MULTIPLICITY OF SOLUTIONS TO A NONLINEAR BOUNDARY VALUE PROBLEM OF CONCAVE–CONVEX TYPE

JOSÉ C. SABINA DE LIS AND SERGIO SEGURA DE LEÓN

To Professor Carlos Fernández Pérez
in recognition of his scientific and academic career

Abstract. Problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{r-1}u & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{s-1}u & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, ν is the unit outward normal at $\partial\Omega$, Δ_p is the p–Laplacian operator and $\lambda > 0$ is a parameter, was studied in [18] and [19]. Among other features, it was shown there that when exponents lie in the regime $1 < s < p < r$, a minimal positive solution exists if $0 < \lambda \le \Lambda$, for a certain finite Λ, while no positive solutions exist in the complementary range $\lambda > \Lambda$. Furthermore, in the radially symmetric case a second positive solution exists for λ varying in the same full range $(0, \Lambda)$ provided $r < p^*$. Our main achievement in this work just asserts that such global multiplicity feature holds true when Ω is a *general* domain. To show such result the well–known Brezis–Nirenberg variational result in [6] must be extended to the framework of (P). This is the second main contribution in the present work.

1. Introduction.

This article is devoted to study positive solutions to problem

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \varphi_r(u) & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda \varphi_s(u) & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a class C^2 bounded domain, ν stands for the unit outward normal to $\partial\Omega$, $\Delta_p u = \operatorname{div}\,(|\nabla u|^{p-2}\nabla u)$ is the p–Laplacian

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operator ($1 < p < \infty$) and λ is a positive parameter. Here, and henceforth, notation $\varphi_q(u) = |u|^{q-2}u$ is used.

Problem (1.1) is a further model of a reaction–diffusion equation. Reaction–Diffusion systems indeed constitutes a long tradition area in Nonlinear Analysis since the seventies (cf. [17] for a recent account). In the present case, source reactions $\varphi_r(u)$, $\varphi_s(u)$ compete with absorption $\varphi_p(u)$ and the diffusion term $\Delta_p u$, and our interest here is stating the existence of equilibrium configurations to such system. In other words, the existence of positive solutions to (1.1) (we refer to [7] as a source on the rôle and further applications of the operator Δ_p). In addition, we are focusing our attention in the exponent range,

$$1 < s < p < r.$$

This means that (1.1) falls in the class of the so–called "concave–convex" problems (see [8], [9], [4] and specially [1] for pioneering works on the subject). Factor λ, modulating the surface reaction intensity, exhibits the status of bifurcation parameter in (1.1). Thus, studying the possible variations in the solution set to (1.1) in response to perturbations of λ becomes imperative.

Let us briefly account for the historical background of our problem. A Dirichlet version of (1.1), i. e.

$$\begin{cases} -\Delta_p u = \lambda\varphi_s(u) + \varphi_r(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

$1 < s < p < r < p^*$, was studied in [2] and [10], providing an extension for the case $p \neq 2$ of the results in [1]. As main features, it was shown the existence of $\Lambda > 0$ such that a minimal solution to (1.2) exists when $0 < \lambda \leq \Lambda$, not existing any positive solution if $\lambda > \Lambda$. Moreover, a *global* multiplicity result was also stated. Namely, the existence of an "extra" positive solution in the full range $0 < \lambda < \Lambda$. The crucial technical point to attain this second solution was to obtain "a priori" L^∞ bounds of the solutions. This was only achieved in [2] in the special case of radial solutions. The case of an arbitrary domain Ω was latter addressed in [10], however avoiding to obtain such L^∞ estimates. It should be mentioned in passing that our approach here is inspired in the alternative pathway to global multiplicity traced in [10].

All these features were observed afterwards to hold in the following "nonlinear boundary conditions" version of (1.2):

$$\begin{cases} -\Delta u + u = \varphi_r(u) & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda\varphi_s(u) & x \in \partial\Omega, \end{cases} \tag{1.3}$$

with $1 < s < 2 < r \leq 2^*$, where diffusion is regarded linear ($\Delta_p = \Delta$). Such results were accomplished in [11] being just this work a starting point for our interest in (1.1). In fact, (1.1) constitutes in turn a nonlinear diffusion version of (1.3) and was first addressed in [18]. The existence of a critical range $0 < \lambda \leq \Lambda$ together with the arising of a minimal positive solution u_λ to (1.1) was stated there (see Theorem 2.1 below). However, a global multiplicity result in $(0, \Lambda)$ could only be later accomplished in [19] in the radially symmetric case. As in [2], our proof relies upon getting L^∞ estimates. Therefore, our main objective here is showing that a further second positive solution to (1.1) exists in a *general* domain for λ varying in the whole interval $(0, \Lambda)$.

It should be stressed that in order to show the existence of a second solution to (1.1) two obstacles must be overcome. A first one, to show a *strong* comparison result between minimal solutions. This task was completed in [19] (see Theorem 2.1 and Remark 2.2). The second one, to produce a suitable uniform Hölder estimate for an auxiliary problem associated to (1.1) (see problem (5.1) below). This is our second main aim here.

The plan of this paper is as follows. Section 2 is devoted to state the main results, Theorems 2.3 and 2.4. Sections 3 and 4 contain the proofs of Theorems 2.4 and 2.3, respectively. In order to achieve the uniform Hölder estimate mentioned above, a suitable uniform L^∞ estimate must be previously obtained, and this is the objective of Section 5 (see Theorem 5.1). Theorem 6.1 providing the main C^α estimate and its proof are delayed to Section 6.

Let us fix now some few notation. As usual, for $1 \leq q \leq \infty$, $L^q(\Omega)$ and $L^q(\partial\Omega)$ stand for the Lebesgue spaces on the domain and its boundary (recall $\partial\Omega$ is smooth), respectively; their norms being designated by $\|\cdot\|_p$ and $\|\cdot\|_{p,\partial\Omega}$. Likewise, $W^{1,p}(\Omega)$ is the Sobolev space of p–integrable functions with first derivatives also in $L^p(\Omega)$, endowed with its natural norm $\|\cdot\|_{1,p}$.

We recall the embeddings $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, $W^{1,p}(\Omega) \subset L^{p_*}(\partial\Omega)$ where $p^* = \dfrac{Np}{N-p}$ and $p_* = \dfrac{(N-1)p}{N-p}$ if $1 < p < N$, while $p^* = p_* = \infty$ otherwise.

The space of uniformly Hölder continuous functions in Ω with exponent $\alpha \in (0,1)$ will be represented, as customary, by $C^\alpha(\overline{\Omega})$ (cf. [14]).

To simplify the writing we are further employing

$$\frac{\partial u}{\partial \nu_p} = |\nabla u|^{p-2} \frac{\partial u}{\partial \nu},$$

to designate the p–Laplacian version of the co–normal derivative at $\partial\Omega$.

2. Statement of main results

Our purpose in this paper is to analyze the existence of positive solutions to problem (1.1):

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = |u|^{r-2}u & x \in \Omega \\ \dfrac{\partial u}{\partial \nu_p} = \lambda |u|^{s-2}u & x \in \partial\Omega, \end{cases}$$

where

$$1 < s < p < r. \tag{2.1}$$

In the linear setting $p = 2$, condition (2.1) implies that problem (1.1) combines a concave source on $\partial\Omega$ with a convex reaction in Ω. That is why we still regard (1.1) as a concave–convex problem.

Several issues on (1.1) have already been analyzed both in [18] and [19]. Some of the relevant results therein are next stated for its use in this work.

Theorem 2.1. *Assume that* $0 < s < p < r$ *are arbitrary. Then, problem* (1.1) *exhibits the following features:*

i) *There exists* $\Lambda > 0$ *such that positive solutions* $u \in C^1(\overline{\Omega})$ *to* (1.1) *are only possible if*

$$0 < \lambda \leq \Lambda, \tag{2.2}$$

for a certain finite $\Lambda > 0$.

ii) *For all* λ *satisfying* (2.2) *there exists a minimal positive solution* $u_\lambda \in C^1(\overline{\Omega})$.

iii) *Limit,*

$$u^* = \lim_{\lambda \to \Lambda} u_\lambda = \sup_{0<\lambda<\Lambda} u_\lambda,$$

holds in $W^{1,p}(\Omega) \cap L^r(\Omega)$ *and defines a positive solution to* (1.1) *for* $\lambda = \Lambda$.

iv) [Strong comparison between minimal solutions]. *For* $0 < \lambda_1 < \lambda_2 < \Lambda$ *strict inequality*

$$u_{\lambda_1} < u_{\lambda_2}$$

holds in $\overline{\Omega}$.

Remarks 2.2.

a) Properties i), ii) and iii) were obtained in [18] under the additional assumption that $r \leq p^*$. However, this restriction is unnecessary for those properties to hold (see [18] and Sect. 1 in [1]).

b) The strong comparison assertion in iv) is a key fact for our purposes here (see Proof of Theorem 5.1 below). It was shown in [19, Theorem 4.1]. It should be remarked in this regard that a *general* strong comparison principle is not available for the p–Laplacian. Moreover, it is shown in [19] that such result fails if terms $\varphi_r(u)$ and $\varphi_s(u)$ is (1.1) are interchanged and $p > 2$.

Our first main theorem furnishes the existence of a *second* positive solution provided that we restrict the growth degree of the nonlinearities in (1.1).

Theorem 2.3. *Assume that* (1.1) *satisfies the extra growth condition*

$$1 < s < p < r < p^*. \tag{2.3}$$

Then, for every $0 < \lambda < \Lambda$, *problem* (1.1) *possesses a second positive solution* $v_\lambda \in C^{1,\alpha}(\overline{\Omega})$.

The proof of the existence of a second positive solution to (1.1) makes use of a variational argument, being the key tool an extension of the analysis in [6] (see also [10]) to the framework of our problem. In fact, to broaden the scope of our results we are dealing with a slightly more general layout than (1.1). Namely,

$$\begin{cases} -\Delta_p u + \varphi_p(u) = f(x,u) & x \in \Omega \\ \dfrac{\partial u}{\partial \nu_p} = g(x,u) & x \in \partial\Omega, \end{cases} \tag{2.4}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions which satisfy the growth conditions

$$|f(x,u)| \le C(1 + |u|^{r-1}), \quad \text{with } 0 < r < p^*, \tag{2.5}$$

and

$$|g(x,u)| \le C(1 + |u|^{s-1}), \quad \text{with } 0 < s < p_*. \tag{2.6}$$

Thus, the associated functional

$$J_p(u) = \frac{1}{p} \int_\Omega |\nabla u|^p + |u|^p \, dx - \int_\Omega F(x,u)\, dx - \int_{\partial\Omega} G(x,u)\, d\sigma,$$

with $F(x,u) = \int_0^u f(x,t)\, dt$ and $G(x,u) = \int_0^u g(x,t)\, dt$, is well defined in $W^{1,p}(\Omega)$.

In our second main result the well–known Brezis–Nirenberg's result in [6] is extended to problem (2.4).

Theorem 2.4. *Assume that $u_0 \in C^1(\overline{\Omega})$ is a $C(\overline{\Omega})$–local minimizer of J_p, i. e.,*

$$J_p(u_0 + v) \geq J_p(u_0),$$

for all $v \in C^1(\overline{\Omega})$ such that

$$\|v\|_\infty \leq \varepsilon_0,$$

for certain $\varepsilon_0 > 0$. Then u_0 is actually a local minimizer of J_p in $W^{1,p}(\Omega)$.

Remarks 2.5.

a) If $u_0 \in W^{1,p}(\Omega)$ is a $C(\overline{\Omega})$–local minimizer of J_p, then it defines a weak solution to (2.4). On the other hand, it is shown in [18] that under the sole conditions (2.5) and (2.6), weak solutions $u \in W^{1,p}(\Omega)$ to (2.4) lie in $L^\infty(\Omega)$. Finally, weak solutions $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (2.4) belong to $C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$, provided $g(x,u)$ satisfy a local Hölder condition (cf. [16, Theorem 2]). Therefore, hypothesis $u_0 \in C^1(\overline{\Omega})$ in Theorem 2.4 is not restrictive. Finally, observe that, under the growth condition (2.3), weak solutions $u \in W^{1,p}(\Omega)$ to (1.1) satisfy $u \in C^{1,\alpha}(\overline{\Omega})$ for certain $0 < \alpha < 1$.

b) It is a consequence of the analysis in Section 6 that every weak solution $u \in W^{1,p}(\Omega)$ to (2.4) satisfies $u \in C^\gamma(\overline{\Omega})$ for certain $0 < \gamma < 1$.

3. Proof of Theorem 2.4

Assume that the conclusion does not hold. Then, for every $0 < \varepsilon \leq \varepsilon_0$, ε_0 small, there exists $w_\varepsilon \in W^{1,p}(\Omega)$ such that

$$\|w_\varepsilon - u_0\|_{W^{1,p}(\Omega)} \leq \varepsilon,$$

and satisfying

$$J_p(w_\varepsilon) < J_p(u_0).$$

Since J_p is weakly lower semicontinuous in $\overline{B}_\varepsilon(u_0)$, the closed ball in $W^{1,p}(\Omega)$ centered at u_0 with radius ε, then $v_\varepsilon \in \overline{B}_\varepsilon(u_0)$ exists such that,

$$J_p(v_\varepsilon) = \min_{v \in \overline{B}_\varepsilon(u_0)} J_p(v) \leq J_p(w_\varepsilon) < J_p(u_0)\,. \tag{3.1}$$

Two options are now possible:

a) There exists $\varepsilon_n \to 0$ such that

$$\|v_n - u_0\|_{W^{1,p}(\Omega)} < \varepsilon_n,$$

for all n, where $v_n = v_{\varepsilon_n}$.

b) Equality

$$\|v_\varepsilon - u_0\|_{W^{1,p}(\Omega)} = \varepsilon,$$

holds true for every $0 < \varepsilon \le \varepsilon_0$.

In case a) every v_n solves (2.4). By choosing $\mu = 0$ in problem (5.1) below, Theorem 6.1 of Section 6 furnishes the existence of an exponent $0 < \alpha < 1$ and a constant $M > 0$ such that,

$$\|v_n\|_{C^\alpha(\overline{\Omega})} \le M,$$

for every n. Passing through a subsequence this means (by Ascoli–Arzelà's Theorem) that $(v_n)_n$ converges uniformly on $\overline{\Omega}$. Thus, $v_n \to u_0$ in $W^{1,p}(\Omega)$ implies $v_n \to u_0$ in $C(\overline{\Omega})$. However (3.1) contradicts the fact that u_0 is a $C(\overline{\Omega})$–local minimizer of J_p. Thus, option a) can not occur.

If remaining option b) holds, Lagrange's multiplier rule implies that $v = v_\varepsilon$ solves the problem,

$$\begin{cases} -\Delta_p v + \varphi_p(v) - \mu\Delta_p(v-u_0) + \mu\varphi_p(v-u_0) = f(x,v) & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu_p} + \mu\dfrac{\partial (v-u_0)}{\partial \nu_p} = g(x,v) & \text{on } \partial\Omega, \end{cases}$$

where $\mu = \mu_\varepsilon$ stands for the multiplier. We point out that the only available information on μ_ε is its sign. Indeed, being v_ϵ a minimum of J_p in $\overline{B}_\epsilon(u_0)$, it follows that

$$\mu_\varepsilon \ge 0.$$

Nevertheless, Theorem 6.1 again provides us with an estimate,

$$\|v_\varepsilon\|_{C^\alpha(\overline{\Omega})} \le M, \tag{3.2}$$

for all $0 < \varepsilon \le \varepsilon_0$, with α and M not depending on $\mu \ge 0$ and hence nor on $\varepsilon > 0$. By arguing as in a) we arrive again to a contradiction. Therefore u_0 is in fact a local minimizer in $W^{1,p}(\Omega)$.

Remark 3.1. To show that an estimate as (3.2) holds uniformly on $\mu \ge 0$ is the crux of the matter (compare with the corresponding case in [10]). Its proof is split in two steps addressed in Sections 5 and 6 below.

4. Proof of Theorem 2.3.

Fix $\lambda \in (0, \Lambda)$ and choose λ_1 and λ_2 satisfying $0 < \lambda_1 < \lambda < \lambda_2 < \Lambda$. Let u_i denote the minimal solution corresponding to λ_i, $i = 1, 2$ and set $u_0 = u_\lambda$. By Theorem 2.1 iv) ([19, Theorem 4.1]) we deduce that $u_1(x) < u_2(x)$ for all $x \in \overline{\Omega}$.

Next observe that $\underline{u} := u_1$ and $\overline{u} := u_2$ constitutes a pair of ordered weak sub and super solutions to (1.1), respectively. Following the proof of Theorem 3 in [12] set

$$\mathcal{M} = \{u \in W^{1,p}(\Omega) : \underline{u} \leq u \leq \overline{u}\}\,;$$

alternatively, $\mathcal{M} = [\underline{u}, \overline{u}]$. Since $\mathcal{M}$ is closed–convex and the functional J_p associated to (1.1),

$$J_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p\,dx - \frac{1}{r}\int_\Omega |u|^r\,dx - \frac{\lambda}{s}\int_{\partial\Omega} |u|^s\,d\sigma\,,$$

is weakly lower semicontinuous, then J_p achieves a global minimizer $\tilde{u}_0$ in $\mathcal{M}$. As shown in [12], $\tilde{u}_0$ provides a solution to (1.1).

It can be assumed now, without loss of generality, that $\tilde{u}_0 = u_0$. Otherwise we would have already obtained a second positive solution to (1.1) which is our main concern. Condition (2.3) says that problem (1.1) satisfies the growth conditions (2.5) and (2.6). Being

$$\underline{u} < u_0 < \overline{u},$$

u_0 becomes a local minimizer of J_p in $C(\overline{\Omega})$ and hence, by Theorem 2.4, a local minimizer of J_p in $W^{1,p}(\Omega)$ as well. It should be pointed out that this is just the crucial step where strong comparison result in Theorem 2.1-iv) is required.

To search for a second solution we are next discussing the Mountain Pass Lemma ('MPL' for short) properties of functional J_p near the local minimum u_0. To catch an insight of the forthcoming difficulties, let us first consider the natural truncation of (1.1) in order to find positive solutions. Namely,

$$\begin{cases} -\Delta_p u + \varphi_p(u) = (u^+)^{r-1} & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_p} = (u^+)^{s-1} & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $u^+ = u$ if $u \geq 0$, $u^+ = 0$ otherwise. Since its associated functional

$$J_{p,+}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p\,dx - \frac{1}{r}\int_\Omega (u^+)^r\,dx - \frac{\lambda}{s}\int_{\partial\Omega} (u^+)^s\,d\sigma\,,$$

agree with J_p in $\mathcal{M}$, then $J_{p,+}$ has a local minimum at u_0 in $W^{1,p}(\Omega)$. This is achieved by means of Theorem 2.4 when applied to problem (4.1). Moreover, it can be checked that any nontrivial critical point $u \in W^{1,p}(\Omega)$ of $J_{p,+}$ defines a positive solution to (1.1). On the other hand, a slight variation of the proof of Lemma 3.6 in [3] shows that $J_{p,+}$ satisfies the Palais–Smale condition.

To employ MPL we fix $\rho > 0$ small enough so that $J_{p,+}(u) \geq J_{p,+}(u_0)$ in $\overline{B}_\rho(u_0)$, while choose a constant $t > 0$ so large as to have $J_{p,+}(v_t) < 0$ where $v_t(x) = t$. Next we set, as customary,

$$\Gamma = \{\eta : [0,1] \to W^{1,p}(\Omega) : \eta \text{ continuous},\ \eta(0) = u_0,\ \eta(1) = v_t\}.$$

Then,

$$c := \inf_{\eta\in\Gamma} \max_{[0,1]} J_{p,+} \circ \eta,$$

defines a *critical* value for $J_{p,+}$. In fact, two possibilities arise: either $c > J_{p,+}(u_0)$ or either $c = J_{p,+}(u_0)$. Former case is the standard scenario for MPL (cf. [3]) ensuring us that c is a critical value. In latter case, the Ghoussoub–Preiss improved version of MPL not only implies that c is critical but even asserts that $J_{p,+}$ possesses a critical point on $\partial B_\rho(u_0)$ (cf. [13]).

Finally, it was shown in [18] that $J_p(u_0) = J_{p,+}(u_0) < 0$. Therefore, if $c = J_{p,+}(u_0)$ we obtain a nontrivial critical point $\hat{u}_0$ and hence a second positive solution to (1.1) (see Remark 4.1 below). Troubles arise when $c > J_{p,+}(u_0)$ since in this case we are not able to “a priori” rule out the trivial case $c = 0$.

To circumvent the obstacle we are introducing a further truncation of (1.1). Namely,

$$\begin{cases} -\Delta_p u + \varphi_p(u) = \underline{f}(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_p} = \underline{g}(x,u) & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where,

$$\underline{f}(x,u) = \begin{cases} \underline{u}(x)^{r-1} & u \leq \underline{u}(x) \\ u^{r-1} & u > \underline{u}(x), \end{cases} \quad \underline{g}(x,u) = \begin{cases} \lambda\underline{u}(x)^{s-1} & u \leq \underline{u}(x) \\ u^{s-1} & u > \underline{u}(x). \end{cases}$$

By weak comparison one checks that a critical point $u \in W^{1,p}(\Omega)$ for the associated functional $\underline{J}_p$ of (4.2),

$$\underline{J}_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p\,dx - \int_\Omega \underline{F}(x,u)\,dx - \int_{\partial\Omega} \underline{G}(x,u)\,d\sigma\,,$$

satisfies $u \geq \underline{u}$; $\underline{F}$ and $\underline{G}$ being the primitives of $\underline{f}$ and $\underline{g}$, respectively. Thus, u defines a nonnegative and nontrivial solution to (1.1). On the other hand, $\underline{J}_p$ agrees with J_p in $\mathcal{M}$ meanwhile problem (4.2) fits the profile of Theorem 2.4. Therefore, u_0 defines a local minimum for $\underline{J}_p$ when observed in $W^{1,p}(\Omega)$. Finally, a suitable variant of the proof of Lemma 3.6 in [3] permits us concluding that $\underline{J}_p$ satisfies the Palais–Smale condition.

By arguing as in the case of problem (4.1) we find now that in either options $c > \underline{J}_p(u_0)$ or $c = \underline{J}_p(u_0)$ we obtain a further and nontrivial critical point $\hat{u}_0 \in W^{1,p}(\Omega)$ of $\underline{J}_p$. Therefore, the existence of a second positive solution to (1.1) is accomplished (see Remark 4.1).

REMARK 4.1. It should be pointed out that previous proof does not furnish a second positive solution to (1.1) of MPL type. In fact, our argument shows that this is just the case when the minimal solution $u_0 = u_\lambda$ coincides with the minimizer $\tilde{u}_0$ of J_p in $\mathcal{M}$.

5. A uniform L^∞–estimate.

Let $u_0 \in C^1(\overline{\Omega})$ be a solution to (2.4) and $\mu \geq 0$ a parameter. We are focusing our interest in weak solutions $u \in W^{1,p}(\Omega)$ to problem,

$$
(5.1) \qquad \begin{cases} -\Delta_p u + \varphi_p(u) - \mu\Delta_p(u-u_0) + \mu\varphi_p(u-u_0) & \\ \qquad\qquad = f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_p} + \mu\dfrac{\partial(u-u_0)}{\partial \nu_p} = g(x,u) & \text{on } \partial\Omega. \end{cases}
$$

Our main purpose here is showing that solutions to (5.1) satisfying in addition

$$
(5.2) \qquad \|u-u_0\|_{W^{1,p}(\Omega)} \leq \varepsilon_0,
$$

with $\varepsilon_0 > 0$ a fixed positive number, fulfill an L^∞–estimate which is uniform with respect to $\mu \geq 0$.

More precisely we are proving the next result.

Theorem 5.1. *Assume that $u_0 \in C^1(\overline{\Omega})$ defines a solution to* (2.4) *and let $u \in W^{1,p}(\Omega)$ be a solution to problem* (5.1) *corresponding to a nonnegative parameter μ and satisfying* (5.2) *for a certain $\varepsilon_0 > 0$. Then there exists $M > 0$, not depending on μ such that*

$$
\|u\|_\infty \leq M.
$$

Proof of Theorem 5.1 is organized by handling separately cases $2 \leq p \leq N$ and $1 < p < 2$. Their analysis is performed in Sections 5.1 and 5.2 below. Conclusion of Theorem 5.1 directly follows from Morrey's estimates (see, for instance, [14]) in the remaining case $p > N$.

In what follows, C will always denote a positive constant only depending on the parameters of our problem (and not depending on μ) and whose value may change in the different expressions where it appears. We will also introduce the auxiliary functions

$$
(5.3) \qquad T_k(t) = \min\{|t|, k\}\text{sign}\,(t)\,, \quad G_k(t) = (|t|-k)^+\text{sign}\,(t)\,,
$$

where $k > 0$ and $t^+ = t$ for $t \geq 0$ and $t = 0$ otherwise.

5.1. **Case $2 \leq p \leq N$.** First of all, we set

$$A(k) = \{x \in \Omega \ : \ |u(x) - u_0(x)| \geq k\}\,, \qquad \text{with } k > 0\,.$$

It follows from (5.2) that

$$|A(k)| \leq \frac{\int_{A(k)} |u - u_0|^p\, dx}{k^p} \leq \frac{\int_\Omega |u - u_0|^p\, dx}{k^p} \leq \frac{C}{k^p}\,,$$

and so

$$\lim_{k\to\infty} |A(k)| = 0,$$

uniformly on u satisfying (5.2). Thus, by taking k large enough, we may assume that $|A(k)|$ is suitably small.

Write now problem (5.1) as follows

$$\text{(5.4)}\qquad \begin{cases} -\Delta_p u - (-\Delta_p u_0) - \mu\Delta_p(u - u_0) \\ \qquad\qquad +\varphi_p(u) - \varphi_p(u_0) + \mu\varphi_p(u - u_0) \\ \qquad\qquad\qquad\qquad = f(x,u) - f(x,u_0) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_p} - \dfrac{\partial u_0}{\partial \nu_p} + \mu\dfrac{\partial (u - u_0)}{\partial \nu_p} \\ \qquad\qquad\qquad\qquad = g(x,u) - g(x,u_0) & \text{on } \partial\Omega\,. \end{cases}$$

Two steps will be addressed in turn:

1) $u \in L^q(\Omega) \cap L^q(\partial\Omega)$ for all q, with $1 \leq q < \infty$.
2) $u \in L^\infty(\Omega)$, and $\|u\|_\infty$ does not depend on μ.

Step 1). We assume that $2 \leq p < N$, since the result is a straightforward consequence of Sobolev's embedding theorem when $p = N$. Following Brezis–Kato approach (see [5] and [21]) we will show that

$$|u|^{p(m+1)} \in L^1(\Omega) \cap L^1(\partial\Omega),$$

for an integer $m \geq 0$ implies,

$$\text{(5.5)}\qquad |u|^{m+1} \in W^{1,p}(\Omega)\,.$$

Once this has been achieved, the embeddings $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and $W^{1,p}(\Omega) \subset L^{p_*}(\partial\Omega)$ imply that $|u|^{p(m+1)\frac{N-1}{N-p}} \in L^1(\Omega) \cap L^1(\partial\Omega)$. So, taking $m_0 = 0$ and iterating the previous argument, we obtain a sequence m_n satisfying

$$|u|^{p(m_n+1)\frac{N-1}{N-p}} \in L^1(\Omega) \qquad\qquad m_n + 1 = \Big(\frac{N-1}{N-p}\Big)^n \to \infty\,,$$

and thus we are done.

We begin by denoting

$$a(x)=\frac{|f(x,u(x))-f(x,u_0(x))|}{1+|u(x)-u_0(x)|^{p-1}},$$

$$b(x)=\frac{|g(x,u(x))-g(x,u_0(x))|}{1+|u(x)-u_0(x)|^{p-1}}.$$

Remark that (2.5) implies

$$\begin{aligned}|f(x,u(x))-f(x,u_0(x))|&\le C(1+|u(x)|^{r-1})+C\\&\le C(1+|u(x)-u_0(x)|^{r-1})\le C(1+|u(x)-u_0(x)|^{p^*-1})\,.\end{aligned}$$

Thus,

$$a(x)\le C(1+|u(x)-u_0(x)|^{p^*-p})\,,$$

for almost all $x\in\Omega$. In particular, $a\in L^{\frac{N}{p}}(\Omega)$. Moreover, the norm $\|a\|_{\frac{N}{p}}$ only depends on u_0 and $\|u-u_0\|_{p^*}$; hence it is independent of the parameter μ. Likewise, we derive from (2.6) that

$$b(x)\le C(1+|u(x)-u_0(x)|^{p_*-p}),$$

a. e. on $\partial\Omega$ and so $b\in L^{\frac{N-1}{p-1}}(\partial\Omega)$, its norm not depending on μ.

In order to prove (5.5), we choose

$$\phi=(u-u_0)T_L(|u-u_0|)^{pm}=(u-u_0)\min\{|u-u_0|,L\}^{pm},$$

as test function in (5.4), where $L>1$. Dropping nonnegative terms, we get

$$\begin{aligned}&\int_\Omega(|\nabla u|^{p-2}\nabla u-|\nabla u_0|^{p-2}\nabla u_0)\cdot\nabla[(u-u_0)T_L(|u-u_0|)^{pm}]\,dx\\&\qquad\le\int_\Omega\big(f(x,u)-f(x,u_0)\big)(u-u_0)T_L(|u-u_0|)^{pm}\,dx\\&\qquad\qquad+\int_{\partial\Omega}\big(g(x,u)-g(x,u_0)\big)(u-u_0)T_L(|u-u_0|)^{pm}\,d\sigma\,.\end{aligned}$$

It follows from the monotonicity of the p–Laplacian that there exists a positive constant $c > 0$ satisfying

$$
\begin{aligned}
(5.6)\quad c\int_\Omega \{T_L(|u-u_0|)^{pm}|\nabla(u-u_0)|^p \\
+pmT_L(|u-u_0|)^{pm}|\nabla T_L(|u-u_0|)|^p\}\,dx \\
\le \int_\Omega \big(f(x,u)-f(x,u_0)\big)(u-u_0)T_L(|u-u_0|)^{pm}\,dx \\
+\int_{\partial\Omega} \big(g(x,u)-g(x,u_0)\big)(u-u_0)T_L(|u-u_0|)^{pm}\,d\sigma \\
\le \int_\Omega a(x)(1+|u-u_0|^{p-1})|u-u_0|T_L(|u-u_0|)^{pm}\,dx \\
+\int_{\partial\Omega} b(x)(1+|u-u_0|^{p-1})|u-u_0|T_L(|u-u_0|)^{pm}\,d\sigma\,.
\end{aligned}
$$

The left–hand side of (5.6) can be estimated from below as follows:

$$
\begin{aligned}
(5.7)\quad c\int_\Omega T_L(|u-u_0|)^{pm}|\nabla(u-u_0)|^p\,dx \\
+\,cpm\int_\Omega T_L(|u-u_0|)^{pm}|\nabla T_L(|u-u_0|)|^p\,dx \ge \\
\ge c\int_\Omega \big|T_L(|u-u_0|)^m\nabla|u-u_0|\big|^p\,dx \\
+\frac{cp}{m^{p-1}}\int_\Omega \big||u-u_0|\nabla T_L(|u-u_0|)^m\big|^p\,dx \\
\ge \eta\int_\Omega \big|\nabla\big(|u-u_0|T_L(|u-u_0|)^m\big)\big|^p\,dx,
\end{aligned}
$$

for some positive constant η not depending on L. Hence, (5.6) becomes

$$
\begin{aligned}
(5.8)\quad \eta\int_\Omega \big|\nabla\big(|u-u_0|T_L(|u-u_0|)^m\big)\big|^p\,dx \\
\le \int_\Omega a(x)(1+|u-u_0|^{p-1})|u-u_0|T_L(|u-u_0|)^{pm}\,dx \\
+\int_{\partial\Omega} b(x)(1+|u-u_0|^{p-1})|u-u_0|T_L(|u-u_0|)^{pm}\,d\sigma\,.
\end{aligned}
$$

We next analyze the right–hand side of (5.8). Recalling that $L > 1$ we observe that

$$\begin{aligned}
&(1+|u-u_0|^{p-1})|u-u_0|T_L(|u-u_0|)^{pm} \\
&\qquad \le 2 + (1+|u-u_0|^{p-1})|u-u_0|T_L(|u-u_0|)^{pm}\Big|_{\{|u-u_0|>1\}} \\
&\qquad\qquad \le 2 + 2|u-u_0|^p T_L(|u-u_0|)^{pm}\,,
\end{aligned}$$

meanwhile an analogous inequality holds for the term on the boundary. Thus,

$$\begin{aligned}
&\int_\Omega a(x)(1+|u-u_0|^{p-1})|u-u_0|T_L(|u-u_0|)^{pm}\,dx \\
&\qquad + \int_{\partial\Omega} b(x)(1+|u-u_0|^{p-1})|u-u_0|T_L(|u-u_0|)^{pm}\,d\sigma \\
&\quad \le 2\|a\|_{L^1(\Omega)} + 2\|b\|_{L^1(\partial\Omega)} + 2\int_\Omega a(x)|u-u_0|^p T_L(|u-u_0|)^{pm}\,dx \\
&\qquad + 2\int_{\partial\Omega} b(x)|u-u_0|^p T_L(|u-u_0|)^{pm}\,d\sigma\,.
\end{aligned}$$

To further estimate the integral terms, we introduce a parameter k (to be chosen later) and split each integral in two parts to finally apply Hölder's inequality. As for the first integral we obtain,

$$\begin{aligned}
&\int_\Omega a(x)\{|u-u_0|T_L(|u-u_0|)^m\}^p\,dx \le \\
&k\int_{\{a<k\}}\{|u-u_0|T_L(|u-u_0|)^m\}^p\,dx + \int_{\{a\ge k\}} a(x)\{|u-u_0|T_L(|u-u_0|)^m\}^p\,dx \\
&\quad \le k\int_\Omega |u-u_0|^{p(m+1)}\,dx + \int_{\{a\ge k\}} a(x)\{|u-u_0|T_L(|u-u_0|)^m\}^p\,dx \\
&\qquad \le C + \int_{\{a\ge k\}} a(x)\{|u-u_0|T_L(|u-u_0|)^m\}^p\,dx\,,
\end{aligned}$$

where C only depends on k and the norm $\|u-u_0\|_{p(m+1)}$. Thus, it follows from Sobolev's embedding that

$$\begin{aligned}
&\int_\Omega a(x)\{|u-u_0|T_L(|u-u_0|)^m\}^p\,dx \\
&\qquad \le C + \|a\|_{L^{N/p}(\{a\ge k\})}\||u-u_0|T_L(|u-u_0|)^m\|_{p^*}^p \\
&\qquad\quad \le C + C\|a\|_{L^{N/p}(\{a\ge k\})}\|\nabla(|u-u_0|T_L(|u-u_0|)^m)\|_p^p\,.
\end{aligned}$$

Finally by choosing $\varepsilon > 0$ small enough we can pick k suitably large so as to have

$$C\|a\|_{L^{N/p}(\{a\geq k\})} \leq \varepsilon.$$

Hence,

$$\int_\Omega a(x)\{|u-u_0|T_L(|u-u_0|)^m\}^p\,dx \\ \leq C + \varepsilon\|\nabla(|u-u_0|T_L(|u-u_0|)^m)\|_p^p\,. \tag{5.9}$$

Similarly, just replacing the Sobolev embedding with the trace embedding $W^{1,p}(\Omega) \subset L^{p_*}(\partial\Omega)$ we arrive to the similar estimate

$$\int_{\partial\Omega} b(x)|u-u_0|^p T_L(|u-u_0|)^{pm}\,d\sigma \\ \leq C + \varepsilon\|\nabla(|u-u_0|T_L(|u-u_0|)^m)\|_p^p, \tag{5.10}$$

where C only depends on k and $\|u-u_0\|_{p(m+1)}$.

By taking into account (5.9), (5.10) and (5.8) we get

$$\eta\int_\Omega \left|\nabla\left(|u-u_0|T_L(|u-u_0|)^m\right)\right|^p dx \\ \leq C + 2\varepsilon\int_\Omega \left|\nabla\left(|u-u_0|T_L(|u-u_0|)^m\right)\right|^p dx\,,$$

where η is a positive constant and ε can be taken as small as desired. Therefore,

$$\int_\Omega \left|\nabla\left(|u-u_0|T_L(|u-u_0|)^m\right)\right|^p \leq C\,,$$

for some constant C which does not depend on L. Finally, by doing $L \to \infty$ in this expression, Fatou's Lemma leads to $|u-u_0|^{m+1} \in W^{1,p}(\Omega)$. This proves (5.5).

Step 2). Set $\tilde{f}(x) = |f(x,u(x)) - f(x,u_0(x))|$ and $\tilde{g}(x) = |g(x,u(x)) - g(x,u_0(x))|$ to brief. Having in mind (2.5) and (2.6), it follows from previous Step 1) that $\tilde{f} \in L^q(\Omega)$ and $\tilde{g} \in L^q(\partial\Omega)$ for all $1 \leq q < \infty$. For definiteness purposes, we observe that $\tilde{f} \in L^{\varrho'}(\Omega)$ with $\varrho < \frac{N}{N-p}$, and $\tilde{g} \in L^{\eta'}(\partial\Omega)$ with $\eta < \frac{N-1}{N-p}$. We can further assume that both parameters are related: $(N-1)\varrho = N\eta$.

We are employing Stampacchia's procedure (see Appendix) to find a uniform L^∞ bound for $u - u_0$, and consequently on u. To this purpose it is enough to obtain an estimate of the form,

$$\int_\Omega |G_k(u-u_0)|^{p^*}\,dx \leq C|A(k)|^\gamma \tag{5.11}$$

provided that the exponent $\gamma > 1$, both C and γ not depending on μ.

As a first remark observe that

$$\|G_k(u-u_0)\|_{p^*} \le C(\|G_k(u-u_0)\|_p + \||\nabla G_k(u-u_0)|\|_p) \le \\ C(|A(k)|^{\frac{1}{p}-\frac{1}{p^*}}\|G_k(u-u_0)\|_{p^*} + \||\nabla G_k(u-u_0)|\|_p).$$

Taking into account that $|A(k)| \to 0$ we obtain,

$$\|G_k(u-u_0)\|_{p^*} \le C\||\nabla G_k(u-u_0)|\|_p, \tag{5.12}$$

for $k \ge k_0$ and k_0 large enough.

By choosing $G_k(u-u_0)$ as a test function in (5.4), we obtain

$$\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_0|^{p-2}\nabla u_0)\cdot \nabla G_k(u-u_0)\; dx \\ + \int_\Omega \big(\varphi_p(u)-\varphi_p(u_0)\big)G_k(u-u_0)\,dx \\ + \mu\int_\Omega |\nabla G_k(u-u_0)|^p\,dx + \mu\int_\Omega \varphi_p(u-u_0)G_k(u-u_0)\,dx \\ = \int_\Omega \tilde f(x)G_k(u-u_0)\,dx + \int_{\partial\Omega}\tilde g(x)G_k(u-u_0)\,d\sigma\,.$$

By employing the monotonicity of the p–Laplacian and dropping non-negative terms, such expression leads to

$$\int_\Omega |\nabla G_k(u-u_0)|^p\,dx \\ \le C\left\{\int_\Omega \tilde f\,|G_k(u-u_0)|\,dx + \int_{\partial\Omega}\tilde g\,|G_k(u-u_0)|\,d\sigma\right\} \\ \le C\left\{\|\tilde f\|_{\varrho'}\|G_k(u-u_0)\|_\varrho + \|\tilde g\|_{\eta',\partial\Omega}\|G_k(u-u_0)\|_{\eta,\partial\Omega}\right\}, \tag{5.13}$$

where integral norms on the boundary have been labeled with $\partial\Omega$ as a subindex.

We now proceed to estimate separately each term on the right–hand side of (5.13). Regarding the first one we find,

$$\|G_k(u-u_0)\|_\varrho \le \|G_k(u-u_0)\|_{p^*}|A(k)|^{\frac{1}{\varrho}-\frac{1}{p^*}} \\ \le C\||\nabla G_k(u-u_0)|\|_p|A(k)|^{\frac{1}{\varrho}-\frac{1}{p^*}}. \tag{5.14}$$

As for the last term, we apply the trace embedding

$$W^{1,q}(\Omega) \hookrightarrow L^{q(N-1)/(N-q)}(\partial\Omega)\,,$$

with

$$\frac{q(N-1)}{N-q} = \eta\,,$$

so that

$$q = \frac{N\eta}{N+\eta-1}\,.$$

Notice that $\eta, p > 1$ imply $1 \le q < p$. Hence,

$$\begin{aligned}
\Big(\int_{\partial\Omega} |G_k(u-u_0)|^\eta\, d\sigma\Big)^{1/\eta} &\le C\Big(\|G_k(u-u_0)\|_q + \||\nabla G_k(u-u_0)|\|_q\Big)\\
&\le C\Big(\|G_k(u-u_0)\|_p + \||\nabla G_k(u-u_0)|\|_p\Big)|A(k)|^{\frac{p-q}{pq}}\\
&\le C\Big(1+|A(k)|^{\frac{1}{p}-\frac{1}{p^*}}\Big)\||\nabla G_k(u-u_0)|\|_p |A(k)|^{\frac{p-q}{pq}}\\
&\le C\||\nabla G_k(u-u_0)|\|_p |A(k)|^{\frac{p-q}{pq}}\,,
\end{aligned}$$

where estimate (5.12) have been used. Moreover,

$$\frac{p-q}{p\,q} = \frac{Np+\eta p - p - N\eta}{Np\eta} = \frac{N-1}{N\eta} - \frac{N-p}{Np} = \frac{1}{\varrho} - \frac{1}{p^*}\,.$$

Therefore, by means of (5.12) we obtain

$$\Big(\int_{\partial\Omega} |G_k(u-u_0)|^\eta\, d\sigma\Big)^{1/\eta} \le C\||\nabla G_k(u-u_0)|\|_p |A(k)|^{\frac{1}{\varrho}-\frac{1}{p^*}}\,. \tag{5.15}$$

Relations (5.13), (5.14) and (5.15) altogether imply that

$$\int_\Omega |\nabla G_k(u-u_0)|^p\, dx \le C\||\nabla G_k(u-u_0)|\|_p |A(k)|^{\frac{1}{\varrho}-\frac{1}{p^*}}\,,$$

and so,

$$\||\nabla G_k(u-u_0)|\|_p^{p-1} \le C|A(k)|^{\frac{1}{\varrho}-\frac{1}{p^*}}.$$

Finally,

$$\|G_k(u-u_0)\|_{p^*}^{p^*} \le C\||\nabla G_k(u-u_0)|\|_p^{p^*} \le C|A(k)|^{\frac{p^*}{p-1}\left(\frac{1}{\varrho}-\frac{1}{p^*}\right)},$$

which is estimate (5.11) with

$$\gamma = \frac{p^*}{p-1}\left(\frac{1}{\varrho} - \frac{1}{p^*}\right) > 1. \tag{5.16}$$

5.2. **Case** $1 < p < 2$**.** We are following the strategy of case $p \geq 2$ and are first proving that $u \in L^q(\Omega) \cap L^q(\partial\Omega)$ for all $q > 1$. To this purpose and borrowing an idea from [10], weak equation for (5.1) is written as:

$$
\begin{aligned}
(1+\mu)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx + (1+\mu)\int_\Omega \varphi_p(u)\phi\,dx \\
= \int_\Omega f\phi\,dx + \int_{\partial\Omega} g\phi\,d\sigma \\
+\mu\int_\Omega \big(|\nabla u|^{p-2}\nabla u - |\nabla(u-u_0)|^{p-2}\nabla(u-u_0)\big)\cdot\nabla\phi\,dx+ \\
\mu\int_\Omega \big(\varphi_p(u)-\varphi_p(u-u_0)\big)\phi\,dx\,,
\end{aligned}
$$

where $\phi \in W^{1,p}(\Omega)$. Such equality is obtained by adding

$$
\mu\int_\Omega \big(|\nabla u|^{p-2}\nabla u\cdot\nabla\phi + \varphi_p(u)\phi\big)\,dx,
$$

to both sides of the original weak equation. Now, estimates

$$
|\varphi_p(u) - \varphi_p(u-u_0)| \leq C|u_0|^{p-1},
$$

and

$$
\big||\nabla u|^{p-1}\nabla u - |\nabla(u-u_0)|^{p-1}\nabla(u-u_0)\big| \leq C|\nabla u_0|^{p-1},
$$

(recall that $1 < p < 2$) allow us to obtain

$$
\begin{aligned}
\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx + \int_\Omega \varphi_p(u)\phi\,dx \leq \\
\leq \frac{1}{1+\mu}\Big\{\int_\Omega |f||\phi|\,dx + \int_{\partial\Omega} |g||\phi|\,d\sigma\Big\} \\
+\frac{C\mu}{1+\mu}\int_\Omega \big|\nabla u_0|^{p-1}|\nabla\phi|\,dx + \frac{C\mu}{1+\mu}\int_\Omega |u_0|^{p-1}|\phi|\,dx.
\end{aligned}
\tag{5.17}
$$

By using the terminology of the case $p \geq 2$ such inequality can be more briefly written as

$$
\begin{aligned}
\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx \leq C\Big\{\int_\Omega a(x)(1+|u|^{p-1})|\phi|\,dx \\
+\int_{\partial\Omega} b(x)(1+|u|^{p-1})|\phi|\,d\sigma + \int_\Omega \big|\nabla u_0|^{p-1}|\nabla\phi|\,dx\Big\},
\end{aligned}
\tag{5.18}
$$

where constant C does not depend on μ, and $a \in L^{\frac{N}{p}}(\Omega)$ and $b \in L^{\frac{N-1}{p-1}}(\partial\Omega)$ are satisfied.

To achieve assertion (5.5) we use now $\phi = uT_L(|u|)^{mp}$ as a test function in (5.18) and first observe that

$$|\nabla\phi| = \big\{T_L(|u|)^{pm} + pm|u|^{pm}\chi_{B_L}\big\}|\nabla u|,$$

where $B_L = \{|u| \leq L\}$ and χ_{B_L} is the characteristic function of B_L. Thus,

$$\int_\Omega |\nabla u_0|^{p-1}|\nabla\phi|\,dx \leq C + \varepsilon\int_\Omega \big\{T_L(|u|)^{pm} + pm|u|^{pm}\chi_{B_L}\big\}|\nabla u|^p\,dx,$$

where C depends on ε and $\||u|^{m+1}\|_p$. We now observe that

$$\big\{T_L(|u|)^{pm} + pm|u|^{pm}\chi_{B_L}\big\}|\nabla u|^p \leq |\nabla(uT_L(|u|)^m)|^p.$$

Hence,

$$\int_\Omega |\nabla u_0|^{p-1}|\nabla\phi|\,dx \leq C + \varepsilon\||\nabla(uT_L(|u|)^m)\|_p^p. \tag{5.19}$$

On the other hand,

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx \geq C_p\||\nabla(uT_L(|u|)^m\|_p^p, \tag{5.20}$$

with $C_p = (mp+1)/(m+1)^p < 1$.

Thus, a suitable choice of ε in (5.19) together with relations (5.18) and (5.20) lead to the estimate:

$$\begin{aligned}&\||\nabla(uT_L(|u|)^m\|_p^p \\ &\quad\leq C\big\{\int_\Omega a(x)(1+|u|^{p-1})|\phi|\,dx + \int_{\partial\Omega} b(x)(1+|u|^{p-1})|\phi|\,d\sigma\big\}.\end{aligned} \tag{5.21}$$

Finally, by proceeding as in Step 1) of case $p \geq 2$ we achieve from (5.21) a bound for $\||\nabla(uT_L(|u|)^m\|_p$ that does not depend on L. This implies that $|u|^{m+1} \in W^{1,p}(\Omega)$ provided that $|u|^{m+1} \in L^p(\Omega)\cap L^p(\partial\Omega)$. Thus, (5.5) is shown.

Let us next prove that $u \in L^\infty(\Omega)$. Accordingly, we insert $\phi = G_k(|u|)$ in (5.18) to obtain,

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx \leq \\ &\qquad C\big\{\int_\Omega |f||\phi|\,dx + \int_{\partial\Omega}|g||\phi|\,d\sigma\big\} + C|A(k)|^{(1-\frac{1}{p})}\||\nabla\phi|\|_p\,,\end{aligned}$$

where now

$$A(k) = \{x\in\Omega\,:\,|u(x)| \geq k\}\,.$$

By handling the integral terms in the right–hand side as in Section 5.1 we arrive to the estimate,

$$\|G_k(u)\|_{p^*}^{p^*} \le \left(|A(k)|^{\gamma} + |A(k)|^{\frac{p*}{p}}\right),$$

γ being defined in (5.16). As both exponents are greater than unity, Stampacchia's approach (see Appendix) furnishes a L^∞ uniform bound for u, which does not depend on μ.

Thus, proof of Theorem 5.1 is finished.

6. C^α–ESTIMATE

In this section we are obtaining an estimate of the solutions $u \in W^{1,p}(\Omega)$ to problem (5.1) in the space $C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, which only depends on the parameters of our problem, of course including the uniform L^∞–bound we have accomplished in Section 5. More importantly, it does not depend on parameter μ.

This feature is properly stated as our next result.

Theorem 6.1. *Assume $u_0 \in C^1(\overline{\Omega})$ solves* (2.4) *and let $u \in W^{1,p}(\Omega)$ be a solution to problem* (5.1) *corresponding to a nonnegative parameter μ and satisfying*

$$\|u - u_0\|_{W^{1,p}(\Omega)} \le \varepsilon_0,$$

for some $\varepsilon_0 > 0$. Then there exist an exponent $\alpha \in (0,1)$ and a constant $M > 0$, both independent on μ, such that

$$\|u\|_{C^\alpha(\overline{\Omega})} \le M.$$

Proof of Theorem 6.1 involves the devices introduced in [15] (Chapter 2, Sections 6 and 7) to state the Hölder continuity of weak solutions to quasilinear problems, with a suitable divergence structure.

To this purpose a few more notation is necessary. For a point $x \in \overline{\Omega}$, B_ρ stands for the open ball with radius $\rho > 0$ centered at x, while $\Omega_\rho = \Omega \cap B_\rho$. In addition, if $u \in W^{1,p}(\Omega)$ and $k \ge 0$,

$$A^{\pm}(k,\rho) = \{x \in \Omega_\rho : \pm u(x) \ge k\}.$$

Cut–off functions $\zeta \in C_0^\infty(B_\rho)$, $0 \le \zeta \le 1$ will also be used.

Next, our main concern will be to prove that any weak solution $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ to problem (5.1) fulfilling estimate (5.2), satisfies an integral estimate of the form

$$\begin{aligned} &\int_{A^{\pm}(k,\rho)} \zeta^p |\nabla u|^p \, dx \le \\ &\qquad C\Big[\int_{A^{\pm}(k,\rho)} |G_k(u)|^p |\nabla \zeta|^p \, dx + \int_{A^{\pm}(k,\rho)} \zeta^p \big(1 + |G_k(u)|^p\big) \, dx\Big], \end{aligned} \tag{6.1}$$

in an arbitrary ball B_ρ, provided that $0 < \rho \le \rho_0$ and

$$\sup_{A^\pm(k,\rho)} u - k < \delta,$$

for a suitably small $\delta > 0$. Moreover, constant C in (6.1) depends on p, Ω, $\|u\|_\infty$ but not on μ.

By fixing $0 < \sigma < 1$ arbitrary, function ζ can be chosen so as $\zeta = 1$ in $B_{(1-\sigma)\rho}$ while $|\nabla\zeta| \le 1/\sigma\rho$. For this election of ζ, estimate (6.1) leads to

$$\int_{A^\pm(k,(1-\sigma)\rho)} |\nabla u|^p \, dx \le C\left\{\frac{1}{\{\sigma\rho\}^p} \sup_{A^\pm(k,\rho)} (\pm u - k)^p + 1\right\} |A^\pm(k,\rho)|, \tag{6.2}$$

provided $0 < \rho \le \rho_0$ is small.

Estimate (6.2) is the key point towards Hölder continuity. Once (6.2) is achieved, results in Sections 6, 7 of Chapter 2 in [15] permit us concluding that $u \in C^\alpha(\overline{\Omega})$ and that $\|u\|_{C^\alpha(\overline{\Omega})} \le M$ with an exponent $0 < \alpha < 1$ and constant M both not depending on μ. It should be stressed that, due to the smoothness of $\partial\Omega$, Hölder character of u up to the boundary does not requieres any "a priori" knowledge of the behavior of u on $\partial\Omega$ ([15], Chapter 2, Theorem 7.2 of Section 7). This is not the case when dealing with Dirichlet rather than flux–type boundary conditions.

As in previous Section 5, regimes $1 < p < 2$ and $p \ge 2$ are separately studied in next subsections.

6.1. **Case** $1 < p < 2$**.** Assume $u \in W^{1,p}(\Omega)$ solves (5.1) and fix a ball B_ρ centered at some $x \in \overline{\Omega}$ with a corresponding cut–off function ζ supported in B_ρ. Set $\phi = \zeta^p G_k(u^+)$ as test function in (5.17). Then,

$$\int_\Omega \zeta^p |\nabla u|^{p-2} \nabla u \cdot \nabla G_k(u^+) \, dx + p \int_\Omega \zeta^{p-1} G_k(u^+) |\nabla u|^{p-2} \nabla u \cdot \nabla\zeta \, dx + \int_\Omega \varphi_p(u) \zeta^p G_k(u^+) \, dx \le$$

$$\frac{1}{1+\mu}\int_\Omega \zeta^p|f(x,u)||G_k(u^+)|\,dx + \frac{1}{1+\mu}\int_{\partial\Omega}\zeta^p|g(x,u)||G_k(u^+)|\,d\sigma$$
$$+\frac{C\mu}{1+\mu}\int_\Omega \zeta^p|\nabla u_0|^{p-1}|\nabla G_k(u^+)|\,dx$$
$$+p\frac{C\mu}{1+\mu}\int_\Omega \zeta^{p-1}|G_k(u^+)||\nabla u_0|^{p-1}|\nabla\zeta|\,dx$$
$$+\frac{C\mu}{1+\mu}\int_\Omega \zeta^p|G_k(u^+)||u_0|^{p-1}\,dx\,.$$

Having in mind Theorem 5.1 and neglecting a nonnegative term, it yields

$$\int_\Omega \zeta^p|\nabla G_k(u^+)|^p\,dx + p\int_\Omega \zeta^{p-1}G_k(u^+)|\nabla u|^{p-2}\nabla u\cdot\nabla\zeta\,dx \le \tag{6.3}$$
$$C\Big[\int_\Omega \zeta^p|\nabla G_k(u^+)|\,dx + p\int_\Omega \zeta^{p-1}|G_k(u^+)||\nabla\zeta|\,dx + \int_\Omega \zeta^p|G_k(u^+)|\,dx\Big]$$

where C depends on u_0 and $\|u\|_\infty$ but not on μ. Let us estimate the terms appearing in (6.3).

We first pass the second term to the right side and apply Young's inequality to obtain

$$-p\int_\Omega \zeta^{p-1}G_k(u^+)|\nabla G_k(u^+)|^{p-2}\nabla G_k(u^+)\cdot\nabla\zeta\,dx$$
$$\le \frac{1}{3}\int_\Omega \zeta^p|\nabla G_k(u^+)|^p\,dx + C\int_\Omega |\nabla\zeta|^p|G_k(u^+)|^p\,dx.$$

Thus, second term can be grouped with the first one in (6.3). As for the terms appearing on the right–hand side of (6.3) first observe that,

$$\int_\Omega \zeta^p|\nabla G_k(u^+)|\,dx \le \frac{1}{3}\int_\Omega \zeta^p|\nabla G_k(u^+)|^p\,dx + C\int_{A^+(k,\rho)}\zeta^p\,dx,$$

and so second term can be grouped with the first one of (6.3). Moreover,

$$\int_\Omega \zeta^{p-1}|G_k(u^+)||\nabla\zeta|\,dx \le \frac{p-1}{p}\int_{A^+(k,\rho)}\zeta^p\,dx + \frac{1}{p}\int_\Omega |G_k(u^+)|^p|\nabla\zeta|^p\,dx$$

while

$$\int_\Omega \zeta^p|G_k(u^+)|\,dx \le \frac{p-1}{p}\int_{A^+(k,\rho)}\zeta^p\,dx + \frac{1}{p}\int_\Omega \zeta^p|G_k(u^+)|^p\,dx\,.$$

Therefore, equation (6.3) becomes

$$\int_{A^+(k,\rho)} \zeta^p |\nabla G_k(u^+)|^p \, dx$$
$$\leq C\Big[\int_{A^+(k,\rho)} |G_k(u^+)|^p |\nabla \zeta|^p \, dx + \int_{A^+(k,\rho)} \zeta^p \big(1 + |\nabla G_k(u^+)|^p\big) \, dx \Big] ,$$

which is (6.1) for the positive sign. A similar argument can be performed by setting $\phi = -\zeta^p G_k(u^-)$ as test function in (5.17) to arrive to

$$\int_{A^-(k,\rho)} \zeta^p |\nabla u|^p \, dx$$
$$\leq C\Big[\int_{A^-(k,\rho)} |G_k(u^-)|^p |\nabla \zeta|^p \, dx + \int_{A^-(k,\rho)} \zeta^p \big(1 + |\nabla G_k(u^-)|^p\big) \, dx \Big] .$$

6.2. **Case** $2 \leq p \leq N$**.** Since $u_0 \in C^1(\overline{\Omega})$ we are now dealing with the difference $u - u_0$ rather than u. Accordingly,

$$A^{\pm}(k,\rho) = \{x \in \Omega_\rho : \pm(u(x) - u_0(x)) \geq k\}$$

in what follows. In order to show that the resulting Hölder estimate does not depend on μ we study separately the cases $0 \leq \mu \leq 1$ and $\mu \geq 1$.

Let us begin by assuming that $0 \leq \mu \leq 1$. Set $\phi = \zeta^p G_k(u - u_0)^+$ as a test function in (5.4), ζ being a cut–off function supported on B_ρ. After removing nonnegative terms we get,

$$\begin{aligned}
(6.4) \quad & \int_\Omega \zeta^p \big[|\nabla u|^{p-2} \nabla u - |\nabla u_0|^{p-2} \nabla u_0 \big] \cdot \nabla G_k(u-u_0)^+ \, dx \\
& p \int_\Omega \zeta^{p-1} G_k(u-u_0)^+ \big[|\nabla u|^{p-2} \nabla u - |\nabla u_0|^{p-2} \nabla u_0 \big] \cdot \nabla \zeta \, dx \\
& + \mu p \int_\Omega \zeta^{p-1} G_k(u-u_0)^+ |\nabla (u-u_0)|^{p-2} \nabla (u-u_0) \cdot \nabla \zeta \, dx \\
& \qquad \leq \int_\Omega \zeta^p [f(x,u) - f(x,u_0)] G_k(u-u_0)^+ \, dx \\
& \qquad\qquad + \int_{\partial\Omega} \zeta^p [g(x,u) - g(x,u_0)] G_k(u-u_0)^+ \, d\sigma .
\end{aligned}$$

We are estimating the five integrals appearing in the inequality. By the monotonicity of the p–Laplacian, first integral I_1 can be estimated from below as

$$I_1 \geq I_1' := C \int_{A^+(k,\rho)} \zeta^p |\nabla (u-u_0)|^p \, dx .$$

The second integral I_2 can be estimated from above as follows:

$$\begin{aligned} |I_2| &\le p\int_\Omega \zeta^{p-1}G_k(u-u_0)^+\left[|\nabla u|^{p-1}+|\nabla u_0|^{p-1}\right]|\nabla\zeta|\,dx \\ &\le \varepsilon\int_{A^+(k,\rho)}\zeta^p|\nabla u|^p\,dx + C_\varepsilon\int_{A^+(k,\rho)}|G_k(u-u_0)|^p|\nabla\zeta|^p\,dx \\ &\quad + (p-1)\int_{A^+(k,\rho)}\zeta^p|\nabla u_0|^p\,dx + \int_{A^+(k,\rho)}|G_k(u-u_0)|^p|\nabla\zeta|^p\,dx \end{aligned}$$

$$\begin{aligned} &\le \varepsilon 2^{p-1}\int_{A^+(k,\rho)}\zeta^p|\nabla(u-u_0)|^p\,dx + \varepsilon 2^{p-1}\int_{A^+(k,\rho)}\zeta^p|\nabla u_0|^p\,dx \\ &\quad + (p-1)\int_{A^+(k,\rho)}\zeta^p|\nabla u_0|^p\,dx + C\int_{A^+(k,\rho)}|G_k(u-u_0)|^p|\nabla\zeta|^p\,dx \end{aligned}$$

$$\begin{aligned} &\le \varepsilon 2^{p-1}\int_{A^+(k,\rho)}\zeta^p|\nabla(u-u_0)|^p\,dx + C\int_{A^+(k,\rho)}\zeta^p\,dx \\ &\qquad + C\int_{A^+(k,\rho)}|G_k(u-u_0)|^p|\nabla\zeta|^p\,dx\,. \end{aligned}$$

By choosing ε small enough, last integral with ε can be grouped with I_1'.

Having in mind $0\le\mu\le 1$, third term I_3 in (6.4) satisfies

$$\begin{aligned} |I_3| &\le p\int_\Omega \zeta^{p-1}G_k(u-u_0)^+|\nabla(u-u_0)|^{p-1}|\nabla\zeta|\,dx \\ &\le \varepsilon\int_{A^+(k,\rho)}\zeta^p|\nabla(u-u_0)|^p\,dx + C_\varepsilon\int_{A^+(k,\rho)}|G_k(u-u_0)|^p|\nabla\zeta|^p\,dx\,. \end{aligned}$$

Due to the boundedness of u, fourth integral I_4 in (6.4) can be estimated as:

$$|I_4| \le C\int_\Omega \zeta^p G_k(u-u_0)^+\,dx \le C\int_\Omega \zeta^p(1+|G_k(u-u_0)|^p)\,dx\,.$$

Finally, boundedness of u and the trace inequality yield

$$
\begin{aligned}
|I_5| \le C\Big[\int_\Omega \zeta^p G_k(u-u_0)^+\,dx + p\int_\Omega \zeta^{p-1} G_k(u-u_0)^+|\nabla\zeta|\,dx \\
+\int_\Omega \zeta^p|\nabla G_k(u-u_0)^+|\,dx\Big] \\
\le \varepsilon\int_{A^+(k,\rho)} \zeta^p|\nabla(u-u_0)|^p\,dx \\
+C\Big[\int_{A^+(k,\rho)} \zeta^p(1+|G_k(u-u_0)|)\,dx+\int_{A^+(k,\rho)} |G_k(u-u_0)|^p|\nabla\zeta|^p\,dx\Big],
\end{aligned}
$$

and again, the term in ε can be absorbed by I_1'. Gathering together all the estimates we conclude,

$$
\begin{aligned}
&\int_{A^+(k,\rho)} \zeta^p|\nabla(u-u_0)|^p\,dx \\
&\le C\Big[\int_{A^+(k,\rho)} \zeta^p(1+|G_k(u-u_0)|)\,dx+\int_{A^+(k,\rho)} |G_k(u-u_0)|^p|\nabla\zeta|^p\,dx\Big],
\end{aligned}
\tag{6.5}
$$

where the constant C depends on p, $\|u\|_\infty$, Ω and u_0, but not on μ. This shows that $u-u_0$ satisfies (6.1).

As for the case $\mu \ge 1$, we begin by manipulating problem (5.4) to get

$$
\begin{cases}
-\dfrac{1}{\mu}\Delta_p u-(-\dfrac{1}{\mu}\Delta_p u_0)-\Delta_p(u-u_0) \\
\qquad +\dfrac{1}{\mu}\big(\varphi_p(u)-\varphi_p(u_0)\big)+\varphi_p(u-u_0) \\
\qquad\qquad =\dfrac{1}{\mu}(f(x,u)-f(x,u_0)) & \text{in } \Omega \\
\dfrac{1}{\mu}\dfrac{\partial u}{\partial \nu_p}-\dfrac{1}{\mu}\dfrac{\partial u_0}{\partial \nu_p}+\dfrac{\partial(u-u_0)}{\partial\nu_p}=\dfrac{1}{\mu}(g(x,u)-g(x,u_0)) & \text{on } \partial\Omega\,.
\end{cases}
$$

Choosing $\zeta^p G_k(u-u_0)^+$ as a test function in this problem, it follows that

$$\begin{aligned}\int_\Omega \zeta^p |\nabla G_k(u-u_0)^+|^p\,dx \\ + p\int_\Omega \zeta^{p-1} G_k(u-u_0)^+ |\nabla(u-u_0)|^{p-2}\nabla(u-u_0)\cdot\nabla\zeta\,dx \\ + \frac{p}{\mu}\int_\Omega \zeta^{p-1}G_k(u-u_0)^+\big[|\nabla u|^{p-2}\nabla u - |\nabla u_0|^{p-2}\nabla u_0\big]\nabla\zeta\,dx \\ \le \frac{1}{\mu}\int_\Omega [f(x,u)-f(x,u_0)]\zeta^p G_k(u-u_0)^+\,dx \\ + \frac{1}{\mu}\int_{\partial\Omega}[g(x,u)-g(x,u_0)]\zeta^p G_k(u-u_0)^+\,d\sigma\,.\end{aligned}$$

Since this equality has the same structure as that of (6.4) for the case $0\le\mu\le 1$, we can handle it in a similar way to deduce (6.5).

Finally, it is straightforward to get an analogous inequality replacing $A^+(k,\rho)$ with $A^-(k,\rho)$. This finishes the proof of Theorem 6.1.

Appendix: a Stampacchia lemma

Assume v is a measurable function in Ω and set $\psi(k)=|A_k|$, where $A_k=\{x:|v(x)|\ge k\}$. We state Stampacchia's result (see [20, Lemme 4.1]) which accounts for the summability degree of v in Ω provided ψ exhibits a suitable profile.

Lemma 6.2. *Let $\psi(t)\ge 0$ be a nondecreasing function such that*

$$(h-k)^\alpha\psi(h)\le C\psi(k)^\beta,$$

for $h>k\ge k_0$ and certain positive constants C,α and β.

i) *If $\beta>1$ then*

$$\psi(k)=0 \qquad \text{for all} \quad k\ge k_0+d,$$

with $d=C[\psi(k_0)]^{\beta-1}2^{\alpha\beta(\beta-1)}$.

ii) *If $\beta=1$ then*

$$\psi(h)\le \exp[1-\theta(h-k_0)]\psi(k_0),$$

where $\theta=(eC)^{-1/\alpha}$.

iii) *For $\beta<1$, $k_0>0$,*

$$\psi(h)\le 2^{\mu/(1-\beta)}[C^{1/(1-\beta)}+2^\mu k_0^\mu\psi(k_0)]h^{-\mu},$$

where $\mu=\alpha/(1-\beta)$.

Next conclusions are readily extracted from Lemma 6.2. In case i), $v \in L^\infty(\Omega)$; $v \in L^q(\Omega)$ for all $1 \le q < \infty$ in case ii) while $v \in L^q(\Omega)$ only for a finite range of values of q in case iii).

In the context of the present work, Lemma 6.2 is used as follows. A function $v \in W^{1,p}(\Omega)$ is given so that estimate

$$\int_\Omega |G_k(v)|^{p^*} \le C|A_k|^\gamma, \tag{6.1}$$

holds for $k \ge k_0 > 0$ and certain positive constants C and γ; here as usual $G_k(v) = (|v| - k)^+ \mathrm{sign}(v)$. To apply Lemma 6.2, let $h > k$ and observe that $\{|v| > h\} = \{|G_k(v)| > h - k\}$. Then, inequality (6.1) implies

$$\begin{aligned} |A_h| &\le \int_{A_h} \frac{|G_k(v)|^{p^*}}{(h-k)^{p^*}}\, dx \\ &\le \frac{1}{(h-k)^{p^*}} \int_\Omega |G_k(v)|^{p^*}\, dx \le \frac{C}{(h-k)^{p^*}} |A_k|^\gamma . \end{aligned}$$

This leads to case i) in Lemma 6.2, provided that $\gamma > 1$. It is worth remarking that the L^∞–bound, $k_0 + d$ in the above Lemma, only depends on the parameters in (6.1): k_0, p^*, C and γ.

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José C. Sabina de Lis
Departamento de Análisis Matemático and IUEA,
Universidad de La Laguna,
C/ Astrofísico Francisco Sánchez s/n, 38271 La Laguna, SPAIN
E-mail address: josabina@ull.es

Sergio Segura de León
Departament d'Anàlisi Matemàtica, Universitat de València,
Dr. Moliner 50, 46100 Burjassot, València, SPAIN.
E-mail address: sergio.segura@uv.es