1D LOGISTIC REACTION AND p–LAPLACIAN DIFFUSION AS p GOES TO ONE

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ABSTRACT. This work discusses the existence of the limit as p goes to 1 of the nontrivial solutions to the one–dimensional problem:

$$\begin{cases} -\left(|u_x|^{p-2}u_x\right)_x = \lambda |u|^{p-2}u - |u|^{q-2}u & 0 < x < 1 \\ u(0) = u(1) = 0, \end{cases}$$

where λ is a positive parameter and the exponents p, q satisfy $1 < p < q$. We prove that solutions do converge to a limit function, which solves in a proper sense a Dirichlet problem involving the 1–Laplacian operator.

1. INTRODUCTION

The logistic equation is a standard in nonlinear analysis, population dynamics and reacting–diffusing systems, among other fields ([5], [22], [13]). According to orthodoxy, the asymptotic density distribution u of a migrating species with intrinsic growth rate $\lambda > 0$, living in a habitat $\Omega \subset \mathbb{R}^N$ (a bounded domain) which is surrounded by a completely hostile medium, is governed by the problem:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u - |u|^{q-2}u & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{1.1}$$

The p–Laplacian operator $\Delta_p u = \text{div}\ (|\nabla u|^{p-2}u)$ acts as the diffusive mechanism describing the migration of u throughout Ω. On the other hand, the power q term in the equation accounts for the population crowding effects. This means that the species is in competition against itself for the available resources. The exponents p, q are assumed to satisfy,

$$1 < p < q. \tag{1.2}$$

Let us review some few traits of (1.1). Existence of nontrivial solutions is only possible when $\lambda > \lambda_1(-\Delta_p)$, the first eigenvalue of $-\Delta_p$, while the best understood issues has to do with positive solutions. In fact, there exists a unique positive solution u_λ, bifurcating from zero at $\lambda = \lambda_1$, whose

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asymptotic profile as $\lambda \to \infty$ has been studied in full detail (see [20], [17], [18], [14] for references dealing with the 'genuine' nonlinear diffusion case $p \neq 2$). As a characteristic feature,

$$\|u_\lambda\|_\infty \leq \lambda^{\frac{1}{q-p}} \qquad \text{and} \qquad \lambda^{-\frac{1}{q-p}} u_\lambda \to 1 \qquad \text{as} \qquad \lambda \to \infty,$$

the last convergence being uniform in compact sets of Ω. Moreover, while first estimate is strict in the case $1 < p \leq 2$, the complementary range $p > 2$ enjoys especial phenomena. In fact, the region $\{u_\lambda(x) = \lambda^{\frac{1}{q-p}}\}$ becomes nonempty and converges to Ω as $\lambda \to \infty$ ([20], [17] and Remark 2 below). On the other hand, by means of variational arguments it can be shown the existence of an arbitrarily large number of further nontrivial (two–signed) solutions to (1.1) when $\lambda \to \infty$ (see [15] for this kind of results in a closely related problem).

In the present work, we are only concerned with the one–dimensional case:

$$\begin{cases} -(|u_x|^{p-2}u_x)_x = \lambda|u|^{p-2}u - |u|^{q-2}u & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{1.3}$$

The emphasis is firstly focussed in studying the existence of the limits $\bar{u}$ of its nontrivial solutions u as $p \to 1$. Secondly, in analyzing the rôle of these limits $\bar{u}$ as solutions to the formal limit problem,

$$\begin{cases} -\left(\dfrac{u_x}{|u_x|}\right)_x = \lambda\dfrac{u}{|u|} - |u|^{q-2}u & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{1.4}$$

According to results going back to [19] (see also [9], [12]), the structure of the nontrivial solutions set to (1.3) is essentially dictated by the eigenvalue problem,

$$\begin{cases} -(|u_x|^{p-2}u_x)_x = \lambda|u|^{p-2}u & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{1.5}$$

More precisely, nontrivial solutions u are organized in symmetric curves (invariant with respect to $u \to -u$). Each of these curves is associated to a fixed eigenvalue λ_n to (1.5). The n–th curve can be regarded as a deformation of the n–th eigenspace which bifurcates from $u = 0$ at $\lambda = \lambda_n$. In this regard, the nonlinear diffusion case reproduces the patterns already observed in the linear diffusion case where $p = 2$ and q satisfies (1.2) (see [7], [24] for pioneering results on the subject).

Our main results here state that a similar picture occurs when we deal with the nontrivial solutions to (1.4). Such solutions are required to satisfy a sort of energy condition providing us a uniqueness criterium. In addition, they are characterized as the limit of solutions to (1.3) as $p \to 1$. We are able to show that solutions are also organized in explicitly computed curves. As in the case of problem (1.3) these curves emanate by bifurcation from zero, at the eigenvalues $\underline{\lambda}_n = 2n$ to one–dimensional 1–Laplacian:

$$\begin{cases} -\left(\dfrac{u_x}{|u_x|}\right)_x = \lambda \dfrac{u}{|u|} & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{1.6}$$

It should be mentioned that a detailed discussion on the nature and distribution of the eigenvalues to (1.6) was addressed in [6] and such results has been recently extended in several directions (see [8], [26] and references therein).

This work is distributed as follows. Section 2 presents a selfcontained analysis of problem (1.3). Proofs included there have been specially adapted to the purposes of this paper. Limits as $p \to 1$ of the solutions to problems (1.3) and (1.5) are studied in Section 3 (Theorem 6). The concept of solution to (1.4) is introduced in Section 4. It belongs to the general theory developed in [2, 3] (see also [10]). The main features concerning the nontrivial solutions to (1.4) are stated in Theorem 8.

2. Preliminary facts

In this section we are concerned with the problem (1.3) where it will be always assumed that exponents p,q satisfy (1.2). As we are interested in letting p go to 1, only the regime $1 < p \le 2$ should be analyzed in detail. However, as already mentioned, the complementary range $p > 2$ enjoys especial phenomena. They are just reviewed at the end of the section (Remark 2).

For a weak solution $u \in W_0^{1,p}(0,1)$ to (1.3) it is understood that relation

$$\int_0^1 |u_x|^{p-2} u_x v_x = \lambda \int_0^1 u^{p-2} uv - \int_0^1 u^{q-2} uv \tag{2.1}$$

holds for every $v \in W_0^{1,p}(0,1)$. Due to the fact that $W_0^{1,p}(0,1) \subset L^\infty(0,1)$ it can be shown that weak solutions become genuine C^2 solutions provided $1 < p \le 2$ ([18]). Thus, we are plainly referring to 'solutions' to (1.3) in the sequel.

For later use the next well–known result is stated. It summarizes the main features on the Dirichlet eigenvalues of the one–dimensinal p–Laplacian. See for instance [11], [23], [19], [9] for background material on the subject.

Theorem 1. *The eigenvalue problem* (1.5) *satisfies the following properties.*

i) *The full set of eigenvalues of* (1.5) *consists in the sequence* $\{\lambda_n\}$*:*

$$\lambda_n = (nt_1(p))^p, \qquad t_1(p) = \frac{2(p-1)^{\frac{1}{p}}}{p}\frac{\pi}{\sin\frac{\pi}{p}}, \qquad n = 1, 2, \dots \tag{2.2}$$

ii) *Every eigenvalue* λ_n *is simple, i. e. eigenfunctions associated to* λ_n *are a scalar multiple of a normalized eigenfunction* $u_n(x)$.

iii) u_n *vanishes exactly at the points* $x_k = \frac{k}{n}$, $k = 0, \dots, n$.

A corresponding "perturbed" version of the preceding result is the next one, of bifurcation–type nature. As pointed out in Section 1, there is a clear difference in the response of problem (1.3) depending on whether $p > 2$ or $1 < p \le 2$. Since we want to let $p \to 1+$, the latter case is the one that most concerns us in this work.

Some of the forthcoming assertions are essentially well–known (see [19]). Nevertheless, an independent self–contained proof is enclosed for our subsequent arguments.

Theorem 2. *Let* $0 < \lambda_1 < \lambda_2 < \cdots$ *be the sequence of eigenvalues to* (1.5). *Then, problem* (1.3) *in the regime* $1 < p \le 2$, *satisfies the following properties.*

i) *Nontrivial solutions are only possible if* $\lambda > \lambda_1$. *Moreover, all solutions to* (1.3) *verify the estimate*

$$\|u\|_\infty < \lambda^{\frac{1}{q-p}}. \tag{2.3}$$

ii) *For every* $\lambda > \lambda_1$ *there exists a* unique *positive solution* $u_\lambda^{(1)}$ *satisfying*

$$\|u_\lambda^{(1)}\|_\infty \to 0 \text{ as } \lambda \to \lambda_1+ \quad \& \quad \lambda^{-\frac{1}{q-p}}\|u_\lambda^{(1)}\|_\infty \to 1 \text{ as } \lambda \to \infty. \tag{2.4}$$

iii) *For every* $\lambda > \lambda_n$, $n \ge 2$, *aside of* $\pm u_\lambda^{(1)}$ *there exist* $n-1$ *pairs* $\pm u_\lambda^{(k)}$, $2 \le k \le n$, *of nontrivial solutions to* (1.3) *where* $u_\lambda^{(k)}$ *is normalized so as* $(u_\lambda^{(k)})_x(0) > 0$. *In addition, for all* $2 \le k \le n$,

$$\|u_\lambda^{(k)}\|_\infty \to 0 \text{ as } \lambda \to \lambda_k+ \quad \& \quad \lambda^{-\frac{1}{q-p}}\|u_\lambda^{(k)}\|_\infty \to 1 \text{ as } \lambda \to \infty. \tag{2.5}$$

Moreover, for $\lambda_n < \lambda \le \lambda_{n+1}$ *the* unique *nontrivial solutions to* (1.3) *are exactly* $\{\pm u_\lambda^{(k)}\}_{1\le k\le n}$.

iv) *For every* k *and* $\lambda > \lambda_k$, *solution* $u_\lambda^{(k)}$ *in the k–th branch vanishes exactly at* $x = \dfrac{l}{k}$, $1 \le l \le k$.

Proof. Let us introduce the scaling

$$u(x) = \lambda^{\frac{1}{q-p}} v(t) \qquad t = \lambda^{\frac{1}{p}} x.$$

Then problem (1.3) is transformed into the equivalent one,

$$\begin{cases} -(|v_t|^{p-2}v_t)_t = |v|^{p-2}v - |v|^{q-2}v, & 0 < t < \lambda^{\frac{1}{p}}, \\ v(0) = v(\lambda^{\frac{1}{p}}) = 0. \end{cases} \tag{2.6}$$

To analyze (2.6) we first discuss the initial value problem,

$$\begin{cases} -(|v_t|^{p-2}v_t)_t = |v|^{p-2}v - |v|^{q-2}v, & t > 0, \\ v(0) = \alpha, \quad v_t(0) = 0. \end{cases} \tag{2.7}$$

The existence and uniqueness of a maximal solution for this and a slightly larger class of problems have been considered in the literature (see [18], [25]). However, we can proceed here in a direct way. In fact, the function $E(v,v_t)$ defined by

$$E(v,v_t) = \frac{1}{p'}|v_t|^p + V(v), \qquad V(v) = \frac{1}{p}|v|^p - \frac{1}{q}|v|^q, \tag{2.8}$$

is conserved through the solutions to (2.7). To ascertain the response of problem (2.7) it is enough to assume that $\alpha \geq 0$ since the equation is invariant with respect to the change $v \to -v$. According to the values of $\alpha \geq 0$ and employing the fact that

$$E(v,v_t) = V(\alpha), \tag{2.9}$$

three cases are possible.

a) $\alpha = 1$ which implies $v = 1$. In this regard, the restriction $1 < p \leq 2$ is crucial (see Remark 2 below).

b) $\alpha > 1$. A unique solution v exists, it is increasing, satisfies $V(v) < V(\alpha)$ and blows–up at $t = \omega(\alpha)$,

$$\omega(\alpha) := \{p'\}^{-\frac{1}{p}} \int_\alpha^\infty \frac{ds}{(V(\alpha) - V(s))^{\frac{1}{p}}} < \infty.$$

c) $0 < \alpha < 1$. Again, a unique solution v exists which decreases from α to $-\alpha$ when $0 \leq t \leq T$, vanishes at $t = \dfrac{T}{2}$, is symmetric with respect to $t = T$ and becomes periodic with period $2T$ where

$$T = T(\alpha) = 2\{p'\}^{-\frac{1}{p}} \int_0^\alpha \frac{ds}{(V(\alpha) - V(s))^{\frac{1}{p}}}. \tag{2.10}$$

Coming back to (2.6), let v be any of its nontrivial solutions. It can be assumed without loss of generality that it verifies $v_t(0) > 0$. Such solution must *necessarily* exhibit a first maximum at $t = t_m > 0$ with value $v(t_m) = \alpha$. Since $\tilde{v} = v(t - t_m)$ solves (2.7) then α must satisfy:

$$0 < \alpha < 1.$$

This assertion means $|v(t)| \leq \alpha < 1$ and so, changing the scale back, $|u(x)| < \lambda^{\frac{1}{q-p}}$, which proves estimate (2.3). Moreover, there must exist $n \in \mathbb{N}$ such that

$$\lambda^{\frac{1}{p}} = nT(\alpha). \tag{2.11}$$

We now claim that $T(\alpha)$ is increasing in $(0,1)$, $\lim_{\alpha\to 0} T = t_1(p)$ where $t_1(p)$ is the value introduced in (2.2) while $\lim_{\alpha\to 1-} T = \infty$. Latter assertion is delayed to Lemma 3 below. To show the increasing character of T, we rather substitute the group $\varphi_p(u) - \varphi_q(u)$ by $\varphi_p(u)g(u)$ where g is a decreasing function in $u > 0$. Notation used means $\varphi_r(u) = |u|^{r-2}u$ $(r > 1)$. Then,

$$T(\alpha) = 2\{p'\}^{-\frac{1}{p}} \int_0^1 \frac{ds}{\left(\int_s^1 \varphi_p(\tau)g(\alpha\tau)\, d\tau\right)^{\frac{1}{p}}},$$

whence the increasing variation becomes evident, so that $T(0) < T(\alpha)$. In our precise example,

$$T(\alpha) = 2\{p'\}^{-\frac{1}{p}} \int_0^1 \frac{ds}{\left(\int_s^1 \tau^{p-1}(1-(\alpha\tau)^{q-p})\, d\tau\right)^{\frac{1}{p}}}.$$

Setting $\alpha = 0$:

$$T(0) = 2(p-1)^{\frac{1}{p}} \int_0^1 \frac{ds}{(1-s^p)^{\frac{1}{p}}} = t_1(p).$$

Appealing to (2.11) and Theorem 1, we deduce

$$\lambda^{1/p} \geq T(\alpha) > T(0) = t_1(p) = \lambda_1^{1/p}.$$

Let us denote by $v(t,\alpha)$ the solution to the equation in (2.6) satisfying $v_t(0) > 0$ and $\|v\|_\infty = \alpha$ with $0 < \alpha < 1$. It has been shown that if u solves (1.3) then necessarily $\lambda > \lambda_1$. In addition, u can be expressed in the form,

$$u(x) = \lambda^{\frac{1}{q-p}} v(\lambda^{\frac{1}{p}} x, \alpha)$$

where $\alpha = \lambda^{-\frac{1}{q-p}} \|u\|_\infty$, λ and α being coupled by equation (2.11). Notice that it follows from this fact that

$$\lambda > (nT(0))^p = \lambda_n,$$

λ_n denoting the n–th eigenvalue of (1.5). On the other hand,

$$u\left(\frac{k}{n}\right) = \lambda^{\frac{1}{q-p}} v(kT(\alpha), \alpha) = 0 \qquad 1 \leq k \leq n.$$

In conclusion, assertions of the theorem hold by defining

$$u_\lambda^{(n)}(x) = \lambda^{\frac{1}{q-p}} v(\lambda^{\frac{1}{p}} x, \alpha), \qquad \lambda^{\frac{1}{p}} = nT(\alpha).$$

Relations (2.4), (2.5) follow from the limit behavior of T at $\alpha = 0$ and $\alpha = 1$. □

Remark 1. Solutions $\pm u_\lambda^{(n)}$ arise by bifurcation from zero at λ_n as λ increases.

Lemma 3. *The behavior of $T(\alpha)$ as $\alpha \to 1$ is dictated by:*

$$\lim_{\alpha\to 1-} T(\alpha) = \begin{cases} \infty & 1 < p \le 2, \\ 2\{p'\}^{-\frac{1}{p}} \displaystyle\int_0^1 \frac{ds}{(V(1)-V(s))^{\frac{1}{p}}} & p > 2. \end{cases}$$

Proof. By choosing $\varepsilon > 0$ small enough, $C_\pm := q - p \pm \varepsilon > 0$, then we find

$$C_-(1-v) \le v^{p-1} - v^{q-1} \le C_+(1-v) \qquad 1-\delta \le v \le 1,$$

for certain $0 < \delta < 1$. Function T can be written as

$$T = 2\{p'\}^{-\frac{1}{p}} \left\{ \int_0^{1-\delta} + \int_{1-\delta}^{\alpha} \right\} \frac{ds}{(V(\alpha)-V(s))^{\frac{1}{p}}}.$$

In addition,

$$\left(\frac{2}{C_+}\right)^{\frac{1}{p}} J \le \int_{1-\delta}^{\alpha} \frac{ds}{(V(\alpha)-V(s))^{\frac{1}{p}}} \le \left(\frac{2}{C_-}\right)^{\frac{1}{p}} J,$$

where

$$J = \int_{1-\delta}^{\alpha} \frac{ds}{\left(\int_s^\alpha 2(1-v)\,dv\right)^{\frac{1}{p}}} = (1-\alpha)^{1-\frac{2}{p}} \int_1^{\frac{\delta}{1-\alpha}} \frac{dt}{(t^2-1)^{\frac{1}{p}}},$$

after performing the change $s = 1-(1-\alpha)t$. It can be checked that $J \to \infty$ as $\alpha \to 1-$ if $1 < p \le 2$ while

$$\lim_{\alpha\to 1-} J = \frac{\delta^{1-\frac{2}{p}}}{1-\frac{2}{p}},$$

if $p > 2$. Therefore when $p > 2$ we obtain

$$\overline{\lim_{\alpha\to 1-}}\, T \le 2\{p'\}^{-\frac{1}{p}} \int_0^{1-\delta} \frac{ds}{(V(1)-V(s))^{\frac{1}{p}}} + M\delta^{1-\frac{2}{p}},$$

M being a constant. By letting $\delta \to 0$ we conclude

$$\overline{\lim_{\alpha\to 1-}}\, T \le 2\{p'\}^{-\frac{1}{p}} \int_0^1 \frac{ds}{(V(1)-V(s))^{\frac{1}{p}}}.$$

A symmetric reasoning yields the complementary estimate. □

Remark 2. In the case $p>2$ the convergence of the integral $T(1)$ referred to in Lemma 3 introduces strong differences regarding the regime $1<p\leq 2$. The main point is that the initial value problem (2.7) exhibits, for $\alpha\in\{-1,1\}$, infinitely many solutions in the strip $-1\leq v\leq 1$. Just for completeness, a result describing the nature of the solutions to (1.3) is presented below. With a slightly different statement it is contained in [19]. We point out that an independent proof can be obtained with the same reasoning as in Theorem 2, complemented with the ideas in [16]. Nevertheless, precise details are omitted.

Theorem 4. *Assume that $p>2$. Then problem* (1.3) *exhibits the following features.*

i) *Nontrivial solutions u only exist if $\lambda>\lambda_1$ while all of them fulfill $\|u\|_\infty\leq\lambda^{\frac{1}{q-p}}$. Moreover, a unique positive solution $u_\lambda^{(1)}$ exists for all $\lambda>\lambda_1$ satisfying,*

$$\|u_\lambda^{(1)}\|_\infty\to 0 \qquad \text{as } \lambda\to\lambda_1,$$

while $u_\lambda^{(1)}=\lambda^{\frac{1}{q-p}}$ in the whole interval $\left[\frac{T(1)}{2}\lambda^{-\frac{1}{p}},1-\frac{T(1)}{2}\lambda^{-\frac{1}{p}}\right]$ if $\lambda>T(1)^p$.

ii) *For $\lambda>\lambda_n$, $n\geq 2$, two symmetric and 'multivalued' families $\pm u_\lambda^{(n)}$ of solutions bifurcate from zero at $\lambda=\lambda_n$. More precisely,*

a) *If $\lambda_n<\lambda\leq(nT(1))^p$, the family reduces to a single solution $u=u_\lambda^{(n)}$ which satisfies $(u_\lambda^{(n)})_x(0)>0$, vanishes at $x=\frac{k}{n}$, $1\leq k\leq n-1$ and*

$$\|u_\lambda^{(n)}\|_\infty\to 0 \quad \text{as} \quad \lambda\to\lambda_n, \quad \lambda^{-\frac{1}{q-p}}\|u_\lambda^{(n)}\|_\infty\to 1 \quad \text{as} \quad \lambda\to(nT(1))^p.$$

b) *For every $\lambda>(nT(1))^p$ and every family $I_1,\dots,I_n$ of disjoint closed subintervals of $(0,1)$ (some of them possibly reduced to a single point) such that*

$$\operatorname{dist}(\{0\},I_1)=\operatorname{dist}(\{1\},I_n)=\frac{T(1)}{2}\lambda^{-\frac{1}{p}},$$

$$\operatorname{dist}(I_{k-1},I_k)=T(1)\lambda^{-\frac{1}{p}}, \qquad \text{for } 2\leq k\leq n,$$

and

$$|I_1|+\cdots+|I_n|=1-nT(1)\lambda^{-\frac{1}{p}},$$

there exists a unique solution u in the family $u_\lambda^{(n)}$ such that $u_x(0)>0$ while for every $k=1,\dots,n$, u achieves the value $(-1)^{k-1}\lambda^{\frac{1}{q-p}}$ in the whole interval I_k. Finally, u vanishes exactly at $n-1$ points of $(0,1)$ the $k-1$–th of them, ξ_{k-1}, located midway between I_{k-1} and I_k.

3. LIMIT PROFILES AS $p \to 1$

It is assumed henceforth that $1 < p \le 2$ and we are going to study the limit as $p \to 1$ of the solutions to (1.3) described in Theorem 2. A first auxiliary result is the following.

Remark 3. *For* $1 < p < 2$ *and* $0 < \alpha < 1$ *let* $T(\alpha)$ *be the integral defined in* (2.10)*. Then,*

$$\lim_{p \to 1+} T(\alpha) = \frac{2}{1 - \alpha^{q-1}}.$$

Proof. After scaling the integral $T(\alpha)$ can be written as

$$T(\alpha) = 2(p-1)^{\frac{1}{p}} \int_0^1 \frac{ds}{h(s)^{\frac{1}{p}} (1 - s^p)^{\frac{1}{p}}},$$

where

$$h(s) = 1 - \alpha^{q-p} \frac{p}{q} \frac{1 - s^q}{1 - s^p}, \qquad 0 \le s < 1.$$

A more suitably expression for h is $h(s) = 1 - \alpha^{q-p} g(u)$, where $u = s^{\frac{1}{p}}$ and

$$g(u) = \frac{p}{q} \frac{1 - u^{\frac{q}{p}}}{1 - u} \qquad 0 \le u < 1.$$

Function g is increasing, $g(0) = \frac{p}{q}$ and $\lim_{u \to 1-} g(u) = 1$. Thus,

$$\frac{T(0)}{\left(1 - \frac{p}{q}\alpha^{q-p}\right)^{\frac{1}{p}}} \le T(\alpha) \le \frac{T(0)}{(1 - \alpha^{q-p})^{\frac{1}{p}}}.$$

Since $T(0) = t_1(p)$, t_1 being the value given in (2.2), it can be shown by direct computation that $\lim_{p \to 1} T(0) = 2$. Thus,

$$\frac{2}{1 - \frac{1}{q}\alpha^{q-1}} \le \varliminf_{p \to 1-} T(\alpha) \le \varlimsup_{p \to 1-} T(\alpha) \le \frac{2}{1 - \alpha^{q-1}}. \tag{3.1}$$

We are next refining the lower estimate in (3.1). Given $\varepsilon > 0$, take $p_\varepsilon > 1$ so that $q - \frac{\varepsilon}{2} < \frac{q}{p_\varepsilon} < q$ and then using

$$\lim_{u \to 1} \frac{1 - u^{\frac{q}{p_\varepsilon}}}{1 - u} = \frac{q}{p_\varepsilon},$$

get $0 < \eta < 1$ satisfying

$$q - \varepsilon < \frac{1 - u^{\frac{q}{p_\varepsilon}}}{1 - u}, \qquad 1 - \eta < u < 1.$$

Observing that for all $0 < u < 1$ the group $\dfrac{1-u^{\frac{q}{p}}}{1-u}$ increases in value as $p \to 1+$, we infer that

$$q-\varepsilon < \frac{1-u^{\frac{q}{p}}}{1-u} < \frac{q}{p} < q, \qquad 1-\eta < u < 1, \qquad 1 < p < p_{\varepsilon}.$$

Hence,

$$\begin{aligned}
\varliminf_{p\to 1+} T(\alpha) &= \varliminf_{p\to 1+} 2(p-1)^{\frac{1}{p}} \int_{(1-\eta)^{\frac{1}{p}}}^{1} \frac{ds}{h(s)^{\frac{1}{p}}(1-s^p)^{\frac{1}{p}}} \\
&\geq \lim_{p\to 1} \frac{2(p-1)^{\frac{1}{p}}}{\left\{1-\frac{p(q-\varepsilon)}{q}\alpha^{q-p}\right\}^{\frac{1}{p}}} \int_{(1-\eta)^{\frac{1}{p}}}^{1} \frac{ds}{(1-s^p)^{\frac{1}{p}}} \\
&= \lim_{p\to 1} \frac{2}{\left\{1-\frac{p(q-\varepsilon)}{q}\alpha^{q-p}\right\}^{\frac{1}{p}}} \int_0^1 \frac{(p-1)^{\frac{1}{p}}}{(1-s^p)^{\frac{1}{p}}}\,ds = \frac{2}{1-\frac{(q-\varepsilon)}{q}\alpha^{q-1}}.
\end{aligned}$$

By taking limits as $\varepsilon \to 0+$ we finally achieve that

$$\varliminf_{p\to 1+} T(\alpha) \geq \frac{2}{1-\alpha^{q-1}}.$$

□

Our next result reviews the limit behavior of the eigenpairs to problem (1.5) as $p \to 1$. Interested reader is referred to [6], [26] for details (see also [8] for further developments in a convective variant of (1.5)).

Theorem 5. *Let* (λ_n, u_n) *be the n–th eigenpair to* (1.5) *where* $\lambda_n = \lambda_n(p)$ *is given in* (2.2) *and let* $u_n = u_{n(p)}$ *be its associated eigenfunction normalized according to* $u_{nx}(0) > 0$ *and* $\sup u_n = 1$. *Then, the following properties are satisfied.*

i) *Limit values of* λ_n *are given by,*

$$\overline{\lambda}_n := \lim_{p\to 1} \lambda_n(p) = 2n, \qquad n \in \mathbb{N}. \tag{3.2}$$

ii) *Limit profiles of eigenfunctions are,*

$$\overline{u}_n := \lim_{p\to 1} u_n = \sum_{k=1}^{n} (-1)^{k-1}\chi_k, \tag{3.3}$$

where χ_k *is the characteristic function of* $I_k = \left(\frac{k-1}{n}, \frac{k}{n}\right)$ *and the sequences* u_n *and* u_{nx} *converge to* $\overline{u}_n$ *and* $\overline{u}_{nx} = 0$, *respectively, uniformly on compact sets of* $(0,1) \setminus \left\{\frac{1}{n}, \dots, \frac{n-1}{n}\right\}$.

Remark 4. A description on the status of $(\overline{\lambda}_n, \overline{u}_n)$ as the natural set of eigenpairs of the 1–Laplacian is contained in Section 4.

Main result of this section can already be stated.

Theorem 6. *Assume that* $1 < p < 2$ *and let* $u_\lambda^{(n)}$ *be the branch of solutions to* (1.3)*, normalized as* $(u_\lambda^{(n)})_x(0) > 0$*, that bifurcates from zero at* $\lambda = \lambda_n$*. Then, for all* $\lambda > \overline{\lambda}_n$

$$\overline{u}_\lambda^{(n)} := \lim_{p\to 1} u_\lambda^{(n)} = \sum_{k=1}^{n} (-1)^{k-1} (\lambda - 2n)^{\frac{1}{q-1}} \chi_k, \qquad \lim_{p\to 1} \frac{du_\lambda^{(n)}}{dx} = 0, \quad (3.4)$$

where both limits hold uniformly on compact sets of $(0,1) \setminus \{\frac{1}{n}, \dots, \frac{n-1}{n}\}$ *and* χ_k *is the characteristic function of the interval* $\frac{k-1}{n} \le x \le \frac{k}{n}$.

Proof of Theorem 6. Fix $\lambda > \overline{\lambda}_n$. Then $\lambda > \lambda_n = \lambda_n(p)$ for p close enough to 1 and the corresponding solution $u_\lambda^{(n)}$ can be expressed in the form,

$$u_\lambda^{(n)}(x) = \lambda^{\frac{1}{q-p}} v(t, \alpha), \qquad t = \lambda^{\frac{1}{p}} x,$$

where $v(\cdot, \alpha)$ is the solution to the equation in (2.6) such that $v_t(0) > 0$, $\|v\|_\infty = \alpha$ where $\lambda^{\frac{1}{p}} = nT(\alpha)$. Notice that $v(\cdot, \alpha)$ also depends on p, but an explicit reference to this parameter has been omitted for short. By setting,

$$\overline{T}(\alpha) = \frac{2}{1 - \overline{\alpha}^{q-1}},$$

and doing $p \to 1$ in $\lambda^{\frac{1}{p}} = nT(\alpha)$ we get the relation $\overline{\alpha} = (1 - \frac{2n}{\lambda})^{\frac{1}{q-1}}$ between λ and the amplitude $\overline{\alpha}$ of $\lim_{p\to 1} v(\cdot, \alpha)$.

On the other hand the autonomous character of (2.7) implies that for every $1 \le k \le n$,

$$v(t, \alpha) = (-1)^{k-1} v(t - (k-1)T(\alpha), \alpha), \quad (k-1)T(\alpha) \le t \le kT(\alpha).$$

Thus, the behavior as $p \to 1+$ of v in the whole interval $[0, nT(\alpha)]$ is dictated by the corresponding behavior in the interval $[0, T(\alpha)]$.

Let us show that $v(t, \alpha) \to \overline{\alpha}$ as $p \to 1+$ uniformly on compact sets of $\left(0, \dfrac{\overline{T}(\overline{\alpha})}{2}\right)$. To this end, for $\varepsilon > 0$ so small as $0 < \overline{\alpha} - \varepsilon < \alpha$, set $t_\varepsilon \in \left(0, \frac{T(\alpha)}{2}\right)$ the point where $v(t, \alpha)$ achieves the value $\overline{\alpha} - \varepsilon$. Then,

$$t_\varepsilon = \{p'\}^{\frac{1}{p}} \int_0^{\overline{\alpha}-\varepsilon} \frac{ds}{(V(\alpha) - V(s))^{\frac{1}{p}}}.$$

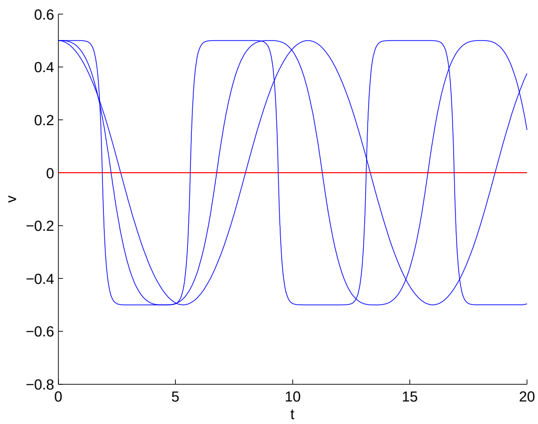

FIGURE 1

Hence $t_\varepsilon \to 0$ as $p \to 1+$. The symmetry of the solution leads to the desired assertion in the whole interval $[0, \overline{T}(\overline{\alpha})]$. This shows (3.4). Second convergence in (3.4) is a consequence of the conservation of E in (2.9). □

Graphics of the solution to (2.7) are shown in Figure 1. Values chosen are $q = 2.5$, $\alpha = 0.5$, together with $p = 2$, $p = 1.5$ and $p = 1.1$. The smaller p, the flatter the graphic.

4. ANALYSIS OF THE LIMIT PROBLEM

In this section we are dealing with problem (1.4),

$$\begin{cases} -\left(\dfrac{u_x}{|u_x|}\right)_x = \lambda \dfrac{u}{|u|} - |u|^{q-2}u & 0 < x < 1 \\ u(0) = u(1) = 0, \end{cases}$$

which is the formal limit of (1.3) as $p \to 1$. The natural setting to study this problem is $BV(0,1)$, the space of functions $u \in L^1(0,1)$ so that its distributional derivative u_x is a finite Radon measure. We point out that every $u \in BV(0,1)$ coincides a. e. with a function $\tilde{u} \in L^\infty(0,1)$ which is of bounded variation in the classical sense ([1]). Thus, by identifying u with $\tilde{u}$, it can be assumed that the set of discontinuities of u is at most denumerable and consists only of jump discontinuities. In particular u possesses finite side limits $u(x\pm)$ at any $x \in [0,1]$. It is also recalled that functions in $W^{1,\infty}(0,1)$ can be identified with Lipschitzian functions on $[0,1]$.

After these remarks, the notion of solution to (1.4) (adapted from [2, 3]) is formulated as follows.

Definition 7. *A function $u \in BV(0,1)$ defines a solution to* (1.4) *if there exist* $\mathbf{z} \in W^{1,\infty}(0,1)$ *and* $\beta \in L^\infty(0,1)$ *satisfying* $\|\mathbf{z}\|_\infty \le 1$ *and* $\|\beta\|_\infty \le 1$ *together with:*
1) $-\mathbf{z}_x = \lambda\beta - |u|^{q-2}u$ *in* $\mathscr{D}'(0,1)$,
2) $(\mathbf{z}, u_x) = |u_x|$ *as measures and* $\beta u = |u|$ *a.e.,*
3) $\mathbf{z}(0) \in \operatorname{sign}(u(0+))$ *and* $-\mathbf{z}(1) \in \operatorname{sign}(u(1-))$.

Remark 5.
1) Condition $\mathbf{z} \in W^{1,\infty}(0,1)$ is coherent with the right hand side of equation in 1).
2) For $v \in BV(0,1)$ and $\mathbf{z} \in W^{1,\infty}(0,1)$, $(\mathbf{z}, v_x)$ stands for the distribution,

$$\langle (\mathbf{z}, v_x), \varphi \rangle = -\int_0^1 v\varphi\mathbf{z}_x - \int_0^1 v\mathbf{z}\varphi_x, \qquad \varphi \in C_0^\infty(0,1). \tag{4.1}$$

Since $\mathbf{z}$ is continuous, it can be shown by an approximation argument that $(\mathbf{z}, v_x) = \mathbf{z}v_x$ as measures. A further reasoning leads to the Green formula,

$$\int_0^1 (\mathbf{z}, v_x) + \int_0^1 v\mathbf{z}_x = v\mathbf{z}\big|_0^1 = v(1-)\mathbf{z}(1) - v(0+)\mathbf{z}(0), \tag{4.2}$$

the first term meaning $\mathbf{z}v_x(0,1)$. Both (4.1) and (4.2) were introduced in [4] in a more general N–dimensional framework. Moreover, by using an arbitrary $v \in B(0,1)$ as a test function in equation 1) we achieve,

$$\int_0^1 (\mathbf{z}, v_x) - v\mathbf{z}\big|_0^1 = \int_0^1 (\lambda\beta - |u|^{q-2}u)v. \tag{4.3}$$

3) Last requirement in Definition 7 is a weak form of the Dirichlet boundary condition. Terminology $\mathbf{z} \in \operatorname{sign} u$ means $\mathbf{z} = \frac{u}{|u|}$ if $u \neq 0$, $\mathbf{z} \in [-1,1]$ otherwise.

A further relevant subject to be reviewed is the eigenvalue problem (1.6) for the one dimensional 1–Laplacian. Namely,

$$\begin{cases} -\left(\dfrac{u_x}{|u_x|}\right)_x = \lambda \dfrac{u}{|u|} & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases}$$

In [6], a definition of the full set of eigenvalues to (1.6) was presented for the first time. They consist of the critical values of the ‘total variation’ functional, constrained under suitable restrictions. A main result in [6] states that the set of eigenpairs $(\lambda, u) \in \mathbb{R} \times BV(0,1)$ to (1.6) just coincides with the limits $(\bar{\lambda}_n, \bar{u}_n)$ given in (3.2) and (3.3) (see Theorem 5 above). It is also shown in [6] that any eigenpair (λ, u) solves (1.6) in the sense of Definition 7 (see also [26], [8] for related problems).

The main statement of the section is the following result. It provides us a genuine extension of Theorem 2 to the 1–Laplacian setting.

Theorem 8. *Let $\overline{\lambda}_n = 2n$ be the sequence of eigenvalues of the* 1*–Laplacian. The structure of the set of nontrivial solutions to problem* (1.4) *can be described in the following terms.*

i) *Nontrivial solutions $u \in BV(0,1)$ are only possible if $\lambda > \overline{\lambda}_1 = 2$. Moreover, all solutions to* (1.3) *verify the estimate*

$$\|u\|_\infty \le \lambda^{\frac{1}{q-1}}. \tag{4.4}$$

ii) *Limit family* (3.4) *obtained in Theorem 6, $\overline{u}_\lambda^{(n)} = \sum_{k=1}^n (-1)^{k-1}(\lambda - 2n)^{\frac{1}{q-1}}\chi_k$, define a branch of nontrivial solutions to* (1.4) *for $\lambda > \overline{\lambda}_n$. In addition,*

$$\|\overline{u}_\lambda^{(n)}\|_\infty \to 0 \text{ as } \lambda \to \overline{\lambda}_n + \quad \& \quad \lambda^{-\frac{1}{q-1}}\|\overline{u}_\lambda^{(n)}\|_\infty \to 1 \text{ as } \lambda \to \infty. \tag{4.5}$$

Moreover, $\overline{u}_\lambda^{(n)}$ changes its sign at $x = \frac{k}{n}$, $1 \le k \le n-1$.

iii) *For $\overline{\lambda}_n < \lambda \le \overline{\lambda}_{n+1}$,*

$$u = \pm \overline{u}_\lambda^{(m)}, \qquad 1 \le m \le n,$$

are the unique *nontrivial solutions to* (1.4) *satisfying the extra condition,*

$$|u| = constant. \tag{4.6}$$

Remark 6. Some observations on the uniqueness requirement (4.6) are in order. As observed in the proof of Theorem 2, functional $E_p(v, v_t)$ defined in (2.8), is conserved through the solutions to (1.3). By formally letting $p \to 1$ we obtain $E_p(v, v_t) \to |v| - \frac{1}{q}|v|^q$. The latter quantity is conserved only when $|v|$ keeps constant. That is why (4.6) seems reasonable and may be regarded as an energy condition.

A further reflection on condition (4.6). It is said that $(\lambda, u) \in \mathbb{R} \times BV(0,1)$, $u \ne 0$, is a weak eigenpair to (1.6) ([26], [8]) provided that u solves (1.6) in the sense of Definition 7, where $-\mathbf{z}_x = \lambda \beta$ replaces the equation in 1). As already pointed out, the $(\overline{\lambda}_n, \overline{u}_n)$'s obtained in Theorem 5 define weak eigenpairs. However, it was discovered in [6] that *all* values $\lambda \ge 2$ are weak eigenvalues. It is amazing that extra condition (4.6) discriminates the genuine 'variational' eigenvalues $\overline{\lambda}_n = 2n$ from the remaining 'artificial' weak eigenvalues in $[2, \infty)$.

Finally, an example of a family of nontrivial solutions to (1.4) which does not satisfy condition (4.6) is presented in Remark 7 below.

Proof of Theorem 8. Let u be a nontrivial solution. By choosing $v = u$ in (4.3), the variational characterization of $\overline{\lambda}_1$ ([6]) leads to:

$$\overline{\lambda}_1 \int_0^1 |u| \le \int_0^1 |u_x| + |u(0)| + |u(1)| < \lambda \int_0^1 |u|.$$

Thus, the existence of a nontrivial solution implies $\lambda > \overline{\lambda}_1$.

To prove (4.4) we set $v = \max\{u-\lambda^{\frac{1}{q-1}},0\}$ as a test function. It can be shown that 2) also entails $(\mathbf{z},v_x) = |v_x|$ ([21, Proposition 2.7]). From equation (4.3) we arrive at:

$$\int_0^1 |v_x| \le \lambda \int_0^1 (\beta - \varphi_q(\lambda^{-\frac{1}{q-1}}u))v,$$

where $\varphi_q(t) = |t|^{q-2}t$. Therefore, $v = 0$ and so $u \le \lambda^{\frac{1}{q-p}}$. The complementary estimate $u \ge -\lambda^{\frac{1}{q-p}}$ is obtained in a similar way and so (4.4) is shown.

We are next checking that $u = \bar{u}_\lambda^{(n)}(x)$ defines a solution to (1.4). By choosing $\beta = \sum_{k=1}^n (-1)^{k-1}\chi_k$ it is clear that $\|\beta\|_\infty \le 1$ and $\beta\bar{u}_\lambda^{(n)} = |\bar{u}_\lambda^{(n)}|$. The scalar field $\mathbf{z}$ can be found by solving 1) separately on each interval $\left[\frac{k-1}{n},\frac{k}{n}\right]$ with the initial condition $z = (-1)^{k-1}$ at $x = \frac{k-1}{n}$, and the restriction $\|\mathbf{z}\|_\infty \le 1$. We arrive in this way at

$$\mathbf{z} = \sum_{k=1}^n (-1)^k 2n\left(x - \frac{2k-1}{2n}\right)\chi_k$$

and so $\bar{u}_\lambda^{(n)}$, β and $\mathbf{z}$ are linked by condition 1). Notice that $\mathbf{z}(0) = 1 = \operatorname{sign} u(0+)$ and $\mathbf{z}(1) = (-1)^n = -\operatorname{sign} u(1-)$ so the boundary conditions 3) are satisfied.

We are checking condition 2). For $u = \bar{u}_\lambda^{(n)}$ and $\varphi \in C_0^\infty(0,1)$, identity (4.1) implies

$$\langle(\mathbf{z},u_x),\varphi\rangle = -\int_0^1 u(\mathbf{z}\varphi)_x = -\sum_{k=1}^n (-1)^{k-1}(\lambda-2n)^{\frac{1}{q-1}}\mathbf{z}\varphi\Big|_{x_{k-1}}^{x_k}$$
$$= \sum_{k=1}^{n-1}(-1)^k(\lambda-2n)^{\frac{1}{q-1}}\mathbf{z}(x_k)\,\varphi(x_k) - \sum_{k=2}^{n}(-1)^k(\lambda-2n)^{\frac{1}{q-1}}\mathbf{z}(x_{k-1})\,\varphi(x_{k-1})$$
$$= \sum_{k=1}^{n-1} 2(\lambda-2n)^{\frac{1}{q-1}}\varphi(x_k) = \sum_{k=1}^{n-1} 2(\lambda-2n)^{\frac{1}{q-1}}\langle\delta_{x_k},\varphi\rangle = \langle |u|_x,\varphi\rangle,$$

where $x_k = \frac{k}{n}$. In the last inequality, δ_{x_0} stands for Dirac's delta located at $x = x_0$. Therefore, $\bar{u}_\lambda^{(n)}$ defines a nontrivial solution to (1.4) for $\lambda > \bar{\lambda}_n$. Remaining properties in ii) are a consequence of the explicit expression of $\bar{u}_\lambda^{(n)}$.

We are next showing the *uniqueness* assertion iii). Thus, let u be a solution to problem (3.2) with constant modulus $|u| = \xi$. No generality is lost if we assume $u(0+) = \xi > 0$.

We claim that $0 < \xi < \lambda^{\frac{1}{q-1}}$. In fact, it follows from $u(0+) = \xi$ that $-\mathbf{z}_x = \lambda - \xi^{q-1}$. Thus, conditions $\mathbf{z}(0) = 1$ and $|\mathbf{z}| \leq 1$ imply $\xi^{q-1} \leq \lambda$. Moreover, if $\lambda = \xi^{q-1}$, then $\mathbf{z}_x = 0$ and consequently $\mathbf{z}(x) = 1$. Condition 2) in Definition 7 implies that $u_x = |u_x|$, u is nondecreasing and so $u(x) = \xi$ for all $x \in (0,1)$. However, this solution is not possible because $\mathbf{z}(1) = 1 \neq -1 = \text{sign }(-u(1-))$ and so the boundary condition is not satisfied at $x = 1$. Therefore, the claim follows.

Next observe that solutions with constant absolute value have only a finite number of changes of sign owing to belong to $BV(0,1)$. Hence, two possibilities must be analyzed.

a) u does not change its sign. Problem $-\mathbf{z}_x = \lambda - \xi^{q-1}$, $\mathbf{z}(0) = 1$ has the solution $\mathbf{z}(x) = -(\lambda - \xi^{q-1})x + 1$. Assume that there exists $x_0 < 1$ such that $\mathbf{z}(x_0) = -1$. Since $\mathbf{z}$ is decreasing, we get $\mathbf{z}(x) < -1$ for all $x \in (x_0, 1)$ contradicting the condition $|\mathbf{z}| \leq 1$. On the other hand, the boundary condition at $x = 1$ reads as $\mathbf{z}(1) = -1$. Hence,

$$-1 = -(\lambda - \xi^{q-1}) + 1$$

and this fact implies $\lambda > 2$ and $\xi = (\lambda - 2)^{\frac{1}{q-1}}$, i. e., $u = \overline{u}_\lambda^{(1)}$.

b) u changes its sign $m-1$ times ($m \geq 2$). In this case we know that the solution u can expressed as

$$u = \sum_{k=1}^{m} (-1)^{k-1} \xi \chi_{J_k},$$

for some intervals $J_k = (x_{k-1}, x_k)$, with $x_0 = 0$ and $x_m = 1$. We are searching for the value of ξ and the endpoints x_k.

In the first interval $J_1 = (0, x_1)$ the solution is positive and so $-\mathbf{z}_x = \lambda - \xi^{q-1}$ holds and it implies

$$\mathbf{z}(x) = -(\lambda - \xi^{q-1})x + 1.$$

Notice that $\mathbf{z}(\tilde{x}_1) = -1$ where $\tilde{x}_1 = \frac{2}{\lambda - \xi^{q-1}}$. Thus, $|\mathbf{z}(x)| < 1$ for all $x \in (0, \tilde{x}_1)$ and, as a consequence of condition $(\mathbf{z}, u_x) = |u_x|$, we get $u = \xi$ in $(0, \tilde{x}_1)$. This means that $\tilde{x}_1 \leq x_1$. However it is not possible that $\tilde{x}_1 < x_1$, otherwise $\mathbf{z} < -1$ in the interval $(\tilde{x}_1, x_1)$ contradicting that $|\mathbf{z}(x)| \leq 1$ for all $0 \leq x \leq 1$. Thus $x_1 = \tilde{x}_1$ and from the very definition of x_1, u jumps from ξ to $-\xi$ at this point.

In the second interval $J_2 = (x_1, x_2)$ the solution is negative and so the problem for $\mathbf{z}$ becomes $\mathbf{z}_x = \lambda - \xi^{q-1}$, $\mathbf{z}(x_1) = -1$ whose solution is $\mathbf{z}(x) = (\lambda - \xi^{q-1})x - 3$. By a similar argument, we infer that u changes again its sign at $x_2 = \frac{4}{\lambda - \xi^{q-1}}$. Proceeding inductively, it is found that changes of sign occur successively at $x_k = \frac{2k}{\lambda-\xi^{q-1}}$, $k = 1,\ldots,m-1$, being the length of the intervals $|J_k| = \frac{2}{\lambda-\xi^{q-1}}$. In particular,

$$\frac{2m}{\lambda-\xi^{q-1}} = 1.$$

Hence, $\xi^{q-1} = \lambda - 2m$ and $x_k = \frac{k}{m}$, $k = 1,\ldots,m-1$. We both conclude that $\lambda > 2m$ and $u = \bar{u}_\lambda^{(m)}(x)$ hold, and so we are done. □

Remark 7. To illustrate the rôle of condition (4.6), we are showing that problem (1.4) exhibits further families of nontrivial solutions aside the ones referred to in Theorem 8. In fact, choose $0 < \alpha < 1$ and set $\lambda^*(\alpha) = \dfrac{2}{1-\alpha^{q-1}}$. Then,

$$u = \lambda^{\frac{1}{q-1}} \alpha \chi_{[0,\frac{\lambda^*}{\lambda}]}(x), \qquad \lambda > \lambda^*(\alpha), \tag{4.7}$$

constitutes a family of nontrivial solutions having $\lambda = \lambda^*(\alpha)$ as the onset critical value. Obviously, solutions in (4.7) do not satisfy (4.6). To check that conditions in Definition 7 are fulfilled it is enough with using,

$$\beta = \chi_{[0,\frac{\lambda^*}{\lambda}]}(x) \qquad \text{and} \qquad z = \begin{cases} 1 - 2\dfrac{\lambda}{\lambda^*}x & 0 \le x \le \dfrac{\lambda^*}{\lambda} \\ -1 & 0 \le \dfrac{\lambda^*}{\lambda} \le x \le 1, \end{cases}$$

as the corresponding functions alluded to in the definition. Other further families, showing more complicated patterns, can be also found out. Of course, none of them satisfies (4.6).

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