THE LIMIT AS $p \to 1$ OF THE HIGHER EIGENVALUES OF THE p–LAPLACIAN OPERATOR $-\Delta_p$

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Abstract. This work provides a direct proof of the existence for each $n \in \mathbb{N}$ of the limit $\lambda_{(1),n} := \lim_{p\to 1} \lambda_{(p),n}$ of the n–th Ljusternik–Schnirelman Dirichlet eigenvalue $\lambda_{(p),n}$ of $-\Delta_p$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$. Most importantly, it is shown that $\lambda_{(1),n}$ defines an eigenvalue of the 1–Laplacian operator $-\Delta_1$, with a well–defined strong associated eigenfunction $u_n \in BV(\Omega)$. In the main results of the paper, the radial LS eigenvalues of $-\Delta_1$ are fully described, together with a detailed account on the profiles of their associated eigenfunctions. Our approach does not involve critical point theory for non–smooth functionals, although the definition of the LS–spectrum of $-\Delta_1$ relies on it.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain having Lipschitz–continuous boundary. It is well–known that, for any fixed $p > 1$, the Dirichlet eigenvalue problem,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p–Laplacian, exhibits an infinite and increasing sequence of eigenvalues $\lambda_{(p),n} \xrightarrow{n\to\infty} \infty$, the so–called Ljusternik–Schnirelman (Dirichlet) eigenvalues of $-\Delta_p$ (see Section 2.1). We refer to [31] for a detailed analysis of problem (1.1).

In this paper, we are interested in letting p go to 1 both in this sequence of eigenvalues and in the associated sequence of eigenfunctions. In this way we arrive at the formal problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \lambda \dfrac{u}{|u|} & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

which involves the 1–Laplacian, an operator studied by many authors since the early nineties. Indeed, it first appeared when analyzing the behavior of solutions to p–Laplacian problems when p goes to 1 ([24]). As we are going to stress in next remarks, to furnish a precise sense to problem (1.2) (see Definition 4 in Section 4 as a reference) is by no means obvious:

Date: August 20, 2019.

2010 *Mathematics Subject Classification.* 35P30, 35J92, 49K15, 49R05, 35B38.

Key words and phrases. Eigenvalues, Eigenfunctions, 1–Laplacian operator, radial solutions, functions of bounded variation.

(1) Since $W_0^{1,1}(\Omega)$ is not reflexive, it is not the suitable energy space to deal with the 1–Laplacian operator. The right one turns out to be $BV(\Omega)$, the space of functions of bounded variation.
(2) A first difficulty occurs when trying to define $\dfrac{Du}{|Du|}$, being Du a Radon measure which can vanish in a subdomain. It is overcome by considering a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \le 1$ and $(\mathbf{z}, Du) = |Du|$, so that $\mathbf{z}$ plays the role of the above quotient. As Du is just a Radon measure, Anzellotti's theory on pairings between L^∞–divergence vector fields and gradients of BV–functions must be applied (see [9]). Moreover, the Dirichlet condition must be formulated in a very weak form. Namely, $[\mathbf{z}, \nu] \in \operatorname{sign}(-u)$, where $[\mathbf{z}, \nu]$ stands for the weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ and 'sign' must be understood as a monotone graph. This approach can be traced back to [7], [8], [17], [18].
(3) Eigenvalues λ of (1.2) are defined in [12] as the critical values, in the sense of the weak slope, of the natural associated variational problem. Moreover, it is shown that eigenpairs (λ, u) satisfy Definition 4 below. However, it is also noticed ([12, Remark 2.5]) that this definition gives rises to an excessive number of (spurious) solutions to (1.2) (see also [37]).
(4) Right hand side of (1.2) is not determined by u where u vanishes. For the case of the principal eigenvalue, this was pointed out in [26]. It is required there that (1.2) be solvable by changing $\dfrac{u}{|u|}$ for any measurable selection $\gamma \in \operatorname{sign}(u)$. So, the necessary condition achieved implies that infinitely many equations must be solved.
(5) Likewise, right hand side of (1.2) does not determine u where u does not vanish. Indeed, if u is a weak non trivial solution to problem (1.2) and $g : \mathbb{R} \to \mathbb{R}$ is a non–decreasing smooth function such that $g(0) = 0$ and $g(u) \neq 0$, then $g(u)$ is also a non trivial solution to problem (1.2).

Therefore, one of the main concerns of any analysis of the spectrum of the 1–Laplacian operator is not only to find out the eigenvalues, but also to propose the suitable eigenfunctions. Of course, these tasks are much easier for the p–Laplacian as $p > 1$.

The first eigenvalue to (1.2) has been analyzed by many authors, beginning in [18]. In [25] the limit as $p \to 1$ of the first eigenvalue to (1.1) is identified as the Cheeger's constant $h(\Omega)$ of Ω. The characteristic function of a Cheeger set defines a minimizer of the functional

$$u \mapsto \frac{\int_\Omega |Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}}{\int_\Omega |u|} \,. \tag{1.3}$$

As shown in [26], this minimizer turns out to be a solution to problem (1.2) and so can be regarded as an associated eigenfunction. Another approach to the first eigenvalue employing a penalization method is developed in [27]. As far as the uniqueness of the first eigenfunction, we refer to [2], [11].

A precise definition of problem (1.2), including higher order eigenvalues, is introduced for the first time in [12]. By employing a nonsmooth version of the critical point theory, a complete sequence of Ljusternik–Schnirelman (LS) eigenvalues $\bar\lambda_n \to \infty$ is found. In addition, eigenpairs (λ, u) so obtained are shown in [12] to satisfy (1.2) in a well–defined sense. This work also contains a detailed study of the one–dimensional case (cf. also [37]). Sequence $\bar{\lambda}_n$ was independently found in [35]. More importantly, the convergence $\lambda_{(p),n} \to \bar{\lambda}_n$ as $p \to 1$, $\lambda_{(p),n}$ being the n–th LS eigenvalue to (1.1) has been recently stated in [32]. It should be also mentioned that [39] introduces the n–th Cheeger constant $h_n(\Omega)$ of a domain Ω while it is shown that $h_2(\Omega) = \lim_{p\to 1} \lambda_{(p),2}$. In addition, the existence for all n of the limit $\lim_{p\to 1} \lambda_{(p),n}$ is stated in [38] but no explicit connection between these limits and problem (1.2) is shown. Moreover, it is now known that $\bar{\lambda}_n \leq h_n(\Omega)$ for all $n \geq 3$ while improving these estimates to an equality seems not likely ([10]).

For future use we are referring to $\{\bar{\lambda}_n\}$ as the LS eigenvalues to (1.2) or plainly the "spectrum" of $-\Delta_1$ (notation $\lambda_{(1),n}$ will replace $\bar{\lambda}_n$ in Section 3). As mentioned above its relation with the LS–spectrum of $-\Delta_p$ has been well–clarified in [12], [35] and [32]. On the other hand, it should be pointed out that, as far as we know, these references do not provide the "right" associated eigenfunctions. To overcome the indetermination mentioned in the above remarks (4) and (5), in this work we regard u as an associated eigenfunction to an eigenvalue $\bar{\lambda}_n$ of $-\Delta_1$ provided that $u = \lim_{p\to 1} u_{(p),n}$ with $u_{(p),n}$ a properly normalized eigenfunction to $\lambda_{(p),n}$.

Our main aim in this work is to furnish a detailed description of both the radial spectrum and the proper eigenfunctions of the 1–Laplace operator in a ball, by avoiding any critical point theory for nonsmooth functionals. As a first step we prove by a direct approach the existence of the limit of the LS spectrum of $-\Delta_p$ as $p \to 1$ in a general domain Ω (we refer to [25, Theorem 8] for the first eigenvalue, to [12, Theorem 3.10] in the one–dimensional case and to [38, Theorem 2.16] and [32, Theorem 2.4] in the general case). This analysis permits us introducing the concept of *strong* eigenvalues of the 1–Laplacian as limits of (non necessarily LS) eigenvalues of the p–Laplacian (Definition 7 in Section 4). Since normalized eigenfunctions of the p–Laplacian converge as p tends to 1, it follows that the limit process allows us to select a "reference" eigenfunction. Finally, we study the radial spectrum of the p–Laplacian in a ball and, letting p goes to 1, we obtain a full description of the strong radial eigenfunctions of the 1–Laplace operator.

This article is organized as follows. Section 2 is devoted to preliminaries: we introduce the p–Laplacian setting, the suitable energy space to deal with $-\Delta_1$ and the Anzellotti theory of pairings. In Section 3, we handle the limit of eigenvalues of $-\Delta_p$ as p goes to 1, while our stability result, furnishing eigenpairs to $-\Delta_1$ by passing to the limit in eigenpairs of $-\Delta_p$, is proved in Section 4. The one–dimensional case is briefly reviewed in Section 5 being the radial spectrum of the p–Laplacian analyzed in detail in Section 6. In this latter section we introduce the Bessel type function $v_p(|x|)$ from whose zeros the radial eigenvalues are produced by scaling. It is also shown there that the radial spectrum of $-\Delta_p$ coincides with the Ljusternik–Schnirelman eigenvalues and so this fact is transferred to the radial eigenvalues of $-\Delta_1$. Section 6 also addresses some preliminary aspects of the limit behavior of v_p as p goes to 1. Finally, Section 7 contains the main results. The convergence process together with the radial spectrum of the 1–Laplacian is addressed there.

2. Preliminaries

In this Section we will introduce some notation and auxiliary results which will be used throughout this paper. In what follows, we will consider $N \geq 2$, and

$\mathcal{H}^{N-1}(E)$ will denote the $(N-1)$–dimensional Hausdorff measure of a set E and $|E|$ its Lebesgue measure.

It will be always understood that Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Thus, an outward unit normal $\nu(x)$ is defined for $\mathcal{H}^{N-1}$–almost every $x \in \partial\Omega$. Usual Lebesgue and Sobolev spaces are denoted by $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively. When dealing with radially symmetric functions, we occasionally need to consider Lebesgue spaces in an interval I with weight $h(r) = r^{N-1}$, then we will write $L^q(I, r^{N-1}dr)$. The subspace of $W_0^{1,p}(B(0,R))$ consisting of radial functions is denoted by $\widetilde{W}_0^{1,p}(B)$; for each $u \in \widetilde{W}_0^{1,p}(B(0,R))$ there exists $v \in L^p((0,R); r^{N-1}dr)$ satisfying $u(x) = v(|x|)$, $\lim_{r\to R} v(r) = 0$ and $\nabla u(x) = v'(|x|)\frac{x}{|x|}$ where $v' \in L^p((0,R); r^{N-1}dr)$ stands for the weak derivative of v.

Finally, for a given measurable function u in Ω, the notation

$$v \in \operatorname{sign}(u)$$

will be used to mean that $v \in L^\infty(\Omega)$ satisfies $\|v\|_\infty \le 1$ and $v(x)u(x) = |u(x)|$ a. e. in Ω. Accordingly, infinitely many v's can be found whenever u vanishes in a positive measure set.

2.1. **The eigenvalues of $-\Delta_p$.** Let $p > 1$ be fixed and consider the real function given by $\varphi_p(t) = |t|^{p-2}t$.

A real number λ is defined to be a weak eigenvalue to problem

$$\begin{cases} -\Delta_p u = \lambda\varphi_p(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.4}$$

with associated eigenfunction $u \in W_0^{1,p}(\Omega)$, $u \ne 0$, if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \; dx = \lambda \int_\Omega \varphi_p(u)v \; dx,$$

for all $v \in W_0^{1,p}(\Omega)$. The first eigenvalue to (2.4) is furnished by

$$\lambda_1 = \lambda_{(p),1} = \inf_{u\in\mathcal{M}} \int_\Omega |\nabla u|^p \; dx,$$

where $\mathcal{M} = \{u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p \; dx = 1\}$. It is well–known that λ_1 is the unique principal eigenvalue (i.e. an eigenvalue with a nonnegative associated eigenfunction) which is in addition simple (in a proper sense) and isolated. This fact was shown in full generality, for the first time, in [4], [5]. The existence of a second eigenvalue λ_2 is a consequence of the isolation of λ_1. A variational characterization of λ_2 was achieved in [6] (see also [13] and Remark 1 for a further characterization of λ_2). Moreover, it was shown in [14] that every eigenfunction u associated to λ_2 exhibits exactly two nodal domains (see [20] for an alternative proof).

The existence of λ_2, together with an increasing sequence of eigenvalues to (2.4) tending to infinity, was first shown in [23], by employing the general eigenvalue theory for nonlinear operators contained in [3] (see [4] and [19] for different approaches). Specifically, define

$$\lambda_n^{LS} = \inf_{A\in\mathcal{A}_n} \sup_{u\in A} \int_\Omega |\nabla u|^p \; dx, \tag{2.5}$$

where

$$\mathcal{A}_n = \{A \subset \mathcal{M} : A \text{ compact in the topology of } W_0^{1,p}(\Omega),\ A = -A,\ \gamma(A) \geq n\}$$

and $\gamma(A)$ stands for the Krasnosel'skii genus of A. Recall that for a nonempty closed symmetric set A of a Banach space X

$$\gamma(A) = \min\{m \in \mathbb{N} : \exists f \in C(A, \mathbb{R}^m \setminus \{0\}),\ f(-x) = -f(x) \text{ for every } x \in X\},$$

whereas $\gamma(A) = \infty$ if no such an integer m exists. In addition $\gamma(\emptyset) = 0$ ([41]).

Sequence λ_n^{LS} constitutes the Ljusternik–Schnirelman (LS) eigenvalues of (2.4) and it still remains an open problem whether they fill or not the whole Dirichlet spectrum of $-\Delta_p$. Moreover, it holds that $\lambda_n^{LS} \to \infty$ ([4], [23]). More importantly, it turns out that $\lambda_2 = \lambda_2^{LS}$ ([6]).

Remark 1.

a) A sequence of eigenvalues to (2.4) is obtained in [4] in the form $\lambda'_n = \dfrac{1}{2\sqrt{-c_n}}$ where $c_n \to 0-$ is an increasing sequence of critical values of the functional given by $J(u) = a(u)^2 - b(u)$, $a(u) = \dfrac{1}{p}\int_\Omega |\nabla u|^p$, $b(u) = \dfrac{1}{p}\int_\Omega |u|^p$, being the sequence of critical values c_n found according the general procedure in [3]. It can be shown that the sequence of eigenvalues so obtained exactly coincides with the one in (2.5).

b) A sequence of higher eigenvalues λ''_n to (2.4) is constructed in [19] by employing the Rayleigh quotients (2.5) but using instead a narrower class $\mathcal{B}_n \subset \mathcal{A}_n$. Namely, elements $A \subset \mathcal{M}$ in $\mathcal{B}_n$ are defined as $A = h(\partial B_1)$ where $h : \partial B_1 \to \mathcal{M}$ is an odd and continuous mapping, $B_1 \subset \mathbb{R}^n$ is the open unit ball. In this case $\lambda''_n \geq \lambda_n^{LS}$ for all n and at the present moment it is an open question whether the sequences λ''_n and λ_n^{LS} are the same. Nevertheless, $\lambda''_i = \lambda_i^{LS}$ for $i = 1, 2$.

2.2. **BV–functions.** The space of all functions of finite variation, that is the space of those $u \in L^1(\Omega)$ whose distributional gradient is a Radon measure with finite total variation, is denoted by $BV(\Omega)$. This is the natural energy space to study problems involving the 1–Laplacian operator. It is endowed with the norm defined by

$$\|u\| = \int_\Omega |u|\, dx + \int_\Omega |Du|\,,$$

for any $u \in BV(\Omega)$. We recall that the notion of trace can be extended to any $u \in BV(\Omega)$ and this fact allows us to interpret it as the boundary values of u and to write $u\big|_{\partial\Omega}$. Using the trace, we have available an equivalent norm, which we will use in the sequel. It is given by

$$\|u\|_{BV(\Omega)} = \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \int_\Omega |Du|\,.$$

A Sobolev type embedding holds in $BV(\Omega)$: there exists a constant $S > 0$ satisfying

$$\left(\int_\Omega |u|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} \leq S\|u\|_{BV(\Omega)}\,, \qquad \text{for all } u \in BV(\Omega)\,. \tag{2.6}$$

This continuous embedding $BV(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$ turns out to be compact when $L^{\frac{N}{N-1}}(\Omega)$ is replaced by any $L^q(\Omega)$, with $1 \leq q < \frac{N}{N-1}$ (see [1]). Thus, every sequence that is bounded in $BV(\Omega)$ has a subsequence which strongly converges in $L^1(\Omega)$ to a certain $u \in BV(\Omega)$.

To pass to the limit we will often apply that some functionals defined on $BV(\Omega)$ are lower semicontinuous with respect to the convergence in $L^1(\Omega)$. The most important are the functionals defined by

$$u \mapsto \int_\Omega |Du|,$$

$$u \mapsto \int_\Omega |Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}.$$

and

$$u \mapsto \int_\Omega \varphi \, |Du| \,,$$

where $\varphi \in C_0^1(\Omega)$, $\varphi \geq 0$, is fixed.

In Sections 6 and 7 we are concerned with radial solutions. That is why we introduce the subspace $\widetilde{BV}(B(0,R)) \subset BV(B(0,R))$ of its radially symmetric elements. If $u \in \widetilde{BV}(B(0,R))$ there exists $v \in L^1((0,R); r^{N-1}dr)$ satisfying $u(x) = v(|x|)$ and $Du(x) = v'(|x|)\frac{x}{|x|}$ where v', the derivative of v in $\mathcal{D}'(0,R)$, is a Radon measure. In addition, $h(r) = r^{N-1}$ is sumable with respect to $|v'|$.

For further information on functions of bounded variation, we refer to [1], [21].

2.3. **A generalized Gauss–Green formula.** Since our concept of solution lies on the Anzellotti theory, we next introduce it. Consider $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ such that its distributional divergence $\operatorname{div} \mathbf{z}$ belongs to $L^N(\Omega)$. For such a vector field $\mathbf{z}$ and any $u \in BV(\Omega)$ we denote by $(\mathbf{z}, Du) : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$ the distribution introduced by Anzellotti ([9]):

$$\langle (\mathbf{z}, Du), \varphi \rangle = -\int_\Omega u \, \varphi \operatorname{div} \mathbf{z} - \int_\Omega u \, \mathbf{z} \cdot \nabla \varphi, \quad \forall \, \varphi \in C_0^\infty(\Omega) \,. \tag{2.7}$$

It is shown in [9] that $(\mathbf{z}, Du)$ defines a Radon measure with finite total variation, and for every Borel set B satisfying $B \subseteq U \subseteq \Omega$ (U open) it holds

$$\left| \int_B (\mathbf{z}, Du) \right| \leq \int_B |(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_{L^\infty(U)} \int_B |Du| \,. \tag{2.8}$$

We recall the notion of weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ defined in [9] as an application $[\mathbf{z}, \nu] : \partial\Omega \to \mathbb{R}$ such that $[\mathbf{z}, \nu] \in L^\infty(\partial\Omega)$ and $\| \, [\mathbf{z}, \nu] \, \|_{L^\infty(\partial\Omega)} \leq \|\mathbf{z}\|_{L^\infty(\Omega;\mathbb{R}^N)}$. In [9] a Green formula involving the measure $(\mathbf{z}, Du)$ and the weak trace $[\mathbf{z}, \nu]$ is established, namely:

$$\int_\Omega (\mathbf{z}, Du) + \int_\Omega u \operatorname{div} \mathbf{z} \, dx = \int_{\partial\Omega} u \, [\mathbf{z}, \nu] \; d\mathcal{H}^{N-1}, \tag{2.9}$$

being $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ satisfying $\operatorname{div} \mathbf{z} \in L^N(\Omega)$ and $u \in BV(\Omega)$.

Finally, we adapt [9, Proposition 2.8] (see also [28, Proposition 2.7]) to our setting: if for some vector field $\mathbf{z}$ and some $u \in BV(\Omega)$ $(\mathbf{z}, Du) = |Du|$ holds as measures, then $(\mathbf{z}, Dg(u)) = |Dg(u)|$ holds for any non–decreasing and Lipschitz–continuous function $g : \mathbb{R} \to \mathbb{R}$.

3. The limit of the eigenvalues as $p \to 1$

From now on, we denote by $\lambda_{(p)}$ any eigenvalue of $-\Delta_p$ and by $\{\lambda_{(p),n}\}_{n=1}^{\infty}$ the sequence of its LS eigenvalues.

The eigenvalues of (2.4) keep bounded away from zero as $p \to 1$. This follows from the rough estimate (which can be checked by a direct computation),

$$p\lambda_{(p),1}^{\frac{1}{p}}(\Omega) \geq \frac{1}{D(\Omega)},$$

where $D(\Omega)$ is the diameter of Ω. However a sharper estimate is known. Namely,

$$p\lambda_{(p),1}^{\frac{1}{p}}(\Omega) \geq h(\Omega),$$

where $h(\Omega)$ is the Cheeger constant of Ω defined by

$$h(\Omega) = \min_{K\subset\Omega} \frac{Pe(K)}{|K|}\,. \tag{3.10}$$

Here $Pe(K)$ denotes the perimeter of K; the minimum in (3.10) being taken over all nonempty sets of finite perimeter contained in Ω (appendix in [29]). In fact, it can be further shown that $\lambda_{(p),1} \to h(\Omega)$ as $p \to 1$ ([25, Theorem 3]).

An immediate consequence is the following result.

Theorem 1. *For all $p > 1$, an arbitrary Dirichlet eigenvalue $\lambda = \lambda_{(p)}$ to* (2.4) *satisfies,*

$$p\lambda_{(p)}^{\frac{1}{p}} \geq h(\Omega)\,. \tag{3.11}$$

In order to see that the limit of $\lambda_{(p),n}$ as $p \to 1$ exists, we begin with the following result.

Theorem 2. *Let $\lambda_{(p),n}$ be the n–th LS eigenvalue of problem* (2.4) *and choose*

$$1 < p < s.$$

Then,

$$p\lambda_{(p),n}^{1/p} \leq s\lambda_{(s)n}^{1/s}.$$

In other words, $p\lambda_{(p),n}^{1/p}$ is an increasing function of p for $p > 1$.

Proof. Fix $1 < p < s$ and introduce the mapping

$$\begin{array}{cccc} \Psi: & W_0^{1,s}(\Omega) & \longrightarrow & W_0^{1,p}(\Omega) \\ & u & \longmapsto & \Psi(u) = \varphi_{\frac{s}{p}+1}(u), \end{array}$$

i. e., $\Psi(u) = |u|^{\frac{s}{p}-1}u$. By using well–known properties of the power–like Nemitskii operators, Ψ is a well–defined and continuous mapping.

Define $\mathcal{M}_s = \{u \in W_0^{1,s}(\Omega) : \int_\Omega |u|^s\, ds = 1\}$. Then Ψ maps $\mathcal{M}_s$ into $\mathcal{M}_p$, i. e. $\Psi(\mathcal{M}_s) \subset \mathcal{M}_p$. We now follow the idea in [30] and observe that for all $u \in \mathcal{M}_s$, $w = \Psi(u)$ satisfies:

$$\left(\int_\Omega |\nabla w|^p\, dx\right)^{\frac{1}{p}} = \frac{s}{p}\left(\int_\Omega |u|^{s-p}|\nabla u|^p\, dx\right)^{\frac{1}{p}} \leq \frac{s}{p}\left(\int_\Omega |\nabla u|^s\, dx\right)^{\frac{1}{s}}.$$

On the other hand $\gamma(\Psi(A)) \geq n$ for all $A \in \mathcal{A}_{n,s}$, where $\mathcal{A}_{n,s}$ stands for the class of compact symmetric sets in $\mathcal{M}_s$ with genus greater or equal than n. Thus:

$$\mathcal{A}'_{n,p} := \{\Psi(A) : A \in \mathcal{A}_{n,s}\} \subset \mathcal{A}_{n,p}.$$

Therefore:

$$\lambda_{(p),n}^{\frac{1}{p}} = \inf_{B\in\mathcal{A}_{n,p}} \sup_{w\in B} \left(\int |\nabla w|^p \, dx\right)^{\frac{1}{p}} \leq \inf_{B\in\mathcal{A}'_{n,p}} \sup_{w\in B} \left(\int |\nabla w|^p \, dx\right)^{\frac{1}{p}}$$

$$= \inf_{A\in\mathcal{A}_{n,s}} \sup_{w\in\Psi(A)} \left(\int |\nabla w|^p \, dx\right)^{\frac{1}{p}} \leq \frac{s}{p} \inf_{A\in\mathcal{A}_{n,s}} \sup_{u\in A} \left(\int |\nabla u|^s \, dx\right)^{\frac{1}{s}} = \frac{s}{p}\lambda_{(s)n}^{\frac{1}{s}},$$

and the proof is concluded. □

Remark 2. Notice that Ψ is one to one and thus $\Psi(A)$ is homeomorphic to A for all $A \in \mathcal{A}_{n,s}$. Therefore it can be even asserted above that $\gamma(\Psi(A)) = \gamma(A)$ for all these $A \in \mathcal{A}_{n,s}$.

Corollary 3. *For all n the limit*

$$\lim_{p\to 1} \lambda_{(p),n}, \tag{3.12}$$

exits and is positive.

Remark 3. A similar result holds for the Neumann eigenvalues $\lambda_{(p),n}^{\mathcal{N}}$ of the p–Laplacian in a bounded domain. Details are left for a future job.

Remark 4. Although the conclusion of Corollary 3 is already known, ours is a direct approach to the existence of the limits (3.12). Indeed, such existence was shown in [38] by a more sophisticated approach, however no explicit connection between such limits and the eigenvalue problem for $-\Delta_1$ was discovered there. In the more recent work [32], limits in (3.12) are identified as variational eigenvalues of $-\Delta_1$.

Next section states, also in a direct way, that $\lim_{p\to 1} \lambda_{(p),n}$ can be regarded as an eigenvalue of $-\Delta_1$.

4. The limit of the eigenfunctions as $p \to 1$

The aim of this section is proving a perturbation result on eigenpairs $(\lambda_{(p)}, u_{(p)})$ of problem (2.4) as $p \to 1$. A main feature is that provided $\lambda_{(p)} \to \lambda_{(1)}$ then, up to a subsequence, $u_{(p)} \to u_{(1)}$ strongly in $L^1(\Omega)$, being $(\lambda_{(1)}, u_{(1)})$ an eigenpair of the limit problem (1.2) in a sense made precise next. Its proof will be later essential to analyze the radial spectrum of $-\Delta_1$ as the limit of the radial spectrum of $-\Delta_p$ (see Proposition 16 below).

To begin with, we introduce our concept of solution to this problem (compare with Section 2 in [12]).

Definition 4. *A real number λ is a weak eigenvalue to problem*

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \lambda \dfrac{u}{|u|} & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{4.13}$$

with associated weak eigenfunction $u \in BV(\Omega)$, $u \neq 0$, if there exists a vector field $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ and a function $\gamma \in L^\infty(\Omega)$ satisfying

(1) $\|\mathbf{z}\|_\infty \leq 1$ *and* $\|\gamma\|_\infty \leq 1$.
(2) *Equation* $-\operatorname{div}\mathbf{z} = \lambda\gamma$ *holds in the sense of distributions.*
(3) $(\mathbf{z}, Du) = |Du|$ *as measures and* $\gamma|u| = u$ *a.e. in* Ω.
(4) $[\mathbf{z}, \nu] \in \operatorname{sign}(-u)$ $\mathcal{H}^{N-1}$*–a.e. on* $\partial\Omega$.

Remark 5.

a) Condition (2) implies $\operatorname{div}\mathbf{z} \in L^\infty(\Omega)$ and so Anzellotti's theory can be applied. Thus, the weak trace $[\mathbf{z},\nu]$ on $\partial\Omega$ of the normal component of $\mathbf{z}$ belongs to $L^\infty(\partial\Omega)$ while the "dot product" $(\mathbf{z}, Dv)$ is a Radon measure for every $v \in BV(\Omega)$. Therefore, every condition in Definition 4 makes sense.

b) If $g : \mathbb{R} \to \mathbb{R}$ is a non–decreasing and Lipschitz–continuous function such that $g(0) = 0$, then conditions (3) and (4) imply $(\mathbf{z}, Dg(u)) = |Dg(u)|$ as measures and $\gamma|g(u)| = g(u)$ a.e. in Ω, and $[\mathbf{z},\nu] \in \operatorname{sign}(-g(u))$ $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$. Therefore, every $g(u) \neq 0$ is also a weak eigenfunction.

c) By Green's formula (2.9) and condition (2), we get

$$\int_\Omega (\mathbf{z}, Dv) = \int_{\partial\Omega} v[\mathbf{z},\nu]\, d\mathcal{H}^{N-1} + \lambda \int_\Omega \gamma v\, dx,$$

for all $v \in BV(\Omega)$. By choosing $v = u$, and having in mind conditions (3) and (4),

$$\lambda = \frac{\int_\Omega |Du| + \int_{\partial\Omega} |u|}{\int_\Omega |u|}.$$

Hence –as should be expected– all possible weak eigenvalues must be positive.

d) In case we choose $v = g(u)$ (where $g : \mathbb{R} \to \mathbb{R}$ is a non–decreasing and Lipschitz–continuous function such that $g(0) = 0$) in the above identity, it follows from conditions (3) and (4) that

$$\int_\Omega |Dg(u)| + \int_{\partial\Omega} |g(u)|\, d\mathcal{H}^{N-1} = \lambda \int_\Omega |g(u)|\, dx\,.$$

First we show that every weak eigenfunction of problem (4.13) is bounded.

Lemma 5. *Let λ be any weak eigenvalue of problem* (4.13) *with associated eigenfunction u. Then $u \in L^\infty(\Omega)$ and*

$$\|u\|_\infty \leq (S\lambda)^N \|u\|_1,$$

S being the embedding constant in $BV(\Omega) \subset L^{\frac{N}{N-1}}(\Omega)$.

Proof. For each $k > 0$, consider the auxiliary real function defined by

$$G_k(s) = (|s| - k)^+ \operatorname{sign}(s)$$

and take $G_k(u)$ as test function in problem (4.13). We use d) of the above remark to get

$$\int_\Omega |DG_k(u)| + \int_{\partial\Omega} |G_k(u)|\, d\mathcal{H}^{N-1} = \lambda \int_\Omega |G_k(u)|\, dx\,.$$

Noting that actually $\int_\Omega |G_k(u)|\, dx = \int_{A_k} |G_k(u)|\, dx$, where $A_k = \{|u| > k\}$, and applying the Hölder and Sobolev inequalities on the left hand side it yields

$$\begin{aligned}\int_\Omega |G_k(u)|\, dx &\leq |A_k|^{\frac{1}{N}} \left(\int_{A_k} |G_k(u)|^{\frac{N}{N-1}}\, dx \right)^{1-\frac{1}{N}} \\ &\leq S|A_k|^{\frac{1}{N}} \left[\int_\Omega |DG_k(u)| + \int_{\partial\Omega} |G_k(u)|\, d\mathcal{H}^{N-1} \right] \\ &\leq S\lambda |A_k|^{\frac{1}{N}} \int_\Omega |G_k(u)|\, dx\,. \qquad (4.14)\end{aligned}$$

Now choose $k > (S\lambda)^N \|u\|_1$. Since $k|A_k| \le \|u\|_1$ by Chebyshev's inequality, it follows that

$$|A_k|^{\frac{1}{N}} \le \left(\frac{\|u\|_1}{k}\right)^{\frac{1}{N}} < (S\lambda)^{-1}$$

wherewith (4.14) implies $\int_\Omega |G_k(u)|\,dx = 0$. Hence, $|u| \le k$ so that $u \in L^\infty(\Omega)$ and our choice of k leads to the desired estimate. □

Theorem 6. *Consider a convergent family $\{\lambda_{(p)}\}$ of eigenvalues of problem* (2.4) *and let $u_{(p)}$ be a family of associated eigenfunctions. Assume that there exist positive constants $0 < \kappa < \Gamma$ satisfying*

$$\kappa \le \int_\Omega |u_{(p)}|^p\,dx \le \Gamma \qquad \text{for all } p > 1$$

Then, up to subsequences, $\{u_{(p)}\}$ converges strongly in $L^1(\Omega)$ and the limits

$$\lambda_{(1)} = \lim_{p\to 1} \lambda_{(p)} \qquad u_{(1)} = \lim_{p\to 1} u_{(p)}$$

define an eigenpair $(\lambda_{(1)}, u_{(1)})$ to problem (4.13).

Proof. Taking $u_{(p)}$ as a test function in problem (2.4) we get

$$\int_\Omega |\nabla u_{(p)}|^p dx = \lambda_{(p)} \int_\Omega |u_{(p)}|^p dx \le \lambda_{(p)}\Gamma\,. \tag{4.15}$$

Then Young's inequality yields

$$\int_\Omega |\nabla u_{(p)}|\,dx + \int_{\partial\Omega} |u_{(p)}|\,d\mathcal{H}^{N-1} \le \frac{1}{p}\int_\Omega |\nabla u_{(p)}|^p dx + \frac{p-1}{p}|\Omega| \le \lambda_{(p)}\Gamma + |\Omega|\,. \tag{4.16}$$

The reasons for including the boundary term will become clear later in the proof. Since $\{\lambda_{(p)}\}$ converges, we obtain

$$\int_\Omega |\nabla u_{(p)}|\,dx + \int_{\partial\Omega} |u_{(p)}|\,d\mathcal{H}^{N-1} \le M\,, \tag{4.17}$$

for all $p > 1$ and a certain constant $M > 0$. Thus, a BV–estimate is attained. Hence, there exists $u_{(1)} \in BV(\Omega)$ such that, up to subsequences,

$$u_{(p)} \to u_{(1)} \qquad \text{strongly in } L^q(\Omega) \qquad \text{for all } 1 \le q < \frac{N}{N-1}\,. \tag{4.18}$$

We may also assume that $u_{(p)} \to u_{(1)}$ pointwise a.e. in Ω.

We next show that $u_{(1)}$ is not trivial. Fix $1 < q < \frac{N}{N-1}$ and consider $p \in (1, q)$. Observe that Young's inequality yields

$$|u_{(p)}|^p \le \frac{p}{q}|u_{(p)}|^q + \frac{q-p}{q} \le |u_{(p)}|^q + 1\,.$$

Since $u_{(p)} \to u_{(1)}$ strongly in $L^q(\Omega)$, it follows that $\{|u_{(p)}|^q\}$ is equi–integrable, and so is $\{|u_{(p)}|^p\}$. On the other hand, $|u_{(p)}|^p \to |u_{(1)}|$ a.e. By Vitali's theorem,

$$\int_\Omega |u_{(1)}|\,dx = \lim_{p\to 1} \int_\Omega |u_{(p)}|^p\,dx\,. \tag{4.19}$$

We point out that it implies

$$\kappa \leq \int_\Omega |u_{(1)}|\, dx \leq \Gamma$$

and so $u_{(1)} \neq 0$.

We now get γ as a consequence of (4.17) by simplifying the argument of [33, Remark 4.1]. Consider the sequence defined by $v_p = |u_{(p)}|^{p-2} u_{(p)}$ and choose $s \in (1, +\infty)$. For every p satisfying $0 < p-1 < \frac{N'}{s}$, $N' = \frac{N}{N-1}$, the Hölder and Sobolev inequalities yield

$$\begin{aligned} \int_\Omega |v_p|^s dx \leq |\Omega|^{1-\frac{(p-1)s}{N'}} \left(\int_\Omega |v_p|^{\frac{N'}{p-1}} \right)^{\frac{(p-1)s}{N'}} &= |\Omega|^{1-\frac{(p-1)s}{N'}} \|u_{(p)}\|_{N'}^{(p-1)s} \\ &\leq |\Omega|^{1-\frac{(p-1)s}{N'}} \left(S\|u_{(p)}\|_{BV} \right)^{(p-1)s} \leq |\Omega|^{1-\frac{(p-1)s}{N'}} (SM)^{(p-1)s}\,. \end{aligned}$$

Thus,

$$\|v_p\|_s \leq (SM)^{p-1} |\Omega|^{\frac{1}{s}-\frac{p-1}{N'}} \leq (1+SM)(1+|\Omega|) \tag{4.20}$$

for $p > 1$, $p-1$ small enough. We deduce the existence of $\gamma_s \in L^s(\Omega)$ such that

$$v_p \rightharpoonup \gamma_s \qquad \text{weakly in } L^s(\Omega)\,.$$

It follows from a diagonal argument that there exists γ (non depending on s) satisfying

$$v_p \rightharpoonup \gamma \qquad \text{weakly in } L^s(\Omega) \qquad \text{for all } 1 \leq s < \infty\,. \tag{4.21}$$

Going back to (4.20), the lower semicontinuity of the s–norm yields $\|\gamma\|_s \leq |\Omega|^{\frac{1}{s}}$, and it holds for all $s \in (1, +\infty)$. Therefore, $\gamma \in L^\infty(\Omega)$ and

$$\|\gamma\|_\infty = \lim_{s\to\infty} \|\gamma\|_s \leq 1\,.$$

Let us show now the connection between γ and $u_{(1)}$. By employing the estimates:

$$\liminf_{p\to 1} e^{(p-1)\log u_{(p)}} \leq \gamma \leq \limsup_{p\to 1} e^{(p-1)\log u_{(p)}}$$

we achieve that $\gamma = 1$ in the set $\{x \in \Omega \,:\, u_{(1)}(x) > 0\}$. Since a similar argument holds in the set $\{x \in \Omega \,:\, u_{(1)}(x) < 0\}$, we conclude that $\gamma \in \operatorname{sign}(u_{(1)})$.

We point out that an analogous procedure beginning from (4.17) (see [34, Theorem 3.5] for details) can be used to prove that there exists $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\|\mathbf{z}\|_\infty \leq 1$ and

$$|\nabla u_{(p)}|^{p-2} \nabla u_{(p)} \rightharpoonup \mathbf{z} \qquad \text{weakly in } L^s(\Omega; \mathbb{R}^N) \text{ for every } 1 \leq s < \infty\,. \tag{4.22}$$

To see that $(\lambda_{(1)}, u_{(1)})$ is an eigenpair of problem (4.13), conditions (2)–(4) in Definition 4 must be checked. Now, it is enough to follow the proof of [8] with minor modifications. □

Remark 6. We may apply Theorem 6 to eigenfunctions normalized as $\int_\Omega |u_{(p)}|^p\, dx = 1$, then obtaining an eigenfunction normalized by the condition $\int_\Omega |u_{(1)}|\, dx = 1$.

The previous result allows us to define strong eigenpairs to problem (4.13) as limits (when p goes to 1) of eigenpairs to problem (2.4).

Definition 7. *Let* $(\lambda_{(p_j)}, u_{(p_j)})$ *be a sequence of eigenpairs to problem* (2.4) *(with* $p = p_j$*) satisfying* $\lim_{j\to\infty} p_j = 1$, $\lambda = \lim_{j\to\infty} \lambda_{(p_j)}$, $u = \lim_{j\to\infty} u_{(p_j)}$ *strongly in* $L^1(\Omega)$ *and* $u \neq 0$*. Then we define* λ *as a strong eigenvalue to problem* (4.13) *with associated eigenfunction* u*.*

Remark 7. It should be stressed that the preceding notion of strong eigenpair does not coincide with the corresponding ones in [12], [36]. On the other hand, and as was recently shown in [32] all of the LS eigenvalues of $-\Delta_1$ define strong eigenvalues in the present sense. Accordingly Theorem 6 associates a strong "reference" eigenfunction to these eigenvalues.

Remark 8. Under the notation of Definition 7, let us highlight now a point of the proof of Theorem 6 that will be applied later on. In order to get condition (3) (we are following the proof of [8]), we fix a nonnegative $\varphi \in C_0^\infty(\Omega)$, take $\varphi u_{(p_j)}$ as test function in (2.4) and apply Young's inequality. Then we arrive at

$$\int_\Omega \varphi|\nabla u_{(p_j)}| \le \frac{1}{p_j}\int_\Omega \varphi|\nabla u_{(p_j)}|^{p_j} + \frac{p_j-1}{p_j}\int_\Omega \varphi$$
$$= -\frac{1}{p_j}\int_\Omega u_{(p_j)}|\nabla u_{(p_j)}|^{p_j-2}\nabla u_{(p_j)}\cdot\nabla\varphi\,dx + \frac{\lambda_{(p_j)}}{p_j}\int_\Omega |u_{(p_j)}|^{p_j}\varphi\,dx + \frac{p_j-1}{p_j}\int_\Omega \varphi\,.$$

Using the lower semicontinuity of the left hand side, we obtain

$$\int_\Omega \varphi|Du_{(1)}| \le \liminf_{j\to\infty}\int_\Omega \varphi|\nabla u_{(p_j)}|^{p_j}$$
$$= -\int_\Omega u_{(1)}\mathbf{z}\cdot\nabla\varphi\,dx + \lambda_{(1)}\int_\Omega |u_{(1)}|\varphi\,dx = \int_\Omega \varphi(\mathbf{z}, Du_{(1)}) \le \int_\Omega \varphi|Du_{(1)}|\,.$$

Thus, the above inequalities become equalities, so that

$$\int_\Omega \varphi|Du_{(1)}| = \liminf_{j\to\infty}\int_\Omega \varphi|\nabla u_{(p_j)}|^{p_j}\,. \tag{4.23}$$

5. The one–dimensional case

This section reviews the main features of the one–dimensional problem,

$$\begin{cases} -(\varphi_p(u'))' = \lambda\varphi_p(u) & x \in J \\ u = 0 & x \in \partial J, \end{cases} \qquad J = (0, L),\ L > 0. \tag{5.24}$$

Let ψ be the function

$$\psi(u) = (p-1)^{\frac{1}{p}}\int_0^u \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} \qquad 0 \le u \le 1,$$

and set,

$$t_1 = t_1(p) = 2(p-1)^{\frac{1}{p}}\int_0^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} = \frac{2}{p}(p-1)^{\frac{1}{p}}\frac{\pi}{\sin\frac{\pi}{p}}. \tag{5.25}$$

We remark that $\psi : [0,1] \to [0, t_1/2]$ is an increasing function whose inverse ψ^{-1} satisfies $(\psi^{-1})'(t_1/2) = 0$ and $-(\varphi_p((\psi^{-1})'))' = \varphi_p(\psi^{-1})$ in $[0, t_1/2]$. Define:

$$\phi_0(t) = \begin{cases} \psi^{-1}(t) & 0 \le t \le \dfrac{t_1}{2} \\ \psi^{-1}(t_1 - t) & \dfrac{t_1}{2} < t \le t_1, \end{cases}$$

which solves (5.24) with $\lambda = 1$ and $L = t_1$. Let also define $\phi_1 : \mathbb{R} \to \mathbb{R}$ as the odd and $2t_1$–periodic extension of ϕ_0 to the whole of $\mathbb{R}$. Function ϕ_1 exhibits a sinus type profile and its zeros are $t = nt_1$ for all $n \in \mathbb{Z}$.

The following properties can be shown by using elementary o.d.e's techniques (see for instance [15]).

Proposition 8. *The full set of eigenvalues of* (5.24) *consists in the sequence* $\{\lambda_n\}$ *given by:*

$$\lambda_n = \left(\frac{t_n}{L}\right)^p = (p-1)\left(\frac{2n}{pL}\right)^p \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \qquad n = 1, 2, \ldots$$

Every eigenvalue λ_n *is simple, i. e. all eigenfunctions associated to* λ_n *are a scalar multiple of the normalized eigenfunction* $u_n(x) = \phi_1\left(\lambda_n^{1/p} x\right)$. *Moreover,* u_n *vanishes exactly at the points* $x_k^{(n)} = kL/n$, $k = 0, \ldots, n$.

An essentially known result is next introduced. Namely, the limit behavior of the eigenpairs (λ_n, u_n) as $p \to 1+$ (see also [12]). An independent direct proof is included for the reader's convenience.

Proposition 9. *Let* $\lambda_n = \lambda_{(p),n}$ *be the* n*–th eigenvalue to* (3.11) *with corresponding normalized eigenfunction* $u_n(x) = \phi_1\left(\lambda_n^{1/p} x\right)$. *Then:*

$$\lim_{p\to 1+} \lambda_n = \frac{2n}{L} \qquad n = 1, 2, \ldots. \tag{5.26}$$

In addition, denoting $x_k^{(n)} = kL/n$, $k = 0, \ldots, n$, *we have*

$$\lim_{p\to 1+} u_n(x) = (-1)^k \qquad \text{for} \qquad x_k^{(n)} < x < x_{k+1}^{(n)} \qquad 0 \le k \le n-1, \tag{5.27}$$

being the convergence in the topology of $C^1[(0, L) \setminus \{x_1^{(n)}, \ldots, x_{n-1}^{(n)}\}]$, *that is,* $u_n \to (-1)^k$ *and* $u_n' \to 0$ *uniformly on compact sets of* $(x_k^{(n)}, x_{k+1}^{(n)})$ *for* $0 \le k \le n-1$.

Proof. It follows from (5.25) by direct computation that $\lim_{p\to 1} t_1(p) = 2$. This yields (5.26).

Suppose now that $x_k^{(n)} \le x \le x_{k+1}^{(n)}$. Then, $kt_1 \le \lambda_n^{\frac{1}{p}} x \le (k+1)t_1$. Since this means that $t_k \le \lambda_n^{\frac{1}{p}} x \le t_{k+1}$ we find that:

$$u_n(x) = (-1)^k \phi_0(\lambda_n^{\frac{1}{p}} x - kt_1) = (-1)^k \phi_0(\lambda_n^{\frac{1}{p}}(x - x_k^{(n)})) = (-1)^k u_n(x - x_k^{(n)}).$$

We now observe that for an arbitrary compact $K \subset (x_k^{(n)}, x_{k+1}^{(n)})$ the corresponding compact $K' = \{\lambda_n^{1/p}(x - x_k^{(n)}) : x \in K\}$ remains uniformly bounded away from the boundary of $(0, 2)$ as $p \to 1$. Thus, to show (5.27) it suffices to prove that $\phi_0 \to 1$ uniformly in compact sets of $(0, 2)$. To this proposal notice that for $0 < \varepsilon < 1$ the inequality $\phi_0(t) > 1 - \varepsilon$ holds true if $0 < t_\varepsilon < t < t_1 - t_\varepsilon$ where t_ε is defined by

$$t_\varepsilon = (p-1)^{\frac{1}{p}} \int_0^{1-\varepsilon} \frac{ds}{(1 - |s|^p)^{\frac{1}{p}}}.$$

Since the integral converges to a finite value as $p \to 1+$ then $t_\varepsilon \to 0$ as $p \to 1+$. This shows the convergence assertion.

On the other hand $u = \phi_0$ satisfies,

$$\varphi_p(u'(t)) = \varphi_p(u'(0)) - \int_0^t \varphi_p(u(s))\ ds \to 1 - t,$$

as $p \to 1$ with fixed $0 < t < 2$. In fact $(u'(0))^{p-1} = (p-1)^{p'} \to 1$ as $p \to 1$ while $\int_0^t \varphi_p(u(s))\ ds \ \to t$ as $p \to 1$ by dominated convergence. Thus,

$$u'(t) = \varphi_{p'}\left(\varphi_p(u'(0)) - \int_0^t \varphi_p(u(s))\ ds\right) \to 0,$$

as $p \to 1$. The uniform character on compact sets of the latter convergence can be easily checked. □

6. Radial spectrum of $-\Delta_p$ and its convergence as p goes to 1

When Ω is the open ball $B(0,R) \subset \mathbb{R}^N$, the radial eigenvalues λ to (2.4) are those ones exhibiting a radial eigenfunction $u \in W_0^{1,p}(B(0,R))$. By writing $u = u(r)$, $r = |x|$, it can be shown that radial eigenpairs (λ, u) are characterized as the solutions to:

$$\begin{cases} -(r^{N-1}\varphi_p(u'))' = \lambda r^{N-1}\varphi_p(u) & 0 < r < R \\ u'(0) = 0 \qquad u(R) = 0, \end{cases} \tag{6.28}$$

where $u, \varphi_p(u') \in C^1[0,R]$ and equation is solved in classical sense. Our main interest in the sequel will be to consider the regime $1 < p < 2$. In this case it can be further asserted that $u \in C^2[0,R]$. By performing the scale change:

$$v(t) = u(\lambda^{-\frac{1}{p}}t),$$

problem (6.28) is equivalent to:

$$\begin{cases} -(t^{N-1}\varphi_p(v'))' = t^{N-1}\varphi_p(v) & 0 < t < \theta \\ v'(0) = 0 \qquad v(\theta) = 0, \end{cases} \tag{6.29}$$

where $\theta = \lambda^{\frac{1}{p}}R$. Thus, a complete solution to the eigenvalue problem (6.28) is furnished by the next lemma. We are providing a self–contained proof for future reference.

Lemma 10. *Assume $p > 1$. The initial value problem,*

$$\begin{cases} -(t^{N-1}\varphi_p(v'))' = t^{N-1}\varphi_p(v) & t > 0 \\ v'(0) = 0 \\ v(0) = 1, \end{cases} \tag{6.30}$$

has a unique solution $v_p \in C^1[0,\infty)$ ($v_p \in C^2[0,\infty)$ if $1 < p \le 2$), which satisfies $\varphi_p({v_p}') \in C^1[0,\infty)$ and the following properties:

i) *For every $\rho > 0$ the mapping*

$$\begin{array}{cccc} \Phi : & (1,\infty) & \longrightarrow & C^1([0,\rho];\mathbb{R}^2) \\ & p & \longmapsto & (v_p, \varphi_p({v_p}')) \end{array}$$

is continuous.

ii) *The set of zeros of v_p consists in an increasing sequence*

$$0 < \theta_{(p),1} < \theta_{(p),2} < \dots$$

so that $\theta_{(p),n} \stackrel{n\to\infty}{\longrightarrow} \infty$.

iii) *Zeros $\theta_{(p),n}$ of v_p are simple and define continuous functions of p.*

iv) *The asymptotic estimate,*

$$\lim_{n\to\infty} (\theta_{(p),n+1} - \theta_{(p),n}) = t_1(p) \tag{6.31}$$

of the distance $\theta_{(p),n+1} - \theta_{(p),n}$ between consecutive zeros of v_p holds true. (Here $t_1(p)$ denotes the value defined in (5.25)*.) Moreover, $(v_p(t), v_p{}'(t)) \to (0,0)$ as $t \to \infty$.*

Remark 9. Function $v_p(t)$ could be regarded as a sort of p–version of the well–known Bessel function corresponding to $p = 2$. Values $\theta_{(p),n}$ play the same rôle as the classical zeros of v_2. Moreover, asymptotic estimate in (6.31) furnishes an interesting extension of the classical result valid for the Laplace operator $p = 2$. In this case the solution to (6.30) is

$$v_2(t) = ct^{-\frac{N-2}{2}} J_{\frac{N-2}{2}}(t),$$

where c is a normalizing constant, $J_{\frac{N-2}{2}}$ is the Bessel function of first class and order $\frac{N-2}{2}$ meanwhile $\theta_{(2)n} = \zeta_n$ where ζ_n is the n–th zero of $J_{\frac{N-2}{2}}$. It is well–known that ([43])

$$\lim_{n\to\infty} (\zeta_{n+1} - \zeta_n) = \pi.$$

Observe that π is just $t_1(2)$ in the expression (5.25).

Remark 10. It can be also shown that the n–th zero $t := \sigma_{(p),n}$ of v_p' in the interval $(\theta_{(p),n}, \theta_{(p),n+1})$ defines a continuous function of $p > 1$.

Proof of Lemma 10. Initial value problem (6.30) can be written in the form,

$$\begin{cases} v' = \varphi_{p'}(w) \\ w' = -\varphi_p(v) - \dfrac{N-1}{t} w, \end{cases} \qquad v(t_0) = v_0 \qquad w(t_0) = w_0, \tag{6.32}$$

where $t_0 = 0$, $v_0 = 1$ and $w_0 = 0$. The existence of a local solution for $t_0 \geq 0$ and arbitrary (v_0, w_0) (with $w_0 = 0$ if $t_0 = 0$) follows from the general ordinary differential equations existence theory. In addition, local uniqueness was fully studied and shown in [40] for this range of values of t_0, v_0 and w_0. From this fact two conclusions can be extracted. First, the existence of a unique non continuable solution $(v(t), w(t)) = \big(v_p(t), \varphi_p(v_p'(t))\big)$ to (6.32) defined in an interval $[0, \omega)$. Second, the continuous dependence of both v_p and $\varphi_p(v_p{}')$ with respect to the parameter p, when these functions are observed as having values in the space $C^1[0, \omega)$. Since by uniqueness the possible zeros θ must be simple, the latter assertion also entails its continuous dependence with respect to p.

On the other hand, for every solution (v, w) to (6.32) the relation

$$\dot{E} := \frac{d}{dt}\left(E(v, w)\right) = -\frac{N-1}{t} |w|^{p'}, \tag{6.33}$$

holds true, where E is the Lyapunov function:

$$E := \frac{1}{p'}|w|^{p'} + \frac{1}{p}|v|^p = \frac{1}{p'}|v'|^p + \frac{1}{p}|v|^p. \tag{6.34}$$

Being E decreasing along trajectories, this implies that v_p can be continued to the whole of $[0,\infty)$ $(\omega = \infty)$.

As for the oscillatory character of v_p in $t > 0$ it can be handled by a direct argument as in [16] or either by an analysis of phase–space nature as in [42]. Nevertheless, we are next giving a self contained proof of this fact which is more appropriate in order to show iv).

Suppose v_p does not oscillate and set $v_{0n} = v_p(n)$, $v'_{0n} = {v_p}'(n)$. By supposing that $v_{0n} \neq 0$ for all n, two options are possible:

A) $\lim_{n\to\infty} \dfrac{v'_{0n}}{v_{0n}} = \zeta$, for some $\zeta \in \mathbb{R}$.

B) $\lim_{n\to\infty} \dfrac{v_{0n}}{v'_{0n}} = 0$.

In both cases, modulus the choice of a suitable subsequence. In addition, observe that $u = v_p$ satisfies,

$$\frac{1}{p'}|u'(t)|^p + \frac{1}{p}|u(t)|^p \le \frac{1}{p'}|v'_{0n}|^p + \frac{1}{p}|v_{0n}|^p,$$

for all $t \geq n$.

Define in case A):

$$\tilde{v}_n(t) = \frac{1}{v_{0n}} v_p(t+n) \qquad t \geq 0.$$

It is clear then that $\tilde{v}_n$ is uniformly bounded in $t \geq 0$. In addition $\tilde{v}_n$ satisfies

$$\varphi_p(\tilde{v}'_n(t)) = \left(\frac{n}{t+n}\right)^{N-1} (\varphi_p(\zeta) + o(1)) - \int_0^t \left(\frac{s+n}{t+n}\right)^{N-1} \varphi_p(\tilde{v}_n(s))\, ds.$$

Hence and for arbitrary $\eta > 0$, $\tilde{v}_n$ admits a subsequence converging in $C^1[0, t_1(p)+\eta]$ towards the solution $v = v_\infty(t)$ to the initial value problem,

$$\begin{cases} (\varphi_p(v'))' + \varphi_p(v) = 0 \\ v(0) = 1 \quad v'(0) = \zeta. \end{cases}$$

But this problem has a unique solution defined in $[0, t_1(p)+\eta]$. Namely, $v_\infty(t) = \kappa\phi_1(t+\tau)$ for a certain choice of k and $\tau > 0$, where ϕ_1 is the function introduced in Section 5. Thus the whole sequence $\tilde{v}_n \to v_\infty$ in $C^1[0, t_1(p)+\eta]$ and therefore $\tilde{v}_n$ must vanish somewhere in $(0, t_1(p)+\eta)$ for large n. Since this is not compatible with the non oscillatory character of v_p then hypothesis A) must be ruled out.

Assume now B) and define similarly

$$\tilde{w}_n(t) = \frac{1}{v'_{0n}} v_p(t+n) \qquad t \geq 0.$$

By arguing in the same way we find that $\tilde{w}_n(t)$ converges in $C^1[0, t_1(p)+\eta]$ towards the unique solution $w = w_\infty(t)$ to problem,

$$\begin{cases} (\varphi_p(w'))' + \varphi_p(w) = 0 \\ w(0) = 0 \quad w'(0) = 1. \end{cases} \tag{6.35}$$

Since $w_\infty(t) = (p-1)^{\frac{1}{p}}\phi_1(t)$ again $\tilde{w}_n$ must vanish in some point of $(0, t_1(p)+\eta)$ for large n, and also this is inconsistent with both B) and the non oscillation assumption on v_p. Therefore, v_p must oscillate in $(0,\infty)$.

Let us show now iv). Let θ_n be the sequence of zeros of v_p and set $\phi'_{0n} = v_p{}'(\theta_n)$. Of course, $\phi'_{0n} \neq 0$ for all n. Define,

$$z_n(t) = \frac{1}{\phi'_{0n}} v_p(t+\theta_n),$$

which solves, for every n, the initial value problem,

$$\begin{cases} (\varphi_p(z'))' + \varphi_p(z) = -\dfrac{N-1}{t+\theta_n}\varphi_p(z') & t>0 \\ z(0)=0 \quad z'(0)=1. \end{cases}$$

By the same arguments as in the discussion of the oscillatory character of v_p it can be proved that, for every $\eta > 0$, the sequence $z_n(t)$ converges in $C^1[0, t_1(p)+\eta]$ towards the unique solution $w_\infty(t) = (p-1)^{\frac{1}{p}}\phi_1(t)$ to problem (6.35). Accordingly, z_n must exhibit a first zero in $(0, t_1(p)+\eta)$ for large n which must be necessarily close to $t_1(p)$. Since this zero is $\theta_{n+1}-\theta_n$ this means that $\lim_{n\to\infty}(\theta_{n+1}-\theta_n) = t_1(p)$ and the proof is concluded.

Finally, to show that $\lim_{t\to\infty} v_p(t) = 0$ let us set $|\alpha_n| = \max_{t\in[\theta_n,\theta_{n+1}]}|v_p(t)|$. It follows from the decreasing character of E along solutions that $|\alpha_n|$ decreases and so $\lim|\alpha_n| = \alpha \geq 0$. It must be $\alpha = 0$ since otherwise we would get a contradiction. Indeed, assume that $\alpha > 0$ and note that

$$\int_{t_0}^{\infty} \frac{N-1}{s}|v_p{}'(s)|^p \, ds \leq E(t_0),$$

for some fixed $t_0 > 0$. From the convergence of the integral we deduce that:

$$\sum_{n=1}^{\infty}\int_{\theta_n}^{\theta_{n+1}} \frac{N-1}{s}|v_p{}'(s)|^p \, ds = \sum_{n=1}^{\infty}\int_0^{\Delta\theta_n} \frac{N-1}{\tau+\theta_n}|v_p{}'(\tau+\theta_n)|^p \, d\tau := \sum_{n=1}^{\infty} c_n < \infty.$$

Now observe that:

$$c_n \sim \frac{\kappa}{\theta_n} \qquad n\to\infty,$$

with $\kappa = (N-1)\alpha^p \int_0^{t_1(p)} |\phi'_0|^p \, d\tau > 0$. From the fact that $\Delta\theta_n = t_1(p) + o(1)$ as $n\to\infty$, Cesaro's theorem permits us concluding that

$$\theta_n \sim n t_1(p) \qquad n\to\infty.$$

This implies that the series $\sum_{n=1}^{\infty} c_n$ diverges, which is not possible. Therefore $\alpha = 0$ and $v_p(t)\to 0$ as $t\to\infty$. □

The main features of the radial spectrum of $-\Delta_p$ are now summarized (see [4], [16] and [42] for early accounts and further details on the subject). Next result is a straightforward consequence of Lemma 10.

Theorem 11. *The radial eigenvalues of problem* (2.4) *in the ball* $B(0,R)$ *consist in the sequence of values,*

$$\lambda_{(p),n} = \left(\frac{\theta_{(p),n}}{R}\right)^p,$$

with

$$u_{(p),n}(x) = v_p\left(\frac{\theta_{(p),n}}{R} r\right), \qquad r = |x|,$$

as an associated eigenfunction. Moreover, the following extra features are satisfied.

i) *Each $\lambda_{(p),n}$ is simple in the sense that any other eigenfunction to $\lambda_{(p),n}$ is a scalar multiple of u_n.*

ii) *Every $\lambda_{(p),n}$ defines a continuous function on the parameter $p > 1$.*

iii) *The sequence of eigenvalues satisfies the following asymptotic relation,*

$$\lim_{n\to\infty} (\lambda_{(p),n+1}^{\frac{1}{p}} - \lambda_{(p),n}^{\frac{1}{p}}) = \frac{t_1(p)}{R}.$$

iv) *Any eigenfunction associated to $\lambda_{(p),n}$ vanishes exactly at the values*

$$r_k = \frac{\theta_{(p)k}}{\theta_{(p),n}} \in (0, R) \qquad k = 1, \ldots, n-1.$$

To study the limit behavior as $p \to 1$ of the radial eigenvalues a comparison between the radial eigenvalues and "LS radial" eigenvalues of $-\Delta_p$ is required. Specially in order to use Corollary 3 and Theorem 6. In fact, the Ljusternik–Schnirelman approach (2.5) can also be employed to obtain a sequence λ_n^{LS} of radial eigenvalues to (2.4) in the ball $B(0, R)$. It suffices to replace $W_0^{1,p}(\Omega)$ in the definition of the classes $\mathcal{A}_n$ (see Section 2.1) by its subspace $\widetilde{W}_0^{1,p}(B)$ of radially symmetric functions. By the same token, a sequence of radial eigenvalues, coined λ_n^{DR}, can be also produced by employing the alternative definition in [19], [20] when it is regarded in $\widetilde{W}_0^{1,p}(B)$ (see Remark 1–b)).

Our next result ensures us that, as expected, the three sequences of eigenvalues coincide. In the general framework of the problem (1.1) and Ω an arbitrary domain, the coincidence of the sequences λ_n^{LS} and λ_n^{DR} is, at the best of our knowledge, an open problem.

Theorem 12. *For a fixed $p > 1$, let $\{\lambda_n\}$, $\{\lambda_n^{LS}\}$ and $\{\lambda_n^{DR}\}$ denote the three sequences of radial eigenvalues obtained as in Theorem 11, by the Ljusternik–Schnirelman procedure and by the Drábek–Robinson approach, respectively. Then,*

$$\lambda_n = \lambda_n^{LS} = \lambda_n^{DR},$$

for all n.

Proof. It follows from the definition (2.5) that $\lambda_1 = \lambda_1^{LS} = \lambda_1^{DR}$.

On the other hand $\lambda_n^{LS} \neq \lambda_{n+1}^{LS}$ for all n. Otherwise, a standard multiplicity result in critical point theory ([41], Chapter II) would imply the existence of infinitely many eigenfunctions associated to λ_n^{LS} in $\mathcal{M}$. Since λ_n^{LS} is in particular a radial eigenvalue, this possibility is ruled out by Theorem 11. Exactly the same argument proves that $\lambda_n^{RD} < \lambda_{n+1}^{RD}$ for all n.

We are now dealing with nodal regions. For an eigenvalue λ to the general problem (1.1) and corresponding eigenfunction $u \in W_0^{1,p}(\Omega)$, the nodal regions of u are the components of $\{x \in \Omega : u(x) \neq 0\}$. It can be shown that the maximum number $N(\lambda)$ of nodal regions that an eigenfunction to λ can exhibit is a finite function of λ ([6]). Moreover, it has been proved in [20] that $N(\lambda_n^{DR}(\Omega)) \leq n$ (Courant's nodal domains theorem) provided that $\lambda_n^{DR}(\Omega) < \lambda_{n+1}^{DR}(\Omega)$ where $\{\lambda_n^{DR}(\Omega)\}$ is the spectrum of $-\Delta_p$ in a bounded domain $\Omega \subset \mathbb{R}^N$, according to [19],[20] (Remark

1). The same argument used in [20] can be applied to show that $N(\lambda_n^{LS}(\Omega)) \le n$ if $\lambda_n^{LS}(\Omega) < \lambda_{n+1}^{LS}(\Omega)$ where $\lambda_n^{LS}(\Omega)$ is given by (2.5) (see [6] for related partial results). Notice that in both cases a sort of simplicity condition is required in $\lambda_n^{LS}(\Omega)$ and $\lambda_n^{DR}(\Omega)$ (this fact contrasts with the case of the Laplacian operator).

Now observe that $N(\lambda_n) = n$ for every radial eigenvalue λ_n (Theorem 11) and so

$$\lambda_n^{LS} \le \lambda_n^{DR} \le \lambda_n,$$

since radial eigenvalues greater than λ_n exhibits more than n nodal domains. Finally, $\lambda_n = \lambda_n^{LS}$ implies $\lambda_{n+1} = \lambda_{n+1}^{LS}$, otherwise we would find the set of inequalities

$$\lambda_n = \lambda_n^{LS} < \lambda_{n+1}^{LS} < \lambda_{n+1},$$

and there not exist further radial eigenvalues in the interval $(\lambda_n, \lambda_{n+1})$. Thus, the proof is concluded. □

We point out that as a consequence of Theorems 2 and 12 we can now assure the existence of the limits $\lim_{p\to 1} \lambda_{(p),n}$ of the radial eigenvalues of $-\Delta_p$. In fact, for any n fixed, $p\theta_{(p),n}$ is an increasing function of p for $p > 1$. Some further information concerning these limits is stated in the following result.

Theorem 13. *Let $\{\lambda_{(p),n}\}$ be the radial spectrum of $-\Delta_p$ in $B(0,R)$. Then the limits*

$$\lambda_{(1),n} := \lim_{p\to 1} \lambda_{(p),n} \qquad \& \qquad \theta_{(1),n} := \lim_{p\to 1} \theta_{(p),n},$$

exist and satisfy the strict inequalities

$$\lambda_{(1),n} < \lambda_{(1),n+1} \qquad \theta_{(1),n} < \theta_{(1),n+1},$$

for all n.

Proof. Let $p > 1$ and take a couple of continuous funtions $q(t), m(t)$ defined and continuous in $\overline{J}$, with $J = (a,b)$ a finite interval, $q(t) \ge q_0 > 0$ for all $t \in J$. Now consider the eigenvalue problem,

$$\begin{cases} -(q(t)\varphi_p(u'(t)))' - m(t)\varphi_p(u(t)) = \sigma\varphi_p(u(t)) & t \in J \\ u = 0 \qquad t \in \partial J. \end{cases} \tag{6.36}$$

The first eigenvalue σ_1 can be variationally characterized by

$$\sigma_1 = \inf \frac{\int_J (q|u'|^p - m|u|^p)}{\int_J |u|^p}, \tag{6.37}$$

the infimum being extended to $W_0^{1,p}(J) \setminus \{0\}$. As it is well–known, σ_1 is the unique eigenvalue with a one–signed associated eigenfunction. Thus, by observing $u = v_p$ is a solution of (6.36) in $J_n = (\theta_{(p),n-1}, \theta_{(p),n})$, $n \ge 2$, with $q(t) = m(t) = t^{N-1}$ we find that

$$\sigma_1(q,m) = 0,$$

where $\sigma_1(q,m)$ has been employed to mean that σ_1 depends on both $q(t)$ and $m(t)$. By using (6.37) and estimating q and m in J_n we arrive at,

$$\sigma_1(\theta_{(p),n-1}^{N-1}, \theta_{(p),n}^{N-1}) < 0 < \sigma_1(\theta_{(p),n}^{N-1}, \theta_{(p),n-1}^{N-1}).$$

This implies that

$$\left(\frac{\theta_{(p),n-1}}{\theta_{(p),n}}\right)^{N-1} < \lambda_{(p),1}(J_n) < \left(\frac{\theta_{(p),n}}{\theta_{(p),n-1}}\right)^{N-1} \qquad n \ge 2,$$

where $\lambda_{(p)1}(J_n)$ stands for the first eigenvalue of problem (5.24) in $J = (0, \Delta\theta_{(p),n})$ with $\Delta\theta_{(p),n} = \theta_{(p),n} - \theta_{(p),n-1}$. We thus arrive at the inequalities,

$$\left(\frac{\theta_{(p),n-1}}{\theta_{(p),n}}\right)^{N-1} < \left(\frac{t_1(p)}{\Delta\theta_{(p),n}}\right)^{p} < \left(\frac{\theta_{(p),n}}{\theta_{(p),n-1}}\right)^{N-1} \qquad n \geq 2, \tag{6.38}$$

where the value of $t_1(p)$ is the one given by (5.25). Since $\lim_{p\to 1} t_1(p) = 2$ and $\theta_{(1),n} = \lim_{p\to 1} \theta_{(p),n} > 0$ for all n then, necessarily,

$$\theta_{(1),n} - \theta_{(1),n-1} = \lim_{p\to 1} \Delta\theta_{(p),n} > 0,$$

as it was wanted to prove. □

Remark 11. It is implicit in the proof of Lemma 10 that for fixed $p > 1$, $\Delta\theta_{(p),n}$ keeps bounded as $n \to \infty$. This fact together with inequalities in (6.38) furnish an extra proof of the asymptotic estimate (6.31).

Remark 12. Taking into account the existence of the limit as $p \to 1$ of the n–th eigenvalue $\lambda^{\mathcal{N}}_{(p),n}$ to $-\Delta_p$ under Neumann conditions (Remark 3) it follows the existence of the limits $\sigma_{(1),n} := \lim_{p\to 1} \sigma_{(p),n}$ with $\sigma_{(p),n}$ the n–th zero of v_p'.

Lemma 14. *The lower estimate,*

$$\theta_{(1),n} - \theta_{(1),n-1} = \lim_{p\to 1} \Delta\theta_{(p),n} \geq 1 \qquad n \in \mathbb{N}, \tag{6.39}$$

holds true. In particular $\theta_{(1),n} \to \infty$.

Proof. Let $\theta_{(p),n} < \theta_{(p),n+1}$ be consecutive zeros of the solution $v(t) = v_p(t)$ to (6.30) and assume without loss of generality that v is positive in $(\theta_{(p),n}, \theta_{(p),n+1})$. By making use of the phase space of equation (6.32) it can be shown that there exists a unique zero t_n' of v' in that interval, being $v_n := v(t_n') = \max_{(\theta_{(p),n}, \theta_{(p),n+1})} v$.

Since v is decreasing in $t_n' < t < \theta_{(p),n+1}$ we obtain

$$(p-1)(-v'(t))^p + v^p < v_n^p,$$

in this range of t. This implies,

$$(p-1)^{\frac{1}{p}} \int_{\frac{v(t)}{v_n}}^{1} \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} < t - t_n'.$$

Thus

$$\theta_{(p),n+1} - t_n' > \frac{t_1(p)}{2},$$

what implies the estimate (6.39). □

Remark 13. By taking limits in (6.38) one finds

$$\left(\frac{\theta_{(1),n-1}}{\theta_{(1),n}}\right)^{N-1} \leq \frac{2}{\Delta\theta_{(1),n}} \leq \left(\frac{\theta_{(1),n}}{\theta_{(1),n-1}}\right)^{N-1} \qquad n \geq 2. \tag{6.40}$$

We achieve

$$\lim_{n\to\infty} \Delta\theta_{(1),n} = 2 \tag{6.41}$$

from the convergence $\theta_{(1),n} \to \infty$ and the fact that $\Delta\theta_{(1),n}$ is bounded from above. A proof of this last assertion is omitted for brevity. Estimate (6.41) is the equivalent of (6.31) for $p = 1$.

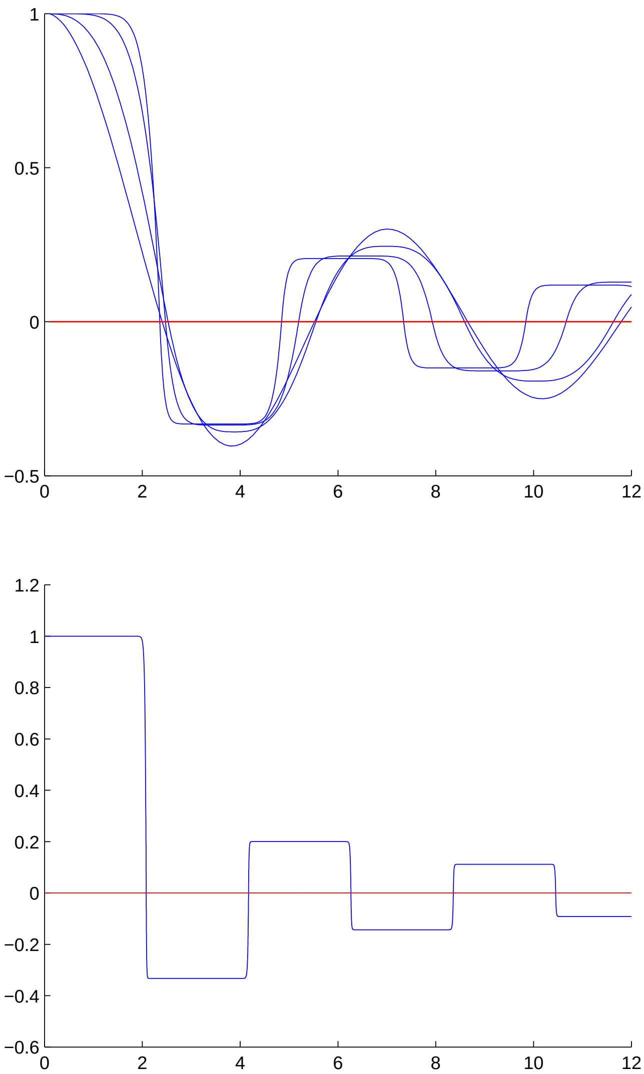

FIGURE 1. Profiles of v_p for $N = 2$ and values $p = 2$, $p = 1.5$, $p = 1.2$ and $p = 1.1$. Last figure shows the case $p = 1.01$.

If $v_p(t)$ stands for the solution to (6.30) and $\theta_{(p),1}$ designates its first zero (Lemma 10), then next result essentially discus the limit profile of v_p as $p \to 1$ in the first interval $[0, \theta_{(1),1}]$.

Lemma 15. *The identity* $\theta_{(1),1} = N$ *holds as well as the following convergences uniformly in compact sets of* $[0, N)$*:*

(1) $v_p(t) \xrightarrow{p\to 1} 1$,

(2) $v_p{}'(t) \xrightarrow{p\to 1} 0$,

(3) $\varphi_p\big({v_p}'(t)\big)\overset{p\to 1}{\longrightarrow} -\frac{t}{N}$.

Proof. The existence and positivity of the limit $\theta_{(1),1} = \lim_{p\to 1}\theta_{(p),1}$ is provided in Theorem 13. Thanks to [25, Corollary 6], we have that

$$\lim_{p\to 1}\left(\frac{\theta_{(1),n}}{R}\right)^p = \frac{N}{R}\,,$$

since Cheeger's constant of a ball of radius R in $\mathbb{R}^N$ is N/R. So, $\theta_{(1),1} = N$.

We are showing the convergences. First note that

$$v_p(t) = 1 - \int_0^t \varphi_{p'}\left(\int_0^s \left(\frac{\sigma}{s}\right)^{N-1} \varphi_p(v_p(\sigma))\ d\sigma\right)\ ds.$$

If $0 \le s \le t < N$, then

$$\varphi_{p'}\left(\int_0^s \left(\frac{\sigma}{s}\right)^{N-1} \varphi_p(v_p(\sigma))\ d\sigma\right) < \varphi_{p'}\left(\frac{s}{N}\right) \le \varphi_{p'}\left(\frac{t}{N}\right) \to 0,$$

as $p \to 1$. Hence,

$$v_p(s) \to 1,$$

uniformly in $0 \le s \le t$. Second,

$$v_p'(s) \to 0 \qquad \text{as } p \to 1,$$

uniformly for $0 \le s \le t < N$. In fact this follows from the estimate $-\varphi_p(v_p') < \frac{s}{N} \le \frac{t}{N}$. Observe in addition that

$$-\varphi_p(v_p'(t)) = \int_0^t \left(\frac{s}{t}\right)^{N-1} \varphi_p(v_p(s))\ ds\,,$$

so that

$$-\varphi_p(v_p'(s)) \to \frac{s}{N} \qquad \text{as } p \to 1,$$

for $0 \le s \le t < N$. □

In Figures 1 and 2, the convergence of v_p as p goes to 1 is illustrated.

7. Radial spectrum of $-\Delta_1$

This section is devoted to a precise analysis of the limit as p goes to 1 of the solution v_p to (6.30). Hence we get the limit profile by means of which strong radial eigenfunctions to $-\Delta_1$ can be expressed. To find out this profile, we are dealing with system (6.32) written in the form,

$$\begin{cases} w = \varphi_p(v') \\ -w' - \dfrac{N-1}{t} w = \varphi_p(v) \end{cases} \qquad v(0) = 1, w(0) = 0.$$

After passing formally to the limit as $p \to 1$ we obtain

$$\begin{cases} w \in \operatorname{sign}(v') \\ -w' - \dfrac{N-1}{t} w \in \operatorname{sign}(v), \end{cases} \qquad v(0) = 1, w(0) = 0\,. \tag{7.42}$$

Observe that v and v' do not appear directly in (7.42), but their signs. The behavior of $E_p(v,w) = \frac{1}{p'}|w|^{p'} + \frac{1}{p}|v|^p$ (see Lemma 10) as p goes to 1 will be most useful

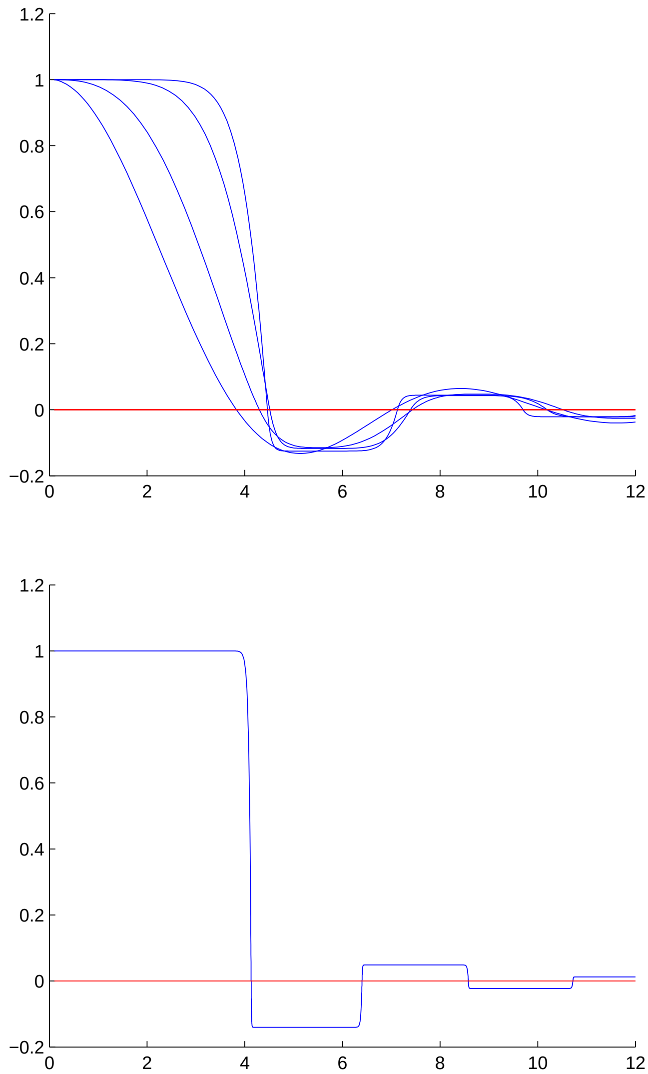

FIGURE 2. Corresponding profiles of v_p now for $N = 4$ and values $p = 2, 1.5, 1.2$ and 1.1. Case $p = 1.01$ is depicted separately.

for determining v. Recall that its derivative along trajectories is $\frac{d}{dt}(E_p(v, w)) = -\frac{N-1}{t}|v'|^p$. When p goes to 1, we formally obtain $E(v, w) = |v|$ and

$$\frac{d|v|}{dt} = -\frac{N-1}{t}\left|\frac{dv}{dt}\right|, \tag{7.43}$$

where this equality should be understood in the sense of measures and $\left|\frac{dv}{dt}\right|$ stands for the total variation of the measure $\frac{dv}{dt}$.

Remark 14. It is worth analyzing the meaning of (7.43) in the one–dimensional setting. In this case, system (6.32) becomes

$$\begin{cases} v' = \varphi_{p'}(w) \\ \\ w' = -\varphi_p(v), \end{cases} \qquad v(t_0) = v_0 \qquad w(t_0) = w_0,$$

and so $\dfrac{dE_p}{dt} = 0$. When p tends to 1, the condition $\dfrac{d|v|}{dt} = 0$ is obtained. Hence, $|v|$ is constant (although function v can change sign). We point out that spurious eigenfunctions shown in [12, Remark 2.5] do not satisfy this condition. Therefore, condition (7.43) will actually be our tool to identify genuine eigenfunctions.

The above formal discussion is properly justified in our next result.

Proposition 16. *Let $(v, w) = (v_p, w_p)$ be the solution to*

$$\begin{cases} v' = \varphi_{p'}(w) \\ \\ w' = -\varphi_p(v) - \dfrac{N-1}{t} w \end{cases} \qquad v(0) = 1 \qquad w(0) = 0. \tag{7.44}$$

Then, for every *fixed $n \in \mathbb{N}$ and up to a subsequence as $p \to 1$, the following properties hold.*

1. *$\{v_p\}$ converges strongly in $L^1((0, \theta_{(1),n}); t^{N-1}dt)$ to v_1.*
2. *$\{w_p\}$ converges weakly in $L^s((0, \theta_{(1),n}); t^{N-1}dt)$ to w_1, for every $1 \le s < \infty$. Furthermore, $w_1 \in L^\infty(0, \theta_{(1),n})$ with $\|w_1\|_\infty \le 1$.*
3. *$\{|v_p|^{p-2}v_p\}$ converges weakly in $L^s((0, \theta_{(1),n}); t^{N-1}dt)$ to β, for every $1 \le s < \infty$, $\beta \in L^\infty((0, \theta_{(1),n})$, with $\|\beta\|_\infty \le 1$ and $\beta v_1 = |v_1|$ holds.*
4. *$v_1 \in BV(\sigma, \theta_{(1),n} - \sigma)$ for every $\sigma > 0$.*
5. *$-w_1' - \frac{N-1}{t} w_1 = \beta$ in the sense of distributions.*
6. *w_1 is Lipschitz–continuous in $(\sigma, \theta_{(1),n})$ for every $\sigma > 0$.*
7. *$|v_1'| = (w_1, v_1')$ as measures.*
8. *The identity*

$$\frac{d|v_1|}{dt} = -\frac{N-1}{t} \left| \frac{dv_1}{dt} \right| \tag{7.45}$$

holds in the sense of distributions.

Proof. Our aim is to employ Theorem 6 subject to radial symmetry in $B(0, R)$. To this purpose observe that

$$\lambda_{(p),n} = \left(\frac{\theta_{(p),n}}{R} \right)^p, \qquad u_{(p),n}(x) = v_p\Big(\lambda_{(p),n}{}^{\frac{1}{p}} r \Big), \qquad r = |x|,$$

defines the normalized radial n–th eigenpair (Theorem 11). In addition,

$$|\nabla u_{(p),n}(x)|^{p-2} \nabla u_{(p),n}(x) \cdot \frac{x}{|x|} = \lambda_{(p),n}{}^{\frac{1}{p'}} w_p\Big(\lambda_{(p),n}{}^{\frac{1}{p}} r \Big).$$

Theorem 13 ensures us the existence of $\lim_{n\to\infty} \lambda_{(p),n}$. Thus, we only have to check the existence of the constants $0 < \kappa < \Gamma$ alluded to in the statement. Observe that changing first to polar coordinates and then putting $t = \lambda_{(1),n} r$ it yields

$$\int_{B(0,R)} |u_{(p),n}|^p = \mathcal{H}^{N-1}(\partial B(0,1)) \int_0^R r^{N-1} \Big| v_p\Big(\lambda_{(p),n}{}^{\frac{1}{p}} r\Big)\Big|^p \, dr$$
$$= \mathcal{H}^{N-1}(\partial B(0,1)) \lambda_{(p),n}{}^{-\frac{N}{p}} \int_0^{\theta_{(p),n}} t^{N-1} |v_p(t)|^p \, dt \,.$$

The upper estimate follows from,

$$\lambda_{(p),n}{}^{-\frac{N}{p}} \int_0^{\theta_{(p),n}} t^{N-1} |v_p(t)|^p \, dt \le \lambda_{(p),n}{}^{-\frac{N}{p}} \int_0^{\theta_{(p),n}} t^{N-1} \, dt \le \frac{R}{N} \,.$$

On the other hand, thanks to Lemma 15, we know that $v_p \to 1$ uniformly in $[0, N/2]$ and so

$$\lambda_{(p),n}{}^{-\frac{N}{p}} \int_0^{\theta_{(p),n}} t^{N-1} |v_p(t)|^p \, dt \ge \lambda_{(p),n}{}^{-\frac{N}{p}} \int_0^{\theta_{(p)1}} t^{N-1} |v_p(t)|^p \, dt$$
$$\ge \lambda_{(p),n}{}^{-\frac{N}{p}} \int_0^{N/2} t^{N-1} \frac{1}{2} \, dt = \lambda_{(p),n}{}^{-\frac{N}{p}} \frac{N^{N-1}}{2^{N+1}} \,.$$

The lower estimate is now a straightforward consequence of $\lim_{n\to\infty} \lambda_{(p),n} = \lambda_{(1),n}$.

We are next separately proving each item.

Proof of 1).

By the arguments of Theorem 6, we have that the family $\{u_{(p),n}\}$ converges strongly in $L^1(B(0,R))$ to $u_{(1),n}$. Setting $u_{(1),n}(x) = v_1(\lambda_{(1),n} r)$ and passing to polar coordinates, we obtain

$$\lim_{p\to 1} \int_0^R r^{N-1} |v_p(\lambda_{(p),n}^{1/p} r) - v_1(\lambda_{(1),n} r)| \, dr = 0 \,.$$

On the other hand, since all functions v_p are normalized, the convergence of eigenvalues $\lambda_{(p),n} \to \lambda_{(1),n}$ implies

$$\lim_{p\to 1} \int_0^R r^{N-1} \big| v_p(\lambda_{(p),n}^{1/p} r) - v_p(\lambda_{(1),n} r)\big| \, dr = 0$$

and consequently we deduce

$$\lim_{p\to 1} \int_0^R r^{N-1} \big| v_p(\lambda_{(1),n} r) - v_1(\lambda_{(1),n} r)\big| \, dr = 0 \,.$$

Finally, the change of variable $t = \lambda_{(1),n} r$ leads to

$$\lim_{p\to 1} \int_0^{\theta_{(1),n}} t^{N-1} |v_p(t) - v_1(t)| \, dt = 0 \,,$$

and we are done.

Before going on, we need to introduce some notation which we will use in the sequel. Given a test $\psi \in C_0^\infty(0, \theta_{(1),n})$, consider

$$\varphi(x) = \begin{cases} \psi\Big(\lambda_{(1),n}|x|\Big) \frac{1}{|x|^{N-1}} & x \ne 0\,; \\ 0 & x = 0\,. \end{cases} \tag{7.46}$$

Obviously, $\varphi \in C_0^\infty(B(0,R))$. Its gradient is given by

$$\nabla\varphi(x) = \left\{\psi'\Big(\lambda_{(1),n}|x|\Big)\,\lambda_{(1),n} - \frac{N-1}{|x|}\psi\Big(\lambda_{(1),n}|x|\Big)\right\}\frac{x}{|x|^N}.$$

Proof of 2) and 3).

The convergence in 2) is a consequence of the weak convergence

$$|\nabla u_{(p),n}(x)|^{p-2}\nabla u_{(p),n}(x) \rightharpoonup \mathbf{z}(x),$$

in $L^s(B(0,R);dx)$, for every $1 \le s < \infty$. To check 2), take $\psi \in C_0^\infty(0,\theta_{(1),n})$ and φ as in (7.46). Denoting $w_1(\lambda_{(1),n}r) = \mathbf{z}(x)\cdot\frac{x}{|x|}$, we obtain $w_1 \in L^\infty(0,\theta_{(1),n})$ with $\|w_1\|_\infty \le 1$. It follows from

$$\lim_{p\to 1}\int_{B(0,R)}\varphi(x)|\nabla u_{(p),n}(x)|^{p-2}\nabla u_{(p),n}(x)\cdot\frac{x}{|x|}\,dx = \int_{B(0,R)}\varphi(x)\mathbf{z}(x)\cdot\frac{x}{|x|}\,dx$$

through polar coordinates that

$$\lim_{p\to 1}\int_0^R \psi(\lambda_{(1),n}r)\lambda_{(p),n}{}^{\frac{1}{p'}}w_p(\lambda_{(p),n}{}^{\frac{1}{p}}r)\,dr = \int_0^R \psi(\lambda_{(1),n}r)w_1(\lambda_{(1),n}r)\,dr\,.$$

Gathering this limit and

$$\lim_{p\to 1}\int_0^R \psi(\lambda_{(1),n}r)\big[\lambda_{(p),n}{}^{\frac{1}{p'}}w_p(\lambda_{(p),n}{}^{1/p}r) - w_p(\lambda_{(1),n}r)\big]\,dr = 0\,,$$

we get

$$\lim_{p\to 1}\int_0^R \psi(\lambda_{(1),n}r)w_p(\lambda_{(1),n}r)\,dr = \int_0^R \psi(\lambda_{(1),n}r)w_1(\lambda_{(1),n}r)\,dr\,,$$

so that the change of variable $t = \lambda_{(1),n}r$ gives

$$\lim_{p\to 1}\int_0^{\theta_{(1),n}}\psi(t)w_p(t)\,dt = \int_0^{\theta_{(1),n}}\psi(t)w_1(t)\,dt\,.$$

Then, we infer 2) by density.

Based upon the fact that $|u_{(p),n}(x)|^{p-2}u_{(p),n}(x) \rightharpoonup \gamma(x)$ weakly in $L^s(B(0,R);dx)$ for every $1 \le s < \infty$, a similar argument yields that $|v_p|^{p-2}v_p$ converges weakly in $L^s((0,\theta_{(1),n});t^{N-1}dt)$ to β, for every $1 \le s < \infty$. Moreover, since $\beta\Big(\lambda_{(1),n}r\Big) = \gamma(x)$, we have $\|\beta\|_\infty \le 1$. To complete the proof of assertion (3), we just observe that:

$$\beta\Big(\lambda_{(1),n}r\Big)\,v_1\Big(\lambda_{(1),n}r\Big) = \gamma(x)u_{(1),n}(x) = |u_{(1),n}(x)| = \Big|v_1\Big(\lambda_{(1),n}r\Big)\Big|.$$

Proof of 4).

The estimate $\int_{B(0,R)}|\nabla u_{(p),n}(x)|^p\,dx \le M$, with M non depending on p, holds true. In particular, it is satisfied in any domain

$$\mathcal{D}(\sigma) = B\left(0,\frac{\theta_{(1),n}-\sigma}{\lambda_{(1),n}}\right)\backslash\overline{B}\left(0,\frac{\sigma}{\lambda_{(1),n}}\right)\qquad \sigma > 0.$$

Then Young's inequality implies

$$\int_{\mathcal{D}(\sigma)} |\nabla u_{(p),n}(x)|\, dx \le \frac{1}{p}\int_{\mathcal{D}(\sigma)} |\nabla u_{(p),n}(x)|^p\, dx + \frac{p-1}{p}\big|\mathcal{D}(\sigma)\big| \\ \le M + |B(0,R)|.$$

Thus the lower semicontinuity of the total variation yields

$$\int_{\mathcal{D}(\sigma)} |Du_{(1),n}| \le \liminf_{p\to\infty}\int_{\mathcal{D}(\sigma)} |\nabla u_{(p),n}|\, dx \le M + |B(0,R)|\,.$$

Passing to polar coordinates, it leads to

$$\left(\frac{\sigma}{\lambda_{(1),n}}\right)^{N-1}\int_{\frac{\sigma}{\lambda_{(1),n}}}^{\frac{\theta_{(1),n}-\sigma}{\lambda_{(1),n}}} |v_1'(\lambda_{(1),n}r)| < \infty\,,$$

and so $\displaystyle\int_\sigma^{\theta_{(1),n}-\sigma} |v_1'(t)| < \infty$, wherewith v_1 is a function of bounded variation in $(\sigma, \theta_{(1),n}-\sigma)$.

Proof of 5) and 6).

To show that equality 5) holds in the sense of distributions, we choose a test $\psi \in C_0^\infty(0,\theta_{(1),n})$, fix $0 < a < b < \theta_{(1),n}$ in such a way that $\operatorname{supp}\psi \subset (a,b)$ and consider φ as in (7.46). Having in mind the identity $-\operatorname{div}\mathbf{z} = \lambda_{(1),n}\gamma$, we obtain

$$\begin{aligned}\lambda_{(1),n}\int_{B(0,R)} \gamma(x)\varphi(x)\, dx &= \int_{B(0,R)} \mathbf{z}(x)\cdot\nabla\varphi(x)\, dx \\ &= \int_{B(0,R)} w_1\Big(\lambda_{(1),n}|x|\Big)\psi'\Big(\lambda_{(1),n}|x|\Big)\,\lambda_{(1),n}\frac{dx}{|x|^{N-1}} \\ &\quad - \int_{B(0,R)} \frac{N-1}{|x|} w_1\Big(\lambda_{(1),n}|x|\Big)\psi\Big(\lambda_{(1),n}|x|\Big)\,\frac{dx}{|x|^{N-1}}.\end{aligned}$$

Passing to polar coordinates, changing the variable and simplifying, this identity becomes

$$\int_0^{\theta_{(1),n}} \beta(t)\psi(t)\, dt = \int_0^{\theta_{(1),n}} w_1(t)\left\{\psi'(t) - \frac{N-1}{t}\psi(t)\right\} dt.$$

That is, the distributional derivative of w_1 satisfies

$$w_1' = -\beta - \frac{N-1}{t} w_1\,.$$

As a direct consequence $w_1' \in L^\infty(\sigma, \theta_{(1),n})$ for all $\sigma > 0$ and so condition 6) also holds.

Proof of 7).

Before checking assertion 7), observe that v_1 is a function of bounded variation and w_1 satisfies that its derivative is bounded in each interval $(\sigma, \theta_{(1),n}-\sigma)$. Thus, the one–dimensional pairing (w_1, v_1') has sense there.

To see 7), consider $\psi \in C_0^\infty(0,\theta_{(1),n})$ and define $\varphi \in C_0^\infty(B(0,R))$ as above. It follows from the identity $|Du_{(1),n}| = (\mathbf{z}, Du_{(1),n})$ as measures that

$$\begin{aligned}\int_{B(0,R)} \varphi|Du_{(1),n}| &= \int_{B(0,R)} \varphi(\mathbf{z}, Du_{(1),n})\\ &= -\int_{B(0,R)} u_{(1),n}\,\varphi \operatorname{div}\mathbf{z}\; dx - \int_{B(0,R)} u_{(1),n}\,\mathbf{z}\cdot\nabla\varphi\; dx\,.\end{aligned}$$

Performing the same manipulations as above, we obtain

$$\begin{aligned}&\int_0^{\theta_{(1),n}} \psi|v_1'|\\ &= \int_0^{\theta_{(1),n}} v_1(t)\psi(t)\beta(t)\,dt + \int_0^{\theta_{(1),n}} v_1(t)\psi(t)w_1(t)\Big(\frac{N-1}{t}\Big)\,dt\\ &\qquad - \int_0^{\theta_{(1),n}} v_1(t)w_1(t)\psi'(t)\,dt\\ &= -\int_0^{\theta_{(1),n}} v_1(t)\psi(t)w_1'(t)\,dt - \int_0^{\theta_{(1),n}} v_1(t)w_1(t)\psi'(t)\,dt\\ &= \int_0^{\theta_{(1),n}} \psi(w_1, v_1')\,,\end{aligned}$$

as desired.

Proof of 8).

Consider a nonnegative $\psi \in C_0^\infty(0,\theta_{(1),n})$ and define now $\varphi \in C_0^\infty(B(0,R))$ by

$$\varphi(x) = \begin{cases} \psi\Big(\lambda_{(1),n}|x|\Big)\frac{N-1}{|x|^N} & x \neq 0\,;\\ 0 & x = 0\,.\end{cases}$$

Our starting point is Remark 8, which (up to subsequences)) yields

$$\int_{B(0,R)} \varphi|Du_{(1),n}| = \liminf_{p\to 1} \int_{B(0,R)} \varphi|\nabla u_{(p),n}(x)|^p dx\,.$$

Passing to a further subsequence, if necessary, and manipulating as above, we deduce

$$\begin{aligned}\int_0^{\theta_{(1),n}} \frac{N-1}{t}\,\psi(t)|v_1'| &= \lim_{p\to 1}\int_0^{\theta_{(1),n}} \frac{N-1}{t}\,\psi(t)|v_p'|^p dt\\ &= \lim_{p\to 1}\int_0^{\theta_{(1),n}} \psi(t)\Big(-\frac{dE_p}{dt}\Big)\,dt = \lim_{p\to 1}\int_0^{\theta_{(1),n}} \psi'(t)E_p(t)\,dt\\ &= \lim_{p\to 1}\frac{1}{p'}\int_0^{\theta_{(1),n}} \psi'(t)|w_p(t)|^{p'}\,dt + \lim_{p\to 1}\frac{1}{p}\int_0^{\theta_{(1),n}} \psi'(t)|v_p(t)|^p\,dt\,. \quad (7.47)\end{aligned}$$

To compute the first integral on the right hand side, recall that we have seen in Theorem 6 the existence of a constant $M > 0$ satisfying

$$\int_{B(0,R)} |\nabla u_{(p),n}|^p dx \le M\,, \qquad \text{for all } p > 1\,.$$

Performing our usual manipulations, we achieve a uniform bound for the family $\int_0^{\theta_{(1),n}} t^{N-1}|w_p(t)|^{p'}\,dt$. As a consequence, if $0 < a < b < \theta_{(1),n}$ satisfy $\operatorname{supp}(\psi) \subset$

(a,b), then the family $\int_a^b |w_p(t)|^{p'}\,dt$ is also uniformly estimated, due to the inequality

$$a^{N-1}\int_a^b |w_p(t)|^{p'}\,dt \le \int_a^b t^{N-1}|w_p(t)|^{p'}\,dt\,.$$

Therefore, there is a certain $M_1 > 0$ such that

$$\int_0^{\theta_{(1),n}} |\psi'(t)||w_p(t)|^{p'}\,dt \le M_1\,, \qquad \text{for all } p>1\,.$$

Then, we arrive at

$$\lim_{p\to 1}\frac{1}{p'}\int_0^{\theta_{(1),n}} |\psi'(t)||w_p(t)|^{p'}\,dt \le \lim_{p\to 1}\frac{p-1}{p}M_1 = 0\,.$$

Going back to (7.47), we conclude that

$$\int_0^{\theta_{(1),n}} \frac{N-1}{t}\,\psi(t)|v_1'| = \lim_{p\to 1}\frac{1}{p}\int_0^{\theta_{(1),n}} \psi'(t)|v_p(t)|^p\,dt\,.$$

This limit can be handled by applying Vitali's Theorem as in the proof of (4.19), so we obtain

$$\int_0^{\theta_{(1),n}} \frac{N-1}{t}\,\psi(t)|v_1'| = \int_0^{\theta_{(1),n}} \psi'(t)|v_1(t)|\,dt\,.$$

Therefore, identity (8) is proved. □

Remark 15. By arguing in the proof of Proposition 16 with higher families of eigenvalues $\lambda_{(p),k}$, with $k>n$, together with a diagonal argument, we can reach any $T>0$ in conditions 4) and 6). On the other hand, it follows from Lemma 15 that $v_1(t)=1$ and $w_1(t)=-\frac{t}{N}$ in $(0,N)$. Thus, assertions 4) and 6) can be strengthened as follows:

(4) $v_1 \in BV(0,T)$ for all $T>0$.
(6) Function w_1 is Lipschitz–continuous on $(0,T)$ for all $T>0$.

Definition 17. *We say that a pair (v,w) is a solution to system (7.42) if the following conditions hold true*

(1) *$v\in BV(0,T)$ and $w\in W^{1,\infty}(0,T)$ for all $T>0$, with $\|w\|_{L^\infty(0,+\infty)}\le 1$.*
(2) *Equations hold in the following sense:*
  (a) *$|v'|=(w,v')$ as measures.*
  (b) *There exists $\beta\in \operatorname{sign}(v)$ such that $-w'-\frac{N-1}{t}w=\beta$ in the sense of distributions*
(3) *$v(0+)=\lim_{t\to 0+}v(t)=1$ and $w(0)=0$.*

Next existence result summarizes the conclusions of Proposition 16 and Remark 15.

Proposition 18. *System (7.42) subject to the condition (7.43) admits a solution (v,w) in $BV(0,T)\times W^{1,\infty}(0,T)$ for all $T>0$.*

Moreover, a solution to (7.42) can be found as the limit, as p goes to 1, of a subsequence of solutions (v_p,w_p) to problems (7.44).

Remark 16. For immediate purposes we observe that for every $A>0$, equation

$$\frac{t^N}{N}-t^{N-1}-A=0$$

has a unique root in the interval $(N-1,+\infty)$. This is a straightforward consequence of being the function given by $h(t)=\frac{t^N}{N}-t^{N-1}-A$ increasing in this interval and $h(N-1)<0$.

The main result of this section is now stated.

Theorem 19. *There exists a unique solution (v,w) to system (7.42) such that condition (7.43) holds. It further satisfies*

i) *w is Lipschitz–continuous in $(0,+\infty)$.*

ii) *There exist an increasing sequence $\{\theta_n\}_{n=0}^{\infty}$, $\theta_0=0$, $\theta_n\to\infty$, and a sign alternating sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $v(t)=\alpha_n$ for all $t\in(\theta_n,\theta_{n+1})$.*

iii) *w is strictly monotone in each interval (θ_n,θ_{n+1}), $n\ge 1$, and it oscillates between -1 and 1.*

iv) *$\{|\alpha_n|\}_{n=0}^{\infty}$ is decreasing.*

v) *α_n and $w(\theta_{n+1})$ have different signs for all n.*

vi) *$\{\theta_n\}$ is recursively defined by $\theta_0=0$ and*

$$\frac{\theta_{n+1}^N}{N}-\theta_{n+1}^{N-1}-\theta_n^{N-1}\left(1+\frac{\theta_n}{N}\right)=0\,, \tag{7.48}$$

while $\{\alpha_n\}$ verifies $\alpha_0=1$ and

$$|\alpha_{n+1}|=\frac{\theta_{n+1}-N+1}{\theta_{n+1}+N-1}\,|\alpha_n|\,. \tag{7.49}$$

vii) *The asymptotic estimates $\lim_{n\to\infty}(\theta_{n+1}-\theta_n)=2$ and $\lim_{n\to\infty}\alpha_n=0$ hold true.*

Proof. We are proceeding by progressively proving a series of partial assertions.

1) *Function v is constant on each connected component of the set $\{|w(t)|<1\}$.*

It follows from $w\in\operatorname{sign}(v')$ that $v'(t)=0$ when $-1<w(t)<1$. In other words, if I is an open interval satisfying $w(t)\in(-1,1)$ for all $t\in I$, then

$$\int_I v(t)\psi'(t)\,dt=0 \quad \text{for all } \psi\in C_0^\infty(I)\,.$$

Hence, v is constant a.e. on I and, by replacing v with a good representative ([1]), there is not loss of generality in assuming v is constant in I. In the course of proof it will be assumed that such substitution has been already performed.

2) *Definition of v and w on $(0,N)$.*

Since $v(0)=1$ and $|w(0)|<1$, then a maximal number $\theta_1>0$ exists such that $v(t)=1$ for all $t\in[0,\theta_1)$. On the other hand, the second equation leads to

$$-w'-\frac{N-1}{t}w=1$$

whose solution satisfying $w(0)=0$ is given by $w(t)=-\frac{t}{N}$. Since $v\equiv 1$ can be extended as long as $|w|<1$ then $|w(\theta_1)|=1$. On account of being w decreasing in $[0,\theta_1)$, we get $w(\theta_1)=-1$, so that $\theta_1=N$. Observe that $w(\theta_1)=-1$ and $\lim_{t\to N^-}v(t)=1$ have different signs.

3) *Proof of i).*

Note that $w'(t) = -\frac{1}{N}$ for all $t \in (0, N)$ and the second equation of (7.42) leads to

$$|w'(t)| \le 1 + \frac{N-1}{t}, \qquad \text{for almost all } t \in (0, +\infty).$$

Hence, $w' \in L^\infty(0, +\infty)$. We can be more precise and estimate $\|w'\|_\infty \le 2$.

4) *Let (a, b) be a component of set $\{|w(t)| < 1\}$ with $a \ge N$. Assume that $v(t) = c$ for all $t \in (a, b)$, where $c \ne 0$ is a constant. Then the following features hold,*

- *(a, b) is finite.*
- $\operatorname{sign}(c) = \operatorname{sign} w(a)$.
- $\operatorname{sign}(c) w(t)$ *is decreasing in $t \in (a, b)$ with $w(a)w(b) = -1$.*
- *The following identity holds,*

$$\frac{1}{N} b^N - b^{N-1} = a^{N-1} + \frac{1}{N} a^N. \tag{7.50}$$

- *$b - a \ge 1$.*

Observe that the finiteness of (a, b) follows by integration from the identity

$$w' + \frac{N-1}{t} w = -\operatorname{sign}(c),$$

together with the fact that $|w| < 1$ in (a, b). In particular $w(t) = \pm 1$ at $t = a, b$.

The second point is a consequence of the relation,

$$(t^{N-1} w)' = -\operatorname{sign}(c) t^{N-1}, \tag{7.51}$$

and the fact that $a \ge N$ implies that $\operatorname{sign}(w(a)) = \operatorname{sign}(c)$. To check it, assume on the contrary that $w(a) = -\operatorname{sign}(c)$. By integrating the last equation between a and t with $t \in (a, b)$ we obtain

$$t^{N-1} w(t) = -\operatorname{sign}(c) \left(a^{N-1} + \frac{1}{N} (t^N - a^N) \right).$$

Being $|w(t)| = -\operatorname{sign}(c) w(t)$ for $t > a$ close to a this expression implies that

$$|w(t)| = \frac{1}{t^{N-1}} \left(a^{N-1} + \frac{1}{N} (t^N - a^N) \right) > 1 \tag{7.52}$$

for $t > a$ near a due to the fact

$$\frac{1}{N} \frac{t^N - a^N}{t^{N-1} - a^{N-1}} \to \frac{a}{N-1} > 1,$$

as $t \to a+$. But (7.52) is not possible since $|w(t)| \le 1$ for all $t \ge 0$.

From $\operatorname{sign}(w(a)) = \operatorname{sign}(c)$ we also conclude that $\operatorname{sign}(w(b)) = -\operatorname{sign}(c)$.

On the other hand, direct integration of the equation (7.51) yields,

$$t^{N-1} w(t) = \operatorname{sign}(c) \left\{ a^{N-1} - \frac{1}{N} (t^N - a^N) \right\},$$

which implies that $\operatorname{sign}(c) w$ is a decreasing function, while setting $t = b$ we obtain,

$$-b^{N-1} = a^{N-1} - \frac{1}{N} (b^N - a^N),$$

what shows (7.50).

The final assertion on $b - a$ is a consequence of being w Lipschitz–continuous with $\|w'\|_\infty \le 2$. In fact,

$$2 = |w(b) - w(a)| \le 2(b - a).$$

5) *Function v cannot jump to 0.*

Assume, to get a contradiction, that there exists $\theta > 0$ satisfying
$$\lim_{t\to\theta^-} v(t) \neq 0 \quad \text{and} \quad \lim_{t\to\theta^+} v(t) = 0\,.$$
Then measures $\frac{dv}{dt}\llcorner\{\theta\}$ and $\frac{d|v|}{dt}\llcorner\{\theta\}$ are nontrivial (for a measurable space (X,μ) and a measurable set $A\subset X$, $\mu\llcorner A$ stands for the restriction of the measure μ to A). In fact, both measures are multiple of $\delta_\theta = \delta(\cdot - \theta)$. Whether $\lim_{t\to\theta^-} v(t)$ is positive or negative, we get from (7.43) that
$$\frac{dv}{dt}\llcorner\{\theta\} = \frac{N-1}{\theta}\frac{dv}{dt}\llcorner\{\theta\}$$
and so $\theta = N-1$, which contradicts $v(t)=1$ for all $t\in(0,N)$. We point out that this step does not exclude the possibility that v reaches the value zero through an infinite staircase with steps shrinking to zero. In addition, the option
$$\lim_{t\to\theta^-} v(t) = 0 \quad \text{and} \quad \lim_{t\to\theta^+} v(t) \neq 0$$
is more obviously discarded.

6) *If $|w(t)|=1$ on an open interval (a,b), then $v(t)=0$ for all $t\in(a,b)$.*

Indeed, we have $w'(t)=0$ for all $t\in(a,b)$, whence we deduce
$$-\frac{N-1}{t}\operatorname{sign}(w(t)) \in \operatorname{sign}(v(t))\,, \quad \text{for all } t\in(a,b)\,.$$
Since $t\ge N > N-1$, we get $v(t)=0$ for all $t\in(a,b)$.

7) *Let (a,b) be a component of $\{|w(t)|<1\}$ where $v(t)=c$, $c\neq 0$. Then there exists a further component (a',b') in $\{|w(t)|<1\}$ such that $a'=b$, $v(t)=c'$, constants c and c' satisfy $cc'<0$, while w exhibits opposite monotone characters in (a,b) and (a',b').*

Since $c\neq 0$ then step 5) implies that $c' := \lim_{t\to b+} v(t)\neq 0$ and so $\operatorname{sign}(v(t)) = \operatorname{sign}(c')$ in an interval $(b,b+\delta)$, $\delta>0$ small. If $w(b)=1$, then $w\equiv 1$ in $(b,b+\delta)$ is ruled out by 6). Furthermore, b can not accumulate points of $\{w(t)=1\}$ from the right. Indeed, this entails the existence of a sequence of components (a_n,b_n) of $\{|w(t)|<1\}$ with $a_n,b_n\to b$ and having $v\neq 0$. But then 4) says that $b_n - a_n\ge 1$ which is impossible. Similar arguments hold if $w(b)=-1$. Thus $|w(t)|<1$ in $(b,b+\delta)$ and we conclude the existence of a unique component (a',b') in $\{|w(t)|<1\}$ with $a'=b$. Since $\operatorname{sign}(c') = w(b) = -\operatorname{sign}(c)$ then $cc'<0$.

8) *Proof of ii), iii) and v).*

Starting with the component $(a,b)=(0,N)$ with $\alpha_0 := 1$ (step 2) we obtain by induction and step 7) a sequence of further components of $\{|w(t)|<1\}$ having the form (θ_n,θ_{n+1}), $\theta_{n+1}-\theta_n\ge 1$ (step 4) wherein $v(t)=\alpha_n$ with $\alpha_n\alpha_{n+1}<0$. This, in combination with 4) completes the proof of ii), iii) and v).

9) *Proof of iv) and vi).*

The recursion identity (7.48) follows from (7.50) in step 4). As for (7.49) observe that,
$$v'(t) = \sum_{n=1}^{\infty}(\alpha_n - \alpha_{n-1})\delta(t-\theta_n),$$

in $\mathcal{D}(0,\infty)'$ with δ the Dirac measure at $t=0$. By employing the restrictions $|v|'\llcorner\{\theta_n\}$, $|v'|\llcorner\{\theta_n\}$ together with equation (7.43) we obtain,

$$|\alpha_n| - |\alpha_{n-1}| = -\frac{N-1}{\theta_n}|\alpha_n - \alpha_{n-1}| = -\frac{N-1}{\theta_n}(|\alpha_n| + |\alpha_{n-1}|),$$

due to the condition $\alpha_{n-1}\alpha_n < 0$. This both show (7.49) and iv).

Proof of vii).

Set $a_n = \frac{\theta_{n+1}}{\theta_n}$. Then $a_n > 1$ and it follows from (7.48) that satisfies the identity

$$a_n^N\left(1 - \frac{N}{\theta_n a_n}\right) = 1 + \frac{N}{\theta_n}\,. \tag{7.53}$$

We claim that $\lim_{n\to\infty} a_n = 1$. In fact, a_n is bounded while any possible limit point must be 1. Going back to (7.48), it yields $\theta_{n+1}^N - \theta_n^N = N(\theta_{n+1}^{N-1} + \theta_n^{N-1})$ and so

$$\theta_{n+1} - \theta_n = \frac{N(\theta_{n+1}^{N-1} + \theta_n^{N-1})}{\displaystyle\sum_{k=0}^{N-1}\theta_{n+1}^{N-1-k}\theta_n^k} = \frac{N(a_n^{N-1}+1)}{\displaystyle\sum_{k=0}^{N-1} a_n^{N-1-k}}\,.$$

Letting n go to ∞, we obtain $\lim_{n\to\infty}(\theta_{n+1} - \theta_n) = 2$.

We finally prove that $\lim_{n\to\infty}|\alpha_n| = 0$. Only the case $N \geq 3$ will be considered as $N = 2$ will be explicitly seen in Example 20 below. Since the sequence $\{|\alpha_n|\}$ is decreasing, there exists $\xi = \lim_{n\to\infty}|\alpha_n| \geq 0$. Assuming $\xi > 0$, we write (7.49) as

$$\theta_n|\alpha_n| - \theta_{n-1}|\alpha_{n-1}| = \left[(\theta_n - \theta_{n-1}) - (N-1)\right]|\alpha_{n-1}| - (N-1)|\alpha_n|\,.$$

The right hand side converges to a negative number. Hence, the sequence $\{\theta_n|\alpha_n|\}$ is eventually decreasing and so it has a finite limit. This fact contradicts $\lim_{n\to\infty}\theta_n = +\infty$. Therefore, $\xi = 0$ as desired. □

Remark 17. Notice that as a consequence of Remark 16 relation (7.48) uniquely determines θ_{n+1} as a function of θ_n. Then identity (7.53) gives uniquely α_{n+1} once α_n is known.

Example 20. In the 2D case, the sequences $\{\theta_n\}$ and $\{\alpha_n\}$ can explicitly be computed. In fact, identity (7.48) becomes

$$\frac{\theta_{n+1}^2}{2} - \theta_{n+1} - \theta_n\left(1 + \frac{\theta_n}{2}\right) = 0\,,$$

and then we get $\theta_n = 2n$ for all $n \geq 0$. As far as the sequence $\{\alpha_n\}$ is concerned, we have the recursive formula

$$|\alpha_n| = \frac{2n-1}{2n+1}|\alpha_{n-1}|$$

from where we deduce

$$\alpha_n = \frac{(-1)^n}{2n+1}\,.$$

Remark 18. From the uniqueness for system (7.42) under assumption (7.43), we conclude that the whole family v_p converges to v as $p \to 1$. We also observe that as a consequence of Theorem 19, v_p can not converge to zero in a whole open interval $J \in \mathbb{R}$.

Next statement gives a complete answer to the structure of the limit profile of v_p as $p \to 1$.

Corollary 21. *Let $v_p(t)$ the solution to* (6.30)*. Then, the limit*

$$v(t) = \lim_{p\to 1} v_p(t),$$

holds a. e. in $t > 0$ where v is the first component of the solution to (7.42)–(7.43)*. In addition,*

$$\theta_{(1),n} = \theta_n$$

for all n, θ_n being the values introduced in (7.48)*.*

We come back now to the limits $\lambda_{(1),n}$ of the radial eigenvalues $\lambda_{(p),n}$ of $-\Delta_p$ (Theorem 13). Setting

$$\lambda_n := \lambda_{(1),n},$$

main result in [32] permits us asserting that $\{\lambda_n\}$ defines the full family of radial LS eigenvalues of $-\Delta_1$. Our final result summarizes the main facts obtained in this section.

Theorem 22. *Let (v, w) be the solution to system* (7.42) *satisfying* (7.43)*. The radial eigenvalues of problem* (4.13) *in the ball $B(0, R)$ consist in the sequence of values,*

$$\lambda_n = \frac{\theta_n}{R},$$

with

$$u_n(x) = v\left(\frac{\theta_n}{R} r\right), \qquad r = |x|,$$

as the associated strong eigenfunction normalized so as $\sup u_n = 1$. Moreover, the vector field $\mathbf{z}$ and function γ required by Definition 4 are

$$\mathbf{z}(x) = \frac{x}{r} w\left(\frac{\theta_n}{R} r\right)$$

and

$$\gamma(x) = -w'\left(\frac{\theta_n}{R} r\right) - \frac{N-1}{(\frac{\theta_n}{R} r)} w\left(\frac{\theta_n}{R} r\right).$$

Moreover, the following additional features hold.

i) *The sequence of eigenvalues satisfies the following asymptotic relation,*

$$\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = \frac{2}{R}.$$

ii) *Eigenfunction u_n associated to λ_n changes its sign exactly at the values*

$$r_k = \frac{\theta_k}{\theta_n} \in (0, R) \qquad k = 1, \dots, n-1.$$

iii) *$u_n(t) = \alpha_n$ for $t \in (\theta_n, \theta_{n+1})$ where $\{\alpha_n\}$ satisfies the relation* (7.49)*.*

Proof. That u_n defines a weak eigenfunction associated to λ_n follows from Theorem 6. Nevertheless, we are checking that Definition 4 is satisfied with the choices of $\mathbf{z}$ and γ given in the statement.

(1) It is straightforward from system (7.42) that

$$|\mathbf{z}(x)| = \left| w\left(\frac{\theta_n}{R} r\right)\right| \le 1$$

owed to $w \in \operatorname{sign}(v')$. Since $\gamma \in \operatorname{sign}(v)$, a similar inequality follows.

(2) It is a consequence of the following computation

$$\begin{aligned} -\operatorname{div}\mathbf{z}(x) &= -\frac{N-1}{r}w\left(\frac{\theta_n}{R}r\right) - \frac{\theta_n}{R}w'\left(\frac{\theta_n}{R}r\right) \\ &= \lambda_n\left[-\frac{N-1}{\left(\frac{\theta_n}{R}r\right)}w\left(\frac{\theta_n}{R}r\right) - w'\left(\frac{\theta_n}{R}r\right)\right] = \lambda_n\gamma(x)\,. \end{aligned}$$

(3) Observe that $Du_n(x) = \frac{\theta_n}{R}\frac{x}{r}v'\left(\frac{\theta_n}{R}r\right)$ at points $r_k = \frac{\theta_k}{\theta_n}$ (for $k = 1, \dots, n-1$) and $Du_n(x) = 0$ at any other point. Then, at $|x| = r_k$, we get

$$\begin{aligned} (\mathbf{z}(x), Du_n(x)) &= \frac{\theta_n}{R}w\left(\frac{\theta_n}{R}r_k\right)v'\left(\frac{\theta_n}{R}r_k\right)\frac{x}{r_k}\cdot\frac{x}{r_k} \\ &= \lambda_n w\left(\frac{\theta_k}{R}\right)v'\left(\frac{\theta_k}{R}\right) = \lambda_n\left|v'\left(\frac{\theta_k}{R}\right)\right| = |Du(x)|\,. \end{aligned}$$

(4) It is inferred by the property that $w(\theta_n)$ and α_{n-1} have different signs. Indeed, since α_{n-1} is the trace value of u_n on the boundary, it follows that

$$\mathbf{z}(x)\cdot\frac{x}{R} = w(\theta_n) \in \operatorname{sign}(-\alpha_{n-1}) = \operatorname{sign}(-u_n(x))$$

holds true on $\partial\Omega$. □

Remark 19. A Weyl type asymptotic estimate of the eigenvalues of $-\Delta_p$ in a bounded domain Ω is proved in [22]. Namely,

$$\frac{c}{|\Omega|} \le \frac{\lambda_n^{N/p}}{n} \le \frac{C}{|\Omega|}$$

as $n \to \infty$, being c, C positive constants depending only on N and p. One may wonder if this estimate can be extended to the radial eigenvalues of $-\Delta_1$. To show that the answer is negative, observe that

$$\lim_{n\to\infty}\frac{\theta_{n+1}-\theta_n}{(n+1)^{1/N}-n^{1/N}} = +\infty,$$

since $\theta_{n+1}-\theta_n \to 2$ and $(n+1)^{1/N}-n^{1/N} \to 0$. Stolz's theorem then yields $\frac{\theta_n}{n^{1/N}} \to 0$ and so

$$\lim_{n\to\infty}\frac{\lambda_n^N}{n} = +\infty.$$

This limit is straightforward when $N = 2$, since an explicit formula is available.

Nevertheless, as a consequence of

$$\lim_{n\to\infty}\frac{\theta_{n+1}/(n+1)^{1/N}}{\theta_n/n^{1/N}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/N}a_n = 1\,.$$

we achieve the following asymptotic estimate

$$\lim_{n\to\infty}\frac{\lambda_{n+1}^N/(n+1)}{\lambda_n^N/n} = 1\,.$$

Acknowledgements

The first author is partially supported by Spanish Ministerio de Ciencia e Innovación and Ministerio de Economía y Competitividad under grant reference MTM2014–52822–P, while the second one is partially supported by MINECO-FEDER under grant PGC2018–094775–B–I00.

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