^{1,\alpha}(\overline{B})$ *for some* $0 < \alpha < 1$.

iii) For each $0 < \lambda < \Lambda$, *problem* (1.23) *admits a second positive radial solution* v_λ *such that*

$$\|v_\lambda\|_\infty \to \infty, \tag{1.24}$$

as $\lambda \to \infty$.

iv) Assume that w_λ *is a family of positive radial solutions verifying* $\|w_\lambda\|_\infty = O(1)$ *as* $\lambda \to 0+$. *Then*

$$w_\lambda = u_\lambda, \tag{1.25}$$

for $0 < \lambda < \lambda_2$. *In particular, any possible family* $\tilde{v}_\lambda$ *of positive radial solutions to* (1.23) *distinct from* u_λ *must satisfy*

$$\|\tilde{v}_\lambda\|_\infty \to \infty,$$

as $\lambda \to 0+$.

v) Suppose that $p > 2$. *Then,*

$$\mathcal{F}_\lambda := \{u_\lambda(x) = 1\},$$

becomes nonempty for $0 < \lambda < \lambda_0$ *while* $\mathcal{F}_\lambda = \overline{B}(0, \rho(\lambda))$ *where*

$$d(\lambda) := 1 - \rho(\lambda) \sim B\lambda^\beta \tag{1.26}$$

as $\lambda \to 0+$ *with*

$$\beta = \frac{p-2}{2(p-1)}, \qquad B = \frac{p}{p-2}\left(\frac{2}{p'(p-q)}\right)^{1/2}. \tag{1.27}$$

Moreover, the asymptotic profile of u_λ *beyond the flat pattern is given by*

$$u_\lambda(x) \sim 1 + C(|x| - \rho(\lambda))^{\frac{1}{\alpha}} \qquad \rho(\lambda) \le |x| \le 1, \tag{1.28}$$

as $\lambda \to 0$, *where:*

$$\alpha = \frac{p-2}{p}, \qquad C = \alpha^{\frac{1}{\alpha}}\left(\frac{p'(p-q)}{2}\right)^{\frac{1}{p-2}}.$$

This paper is organized as follows. Section 2 introduces some basic aspects of problems (1.5) and (1.16) together with further auxiliary tools. Section 3 is devoted to obtain L^∞ estimates of the solutions of such problems in the radial case. Strong comparison between minimal solutions to (1.5) at different values of λ is the objective of Section 4. A corresponding result for (1.16), in the nondegenerate regime $1 < p \le 2$, is also provided. Multiplicity statements in Theorems 1.1 and 1.4 (case $1 < p \le 2$ in problem (1.16)) are shown in Section 5. Proofs of Theorems 1.2, 1.3 and 1.4 with especial emphasis in the phenomenon of flat pattern formation are located in Section 6. Features distinct from multiplicity and stated in Theorem 1.1, mainly uniqueness of small solutions to (1.5) for λ near zero, are also shown in that Section.

2 Preliminary results

In this section we are discussing some basic properties of solutions to problems (1.5) and (1.16). In addition some auxiliary problems to be employed in parts of the work are introduced and studied. We remark that in most situations, features considered are treated in the more general setting of Ω a general domain, since radial symmetry has no influence for their validity. In that case it will be assumed that Ω is a class $C^{1,\gamma}$ bounded domain for some $0<\gamma<1$.

We begin with smoothness of weak solutions to both (1.5) and (1.16). A function $u\in W^{1,p}(\Omega)$ is a weak solution to (1.5) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v+|u|^{p-2}uv=\int_\Omega |u|^{r-2}uv+\lambda\int_{\partial\Omega}|u|^{q-2}uv, \tag{2.1}$$

for all $v\in W^{1,p}(\Omega)$. Here we assume that $1<q<p<r\le p^*$. Weak solutions to (1.16) are defined similarly but interchanging the terms $|u|^{r-2}u$ and $|u|^{q-2}u$ in the integrals of the right hand side of (2.1), and assuming also that $1<q<p<r\le p_*$. A detailed proof of the next result can be found in [27].

Proposition 2.1 *Let Ω be a bounded smooth domain and assume that $1<q<p<r\le p^*$. Then, every weak solution $u\in W^{1,p}(\Omega)$ to (1.5) satisfies*

$$u\in L^\infty(\Omega).$$

In particular such a solution u lies on $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$ which depends on $\|u\|_{L^\infty(\Omega)}$. The same features hold for weak solutions to (1.16) provided $1<q<p<r\le p_*$.

Remark 2.1 A substantial part of the proof of Proposition 2.1 consists in showing the boundedness of weak solutions $u\in W^{1,p}(\Omega)$. Their $C^{1,\alpha}$ regularity then follows from the results in [21].

Since a main part of this work deals with radial solutions to problems (1.5) and (1.16), some of their properties are going to be reviewed. A function $\tilde u\in W^{1,p}(B)$ is said to be radial if $\tilde u(Rx)=\tilde u(x)$ a. e. in B for all orthogonal transformations R in $\mathbb{R}^N$. In that case, there exists a function $u=u(\mathrm{r})$ which is absolutely continuous in $[\varepsilon,1]$ for all $\varepsilon>0$ such that:

$$\tilde u(x)=u(\mathrm{r})\qquad \nabla\tilde u(x)=u'(\mathrm{r})\frac{x}{|x|}\qquad \text{a. e. in } B,$$

where $\mathrm{r}=|x|$. Moreover, both $u,u'\in L^p((0,1),\mathrm{r}^{N-1}d\mathrm{r})$ with

$$\|\tilde u\|^p_{W^{1,p}(B)}=\sigma_N\int_0^1\{|u|^p+|u'|^p\}\ \mathrm{r}^{N-1}\ d\mathrm{r},$$

σ_N being the $(N-1)$–dimensional measure of ∂B. Assume now that $\tilde u\in W^{1,p}(B)$ is a radial weak solution to (1.5), q,r satisfying the conditions of Proposition 2.1. Then, it can be shown that $\tilde u(x)=u(|x|)$ a. e. in B such that,

i) $u \in C^1(I)$, $\mathrm{r}^{N-1}|u'|^{p-2}u' \in C^1(I)$ with $I = [0,1]$,

ii) u solves the problem

$$\begin{cases} -(\mathrm{r}^{N-1}|u'|^{p-2}u')' = \mathrm{r}^{N-1}(|u|^{r-2}u - |u|^{p-2}u) & \mathrm{r} \in I \\ u'(0) = 0 \\ |u'(1)|^{p-2}u'(1) = \lambda |u(1)|^{q-2}u(1). \end{cases} \tag{2.2}$$

Conversely, it can be proved that if u satisfies conditions i) and ii) above then $\tilde{u}(x) = u(|x|)$ gives rise to a radial weak solution to (1.5). The same assertions hold true for problem (1.16) after interchanging the terms involving $|u|^{q-2}u$ and $|u|^{r-2}u$ in the previous expressions. A detailed account of these facts is here omitted for brevity. When dealing with radial solutions to both (1.5) and (1.16) these features will be assumed without further comments.

Another relevant remark concerning (1.5) is that restriction $r \le p^*$ can be relaxed in the radial case if we admit instead that weak solutions satisfy from the start $\tilde{u} \in W^{1,p}(B) \cap L^r(B)$. Under this assumption and by using a radial test function in (2.1) we obtain,

$$\int_0^1 |u'|^{p-2}u'\psi'\mathrm{r}^{N-1}\, d\mathrm{r} = \int_0^1 (|u|^{r-2}u - |u|^{p-2}u)\mathrm{r}^{N-1}\psi\, d\mathrm{r} + \\ \lambda |u'(1)|^{q-2}u'(1)\psi(1), \tag{2.3}$$

for all $\psi \in C^1[0,1]$, $\psi'(0) = 0$. This implies that $\mathrm{r}^{N-1}|u'|^{p-2}u' \in W^{1,1}(0,1)$ and solves the equation in (2.2) a. e. in $(0,1)$. Moreover, after redefining $\mathrm{r}^{N-1}|u'|^{p-2}u'$ in a zero set ([5]) we find that

$$|u'(1)|^{p-2}u'(1)\psi(1) - c_0\psi(0) = \lambda |u(1)|^{q-2}u(1)\psi(1),$$

where $c_0 = \lim_{\mathrm{r}\to 0+} \mathrm{r}^{N-1}|u'|^{p-2}u'$. Suitable choices of ψ lead to

$$|u'(1)|^{p-2}u'(1) = \lambda |u(1)|^{q-2}u(1) \qquad \text{and} \qquad c_0 = 0.$$

In particular,

$$|u'(\mathrm{r})|^{p-2}u'(\mathrm{r}) = -\int_0^{\mathrm{r}} \left(\frac{t}{\mathrm{r}}\right)^{N-1} (|u|^{r-2}u - |u|^{p-2}u)\, dt. \tag{2.4}$$

This entails that in addition $u \in C^1(I)$ with $u'(0) = 0$ and so $u \in L^\infty$.

As for the case of problem (1.16) observe that radial solutions $\tilde{u} \in W^{1,p}(B)$ are bounded on ∂B. Therefore, no restrictions are needed in the size of exponent r in this case.

We are now dealing with a nonlinear eigenvalue problem of Steklov type. Consider

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \mu a(x)|u|^{p-2}u & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda b(x)|u|^{p-2}u & x \in \partial\Omega, \end{cases} \tag{2.5}$$

where $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$, μ has the status of an eigenvalue while λ is regarded as a parameter. Some relevant properties exhibited by (2.5) are next listed. We refer to [15] for the case $p = 2$ and to [27], [20] for an account of their proofs in the p-Laplacian case. Such properties are:

i) For all $\lambda \in \mathbb{R}$, (2.5) admits a unique eigenvalue $\mu := \mu_1(\lambda)$ with the property of admitting a positive eigenfunction, written $u := \phi_1(x, \lambda)$ when normalized as $\|u\|_\infty = 1$. Moreover, $\phi_1 \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ and continuously varies with λ when regarded as taking values in $C^{1,\alpha}(\overline{\Omega})$.

ii) $\mu_1(\lambda)$ is a C^1, concave and decreasing function of λ such that,

$$\lim_{\lambda\to-\infty} \mu_1 = \lambda_{1,D} \qquad \lim_{\lambda\to\infty} \mu_1 = -\infty,$$

where $\lambda_{1,D}$ is the first Dirichlet eigenvalue of the operator $\mathcal{L}u := -\Delta_p u + |u|^{p-2}u$ in Ω.

Since $\mu_1(0) = 1$, ii) implies in particular the existence of a unique positive zero λ_1 of μ_1. Such value provides the first Steklov eigenvalue to the problem,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda b(x)|u|^{p-2}u & x \in \partial\Omega. \end{cases} \tag{2.6}$$

For later use in Section 5 we are now analyzing some properties of the solution operator associated to the problem,

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = f & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = g & x \in \partial\Omega, \end{cases} \tag{2.7}$$

where $f \in (W^{1,p}(\Omega))^*$, $g \in (W^{1-\frac{1}{p},p}(\partial\Omega))^*$ are data, "$*$" meaning dual space. A weak solution $u \in W^{1,p}(\Omega)$ is defined through the equality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv = \langle f, v\rangle + \langle g, v\rangle, \tag{2.8}$$

which must be satisfied for all $v \in W^{1,p}(\Omega)$, $\langle\cdot,\cdot\rangle$ standing for the corresponding duality pairings in $W^{1,p}(\Omega)$ and $W^{1-\frac{1}{p},p}(\partial\Omega)$ and where in the last term, functional g is composed with the trace operator. By the way, it is said that a functional $f \in (W^{1,p}(\Omega))^*$ (respectively, $g \in (W^{1-\frac{1}{p},p}(\partial\Omega))^*$) satisfies $f \geq 0$ ($g \geq 0$) provided that

$$\langle f, v\rangle \geq 0 \qquad \forall v \in W^{1,p}(\Omega),\ v \geq 0,$$

($\langle g, v\rangle \geq 0$ for all $v \in W^{1-\frac{1}{p},p}(\partial\Omega)$, $v \geq 0$). If $\Omega = B$, f is said to be rotationally invariant if $\langle f, v\rangle = \langle f, v \circ R\rangle$ for all orthogonal transformations R in $\mathbb{R}^N$, where $v \circ R(x) = v(Rx)$ (same definition works for g).

A first result concerning (2.7) is now stated. For brevity, we are writing $X := W^{1,p}(\Omega)$ and $Y := W^{1-\frac{1}{p},p}(\partial\Omega)$.

Theorem 2.1 *Assume $\Omega \subset \mathbb{R}^N$ is a class $C^{1,\gamma}$ bounded domain for a certain $0 < \gamma < 1$. Then the following properties hold.*

a) To every pair $(f, g) \in X^* \times Y^*$ corresponds a unique solution $u \in W^{1,p}(\Omega)$.

b) Let $u_i \in W^{1,p}(\Omega)$ be the solutions corresponding to $(f_i, g_i) \in X^* \times Y^*$, $i = 1, 2$. If it is assumed that $f_1 \leq f_2$ and $g_1 \leq g_2$ then

$$u_1(x) \leq u_2(x),$$

a. e. in Ω.

c) If $\Omega = B$, and both $f \in X^*$ and $g \in Y^*$ are rotationally invariant, then the corresponding solution $u \in W^{1,p}(B)$ to (2.7) is radial.

Proof. We are sketching the proof for the sake of completeness. As for the existence statement in a), a weak solution $u \in W^{1,p}(\Omega)$ to (2.7) is achieved by solving the minimization problem

$$\inf_{v \in W^{1,p}(\Omega)} J(v),$$

where $J(v) = \dfrac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} + \langle f, v\rangle + \langle g, v\rangle$. Functional J is convex and coercive in $W^{1,p}(\Omega)$ and therefore it admits a unique global minimizer u which furnishes the unique solution to (2.7).

Regarding b), by substraction of the weak equations for u_1 and u_2 we arrive to

$$\int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla v + (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)v = \\ \langle f_1 - f_2, v\rangle + \langle g_1 - g_2, v\rangle,$$

for all $v \in W^{1,p}(\Omega)$. By putting $v = (u_1 - u_2)_+$ (v_+ denoting the positive part of the function v) we find

$$\int_\Omega (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla (u_1 - u_2)_+ \\ + (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2)_+ = 0.$$

Monotonicity of the p–Laplacian ([30]) then implies that $\nabla(u_1 - u_2)_+ = 0$ what in turns implies that $(u_1 - u_2)_+ = 0$. This shows b).

As for c) and by employing the definition of weak solution it is easily seen that for an arbitrary orthogonal transformation R of $\mathbb{R}^N$, $u_R := u \circ R$ also solves (2.7) provided that u is a solution and f, g are rotationally invariant. Therefore $u = u_R$ which amounts to the radial character of u.

Theorem 2.1 permits us introducing the solution operator,

$$\begin{array}{rccl} S: & X^* \times Y^* & \to & W^{1,p}(\Omega) \\ & (f, g) & \mapsto & S(f, g) = u, \end{array}$$

mapping data (f, g) to the solution u to (2.7).

Proposition 2.2 *Solution operator S to (2.7) exhibits the following properties.*

a) S is continuous.

b) S maps bounded sets into bounded sets.

c) S is order preserving, i.e. $f_1 \leq f_2$ and $g_1 \leq g_2$ in X^* and Y^*, respectively, implies that

$$S(f_1, g_1) \leq S(f_2, g_2).$$

Proof. Relation (2.8) implies that

$$\|u\|^{p-1}_{W^{1,p}(\Omega)} \leq \|f\| + \|g\|,$$

what directly entails b).

To show a) assume that $(f_n, g_n) \to (f, g)$ in $X^* \times Y^*$ and set $u_n = S(f_n, g_n)$. We learn from b) that u_n is bounded in $W^{1,p}(\Omega)$ and thus, after extracting a subsequence, we obtain that $u_n \to u$ both in $L^p(\Omega)$ and $L^p(\partial\Omega)$. On the other hand,

$$\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m) \cdot \nabla(u_n - u_m) \leq \\ \langle f_n - f_m, u_n - u_m\rangle + \langle g_n - g_m, u_n - u_m\rangle = o(1), \quad (2.9)$$

since both $f_n - f_m \to 0$, $g_n - g_m \to 0$ and u_n is bounded. We claim that $\nabla u_n - \nabla u_m \to 0$ in $L^p(\Omega)$. This means that u solves (2.7) and by the uniqueness of the solution, the whole u_n converges to u. Since $u = S(f, g)$, a) is shown.

Let us show the claim. That $\| \, |\nabla u_n - \nabla u_m| \, \|_{L^p(\Omega)} \to 0$ is a direct consequence of (2.9) and the monotonicity of Δ_p if $p \geq 2$ ([30]). When $1 < p < 2$ the inequality

$$||\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2| \geq c_p(1 + |\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|^2,$$

holds for all $\xi_1, \xi_2 \in \mathbb{R}^N$ and a certain constant c_p ([30]). By using Hölder's converse inequality to the left hand side of (2.9) the integral can be estimated from below by

$$\frac{1}{\left[\int_\Omega (1 + |\nabla u_n| + |\nabla u_m|)^p\right]} \| \, |\nabla(u_n - u_m)| \, \|^2_{L^p(\Omega)}. \quad (2.10)$$

Taking into account that the integral in the denominator is bounded above, (2.10) can be estimated from below by

$$A\| \, |\nabla(u_n - u_m)| \, \|^2_{L^p(\Omega)},$$

A being a constant. By combining this with (2.9) we achieve again that $\| \, |\nabla(u_n - u_m)| \, \|_{L^p(\Omega)} \to 0$, as desired.

Finally, c) is a direct consequence of Theorem 2.1, b).

3 L^∞ estimates

Our goal in the present section is obtaining uniform L^∞ estimates of the radial positive solutions to both problems (1.7) y (1.23).

Theorem 3.1 *Assume that $1 < q < p < r < p^*$ and fix values $0 \le \underline{\lambda} < \overline{\lambda}$. Then, there exists a constant M, possibly depending on $\underline{\lambda}, \overline{\lambda}$ such that all positive radial solutions $\in W^{1,p}(\Omega)$ to* (1.7)

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{r-2}u & x \in B \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & x \in \partial B, \end{cases}$$

corresponding to values $\underline{\lambda} \le \lambda \le \overline{\lambda}$ satisfy

$$\|u\|_{L^\infty(\Omega)} \le M. \tag{3.1}$$

The same result holds true for positive radial solutions to (1.23)

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & x \in B \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{r-2}u & x \in \partial B, \end{cases}$$

corresponding to $0 < \underline{\lambda} \le \lambda \le \overline{\lambda}$ under the restriction $1 < q < p < r$ without extra requirements on the size of r.

Remark 3.1 It should be noticed that in the case of problem (1.23), finiteness of M in (3.1) strictly requires $\underline{\lambda} > 0$. On the contrary, problem (1.5) admits $\underline{\lambda} = 0$.

For the proof of that part of Theorem 3.1 concerning problem (1.7) we use the blow-up approach developed in [3]. The key is the following Lemma whose proof is included for completeness.

Lemma 3.1 *Assume that exponents q, r satisfy*

$$1 < q < p < r,$$

and consider the initial value problem,

$$\begin{cases} -(\mathrm{r}^{N-1}|u'|^{p-2}u')' = \mathrm{r}^{N-1}|u|^{r-2}u & \mathrm{r} \ge 0 \\ u(0) = u_0 \\ u'(0) = 0, \end{cases} \tag{3.2}$$

$u_0 \ge 0$. The following properties hold true.

i) For every $u_0 \ge 0$ problem (3.2) admits a unique solution $u(\mathrm{r})$ defined in $0 \le \mathrm{r} < \infty$.

ii) If $r < p^*$ then for every $u_0 > 0$, solution $u(\mathrm{r})$ to (3.2) vanishes at a first zero $\mathrm{r}_0 > 0$ where $u'(\mathrm{r}_0) < 0$.

iii) On the contrary, if $r \geq p^*$ then solution $u(\mathrm{r})$ never vanishes provided $u_0 > 0$.

Proof. Local existence to the Cauchy problem follows, say by the results in [14]. Since $r > p$, equation falls in the non degenerate regime and local uniqueness is consequence of either the results in [14] or [26]. Local uniqueness together with a globalizing argument furnishes a unique "not continuable" solution $u(\mathrm{r})$ to (3.2) which is defined in a maximal interval of the form $[0, \omega)$, $0 < \omega \leq \infty$. Now, observe that the energy

$$\frac{1}{p'}|u'|^p + \frac{|u|^r}{r},$$

$\dfrac{1}{p'} + \dfrac{1}{p} = 1$, decreases on solutions to (3.2). Therefore, $\omega = \infty$ and i) is shown.

To prove the nodal property of solutions announced in ii) we have to resort to an "outer" pde's argument. In fact, the well-known Dirichlet problem

$$\begin{cases} -\Delta_p u = |u|^{r-2}u & x \in B \\ u = 0 & x \in \partial B, \end{cases} \tag{3.3}$$

exhibits at least a positive and bounded weak solution of class $C^{1,\alpha}$ provided $r < p^*$. To see it, we recall that such solution can be found by solving the variational problem

$$\inf_{v \in W^{1,p}_{0,s}(B)} \left\{ \frac{1}{p} \int_B |\nabla u|^p - \frac{1}{r} \int_B |u|^r \right\},$$

where $W^{1,p}_{0,s}(B)$ stands for the space of radial functions in $W^{1,p}_0(B)$ (see related results in [29]). Set $\overline{u}(\mathrm{r})$ any of those solutions. Strong maximum principle ([31]) implies that $\overline{u}(\mathrm{r}) > 0$ for $0 \leq \mathrm{r} < 1$ with $\overline{u}'(1) < 0$. We can now use the analysis on radial solutions in Section 2 to conclude that $u(\mathrm{r})$ solves (3.3) with initial datum $\overline{u}(0) := \overline{u}_0$. A scaling argument permits writing the solution to (3.3) as

$$u(\mathrm{r}) = \sigma^{\frac{p}{r-p}} \overline{u}(\sigma \mathrm{r}) \qquad \sigma = \left(\frac{u_0}{\overline{u}_0}\right)^{\frac{r-p}{p}}.$$

Thus, such solution exhibits a first zero at $\mathrm{r} = \sigma^{-1}$ where u' is negative, and ii) is shown.

A pde's argument is again involved in the proof of iii). It is well–known, by a Pohozaev type equality ([17]), that Dirichlet problem (3.3) does not admit a positive solution when $r \geq p^*$. This means that the solution to (3.3) corresponding to any $u_0 > 0$ keeps positive and decreasing for all $r > 0$ no matter the size of u_0 is. It can be further shown that $\lim_{\mathrm{r}\to\infty} u(\mathrm{r}) = \lim_{\mathrm{r}\to\infty} u'(\mathrm{r}) = 0$. This concludes the proof.

We can already proceed to prove Theorem 3.1

Proof. [Proof of Theorem 3.1] We begin with problem (1.7) and assume that a sequence $u_n(\mathrm{r})$ of solutions, corresponding to $\lambda = \lambda_n$ with $\underline{\lambda} \leq \lambda_n \leq \overline{\lambda}$, exists and satisfies

$$\sup_B u_n \to \infty. \tag{3.4}$$

We are showing that this is not possible and so (3.1) must be true. It should be observed at this stage that a main difference with regard the similar argument in [3] is the fact that, in general, positive solutions to (1.7) do not have to achieve their maximum at $\mathrm{r} = 0$ (see in fact Remarks 3.1 below).

Nevertheless, we claim that

$$M_n := \sup_B u_n = u_n(0),$$

for n large. If this is assumed, we set the standard scaling ([18], [3])

$$v_n = \frac{1}{M_n} u_n(\theta_n \mathrm{r}) \qquad \theta_n = M_n^{-\frac{r-p}{p}},$$

and $v = v_n(\mathrm{r})$ solves

$$-(\mathrm{r}^{N-1}|v'|^{p-2}v')' = \mathrm{r}^{N-1}(|v|^{r-2}v - \theta_n^p|v|^{p-2}v) \qquad 0 \le \mathrm{r} \le \theta_n^{-1},$$

together with $v_n(0) = 1$, $v_n'(0) = 0$ and $0 < v_n(\mathrm{r}) \le 1$ for $0 \le \mathrm{r} \le \theta_n^{-1}$.

We next fix $R > 0$, write

$$|v_n'(\mathrm{r})|^{p-2}v_n'(\mathrm{r}) = -\int_0^{\mathrm{r}} \left(\frac{t}{\mathrm{r}}\right)^{N-1} (|v_n(t)|^{r-2}v_n(t) - \theta_n^p|v_n(t)|^{p-2}v_n(t))\, dt,$$

and conclude that v_n is equicontinuous in $[0, R]$. By extracting a subsequence, $v_n \to v$ uniformly in $[0, R]$ and so (observe that $\theta_n \to 0+$),

$$\lim v_n'(\mathrm{r}) = -\left(\int_0^{\mathrm{r}} \left(\frac{t}{\mathrm{r}}\right)^{N-1} v(t)^{r-1}\, dt\right)^{\frac{1}{p-1}},$$

$\mathrm{r} \in [0, R]$. On the other hand,

$$|v_n'(\mathrm{r}) - v_m'(\mathrm{r})| \le C \left|\int_0^{\mathrm{r}} \left(\frac{t}{\mathrm{r}}\right)^{N-1} |v_n(t)^{r-1} - v_m(t)^{r-1}|\, dt\right|^{\frac{1}{p-1}}$$

if $p > 2$ or either,

$$|v_n'(\mathrm{r}) - v_m'(\mathrm{r})| \le C \int_0^{\mathrm{r}} \left(\frac{t}{\mathrm{r}}\right)^{N-1} |v_n(t)^{r-1} - v_m(t)^{r-1}|\, dt,$$

if $1 < p \le 2$, C being certain positive constant. In both cases v_n' converges uniformly and hence, $v \in C^1[0, R]$ with

$$v'(\mathrm{r}) = -\left(\int_0^{\mathrm{r}} \left(\frac{t}{\mathrm{r}}\right)^{N-1} v(t)^{r-1}\, dt\right)^{\frac{1}{p-1}}, \tag{3.5}$$

$\mathrm{r} \in [0, R]$. This argument can be globalized by choosing an increasing sequence of intervals $[0, R_n]$, $R_n \to \infty$ and selecting a sequence of v_n so that $v_n \to v$ uniformly on compacts of $\overline{\mathbb{R}}^+$, with $v(0) = 1$, $v(\mathrm{r}) > 0$ and satisfying (3.5) in $\mathbb{R}^+$. Since this contradicts Lemma 3.1–ii) then (3.4) is impossible and so (3.1) must be true.

Let us show now the claim. As observed in Section 2 solution $u_n(\mathrm{r})$ define a local solution to the initial value problem

$$\begin{cases} -(\mathrm{r}^{N-1}|u'|^{p-2}u')' = \mathrm{r}^{N-1}(|u|^{r-2}u - |u|^{p-2}u) \\ u(0) = u_{0n} \\ u'(0) = 0, \end{cases} \tag{3.6}$$

for a certain $u_{0n} > 0$, which can be uniquely continued as a maximal solution defined for $0 \le \mathrm{r} < \omega_n$, $0 < \omega_n \le \infty$ ([14]). On the other hand $u = 0$, $u = 1$ are zeros of the right hand side of the equation, the former with multiplicity $p-1$, the later making a local minimum of the potential

$$F(u) := \frac{|u|^r}{r} - \frac{|u|^p}{p}.$$

According to Theorem 2.2 in [14], $(u, u') = (u_n(\mathrm{r}), u_n'(\mathrm{r}))$ can never cross the point $(0, 0)$ nor $(1, 0)$ at finite r. This means that the zeros of u_n' are isolated. Let us remove in the sequel subindex n to brief. Since

$$\frac{d}{d\mathrm{r}}(E(u, u')) = -\frac{N-1}{\mathrm{r}}|u'|^p,$$

with

$$E(u, u') = \frac{1}{p'}|u'|^p + \frac{|u|^r}{r} - \frac{|u|^p}{p},$$

then $E(u, u')$ is strictly decreasing. We next observe that solution u to (3.6) initially increases if $0 < u_0 < 1$ meanwhile decreases if $u_0 > 1$.

On the other hand, boundary condition in (1.7) reads

$$u'(1) = \lambda^{\frac{1}{p-1}} u(1)^{\frac{q-1}{p-1}}.$$

Thus, if u solves (1.7) with $u(0) > 1$ then, since $u'(1) > 0$, there must exist a first $0 < \mathrm{r}_1 < 1$ such that $u'(\mathrm{r}_1) = 0$. Moreover:

$$0 < u_1 := u(\mathrm{r}_1) < 1.$$

Otherwise

$$(\mathrm{r}^{N-1}|u'|^{p-2}u')'_{\big|\mathrm{r}=\mathrm{r}_1} < 0.$$

Since $|u'|^{p-2}u' = 0$ at $\mathrm{r} = \mathrm{r}_1$ this would imply $u'(\mathrm{r}) > 0$ for $\mathrm{r} < \mathrm{r}_1$ close to r_1 what is not possible.

Let us put u^* the positive zero of $F(u)$. We are proving that

$$u_0 > u^* \qquad \Rightarrow \qquad u(\mathrm{r}) < u_0 \qquad \text{for all} \qquad 0 < \mathrm{r} \le 1.$$

Indeed, by the choice of r_1 it follows that

$$u_1 < u(\mathrm{r}) < u_0,$$

for $0 < \mathrm{r} < \mathrm{r}_1$. Since

$$0 < u_1 < 1 < u^*$$

and $F < 0$ in $(0, u^*)$, we find

$$E(u, u')_{|\mathrm{r}=\mathrm{r}_1} = F(u_1) < 0\,.$$

Due to the decreasing character of E, with respect to r, we have

$$E(u, u') < 0 \quad \Rightarrow \quad F(u(\mathrm{r})) < 0 \quad \Rightarrow \quad 0 < u(\mathrm{r}) < u^*,$$

for $\mathrm{r}_1 \le \mathrm{r} \le 1$. Therefore $u(\mathrm{r}) < u_0$ for all $0 < \mathrm{r} \le 1$.

The proof of the claim follows from the fact u_{0n} must be greater than u^* for large n if $M_n \to \infty$. In fact observe that $u_n(\mathrm{r}) \le u^*$ for all r if $u_n(0) \le u^*$. Therefore $u_n(0) = M_n$ for n large.

We next proceed to show (3.1) for problem (1.23). A first remark is that positive solutions u to that problem satisfies

$$u(\mathrm{r}) \ge 1 \qquad 0 \le \mathrm{r} \le 1.$$

A proof of this is delayed until Section 6. Since $u(\mathrm{r})$ solves

$$\begin{cases} -(\mathrm{r}^{N-1}|u'|^{p-2}u')' = \mathrm{r}^{N-1}(|u|^{q-2}u - |u|^{p-2}u) \\ u(0) = u_0 \\ u'(0) = 0, \end{cases} \tag{3.7}$$

with $u_0 \ge 1$ we find that $u(\mathrm{r})$ is nondecreasing while $u(1) = \sup_{[0,1]} u$. On the other hand since

$$E(u, u') = \frac{1}{p'}|u'|^p + G(u) \qquad G(u) := \frac{|u|^q}{q} - \frac{|u|^p}{p},$$

is nonincreasing on solutions to (3.7) then

$$\frac{1}{p'}u'^p + G(u) \le G(u_0) \le G(1),$$

for $\mathrm{r} \ge 0$. In particular,

$$u' \le \{p'\}^{\frac{1}{p}}(G(1) - G(u))^{\frac{1}{p}}.$$

Since

$$\{p'\}^{\frac{1}{p}}(G(1) - G(u))^{\frac{1}{p}} \le Cu,$$

for some $C > 0$ and large u, solutions u to (3.7) are defined and nondecreasing for all $\mathrm{r} \ge 0$.

We now observe that $h(u,\lambda) := \lambda^{\frac{1}{p-1}} u^{\frac{r-1}{p-1}}$ is superlinear and so there exists $\Lambda > 0$ such that equation

$$h(u,\lambda) = \{p'\}^{\frac{1}{p}} (G(1) - G(u))^{\frac{1}{p}} \tag{3.8}$$

has either two, one or no solutions depending on whether $0 < \lambda < \Lambda$, $\lambda = \Lambda$ or $\lambda > \Lambda$, respectively.

Fix $\underline{\lambda} \le \Lambda$ and define $\underline{u}_{\max}$ the maximum value where equality (3.8) is attained when $\lambda = \underline{\lambda}$. We then observe that if u solves (1.23) with $\lambda \ge \underline{\lambda}$ then necessarily

$$u(1) \le \underline{u}_{\max}, \tag{3.9}$$

otherwise,

$$u'(1) \le \{p'\}^{\frac{1}{p}} (G(1) - G(u))^{\frac{1}{p}} < \underline{\lambda}^{\frac{1}{p-1}} u(1)^{\frac{r-1}{p-1}} < \lambda^{\frac{1}{p-1}} u(1)^{\frac{r-1}{p-1}},$$

against the assumption. On the other hand observe that

$$u'(\mathrm{r}) < \lambda^{\frac{1}{p-1}} u(\mathrm{r})^{\frac{r-1}{p-1}} \qquad \mathrm{r} \ge 0,$$

for all solution $u(\mathrm{r}) \ge 1$ to (3.6) provided that $\lambda > \Lambda$. In other words, no solution to (1.23) are possible if $\lambda > \Lambda$. Therefore, it has been shown that solutions to (1.5) corresponding to $\lambda \ge \underline{\lambda}$, if any, must satisfy (3.9). Hence, (3.1) is proved.

Remark 3.1

a) Since $u = 1$ is a simple zero of equation in (3.7), solution $u(\mathrm{r}) = 1$ is the unique one corresponding to $u_0 = 1$ provided $1 < p \le 2$ (see [14, Theorem 2.2]). On the other hand, infinitely many solutions to this specific problem arise when $p > 2$ ([14], Theorem 2.3 and Corollary 2.4).

b) Consider the one dimensional version

$$\begin{cases} -(|u'|^{p-2}u')' + |u|^{p-2}u = |u|^{r-2}u & 0 \le t \le L \\ u'(0) = 0 & \\ |u'(L)|^{p-2}u'(L) = \lambda |u(L)|^{q-2}u(L), & \end{cases} \tag{3.10}$$

of (1.7) in the interval $[0, L]$, L being considered as a parameter. A careful phase plane analysis reveals several features. First, that some $\Lambda > 0$ exists such that no positive solution to (3.10) exists in *any* interval $[0, L]$ if $\lambda > \Lambda$. Moreover, fixed $0 < \lambda < \Lambda$ and an arbitrary integer n, a number $L_n > 0$ exists such that:

i) For $L > L_n$ problem (3.10) admits n pairs of solutions $u^{\pm}_{n,k}(x)$, $1 \le k \le n$ in the interval $[0, L]$.

ii) $\min u^{\pm}_{n,k} = u^{\pm}_{n,k}(0)$ for all $1 \le k \le n$.

iii) Every $u^{\pm}_{n,k}$ undergoes at least $k-1$ oscillations in the sense that $u^{\pm}_{n,k}$ takes on the value $u^{\pm}_{n,k}(0)$ at $k-1$ different points in $(0, L)$.

All these facts suggest that problem (1.5), when observed in a large domain, has a strong tendency to exhibit multiplicity of solutions (the larger the domain, the greater the number of solutions).

4 Comparison between minimal solutions

Existence of a minimal positive solution $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ to (1.5) was stated in [27] (see Theorem 1.1) provided $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, exponents q, r satisfy

$$1 < q < p < r \le p^*$$

and $0 < \lambda \le \Lambda$, $\Lambda > 0$ the maximum value of λ for the existence of positive solution (finiteness of Λ was also ensured there). For its use in next section we are stating a strong comparison result between minimal solutions to (1.5). It is stressed that due to the p–Laplacian, such fact does not follow from any general comparison principle. In addition, result is achieved in a more general context, in spite of our main objective are radial solutions.

For the purposes of our next result it should be recalled that the minimal solution u_λ to (1.5) in a general domain Ω is increasing with respect to λ for $\lambda \in (0, \Lambda]$, i. e.,

$$u_{\underline{\lambda}}(x) \le u_{\overline{\lambda}}(x) \qquad x \in \overline{\Omega}$$

provided $\underline{\lambda} \le \overline{\lambda}$ ([27], Theorem 1.1-iii)).

Theorem 4.1 *Under the previous assumptions on Ω, q, r, let $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ be the minimal solution to problem*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{r-2}u & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & x \in \partial\Omega, \end{cases}$$

corresponding to $0 < \lambda \le \Lambda$. Suppose that $0 < \underline{\lambda} < \overline{\lambda} \le \Lambda$. Then,

$$u_{\underline{\lambda}}(x) < u_{\overline{\lambda}}(x) \qquad x \in \overline{\Omega}. \tag{4.1}$$

Proof. By performing in (1.5) the scaling

$$v = \lambda^{-\frac{1}{p-q}} u,$$

we arrive to problem

$$\begin{cases} -\Delta_p v + v^{p-1} = \lambda^{\frac{r-p}{p-q}} v^{r-1} & x \in \Omega \\ \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} = v^{q-1} & x \in \partial\Omega. \end{cases} \tag{4.2}$$

Set $v_{\underline{\lambda}}$ and $v_{\overline{\lambda}}$ the solutions to (4.2) corresponding to $u_{\underline{\lambda}}$ and $u_{\overline{\lambda}}$, respectively. Since the transformation $u \mapsto \lambda^{-\frac{1}{p-q}} u$ preserves "minimal solutions" they are minimal solutions to (4.2) at the corresponding values of the parameter. We show next that:

$$v_{\underline{\lambda}}(x) \le v_{\overline{\lambda}}(x) \qquad x \in \overline{\Omega}.$$

In fact, $v_{\overline{\lambda}}$ defines a supersolution to $(4.2)_{\lambda=\underline{\lambda}}$. On the other hand, a small ε_0 can be found so that $\underline{v} := \varepsilon\phi$ defines a subsolution to $(4.2)_{\lambda=\underline{\lambda}}$ for all $0 < \varepsilon < \varepsilon_0$, where ϕ is the positive solution to (1.11) in Ω, i. e.,

$$\begin{cases} -\Delta\phi + \phi^{p-1} = 0 & x \in \Omega \\ \phi = 1 & x \in \partial\Omega. \end{cases}$$

By choosing ε so that $\underline{v} = \varepsilon\phi \le v_{\overline{\lambda}}$ a solution v to $(4.2)_{\lambda=\underline{\lambda}}$ exists so that $\varepsilon\phi \le v \le v_{\overline{\lambda}}$. Since $v_{\underline{\lambda}}$ is the miniminal solution to such problem, the desired inequality is achieved. Thus,

$$\big(\underline{\lambda}\big)^{-\frac{1}{p-q}} u_{\underline{\lambda}}(x) \le \big(\overline{\lambda}\big)^{-\frac{1}{p-q}} u_{\overline{\lambda}}(x) \qquad x \in \overline{\Omega}, \tag{4.3}$$

and so,

$$u_{\underline{\lambda}}(x) \le \big(\underline{\lambda}/\overline{\lambda}\big)^{\frac{1}{p-q}} u_{\overline{\lambda}}(x) < u_{\overline{\lambda}}(x),$$

for all $x \in \overline{\Omega}$. This implies (4.1).

Remark 4.1 Existence of a minimal positive solution u_λ to problem (1.16) on a smooth bounded domain and exponents satisfying

$$1 < q < p < r \le p_*,$$

was also shown in [27] (see Section 4), provided $0 < \lambda \le \Lambda$, Λ having the same status mentioned at the beginning of this section. Nevertheless, unexplored aspects of this problem that remained hidden in that work will be studied in next Section 6.

We are now proving a strong comparison result for (1.16). It should be stressed that the approach employed in the proof of Theorem 4.1 does not works for (1.16). In fact, a similar scaling argument would lead to the inequality,

$$u_{\underline{\lambda}}(x) \le \big(\underline{\lambda}/\overline{\lambda}\big)^{-\frac{1}{r-p}} u_{\overline{\lambda}}(x),$$

which does not permit to conclude the strong comparison. Moreover, at the light of Section 6 it "must" necessarily fail in the regime $p > 2$!. Next statement focuss on the complementary range $1 < p \le 2$ and the radial case.

Theorem 4.2 *Assume that* $1 < p \le 2$ *and suppose* $0 < \underline{\lambda} < \overline{\lambda} \le \Lambda$. *Then the minimal radial solutions* $u_{\underline{\lambda}}$, $u_{\overline{\lambda}}$ *to* (1.22),

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & x \in B \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda|u|^{r-2}u & x \in \partial B, \end{cases}$$

corresponding to values $\lambda = \underline{\lambda}$ *and* $\lambda = \overline{\lambda}$, *respectively, satisfy,*

$$u_{\underline{\lambda}}(x) < u_{\overline{\lambda}}(x) \qquad x \in \overline{B}.$$

Proof. As will be proved in Section 6 positive solutions u to (1.14) verify $u \geq 1$. However, as in the present case p falls in the "nondegenerate" case $1 < p \leq 2$, minimal solution u_λ to (1.22) satisfies ([14], Theorem 2.2)

$$u_\lambda(x) > 1 \qquad x \in \overline{B}.$$

Set $\underline{u} = u_{\underline{\lambda}}$, $\overline{u} = u_{\overline{\lambda}}$ to brief. Then $\underline{u}(\mathrm{r}) \leq \overline{u}(\mathrm{r})$ for $0 \leq \mathrm{r} \leq 1$. Moreover, the inequality $u_{\underline{\lambda}} \leq u_{\overline{\lambda}}$ holds true for problem (1.22) regarded in a general domain Ω. In fact $\overline{u} \geq 1$ defines a supersolution to $(1.22)_{\lambda=\underline{\lambda}}$ while $u = 1$ is a strict subsolution to such problem. Thus, a solution $1 \leq u \leq \overline{u}$ to $(1.22)_{\lambda=\underline{\lambda}}$ exists. Being $\underline{u}$ its minimal solution we obtain $\underline{u} \leq u \leq \overline{u}$ and the inequality is shown.

We next observe that $\overline{u}'(1) \leq \underline{u}'(1)$ and hence the boundary condition implies that $\underline{u}(1) < \overline{u}(1)$. Finally if some $0 \leq \mathrm{r}_0 < 1$ exists so that $\underline{u}(\mathrm{r}_0) = \overline{u}(\mathrm{r}_0)$ then it follows that $\underline{u}'(\mathrm{r}_0) = \overline{u}'(\mathrm{r}_0)$. However, the initial value problem

$$\begin{cases} -(\mathrm{r}^{N-1}|u'|^{p-2}u')' = \mathrm{r}^{N-1}(|u|^{q-2}u - |u|^{p-2}u) \\ u(\mathrm{r}_0) = u_0 \\ u'(\mathrm{r}_0) = u_0', \end{cases}$$

has the unique solution property in the region $u > 1$ ([14], [26]). This means that $\underline{u}(\mathrm{r}) = \overline{u}(\mathrm{r})$ for $\mathrm{r}_0 \leq \mathrm{r} \leq 1$ what contradicts our previous inequality near $\mathrm{r} = 1$. Therefore, $\underline{u}(\mathrm{r}) < \overline{u}(\mathrm{r})$ in the whole interval $[0, 1]$.

5 Existence of a second solution

This section is devoted to the proof of Theorem 1.1. We are also taking ventage of the arguments to show the multiplicity assertion ii) of Theorem 1.4 in the nondegenerate case $1 < p \leq 2$.

Proof. [Proof of Theorem 1.1] By applying Theorem 1.1 in [27] to problem (1.7) we obtain a value $\Lambda > 0$ such that no positive solutions exists if $\lambda > \Lambda$, meanwhile a minimal positive solution u_λ arises for $0 < \lambda \leq \Lambda$. Since u_λ is rotationally invariant (Section 2) the proofs of both i) and ii) are concluded (smoothness of u_λ with respect λ, $\lambda \sim 0$, is explained in Section 6).

We proceed next with iv), also postponing iii) until Section 6. First observe that it is a consequence of the proof of Theorem 3.1 that the L^∞ estimate (3.1) holds for any positive solution u no matter the size of $\lambda \geq 0$ is. Thus, $C^{1,\alpha}$ estimate (1.13) follows the results in [21].

Let us address the existence of a second solution by a topological degree argument in the line of [3]. Consider the operator,

$$\begin{array}{rccc} H: & \overline{\mathbb{R}}^+ \times C_s(\overline{B}) & \to & C_s(\overline{B}) \\ & (\lambda, u) & \mapsto & v = S(|u|^{r-2}u, \lambda|u|^{q-2}u), \end{array}$$

where S is the solution operator defined in Section 2, $C_s(\overline{B})$ stands for the space of radially symmetric functions in $C(\overline{B})$ and we set $\langle |u|^{r-2}u, \psi\rangle = \int_B |u|^{r-2}u\psi \, dx$, $\langle \lambda|u|^{q-2}u, \psi\rangle = \lambda \int_{\partial B} |u|^{q-2}u\psi \, ds$ for $v \in W^{1,p}(B)$.

In view of expression (2.4) it is clear that H maps bounded sets in $\overline{\mathbb{R}}^+ \times C_s(\overline{B})$ onto bounded sets in $C^1(\overline{B})$. This, together with Proposition 2.2, implies that H is compact.

On the other hand, a radially symmetric solution $u \in W^{1,p}(B)$ to (1.7) is characterized as a fixed point of H,

$$u = H(\lambda, u).$$

We fix $0 < \lambda < \Lambda$ and are proving the existence of a further positive solution different from u_λ. Choose $\underline{\lambda} < \lambda < \overline{\lambda} < \Lambda$, set $\underline{u} = u_{\underline{\lambda}}$, $\overline{u} = u_{\overline{\lambda}}$ and define $[\underline{u}, \overline{u}] = \{u \in C_s(\overline{B}) : \underline{u}(x) \leq u(x) \leq \overline{u}(x),\ x \in \overline{B}\}$. A first remark is that, in view of Proposition 2.2, operator $H(\lambda, \cdot)$ leaves $[\underline{u}, \overline{u}]$ invariant, i. e.,

$$H(\lambda, \cdot)([\underline{u}, \overline{u}]) \subset [\underline{u}, \overline{u}].$$

A second observation is that, strong comparison in Theorem 4.1 permits finding an open ball $B_\varepsilon := B(u_\lambda, \varepsilon)$ in $C_s(\overline{B})$, centered at u_λ with radius $\varepsilon > 0$, such that

$$B(u_\lambda, \varepsilon) \subset [\underline{u}, \overline{u}].$$

We are getting a contradiction if it is assumed that u_λ is the unique positive solution. In fact, if such assertion holds true, the Leray-Schauder index,

$$d(I - H(\lambda, \cdot), B_\varepsilon, 0),$$

is well defined. In addition, since $C = [\underline{u}, \overline{u}]$ is a closed convex subset of $C_s(\overline{B})$, the fixed point index $i(H(\lambda, \cdot), C, C)$ of $H(\lambda, \cdot)$ relative to C ([1], Section 11) is well defined and

$$i(H(\lambda, \cdot), C, C) = 1.$$

Since we are supposing that $H(\lambda, \cdot)$ has not fixed points aside u_λ, excision property of the index implies that,

$$i(H(\lambda, \cdot), C, C) = i(H(\lambda, \cdot), B_\varepsilon, C).$$

On the other hand, C is a retract of $C_s(\overline{B})$ and thus a retraction $\mathbf{r} : C_s(\overline{B}) \to C$ exists so that $\mathbf{r}_{|B_\varepsilon}$ is the identity in B_ε. Moreover, $\mathbf{r}^{-1}(B_\varepsilon) = B_\varepsilon$. Therefore,

$$i(H(\lambda, \cdot), B_\varepsilon, C) = d(I - H(\lambda, \cdot), \mathbf{r}^{-1}(B_\varepsilon), 0) = d(I - H(\lambda, \cdot), B_\varepsilon, 0).$$

In particular,

$$d(I - H(\lambda, \cdot), B_\varepsilon, 0) = 1.$$

Let us define

$$Q = \{u \in C_s(\overline{B}) : \underline{u}(x) - \delta < u(x) < M + \delta\},$$

M being the constant in estimate (3.1) and $\delta > 0$ is small enough. By Theorem 3.1, operator $H(\lambda', \cdot)$ has not fixed points on ∂Q for every $\lambda \leq \lambda' \leq \Lambda + 1$. In fact, it has not fixed points at all in $\overline{Q}$ if $\lambda' = \Lambda + 1$. The homotopy invariance of the Leray–Schauder degree then implies that

$$d(I - H(\lambda', \cdot), Q, 0)_{|\lambda'=\lambda} = d(I - H(\lambda', \cdot), Q, 0)_{|\lambda'=\Lambda+1} = 0.$$

In addition, excision property leads to

$$d(I - H(\lambda, \cdot), Q, 0) = d(I - H(\lambda, \cdot), B_\varepsilon, 0) = 0,$$

what furnishes the desired contradiction. Therefore, (1.7) admits a second positive solution $\hat{u} \geq u_\lambda$.

Proof. [Proof of Theorem 1.4–ii), case $1 < p \leq 2$] Thanks to Theorems 3.1 and 4.2 previous proof can be adapted word for word to achieve the existence of a second positive solution to (1.23). It is enough with changing operator H to $H_1(\lambda, u) = S(|u|^{q-2}u, \lambda|u|^{r-2}u)$.

6 Flat patterns and further results

Proof. [Proof of Theorem 1.2] Point i) was proved in detail in [27]. In fact, under restriction (1.18), there exists a minimal solution $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover, mapping $\lambda \to u_\lambda(\cdot)$ is furthermore increasing and continuous from the left when observed in $C^{1,\alpha}(\overline{\Omega})$ ([27], Theorem 1.1-iii)).

To show ii) observe that any possible positive solution u to (1.16) defines a strict supersolution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases} \tag{6.1}$$

In addition, it was observed in [27] that (6.1) posses a unique positive solution. Thus, such a solution must be

$$u_1 = 1.$$

Since (6.1) admits arbitrarily small positive subsolutions then, necessarily,

$$u(x) \geq 1 \qquad x \in \Omega,$$

and (1.19) is shown.

On the other hand, assume that w_λ is the family of positive solutions to (1.16) described in iii). Condition $\|w_\lambda\|_\infty \leq M$ for λ small and the estimates in [21] permit us asserting that $\|w_\lambda\|_{C^{1,\alpha}(\overline{\Omega})}$ becomes bounded as $\lambda \to 0$. But then, after passing through a subsequence, w_λ converges in $C^{1,\alpha}(\overline{\Omega})$ –maybe reducing α– to a positive solution to (6.1). Therefore, the whole family w_λ fulfills (1.20). In particular, this holds true for u_λ.

To proceed to prove the convergence assertion in iii) we resort to the eigenvalue problem (2.5) with $a = b = 1$. We set $(\mu_1(\lambda), \phi_1(\cdot,\lambda))$ the principal normalized eigenpair and observe that

$$(\mu_1(\lambda), \phi_1(\cdot,\lambda))_{|\lambda=0} = (1,1),$$

meanwhile $\mu_1(\lambda)$ decreases from 1 to 0 when $\lambda \in [0,\lambda_1]$, $\lambda_1 > 0$ being the first Steklov eigenvalue to (2.6) ($a = b = 1$). Fix now $0 < \overline{\lambda} < \lambda_1$ small. A direct computation shows that

$$\overline{u}_{\overline{\lambda}}(x) = A(\overline{\lambda})\phi_1(x,\overline{\lambda}) \qquad A(\overline{\lambda}) = \mu_1^{-\frac{1}{p-q}}\{\inf_{\Omega}\phi_1(\overline{\lambda})\}^{-1} > 1,$$

becomes a supersolution to (1.16) provided that

$$0 < \lambda \le \overline{\lambda}A(\overline{\lambda})^{-(r-p)}.$$

Since $\overline{u}_{\overline{\lambda}} > 1$ in $\overline{\Omega}$ and $\underline{u} = 1$ defines a subsolution to (1.16), it follows that

$$1 \le \overline{\lim_{\lambda\to 0+}}\, u_\lambda \le \overline{u}_{\overline{\lambda}}.$$

Since both $A(\overline{\lambda}) \to 1$ and $\phi_1(\cdot,\overline{\lambda}) \to 1$ as $\overline{\lambda} \to 0$, we achieve the desired result.

Proof of iv) is postponed until the one of Theorem 1.4.

Proof. [Proof of Theorem 1.4] As already pointed out, minimal solution u_λ to (1.23) must be radial. This fact, i) and ii) follow from the above discussion.

To proceed ahead we introduce, in a smooth bounded domain, the Dirichlet problem:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{q-2}u & x \in \Omega \\ u = c & x \in \partial\Omega, \end{cases} \tag{6.2}$$

where $c \ge 1$ is a parameter. For every $c \ge 1$, (6.2) has a unique solution $u \in C^{1,\alpha}(\Omega)$ in the range $u \ge 1$, such that $1 \le u \le c$. In fact, $\underline{u} = 1$, $\overline{u} = c$ yield a pair of ordered sub and supersolutions, respectively, while uniqueness is provided by the comparison result in [30]. Let $u = \tilde{u}_\lambda(\cdot,c)$ be such solution.

When Ω is the ball $B = \{x : |x| < 1\}$, we have that $\tilde{u}_\lambda$ is radial and $\tilde{u}_\lambda(x) = u(|x|)$, where u solves the initial value problem (3.7),

$$\begin{cases} -(\mathrm{r}^{N-1}|u'|^{p-2}u')' = \mathrm{r}^{N-1}(|u|^{q-2}u - |u|^{p-2}u) & \mathrm{r} \ge 0, \\ u(0) = u_0 \\ u'(0) = 0, \end{cases}$$

with $u_0 \ge 1$. Observe that infinitely many solutions to (3.7) arise if $u_0 = 1$ and $p > 2$ ([14], Corollary 2.4). Nevertheless, as was pointed out in Section 3, all possible solutions $u(\mathrm{r})$ in the range $u \ge 1$ (henceforward, this restriction is assumed) are defined for all $\mathrm{r} \ge 0$ and are nondecreasing (increasing whenever $u > 1$).

To complete the discussion of (3.7), cases $1 < p \le 2$ and $p \ge 2$ must be distinguished.

For $1 < p \le 2$, (3.7) has a unique solution $u = u(\mathrm{r}, u_0)$ for every $u_0 \ge 1$. Moreover, mapping $u_0 \to u(\cdot, u_0)$ is smooth when observed as taking values in $C^1[0, b]$, for every prefixed value $b > 0$. In fact, this is a much simpler situation than the one treated in [14, Theorem 2.5]. Moreover, a direct computation shows that

$$u(\mathrm{r}, u_0) < u(\mathrm{r}, u_1) \qquad \mathrm{r} \ge 0,$$

when $u_0 < u_1$. Therefore, function $h : [1, \infty) \to [1, \infty)$ given by $h(u_0) = u(1, u_0)$ defines a diffeomorphism. For immediate use, derivatives

$$\dot{u} = \frac{\partial u}{\partial u_0} \qquad \dot{v} = \frac{\partial v}{\partial u_0},$$

with $v := |u'|^{p-2}u'$, are going to be computed. In fact, equation can be written as

$$\begin{cases} u' = |v|^{p'-2}v & u(0) = u_0 \\ v' = |u|^{p-2}u - |u|^{q-2}u - \dfrac{N-1}{\mathrm{r}}v & v(0) = 0, \end{cases} \tag{6.3}$$

where $p' \ge 2$. Taking derivatives with respect u_0 and and observing that $u(\cdot, u_0) = 1$, $v(\cdot, u_0) = 0$ at $u_0 = 1$, we arrive to

$$\begin{cases} (\dot{u})' = 0 & \dot{u}(0) = 1 \\ (\dot{v})' = (p-q)\dot{u} - \dfrac{N-1}{\mathrm{r}}\dot{v} & \dot{v}(0) = 0, \end{cases}$$

at $u_0 = 1$. Hence:

$$\dot{u}(\mathrm{r}, 1) = 1 \qquad \dot{v}(\mathrm{r}, 1) = \frac{p-q}{N}\mathrm{r},$$

$\mathrm{r} \in [0, 1]$.

We just return to the solvability of (1.23) in the regime $1 < p \le 2$. A positive radial solution to such problem ($\lambda > 0$) has necessarily the form $\tilde{u}_\lambda(\cdot, c)$ for some $c > 1$. Therefore, c is characterized as a solution to the escalar equation:

$$\frac{u'(1, g(c))^{p-1}}{c^{r-1}} = \lambda \qquad g = h^{-1}. \tag{6.4}$$

Left hand side in (6.4) is a positive continuous function that goes to zero as $c \to 1+$. In fact, since $u(\cdot, 1) = 1$ then $u'(\cdot, g(c)) \to 0$ uniformly in $[0, 1]$ as $c \to 1+$. On the other hand,

$$\mathrm{r}^{N-1}|u'(\mathrm{r})|^{p-2}u'(\mathrm{r}) = \int_0^{\mathrm{r}} t^{N-1}(|u|^{p-2}u - |u|^{q-2}u)\, dt \le \frac{\mathrm{r}^N}{N}(|c|^{p-2}c - |c|^{q-2}c),$$

for all $0 \le \mathrm{r} \le 1$. Thus,

$$u'(1, g(c)) \le \frac{1}{N}(|c|^{p-2}c - |c|^{q-2}c),$$

what clearly implies that

$$\frac{u'(1,g(c))^{p-1}}{c^{r-1}} \to 0 \qquad \text{as} \quad c \to \infty.$$

By discussing equation (6.4) we find that no solutions to (1.23) are possible for all $\lambda > \Lambda$ where

$$\Lambda = \sup_{c\geq 1} \frac{u'(1,u_0)^{p-1}}{c^{r-1}} > 0.$$

For $0 < \lambda < \Lambda$, equation (6.4) exhibits a minimal root $c_-(\lambda)$ and a maximal one $c_+(\lambda)$, $0 < c_-(\lambda) < c_+(\lambda)$, so that $c_-(\lambda) \to 1$ and $c_+(\lambda) \to \infty$ as $\lambda \to 0+$. This shows iii) in the case $1 < p \leq 2$ and furnishes an alternative existence proof to the one given in Section 5. Moreover, no family $c(\lambda)$ of roots satisfies $c(\lambda_n) \to c_0$ for certain sequence $\lambda_n \to 0$ and positive value c_0.

As for uniqueness assertion (1.25) in iv) when $1 < p \leq 2$, it suffices with showing that (6.4) is uniquely solvable for c close 1 and λ small. In fact, by setting $A(c)$ the left hand side of that equation and taking derivatives in c we obtain

$$\begin{aligned}\frac{dA}{dc} = (1-r)c^{-r}\int_0^1 & t^{N-1}(|u|^{p-2}u - |u|^{q-2}u)\ dt + \\ c^{1-r}\int_0^1 & t^{N-1}((p-1)u^{p-2} - (q-1)u^{q-2})\dot{u}g'(c)\ dt = \\ c^{-r}\Big\{\int_0^1 & t^{N-1}\big[c((p-1)u^{p-2} - (q-1)u^{q-2})\dot{u}g'(c) \\ & -(r-1)(|u|^{p-2}u - |u|^{q-2}u)\big]\ dt\Big\}.\end{aligned}$$

Since $u(1,g(c)) = c$ then the group $\dot{u}(1,g(c))g'(c) = 1$ and so $g'(1) = 1$. Doing $c \to 1$ we conclude that dA/dc becomes positive near $c = 1$. This means that $c_-(\lambda)$ is the *unique* root to equation (6.4) which keeps close to 1 as $\lambda \to 0$. The proof of iv) (case $1 < p \leq 2$) is concluded.

Let us study next both problems (1.23) and (3.7) in the degenerate regime $p > 2$. Firstly, problem (3.7) admits a unique solution $u = u(\mathrm{r}, u_0)$ for all $u_0 > 1$ which is defined and increasing in $\mathrm{r} \geq 0$. On the contrary, infinitely many solutions $u(\mathrm{r}) \geq 1$ arise if $u_0 = 1$. To describe all of them, next problem is of great help. Namely,

$$\begin{cases} -((\mathrm{r}+\rho)^{N-1}|u'|^{p-2}u')' = (\mathrm{r}+\rho)^{N-1}(|u|^{q-2}u - |u|^{p-2}u) & \mathrm{r} \geq 0 \\ u(0) = 1 \\ u'(0) = 0, \end{cases} \tag{6.5}$$

$\rho \geq 0$ a parameter. This is a particular case of a broader class of degenerate problems studied in [14]. It is shown there that (6.5) exhibits a maximal local solution $u := u_+(\mathrm{r}, \rho)$ which satisfies $u_+(\mathrm{r}, \rho) > 1$ for $\mathrm{r} > 0$ and smoothly depends on ρ when observed as taking values in $C^1[0, \delta]$ (see Theorems 2.3 in [14] for existence, Theorems 2.5 and 2.6 for smooth dependence). In our present problem (6.5) such solution can be continued to the whole of $[0, \infty)$ still retaining such properties.

As a consequence of [14, Corollary 2.4], all possible solutions u to (3.7) with $u_0 = 1$ and satisfying $u \geq 1$ are exactly of the form

$$u_1(\mathrm{r},\rho) = \begin{cases} 1 & 0 \leq \mathrm{r} < \rho \\ u_+(\mathrm{r}-\rho,\rho) & \mathrm{r} \geq \rho, \end{cases}$$

for some $\rho \geq 0$.

Consider the function $h : [0,\infty) \to [1,\infty)$ defined as

$$h(t) = \begin{cases} u(1,t) & t > 1 \\ u_+(t,1-t) & 0 \leq t \leq 1. \end{cases}$$

Then h is a homeomorphism with inverse $d := h^{-1}(c)$. By construction, solution $\tilde{u}(\mathrm{r},c)$ to Dirichlet problem (6.2) is given by

$$\tilde{u}(\mathrm{r},c) = u(\mathrm{r},u_0) \qquad u_0 = d(c) \quad \text{if} \quad c > c_1,$$

where $c_1 = h(1)$, while

$$\tilde{u}(\mathrm{r},c) = u_1(\mathrm{r},1-d(c)) \qquad \text{if} \qquad 1 \leq c \leq c_1. \tag{6.6}$$

Existence of solutions to problem (1.23) consists in discussing the equivalent to equation (6.4),

$$\frac{\tilde{u}'(1,c)^{p-1}}{c^{r-1}} = \lambda, \tag{6.7}$$

which, by the same reasons, exhibits a minimal and a maximal root, $0 < c_-(\lambda) < c_+(\lambda)$, $c_-(\lambda) \to 1+$ and $c_+(\lambda) \to \infty$ as $\lambda \to 0+$. Existence is only possible provided $\lambda \leq \Lambda$ with Λ the supremum of the left hand side in (6.7). In addition, no family of roots $c(\lambda)$ accumulates to a positive value c_0 as $\lambda \to 0+$. This completes the proofs of iii) and iv) for the case $p > 2$ with the sole exception of the uniqueness statement (1.25).

However, a new feature appears when $p > 2$. Namely, that solutions u to (1.23) corresponding to roots $1 \leq c \leq c_1$ develop a flat region

$$\mathcal{F}_c := \{u = 1\} = B_{1-d(c)}(0).$$

In view of iii) of Theorem 1.2, it follows that any family of positive solutions w_λ such that $\|w_\lambda\|_\infty = O(1)$ as $\lambda \to 0+$, exhibits flat regions when λ becomes small.

We proceed to estimate d as $\lambda \to 0+$. This is performed in two steps, firstly we compare d with c and here we only need to deal with the solution $\tilde{u}(\mathrm{r},c)$ to (6.2). Then we estimate c in terms of λ.

Suppose $\tilde{u}(\mathrm{r}) \geq 1$ solves (6.2) with $1 \leq c \leq c_1$. By adapting the argument in the proof of Theorem 1 in [19] we achieve the inequalities (tildes are removed to brief),

$$\frac{\mathrm{r}_2-\mathrm{r}_1}{N^{1/p}} \leq \frac{1}{\{p'\}^{1/p}} \int_{u(r_1)}^{u(r_2)} \frac{ds}{(G(1)-G(s))^{1/p}} \leq \mathrm{r}_2-\mathrm{r}_1,$$

with $1-d(c) \le \mathrm{r}_1 < \mathrm{r}_2 \le 1$. Thus

$$\frac{d(c)}{N^{1/p}} \le \frac{1}{\{p'\}^{1/p}} \int_1^c \frac{ds}{(G(1)-G(s))^{1/p}} \le d(c). \tag{6.8}$$

Set $J(c)$ the integral above. By noting that $G(1)-G(s) = \frac{p-q}{2}(s-1)^2(1+o(1))$ as $s \to 1+$ it can be shown that

$$J(c) \sim B_1(c-1)^{\alpha} \qquad \text{as} \quad c \to 1+,$$

where $B_1 = \frac{p}{p-2}\left(\frac{2}{p'(p-q)}\right)^{1/p}$ and $\alpha = \frac{p-2}{p}$. By means of (6.8) this implies that,

$$\frac{1}{N^{1/p}} \overline{\lim_{c\to 1+}} \frac{d(c)}{(c-1)^{\alpha}} \le B_1 \le \underline{\lim_{c\to 1+}} \frac{d(c)}{(c-1)^{\alpha}}. \tag{6.9}$$

We are now refining (6.9) to get

$$\overline{\lim_{c\to 1+}} \frac{d(c)}{(c-1)^{\alpha}} \le B_1. \tag{6.10}$$

Solution $u(\mathrm{r}) = \tilde{u}(\mathrm{r}, c)$ solves

$$\begin{cases} -(\mathrm{r}^{N-1}|u'|^{p-2}u')' = \mathrm{r}^{N-1}(|u|^{q-2}u - |u|^{p-2}u) & 1-d \le \mathrm{r} \le 1, \\ u(1-d) = 1 \qquad u(1) = c & \\ u'(1-d) = 0. & \end{cases} \tag{6.11}$$

Introducing the change of variable

$$t = \begin{cases} \frac{p-1}{p-N}[\mathrm{r}^{\frac{p-N}{p-1}} - (1-d)^{\frac{p-N}{p-1}}] & p \ne N \\ \\ \log(\frac{\mathrm{r}}{1-d}) & p = N, \end{cases} \tag{6.12}$$

the function $u = u(t)$ satisfies

$$-(|\dot{u}|^{p-2}\dot{u})^{\cdot} \le (1-d)^{p'(N-1)}(|u|^{q-2}u - |u|^{p-2}u)\,, \quad \text{denoting}\ ^{\cdot} = \frac{d}{dt}, \tag{6.13}$$

with $0 \le t \le t_d$ and $t_d = t(1)$ ($t(\cdot)$ the function above). Thus,

$$(1-d)^{\frac{N-1}{p-1}} \le \frac{1}{\{p'\}^{1/p}} \frac{\dot{u}}{G(1)-G(u)},$$

what after integration yields

$$(1-d)^{\frac{N-1}{p-1}} t_d \le J(c),$$

with $J(c)$ the integral in (6.8). Hence

$$(1-d)^{\frac{N-1}{p-1}} \frac{t_d}{d} \frac{d}{(c-1)^{\alpha}} \le \frac{J(c)}{(c-1)^{\alpha}},$$

and taking limits as $c \to 1+$ we get

$$\overline{\lim_{c\to 1+}} \frac{d}{(c-1)^\alpha} \le B_1,$$

as desired. This, together with (6.9) proves

$$\lim_{c\to 1+} \frac{d}{(c-1)^\alpha} = B_1. \tag{6.14}$$

We are now measuring c in terms of λ and so assume that $u(\mathrm{r}) = \tilde{u}(\mathrm{r}, c)$ solves (1.23) for $c < c_1$. By using the boundary condition,

$$u'(1) = \lambda^{\frac{1}{p-1}} c^{\frac{r-1}{p-1}},$$

we arrive to

$$\frac{1}{p'} \lambda^{p'} c^{(r-1)p'} \le G(1) - G(c),$$

what implies

$$B_1 \left(\frac{2}{p'(p-q)}\right)^{\frac{\alpha}{2}} (\lambda^\beta + o(\lambda^\beta)) \le B_1 \left[\frac{2}{p-q}(G(1) - G(c))\right]^{\frac{\alpha}{2}},$$

where β and $B = B_1 \left(\frac{2}{p'(p-q)}\right)^{\frac{\alpha}{2}}$ are the values announced in the statement of v). Since both $d \to 0$ and $c \to 1+$ as $\lambda \to 0+$ while,

$$d(\lambda) \sim B_1 \left[\frac{2}{p-q}(G(1) - G(c))\right]^{\frac{\alpha}{2}},$$

as $\lambda \to 0+$, we conclude

$$\underline{\lim_{\lambda\to 0+}} \frac{d(\lambda)}{\lambda^\beta} \ge B.$$

The complementary estimate

$$\overline{\lim_{\lambda\to 0+}} \frac{d(\lambda)}{\lambda^\beta} \le B. \tag{6.15}$$

is obtained by performing the change $t = t(\mathrm{r})$ given by (6.12) and then using as above the resulting equation (6.13) to obtain the inequality,

$$(1-d)^{(N-1)p'}(G(1) - G(u)) \le \frac{1}{p'} \lambda^{p'} c^{p'(r-1)}.$$

This estimate leads to (6.15) and we can conclude that

$$c - 1 \sim \left(\frac{B}{B_1}\right)^{\frac{1}{\alpha}} \lambda^{\frac{\beta}{\alpha}},$$

as $\lambda \to 0$. Thus, any family w_λ of solutions fulfilling $\|w_\lambda\|_\infty = O(1)$ as $\lambda \to 0$ satisfies in addition

$$\sup w_\lambda - 1 \sim B_2\lambda^\gamma \qquad d = 1-\rho \sim B\lambda^\beta, \tag{6.16}$$

as $\lambda \to 0+$ where ρ is the radius of the flat region $\{w_\lambda = 1\}$, $\gamma = \frac{p'}{2}$ and $B_2 = \left(\frac{2}{p'(p-q)}\right)^{1/2}$.

Let us address the remaining uniqueness issue in iv) for the case $p > 2$. Next argument is patterned on ideas in [16] (see proof of Theorem 1.4 there). Assume that w_λ is as in the statement. Then, estimates (6.16) suggest setting the scalings

$$u(\mathrm{r}) - 1 = \lambda^\gamma z(t-\sigma), \qquad t = \lambda^{-\beta}\mathrm{r}, \qquad \lambda^{-\beta}\rho,$$

in the problem

$$\begin{cases} -(\mathrm{r}^{N-1}|u'|^{p-2}u')' = \mathrm{r}^{N-1}(|u|^{q-2}u - |u|^{p-2}u) & \mathrm{r} \geq \rho, \\ u(\rho) = 1 \\ u'(\rho) = 0. \end{cases} \tag{6.17}$$

Thus, (6.17) is transformed into

$$\begin{cases} -((t+\sigma)^{N-1}|z'|^{p-2}z')' = (t+\sigma)^{N-1}f(z,\lambda)z & t \geq 0, \\ z(0) = 0 \\ z'(0) = 0, \end{cases} \tag{6.18}$$

where,

$$f(z,\lambda) = (p-1)\int_0^1 (1+\lambda^\gamma zs)^{p-2}\, ds - (q-1)\int_0^1 (1+\lambda^\gamma zs)^{q-2}\, ds.$$

Problem (6.18) is next separately discussed, regarding $\sigma \gg 1$ and $0 \leq \lambda \ll 1$ as parameters.

As shown in [14], (6.18) admits a maximal solution $z(t,\lambda,\sigma)$ which is positive in $t > 0$, while $\sigma \mapsto z(\cdot,\lambda,\sigma) \in C^1[0,b]$ is differentiable for every positive b.

Fulfillment of the boundary condition in (1.23) amounts to solve the equation

$$z(T,\lambda,\sigma)' - (1+\lambda^\gamma z(T,\lambda,\sigma))^{r-1} = 0, \tag{6.19}$$

for some $T > 0$ that it is expected to be close B as $\lambda \to 0$ and $\sigma \to \infty$. On the other hand

$$z(\cdot,\lambda,\sigma) \to z^*(\cdot) \qquad \text{as} \quad \lambda \to 0, \sigma \to \infty, \tag{6.20}$$

in $C^1[0,\infty)$ where $z = z^*(t)$ is the maximal solution to

$$\begin{cases} -(|z'|^{p-2}z')' = (p-q)z & t \geq 0, \\ z(0) = 0 \\ z'(0) = 0, \end{cases} \tag{6.21}$$

whose explicit expression is

$$z^* = Ct^{\frac{1}{\alpha}},$$

with $\alpha = \frac{p-2}{p}$ and

$$C = \left(\frac{p-2}{p}\right)^{\frac{p-1}{p-2}} \left[\frac{(p-2)(p-q)}{2(p-1)}\right]^{\frac{1}{p-2}}.$$

Since $\dfrac{dz^*}{dt}_{|t=B} = 1$ this means that $T = B$ solves

$$\mathcal{G}(T,\lambda,\sigma) = 0,$$

for $\lambda = 0$, $\sigma = \infty$, where $\mathcal{G}$ is the left hand side of (6.19). In addition,

$$\frac{\partial \mathcal{G}}{\partial T}(T,\lambda,\sigma)_{|(T,\lambda,\sigma)=(B,0,\infty)} = \frac{d^2 z^*}{dt^2}_{|t=B} > 0.$$

Therefore, unique solutions (T,λ,σ) to (6.19) with both $T-B$ and λ small, σ large, are furnished in the form

$$T = g(\lambda,\sigma),$$

for a certain continuous function g defined for $\lambda \ge 0$ small and σ large. Moreover $\dfrac{\partial g}{\partial \sigma}$ is continuous.

Let us assume now that (T,λ,σ) solves (6.19) with $T-B$, λ both small and σ large. Then,

$$u(\mathrm{r},\lambda,\sigma) := \begin{cases} 1 & 0 \le \mathrm{r} \le \lambda^\beta \sigma \\ 1 + \lambda^\gamma z(\lambda^{-\beta}\mathrm{r} - \sigma, \lambda, \sigma) & \lambda^\beta\sigma \le \mathrm{r} \le \lambda^\beta\sigma + \lambda^\beta T, \end{cases} \tag{6.22}$$

defines a solution to (1.23) if and only if

$$1 = \lambda^\beta \sigma + \lambda^\beta T,$$

or equivalently,

$$\lambda^{-\beta} = \sigma + g(\lambda,\sigma). \tag{6.23}$$

Of course, for any family of solutions w_λ as in the statement of iv), $(\lambda,\sigma) = (\lambda, \lambda^{-\beta}\rho(\lambda))$ satisfies (6.23). We claim that such equation is uniquely solvable in the form $\sigma = \sigma(\lambda)$ for λ small and σ large. This implies that the *unique* family w_λ of solutions to (1.5) approaching 1 as $\lambda \to 0$ is just

$$w_\lambda = u(\mathrm{r},\lambda,\sigma) \qquad \sigma = \sigma(\lambda),$$

with $u(\mathrm{r},\lambda,\sigma)$ given by (6.22).

To show the claim it suffices with proving that $\dfrac{\partial g}{\partial \sigma}$ becomes small as $\lambda \to 0$, $\sigma \to \infty$. To measure $\dfrac{\partial g}{\partial \sigma}$ we set $T = g(\lambda, \sigma)$ in (6.19), differentiate with respect to σ and let $\lambda \to 0$, $\sigma \to \infty$ to obtain

$$\lim_{\lambda \to 0, \sigma \to \infty} \frac{\partial g}{\partial \sigma} = - \left(\frac{d^2 z^*}{dt^2}_{|t=B} \right)^{-1} \lim_{\lambda \to 0, \sigma \to \infty} z'_\sigma(g, \lambda, \sigma),$$

where z_σ is the derivative of z with respect to σ. According [14], Theorem 2.5,

$$\zeta(t) := \lim_{\lambda \to 0, \sigma \to \infty} z_\sigma(t, \lambda, \sigma),$$

solves the problem

$$\begin{cases} ((p-1) \left| \dfrac{dz^*}{dt} \right|^{p-2} \zeta')' = (p-q)\zeta & t \geq 0, \\ \zeta(0) = \zeta'(0) = 0. \end{cases}$$

It can be checked that $\zeta(t) \equiv 0$ and so

$$\lim_{\lambda \to 0, \sigma \to \infty} \frac{\partial g}{\partial \sigma} = 0.$$

This shows the claim.

On the other hand, asymptotic estimate (1.28) is a consequence of (6.20) and the representation (6.22) for u_λ corresponding to $\sigma = \sigma(\lambda)$.

The proof of Theorem 1.4 has now been completed.

Proof. [Proof of Theorem 1.1 concluded] To finish the proof of Theorem 1.1 we are showing iii) by following the ideas in the previous one. Accordingly, we consider two problems. On one hand, the Dirichlet problem:

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = |u|^{r-2} u & x \in \Omega \\ u = c & x \in \partial\Omega, \end{cases} \tag{6.24}$$

where Ω is a smooth bounded domain and $0 \leq c \leq 1$ is a parameter; on the other hand, the initial value problem:

$$\begin{cases} -(\mathrm{r}^{N-1} |u'|^{p-2} u')' = \mathrm{r}^{N-1} (|u|^{r-2} u - |u|^{p-2} u) & \mathrm{r} \geq 0, \\ u(0) = u_0 \\ u'(0) = 0, \end{cases} \tag{6.25}$$

whose solution is termed again $u(\cdot, u_0)$.

For every $0 \leq c \leq 1$, problem (6.24) has a positive solution $u \in C^{1,\alpha}(\overline{\Omega})$ satisfying $0 \leq u \leq c$ (just take $\overline{u} = c$ as a supersolution). Moreover, setting $c^* = (p - 1/r - 1)^{-1/r-p}$ it can be checked that (6.24) has a unique solution in the class,

$$0 \leq u(x) \leq c^*,$$

for $0 \le c \le c^*$. Set $\tilde{u}(\cdot, c)$ such solution when $\Omega = B$. By keeping the notations in the proof of Theorem 1.4, especially regarding function g, small L^∞ solutions to (1.7) when λ is small are characterized as the roots c to

$$\frac{u'(1, g(c))^{p-1}}{c^{q-1}} = \lambda,$$

which are close zero. Defining $A(c) = u'(1, g(c))^{p-1}/c^{q-1}$, assertion iii) reduces to show that A is increasing for c small. For this purpose it is convenient to normalize (6.24) by setting

$$\tilde{u}(\cdot, c) = cv(\cdot, c),$$

and thus v solves,

$$\begin{cases} -\Delta_p v + |v|^{p-2}v = c^{r-p}|v|^{r-2}v & x \in \Omega \\ v = 1 & x \in \partial\Omega. \end{cases}$$

Moreover, $v \to v_0$ as $c \to 0$ in $C^{1,\alpha}(\overline{\Omega})$ where v_0 is the positive solution to (1.11).

When $\Omega = B$ and c is small, definition of function $g(c)$ means that

$$\tilde{u}(\mathrm{r}, c) = u(\mathrm{r}, g(c)).$$

Hence, $A(c)$ can be written as:

$$A = c^{p-q} I_p(c) - c^{r-q} I_r(c), \tag{6.26}$$

where $I_s(c) := \int_0^1 t^{N-1} v(t, c)^{s-1}\, dt$. We are showing that:

$$\lim_{c \to 0} c v_c = 0, \tag{6.27}$$

where $v_c = \partial v / \partial c$, which implies that:

$$A'(c) = c^{p-q-1}((p-q)I_p(0) + o(1)) \qquad \text{as } c \to 0.$$

This leads to the monotonicity of A near $c = 0$ and iii) is shown.

To prove (6.27), set $\dot{u} = \partial u / \partial u_0$. By taking derivatives in (6.25) with respect u_0 we find,

$$\begin{cases} -((p-1)\mathrm{r}^{N-1}|v'|^{p-2}\dot{u}')' = \\ \qquad\qquad \mathrm{r}^{N-1}((r-1)c^{r-p}v^{r-2} - (p-1)v^{p-2})\dot{u} \\ \dot{u}(0) = 1 \\ \dot{u}'(0) = 0, \end{cases} \tag{6.28}$$

$\mathrm{r} \in (0, 1]$. As explained below, such (linear) problem in $\dot{u}$ has a unique solution at $c = 0$. On the other hand, when $c = 0$, v_0 solves such problem with 1 replaced in the initial condition by $k := v_0(0)$. Hence,

$$v_0 = k\dot{u}(\cdot, 0).$$

Since $\dot{u}(1, g(c))g'(c) = 1$ then $k = g'(0)$. Finally,

$$\lim_{c\to 0} cv_c = \dot{u}(\cdot, 0)g'(0) - v_0 = 0,$$

and so (6.27) is shown.

On the other hand, concerning the uniqueness of solutions to (6.28) as $c = 0$ it only need be proved that problem

$$\begin{cases} (\mathrm{r}^{N-1}|v_0'|^{p-2}z')' = \mathrm{r}^{N-1}v_0^{p-2}z & \mathrm{r} \in (0, 1] \\ z(0) = 0 \\ z'(0) = 0, \end{cases}$$

has $z = 0$ as unique solution. To this purpose, any possible solution $z(\mathrm{r})$ can be written as,

$$z(\mathrm{r}) = \int_0^{\mathrm{r}} \frac{1}{|v_0'|^{p-2}} \int_0^{\rho} \left(\frac{t}{\rho}\right)^{N-1} |v_0|^{p-2} z(t)\ dt\ d\rho.$$

Set $M = \sup_{t\in[0,1]} |z(t)|$. From such expression for z it follows that:

$$M \leq M(v_0(1) - k),$$

which clearly implies $M = 0$, as desired.

As final remarks, a proof of the assertion in ii) on the smoothness of u_λ with respect λ, when $\lambda \sim 0$, is implicit in the previous discussion. Regarding estimate (1.10) notice that equation

$$A(c) = \lambda,$$

together with (6.26) imply that

$$c = I^{-\frac{1}{p-q}} \lambda^{\frac{1}{p-q}} + o(\lambda^{\frac{1}{p-q}}),$$

as $\lambda \to 0$. Since $u_\lambda = cv(\cdot, c) = c(v_0(\cdot) + o(1))$ as $c \to 0$, then the desired estimate is shown.

Remark 6.1 If the concept of weak solution $u \in W^{1,p}(B)$ to (1.7) is subject to the extra condition $u \in L^r(B)$ (see Section 2), then limitation $r \leq p^*$ can be removed from the statement of Theorem 1.1 with the sole exception of iv). In fact, such change in the concept of solution still permits us to employ the ode's approach to (1.7) (see Section 2). On the other hand, it should be pointed out that due to the lack of uniqueness of solutions in problem (6.24) for large c, existence of a second solution to (1.7) can not be attained by the arguments in the proof of Theorem 1.4 above. Thus, and at the best of our knowledge, degree treatment in Section 5 can not be bypassed.

Proof. [Proof of Theorem 1.2 concluded] Let $\tilde{u}(x, \cdot)$ be the solution to the Dirichlet problem (6.2) in the range $u \geq 1$. Since Ω is a smooth domain, a fixed radius

$R > 0$ exists so that for every $\bar{x} \in \partial\Omega$ there exists a ball $B_R \subset \Omega$ with radius R and $\bar{x} \in \partial B_R$. Set $\tilde{u}_B(\cdot, c)$ the solution to (6.2) in the ball B_R. Then

$$\tilde{u}(x,c) \le \tilde{u}_B(x,c) \qquad x \in B_R. \tag{6.29}$$

For $c \le c_1$, $\tilde{u}_B = 1$ in $B_{R-d(c)}$ (the ball with same center as B_R but radius $R-d(c)$), $d(c)$ being the function introduced in the proof of Theorem 1.4. Since (6.2) is translation invariant, (6.29) implies that

$$\tilde{u}(\cdot, c) = 1 \qquad \text{on} \qquad \partial\Omega_{d(c)},$$

for all $c \ge c_1$, $\Omega_{d(c)} = \{x \in \Omega : d(x) > d(c)\}$, $d(x) = \operatorname{dist}(x, \partial\Omega)$. Then, the maximum principle yields

$$\{\tilde{u}(x,c) = 1\} \supset \Omega_{d(c)},$$

for $c \ge c_1$. Note that $d(c)$ satisfies (6.14) and in particular $d(c) \to 0+$ as $c \to 1+$.

Let now w_λ be any family of positive solutions to (1.14) such that $w_\lambda \ge 1$ and $c(\lambda) := \sup_\Omega w_\lambda \to 1$ as $\lambda \to 0+$. Then

$$w_\lambda \le \tilde{u}(\cdot, c)_{|c=c(\lambda)},$$

and therefore

$$\{w_\lambda(x) = 1\} \supset \{x : d(x) \ge d(\lambda)\},$$

where $d(\lambda) := d(c(\lambda))$, provided $c(\lambda) \le c_1$. This completes the proof of Theorem 1.2.

Proof. [Proof of Theorem 1.3] Observe that i) already implies ii). Thus, we only have to prove i). On the other hand, the same argument leading to iii) in Theorem 1.2 permits us concluding that $\tilde{u}_\lambda$ satisfies

$$\tilde{u}_\lambda \to 1,$$

in $C^{1,\alpha}(\overline{\Omega})$ (since $p = 2$ this convergence can be further improved). Accordingly the only fact to be proved is that u_λ is the unique solution near $u = 1$ for λ small. But this can be immediately achieved by the implicit function theorem. In fact, by employing the terminology of Section 2, (1.14) can be written as

$$u - S(|u|^{q-2}u, \lambda|u|^{r-2}u) = 0,$$

where $u \in C^1(\overline{\Omega})$. If $\mathcal{H} : (-\varepsilon, \varepsilon) \times U \subset \mathbb{R} \times C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ stands for the left hand side, U being a neighborhood of $u = 1$, it is clear that $\mathcal{H}(1,0) = 0$. In addition,

$$D\mathcal{H}(1,0)(\hat{u}) = 0,$$

$D\mathcal{H}$ standing for the differential of $\mathcal{H}$ with respect to u, amounts to

$$\begin{cases} -\Delta\hat{u} + (2-q)\hat{u} = 0 & x \in \Omega \\ \dfrac{\partial\hat{u}}{\partial\nu} = 0. & \end{cases}$$

Indeed observe that $\hat{u} \in C^2(\overline{\Omega})$ since Ω is smooth. Then it follows that $\hat{u} = 0$. Since $D\mathcal{H}(\overline{\Omega})$ is a compact perturbation from identity it actually defines an isomorphism. Therefore the desired conclusion is attained.

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