LOGISTIC REACTION COUPLED TO p–LAPLACIAN DIFFUSION AS p GOES TO 1

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Abstract. This work discusses the limit as p goes to 1 of solutions to problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u - |u|^{q-2}u, & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $\lambda > 0$ is a parameter and the exponents p, q satisfy $1 < p < q$.

Our interest is focused on the radially symmetric case. We prove in this radial setting that solutions u_p to (P) converge to a limit u as $p \to 1+$. Moreover, the limit function u defines a solution to the natural 'limit problem' which involves the 1–Laplacian operator. In addition, a precise description of the structure of the set of all possible solutions to such a problem is achieved. This is accomplished by means of the the introduction of a suitable energy condition. Furthermore, a detailed analysis of the profiles of all these solutions is also performed.

1. Introduction

Since the late seventies, reaction–diffusion systems has been one of the more active areas in nonlinear analysis ([17], [42], [12], [36], [7]). The so–called *logistic* problem is a reference model in the field where a wide variety of techniques have been tested (sub and super solutions, degree and bifurcation theory, critical point theory). Under such a term it is understood the nonlinear eigenvalue problem,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u - |u|^{q-2}u & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and the diffusion is governed by the p–Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Exponents p, q are assumed to satisfy,

$$1 < p < q.$$

The number $\lambda > 0$ plays the rôle of a bifurcation parameter. In fact, a well–extended insight in the theory from the very beginning is just observing (1.1) as a crude perturbation of the "pure" eigenvalue problem,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{1.2}$$

The main objective of the present work is analyzing the fine aspects of the asymptotic behavior of problem (1.1) as $p \to 1+$. In the first place, this involves discussing

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the existence of the limit $u = \lim_{p\to 1+} u_p$ of a given family u_p of solutions to (1.1). In the second place, it should be decided whether such possible limits u solve in some weak sense the natural "limit problem". In other words, that one obtained by directly inserting $p = 1$ in (1.1),

$$\begin{cases} -\Delta_1 u = \lambda \dfrac{u}{|u|} - |u|^{q-2}u & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_1 = \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right)$ is the one–Laplacian operator. To complete the analysis, a third task to be faced is that of describing all of the possible nontrivial solutions to (1.3).

Previous experiences on the "natural" associated eigenvalue problem,

$$\begin{cases} -\Delta_1 u = \lambda \dfrac{u}{|u|} & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.4}$$

strongly suggests that characterizing the solutions to (1.3) requires imposing certain restrictions. As a matter of fact, the higher eigenvalues to (1.4) have not been studied until few years ago ([8], [31], [33], [41]). It was just discovered in [8] that infinitely many anomalous eigenpairs to (1.4) arise if the corresponding Euler–Lagrange inclusion is not suitably constrained. The very same phenomenon occurs in the $1D$–version of our problems (1.1), (1.3) as recently remarked in [40]. On the other hand, our analysis in the present work is an extended nontrivial continuation of [41]. The radial spectrum of (1.4) is analyzed for the first time in this work.

As for applications, the linear diffusion case $p = 2$ of (1.1) arises in population dynamics, where it describes the equilibrium regime of a species subject to logistic self–regulation and spatial migration ([7], [34, 35]). In reaction dynamics, a solution to (1.1) furnishes the stationary concentration u of a chemical substance, which diffuses throughout a reactor $\Omega \subset \mathbb{R}^N$ and is subject to parallel competing reactions ([18]). That is why major emphasis has been put on studying its positive solutions (see [7], [32] for a comprehensive overview in population dynamics). The nonlinear diffusion case $p \neq 2$ is comparatively less understood. Most of the results have to do with positive solutions to (1.1) which has been analyzed in a series of works ([13], [25], [26], [23],[22]).

On the other hand, problems involving the 1–Laplacian are deserving a growing interest in the literature. Specially after the pioneering works [3], [4], [10]. To formulate a proper notion of solution to problems as (1.3) counts among the challenges achieved in these references (see Section 2.3). From the very beginning, the applications of Δ_1 range from image processing ([5], [38]) to elasticity ([27]).

However, the structure of the whole set of nontrivial solutions to (1.1) still remains unknown in many concerns, with the exception of the case $N = 1$ ([25], [40]). The problem in a general N–dimensional domain Ω is plagued of obstacles. To quote only a few, there are not any kind of bifurcation results available from the higher eigenvalues $\lambda_{n,p}$ of $-\Delta_p$ (bifurcation at the first eigenvalue $\lambda_{1,p}$ has been studied in [9], [15]). The only exception is the radially symmetric case where Ω is a ball ([24], [39]). What is worse, the complete spectrum of $-\Delta_p$ remains nowadays undetermined ([30], [16]). That is why there hardly exist results providing the existence of two signed solutions to (1.1) when λ growths (see [20] where such a kind of existence issues are addressed in a problem with the same structure).

After these considerations, it seems reasonable that an analysis of problems (1.1) and (1.3) can only be undertaken in the radially symmetric case. In a first step, a detailed account of the set of all possible nontrivial radial solutions to (1.1) is presented in this work. Solutions to this problem in a ball $B_R \subset \mathbb{R}^N$ are shown to be organized in *continuous* curves emanating from the radial eigenvalues $\tilde{\lambda}_{n,p}$ to (2.21). More importantly, it is shown that the interval $\lambda > \tilde{\lambda}_{n,p}$ is the precise existence domain for each of these curves. In this regard, global existence results in [24] (valid in the case $p > 2$) are substantially sharpened for the particular case of (1.1).

Once the nontrivial solutions to (1.1) are known, two main objectives are pursued in this work. First, to analyze the limit of these solutions as $p \to 1+$. Second, to characterize such limits as properly defined solutions to (1.3). It turns out that both problems are deeply connected. On one hand a compactness type result permits us extracting limits u of families of solutions u_p to (1.1) as $p \to 1+$. Moreover, every such a limit u defines a solution to (1.3) and so this statement actually constitutes a true existence tool. In fact, the result is also valid in a general smooth domain $\Omega \subset \mathbb{R}^N$. On the other hand, an uniqueness result allows us concluding the validity of the full limit $u = \lim_{p\to 1} u_p$. In addition, it furnishes a quite detailed description of the profile of the limit u. This stage of the analysis heavily rests upon the radial requirement. It is worth to point out that solutions comprised under the uniqueness result must satisfy suitable symmetry and energy conditions which are revealed in this work. In fact, without restrictions, problem (1.3) could exhibit an uncontrolled number of solutions (Section 5.4).

As a final conclusion, we are able to furnish a rather complete picture of the nontrivial solutions to (1.3) in a ball B_R. It is shown that its radial solutions satisfying an energy condition are organized in continuous curves. Every such a curve emanates from a radial eigenvalue $\bar{\lambda}_n$ to $-\Delta_1$. Moreover, the structure of solutions lying in the same curve is explicitly described. In particular, solutions belonging to the same curve undergo the same number of jumps. Of course, this feature is reminiscent of the nodal properties exhibited by the solutions to (1.1) lying in a fixed branch.

This work is organized as follows. Next section deals with the preliminaries. Subsection 2.2 discusses the basic properties of problem (1.1), while the concept of solution to (1.3) together with the compactness principle satisfied for this problem (Theorem 5) are presented in Subsection 2.3. It is remarked that the material in these subsections is valid on a general domain $\Omega \subset \mathbb{R}^N$. The main features reported here were firstly tested in the one dimensional case ([40]). Due to its intrinsic interest for our purposes in the present work, a partial overview of the later paper is contained in Subsection 2.4. Theorem 10 describing the nontrivial radial solutions to (1.1) is shown in Section 3. It includes important Lemma 9 which introduces and studies the zeros θ_n of the solutions to the initial value problem associated to (1.1). The analysis of the asymptotic behavior of problem (1.1) as $p \to 1$ is launched in Section 4. Two preliminary results stating the finiteness of the limits $\underline{\lim}_{p\to 1} \theta_n$, $\overline{\lim}_{p\to 1} \theta_n$ (Theorem 13) and proving the validity of the strict inequality $\overline{\lim}_{p\to 1} \theta_n < \underline{\lim}_{p\to 1} \theta_{n+1}$ (Theorem 14) are introduced in this section. Proving the main result of this work, Theorem 16, is the objective of Section 5. This task is performed in two steps. The first one discusses the existence of solutions to the initial value problem connected to (1.3) (Theorem 18). Relevant Theorem 19 is the keystone on which the uniqueness feature is built. This second step permits us obtaining the proof of our main statement.

2. Preliminary facts

2.1. **Notation.** In what follows we assume $N \geq 2$ and denote $\mathcal{H}^{N-1}$ the $(N-1)$–dimensional Hausdorff measure in $\mathbb{R}^N$. Bounded domains $\Omega \subset \mathbb{R}^N$ are supposed to be of class $C^{1,\alpha}$. Thus, an outward unit normal $\nu(x)$ is defined for all $x \in \partial\Omega$.

Lebesgue and Sobolev spaces are denoted by $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively. The space of functions of bounded variation is denoted by $BV(\Omega)$. It consists of those L^1–functions whose distributional gradient is a Radon measure with finite total variation. Even though derivatives of members in $BV(\Omega)$ are not functions, they exhibit traces in $L^1(\partial\Omega)$, while this space enjoys the same ranges of continuous and compact embeddings than $W^{1,1}(\Omega)$. We regard $BV(\Omega)$ endowed with the norm

$$\|u\| = \int_\Omega |Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} ,$$

and refer to [1] for a comprehensive account on the theory of functions of bounded variation.

A substantial part of this work is focused on radial solutions. So we deal with a ball in $\mathbb{R}^N$ centered at the origin and of radius $R > 0$, it will be denoted by B_R. Observe that a radial function $u \in W_0^{1,p}(B_R)$ can be represented as $u(x) = v(|x|)$ where $v, v' \in L^p((0,R), r^{N-1}dr)$, v' being the weak derivative of v, while $\nabla u(x) = v'(|x|)\dfrac{x}{|x|}$ (see further details in Section 3). In the same vein, a radial function $u \in BV(B_R)$ satisfies $u(x) = v(|x|)$ where $v \in L^1((0,R), r^{N-1}dr)$. However, v' is now a Radon measure in $(0,R)$ with total variation $|v'|$ so that the measure $r^{N-1}|v'|$ is finite. Moreover, the identity

$$\int_{B_R} \varphi(|x|)|Du| = N\omega_N \int_0^R \varphi(r) r^{N-1} |v'| \,, \tag{2.5}$$

where $N\omega_N = \mathcal{H}^{N-1}(\partial B(0,1))$, holds true for all radial test functions $\varphi(|x|)$ in $C_0^\infty(B_R)$ (precise details are omitted for brevity).

The space of continuous functions $C(J)$ on an interval J is regarded with the uniform convergence on compacta (a similar remark applies to $C^1(J)$).

Finally, for a given measurable function u in Ω, the notation

$$v \in \text{sign } (u)$$

will be used to mean that $v \in L^\infty(\Omega)$ satisfies $\|v\|_\infty \leq 1$ and $v(x)u(x) = |u(x)|$ a. e. in Ω. Accordingly, infinitely many v's can be found whenever u vanishes in a positive measure set.

2.2. **Logistic p–Laplacian problems.** Although we are mainly interested in the radial case, the introduction of some general properties of the nonlinear problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u - |u|^{q-2}u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.6}$$

is quite convenient for later reference. Henceforth, exponents p, q fall in the range,

$$1 < p < q. \tag{2.7}$$

For its use in this section we introduce the notion of weak solution to (2.6).

Definition 1. *A weak solution to* (2.6) *is defined as a function* $u \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ *such that equality*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \lambda \int_\Omega |u|^{p-2}uv - \int_\Omega |u|^{q-2}uv, \tag{2.8}$$

is satisfied for all functions $v \in C_0^1(\Omega)$.

The requirement $u \in L^q(\Omega)$ is natural if one thinks of the variational formulation of (2.6). In addition, since elements $v \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ can be approximated in this space by functions of $C_0^1(\Omega)$ then test functions in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ can be also inserted in (2.8). Finally we are next showing that weak solutions lie on $L^\infty(\Omega)$ and so we can test in (2.6) with arbitrary $v \in W_0^{1,p}(\Omega)$.

Some important features of (2.6) are the goal of the following result.

Theorem 2. *Problem* (2.6) *exhibits the next features.*

i) *All possible solutions* u *belong to* $L^\infty(\Omega)$ *and satisfy the estimate*

$$\|u\|_\infty \le \lambda^{\frac{1}{q-p}}. \tag{2.9}$$

ii) *Nontrivial solutions are only possible for* $\lambda > \lambda_{1,p}$, $\lambda_{1,p}$ *being the first Dirichlet eigenvalue of* $-\Delta_p$.

iii) *For fixed* $\lambda > \lambda_{1,p}$ *there exists* $0 < \beta < 1$ *not depending on* λ *varying in bounded intervals such that the whole set of nontrivial solutions to* (2.6) *constitutes a compact set in* $C^{1,\beta}(\overline{\Omega})$.

iv) *For every* $\lambda > \lambda_{1,p}$ *there exists a unique* positive *solution* u_λ *to* (2.6). *Family* u_λ *is smooth and increasing in* λ *while*

$$\lim_{\lambda \to \lambda_{1,p}} \|u_\lambda\|_\infty = 0, \qquad \lambda^{-\frac{1}{q-p}} u_\lambda \to 1 \quad \lambda \to \infty, \tag{2.10}$$

uniformly on compact sets of Ω.

Proof. We first observe that $v = (|u| - \lambda^{\frac{1}{q-p}})^+ \text{sign } u \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$. Then it can be inserted in (2.8) as a test function leading to:

$$\begin{aligned}\int_\Omega |\nabla(|u| - \lambda^{\frac{1}{q-p}})^+|^p &= \int_\Omega \left(\lambda|u|^{p-1} - |u|^{q-1}\right)(|u| - \lambda^{\frac{1}{q-p}})^+ \\ &= \int_{\{|u| \ge \lambda^{\frac{1}{q-p}}\}} |u|^{p-1}\left(\lambda - |u|^{q-p}\right)(|u| - \lambda^{\frac{1}{q-p}})^+ \le 0.\end{aligned}$$

Thus $(|u| - \lambda^{\frac{1}{q-p}})^+ = 0$ which amounts to $|u| \le \lambda^{\frac{1}{q-p}}$.

By choosing $v = u$ in (2.8) we obtain:

$$\int_\Omega |\nabla u|^p - \lambda \int_\Omega |u|^p = -\int_\Omega |u|^q < 0,$$

and so we deduce

$$\lambda_{1,p} \int_\Omega |u|^p < \lambda \int_\Omega |u|^p.$$

Hence, $\lambda > \lambda_{1,p}$.

The assertion of the $C^{1,\beta}$ smoothness of solutions follows from the estimate (2.9) and the classical results in [43], [14].

The existence of a positive solution when $\lambda > \lambda_{1,p}$ is obtained by using, say the method of sub and super solutions. See for instance [11] and [23] (see also [12] provided that $p \geq 2$). It is sufficient to choose $u^- = \varepsilon\phi_1(\cdot)$, $\varepsilon > 0$ small enough, ϕ_1 a first positive eigenfunction, as a sub solution and $u^+ = \lambda^{\frac{1}{q-p}}$ as a super solution.

Uniqueness of a positive solution is a consequence of [13]. The family u_λ is increasing in λ. Indeed, it is implicit in the fact that u_{λ_0} becomes a sub solution of (2.6) for $\lambda > \lambda_0$. Finally, asymptotic estimate (2.10) and further features on (2.6) are addressed in [22]. □

Remark 1. Only the regime $1 < p \leq 2$ is our main concern in this work. However, the complementary range $p > 2$ enjoys especial phenomena, the most relevant being that the flat core $\mathcal{O}_\lambda = \{u_\lambda(x) = \lambda^{\frac{1}{q-p}}\}$ becomes nonempty and converges to Ω as $\lambda \to \infty$ ([26], [22]).

Remark 2. By means of variational methods one can show the existence of further nontrivial (two–signed) solutions to (2.6), for λ as large as desired. In fact, the number of these solutions growths beyond any bound as $\lambda \to \infty$. See for instance [20] for this kind of results.

2.3. **The 1–Laplacian limit problem.** The main objective of this work is to let p go to 1 in problem (2.6) and obtaining limits of solutions. Accordingly, an important part of our endeavor will be to analyze the resulting Dirichlet problem deduced from (2.6) as $p \to 1$. Namely:

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \lambda\dfrac{u}{|u|} - |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.11}$$

The concept of solution to this problem relies on Anzellotti's theory (see [6]), which we next recall. Given $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ and $u \in BV(\Omega)$, it was introduced a distribution in [6] which resembles the dot product $\mathbf{z} \cdot Du$ for pairs $(\mathbf{z}, u)$ satisfying certain compatibility conditions. For instance, $\operatorname{div}\mathbf{z} \in L^N(\Omega)$ and $u \in BV(\Omega)$ or $\operatorname{div}\mathbf{z} \in L^r(\Omega)$ and $u \in BV(\Omega) \cap L^{r'}(\Omega)$. The distribution $(\mathbf{z}, Du) : C_0^\infty(\Omega) \to \mathbb{R}$ is defined by:

$$\langle(\mathbf{z}, Du), \varphi\rangle = -\int_\Omega u\,\varphi\operatorname{div}\mathbf{z} - \int_\Omega u\,\mathbf{z}\cdot\nabla\varphi, \quad \forall\,\varphi \in C_0^\infty(\Omega)\,. \tag{2.12}$$

When $\mathbf{z}$ and u are compatible every integral in (2.12) is well–defined. It is proved in [6] that $(\mathbf{z}, Du)$ is a Radon measure with finite total variation. More precisely, it is shown that for every Borel B set with $B \subseteq U \subseteq \Omega$ (U open) it holds

$$\left|\int_B (\mathbf{z}, Du)\right| \leq \int_B |(\mathbf{z}, Du)| \leq \|\mathbf{z}\|_{L^\infty(U)} \int_B |Du|\,. \tag{2.13}$$

A further feature of the theory in [6] is the notion of weak trace on $\partial\Omega$ of the normal component, denoted $[\mathbf{z}, \nu]$, of a field $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$. In fact, under the assumption that $\operatorname{div}\mathbf{z}$ is a finite Radon measure, the trace is appropriately defined, satisfies $[\mathbf{z}, \nu] \in L^\infty(\partial\Omega)$ and $\|\,[\mathbf{z}, \nu]\,\|_{L^\infty(\partial\Omega)} \leq \|\mathbf{z}\|_{L^\infty(\Omega,\mathbb{R}^N)}$. Most importantly, a Green formula connecting the measure $(\mathbf{z}, Du)$ and the weak trace $[\mathbf{z}, \nu]$ is established in [6]. Namely:

$$\int_\Omega (\mathbf{z}, Du) + \int_\Omega u \operatorname{div} \mathbf{z} = \int_{\partial\Omega} u\,[\mathbf{z}, \nu]\, d\mathcal{H}^{N-1}, \tag{2.14}$$

for those pairs $(\mathbf{z}, u)$ satisfying the conditions already mentioned (see [6]).

We are now ready to introduce the notion of solution to (2.11) which is based on that introduced in [4].

Definition 3. *A function $u \in BV(\Omega) \cap L^q(\Omega)$ is said to be a solution to problem* (2.11) *if there exist $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ and $\beta \in L^\infty(\Omega)$ satisfying:*

1) $\|\mathbf{z}\|_\infty \le \mathbf{1}$ *and* $\|\beta\|_\infty \le 1$,
2) $-\operatorname{div} \mathbf{z} = \lambda\beta - |u|^{q-2}u$ *in* $\mathcal{D}'(\Omega)$,
3) $(\mathbf{z}, Du) = |Du|$ *as measures and* $\beta u = |u|$ *a.e. in* Ω,
4) $[\mathbf{z}, \nu] \in \operatorname{sign}\,(-u)$ $\mathcal{H}^{N-1}$*–a.e. on* $\partial\Omega$.

Remark 3. Conditions $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$, with $\|\mathbf{z}\|_\infty \le \mathbf{1}$, and $(\mathbf{z}, Du) = |Du|$ indicate that the vector field $\mathbf{z}$ plays the rôle of $\dfrac{Du}{|Du|}$. In fact, they are equal when $u \in W^{1,1}(\Omega)$ and $\{\nabla u = 0\}$ is a set of measure zero since then $\|\mathbf{z}\|_\infty \le \mathbf{1}$ and $\mathbf{z} \cdot \nabla u = |\nabla u|$ imply $\mathbf{z} = \dfrac{\nabla u}{|\nabla u|}$. For a general $u \in BV(\Omega)$, $\dfrac{Du}{|Du|}$ cannot belong to $L^\infty(\Omega, \mathbb{R}^N)$. A similar observation applies to β which plays the rôle of $\dfrac{u}{|u|}$ and they have the same value when $\{u = 0\}$ is a null set.

Remark 4. We point out that the Radon measure $(\mathbf{z}, Du)$ is well–defined since $\operatorname{div} \mathbf{z} \in L^{q'}(\Omega)$ and $u \in BV(\Omega) \cap L^q(\Omega)$. Moreover, $(\mathbf{z}, Dv)$ is defined too whenever $v \in BV(\Omega) \cap L^q(\Omega)$ and so equation in 2) together with (2.14) imply that the equality

$$\int_\Omega (\mathbf{z}, Dv) - \int_{\partial\Omega} v\,[\mathbf{z}, \nu]\, d\mathcal{H}^{N-1} = \int_\Omega (\lambda\beta - |u|^{q-2}u)v, \tag{2.15}$$

holds for all these test functions v in $BV(\Omega) \cap L^q(\Omega)$. For the moment, we are not allowed to consider $(\mathbf{z}, Dv)$ for an arbitrary $v \in BV(\Omega)$. Nevertheless, the next result implies that actually $\operatorname{div} \mathbf{z} \in L^\infty(\Omega)$, so that $(\mathbf{z}, Dv)$ has always a meaning for every $v \in BV(\Omega)$.

In the next statement λ_1 denotes the first Dirichlet eigenvalue of $-\Delta_1$ in Ω ([28], [8], [41]). As shown in [28], λ_1 coincides with the Cheeger constant of Ω and is variationally expressed by,

$$\lambda_1 = \min_{v \in BV(\Omega)\setminus\{0\}} \frac{\int_\Omega |Dv| + \int_{\partial\Omega} |v|\, d\mathcal{H}^{N-1}}{\int_\Omega |v|\, dx}\,.$$

Theorem 4. *Let $q > 1$. Then problem* (2.11) *exhibits the following features.*

i) *All possible solutions u belong to $L^\infty(\Omega)$ and satisfy the estimate*

$$\|u\|_\infty \le \lambda^{\frac{1}{q-1}}. \tag{2.16}$$

ii) *Nontrivial solutions are only possible for $\lambda > \lambda_1$.*

Proof. i) Set $G_k(t) = (|t| - k)^+ \text{sign}\,(t)$, $k > 0$, and choose $v = G_k(u) \in BV(\Omega) \cap L^q(\Omega)$ as a test function in (2.15). Then,

$$\int_\Omega (\mathbf{z}, DG_k(u)) - \int_{\partial\Omega} G_k(u)[\mathbf{z}, \nu]\; d\mathcal{H}^{N-1} = \int_\Omega (\lambda\beta - |u|^{q-2}u) G_k(u).$$

Now, it follows as a consequence of [29, Proposition 2.7] (see also [6, Proposition 2.8]) that that equality $(\mathbf{z}, Du) = |Du|$ as measures implies that $(\mathbf{z}, DG_k(u)) = |DG_k(u)|$ and so,

$$\int_\Omega |DG_k(u)| + \int_{\partial\Omega} |G_k(u)|\; d\mathcal{H}^{N-1} = \lambda \int_{\{|u|>k\}} \left(1 - \frac{|u|^{q-1}}{\lambda}\right)(|u| - k).$$

Observe that $|u| \geq \lambda^{\frac{1}{q-1}}$ implies $1 - \frac{|u|^{q-1}}{\lambda} \leq 0$; in this case, the right hand side becomes nonpositive, while the left hand side is always nonnegative. So it is enough with choosing $k = \lambda^{\frac{1}{q-1}}$ to conclude that $\|G_k(u)\|_{BV(\Omega)} = 0$ what entails the desired estimate.

ii) Let u be a nontrivial solution to (2.11). By using u as a test function in Green's formula (2.15), it yields

$$\int_\Omega (\mathbf{z}, Du) - \int_{\partial\Omega} u[\mathbf{z}, \nu]\, d\mathcal{H}^{N-1} = \lambda \int_\Omega |u|\, dx - \int_\Omega |u|^q\, dx < \lambda \int_\Omega |u|\, dx.$$

Resorting to conditions 3) and 4) of Definition 3, we get

$$\int_\Omega |Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} < \lambda \int_\Omega |u|\, dx\,. \tag{2.17}$$

Thus we infer from (2.17) that

$$\lambda_1 \int_\Omega |u|\, dx < \lambda \int_\Omega |u|\, dx\,,$$

and the result follows. □

We are next stating that solutions (λ_p, u_p) to (2.6) converge as $p \to 1$ and up to subsequences, to a solution (λ, u) to (2.11), provided that $\lambda_p \to \lambda$.

Theorem 5. *Let $\{(\lambda_p, u_p)\}_{p>1}$ be a family of nontrivial solutions to* (2.6) *with $\lambda_p > \lambda_{1,p}$, the first Dirichlet eigenvalue of $-\Delta_p$, such that* $\lim_{p\to 1+} \lambda_p = \lambda$. *Then, up to a sequence, there exist $u \in BV(\Omega)$, $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ and $\beta \in L^\infty(\Omega)$ with $\|\mathbf{z}\|_\infty \leq 1$, $\|\beta\|_\infty \leq 1$ such that the following properties hold.*

1) *$u_p \to u$ strongly in $L^s(\Omega)$ for all $1 \leq s < \infty$.*
2) *$|u_p|^{p-2}u_p \rightharpoonup \beta$ weakly in $L^s(\Omega)$ for all $1 \leq s < \infty$. Moreover $\beta u = |u|$ a. e. in Ω.*
3) *$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup \mathbf{z}$ weakly in $L^s(\Omega, \mathbb{R}^N)$ for all $1 \leq s < \infty$.*
4) *$\lim_{p\to 1+} \int_\Omega \varphi |\nabla u_p|^p = \int_\Omega \varphi |Du|$ for every nonnegative $\varphi \in C_0^\infty(\Omega)$.*

Furthermore, u defines a solution to problem (2.11) *by choosing* $\mathbf{z}$ *and β as the functions referred to in Definition 3.*

Remark 5. It is worth remarking that the above theorem could yield the trivial solution. This occurs, for instance, when $\lim_{p\to 1} \lambda_p = \lambda_1$. Notice that $\lim_{p\to 1} \lambda_{1,p} = \lambda_1$ ([28, Corollary 6]). Accordingly, obtaining a nontrivial solution u requires some extra computations. Indeed, it can be shown that for every $\lambda > \lambda_1$ the limit as $p \to 1$ of the family of positive solutions u_λ to (2.6) defines a nonnegative and nontrivial solution u to (2.11). Details are omitted for brevity.

Proof. By setting $v = u_p$ in (2.8) and taking into account (2.9) we achieve a uniform estimate of the form

$$\int_\Omega |\nabla u_p|^p \leq M, \tag{2.18}$$

for a no depending on p positive constant M. This implies that u_p is bounded in $BV(\Omega)$ and modulus a subsequence we find $u \in BV(\Omega)$ such that $u_p \to u$ both a. e. and in $L^r(\Omega)$ as $p \to 1$, provided that $r < \frac{N}{N-1}$. However, since u_p is uniformly bounded in $L^\infty(\Omega)$ such a convergence is upgraded to $L^s(\Omega)$ for all $s \geq 1$.

The remaining assertions of the theorem are shown by employing similar arguments as in [4] (see also [41, Theorem 6]). Accordingly, their proof are omitted. □

2.4. **Review of the one–dimensional case.** For future reference as auxiliary tools, some features of the one–dimensional version of problem (2.6),

$$\begin{cases} -(|u_x|^{p-2}u_x)_x = \lambda|u|^{p-2}u - |u|^{q-2}u, & 0 < x < 1, \\ \quad u(0) = 0 = u(1), \end{cases} \tag{2.19}$$

are next reported (see [40] for a detailed account and [21] for related one dimensional problems).

To begin with, the one–dimensional version the the eigenvalue problem is

$$\begin{cases} -(|u_x|^{p-2}u_x)_x = \lambda|u|^{p-2}u & x \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{2.20}$$

Its full set of eigenvalues consists in the sequence $\{\hat{\lambda}_{n,p}\}$:

$$\hat{\lambda}_{n,p} = (nt_1(p))^p, \qquad t_1(p) = \frac{2(p-1)^{\frac{1}{p}}}{p}\frac{\pi}{\sin\frac{\pi}{p}}, \qquad n = 1, 2, \ldots \tag{2.21}$$

Notice that $\lim_{p\to1} t_1(p) = 2$, hence $\lim_{p\to1}\hat{\lambda}_{n,p} = 2n$ for every $n \in \mathbb{N}$.

As for (2.19), the scaling $u(x) = \lambda^{\frac{1}{q-p}}v(t)$, $t = \lambda^{\frac{1}{p}}x$ leads to the equivalent form,

$$\begin{cases} -(|v_t|^{p-2}v_t)_t = |v|^{p-2}v - |v|^{q-2}v, & 0 < t < \lambda^{\frac{1}{p}}, \\ \quad v(0) = v(\lambda^{\frac{1}{p}}) = 0. \end{cases} \tag{2.22}$$

Solutions to (2.19) satisfy the estimate $\|u\|_\infty < \lambda^{\frac{1}{q-p}}$ and hence corresponding solutions to (2.22) verify $\|v\|_\infty < 1$.

To study (2.22) it is quite convenient to consider the following initial value problem:

$$\begin{cases} -(|v_t|^{p-2}v_t)_t = |v|^{p-2}v - |v|^{q-2}v, & t > 0, \\ \quad v(0) = \alpha, \quad v'(0) = 0, \end{cases} \tag{2.23}$$

where $0 < \alpha < 1$ plays the rôle of $\|v\|_\infty$ and is regarded as a parameter. It can be shown that to every α in this range corresponds a unique solution $v_0(t)$. Such a solution is described in terms of the function:

$$T(\alpha) = 2\{p'\}^{-\frac{1}{p}}\int_0^\alpha \frac{ds}{(V(\alpha) - V(s))^{\frac{1}{p}}}, \tag{2.24}$$

where $V(v) = \frac{1}{p}|v|^p - \frac{1}{q}|v|^q$. As key properties, $v_0(t)$ decreases from α to $-\alpha$ when $0 \leq t \leq T$, vanishes at $t = \dfrac{T}{2}$, is symmetric with respect to $t = T$ and becomes periodic with period $2T$.

Going back to (2.22), all the relevant information concerning this problem can be now expressed in terms of $v_0(t)$. In this regard, notice that

$$v_0\Big(-\frac{T(\alpha)}{2}\Big) = v_0\Big(-\frac{T(\alpha)}{2} + nT(\alpha)\Big) = 0\,, \qquad n \geq 1\,,$$

so that v_0 vanishes exactly at the points $t = -\frac{T(\alpha)}{2} + nT(\alpha)$. Hence, solutions to problem (2.22) can be viewed as a shift of v_0. It should be remarked that this solution v_0 depends on α, which plays the rôle of the amplitude of v_0. Taking these facts into account, one deduces the following features.

1) Function

$$v(t) = v_0\left(t - \frac{T(\alpha)}{2}\right), \tag{2.25}$$

solves (2.22) if and only if there exists $n \in \mathbb{N}$ such that α solves the equation:

$$\lambda^{\frac{1}{p}} = nT(\alpha)\,. \tag{2.26}$$

Moreover, (2.25) is the unique solution to (2.22) normalized so as $v_t(0) > 0$, fulfilling $\max v = \alpha$ and vanishing $n-1$ times in $(0, \lambda^{\frac{1}{p}})$.

2) Zeros of v are exactly $t = kT(\alpha)$, $0 \leq k \leq n$, v attains its maximum α at $t = \left(\frac{1}{2} + 2k\right)T(\alpha)$, $0 \leq k \leq \left[\frac{1}{2}(n - \frac{1}{2})\right]$ and its minimum $-\alpha$ at $t = \left(\frac{3}{2} + 2k\right)T(\alpha)$, $0 \leq k \leq \left[\frac{1}{2}(n - \frac{3}{2})\right]$ (here $[\cdot]$ denotes the integer part).

3) Function v is increasing in $\left[0, \frac{T(\alpha)}{2}\right]$ and is expressed in this interval by

$$(p')^{-\frac{1}{p}} \int_0^{v(t)} \frac{ds}{(V(\alpha) - V(s))^{\frac{1}{p}}} = t\,. \tag{2.27}$$

The left hand side can be alternatively written as $\psi_0(v(t))$ where $\psi_0 : [0,\alpha] \to \left[0, \frac{T(\alpha)}{2}\right]$ is the inverse of v.

Property 1) asserts that solving (2.22) amounts to discuss the solutions to (2.23). Next result is just introduced for this and further purposes of the present paper (see [40, Lemma1]).

Proposition 6. *Assume that* $1 < p \leq 2$. *Then function* $T : (0,1) \to \mathbb{R}$ *is continuous and increasing. In addition,*

$$T(0) := \lim_{\alpha \to 0+} T(\alpha) = t_1(p),$$

$t_1(p)$ *being the value in* (2.21), *while*

$$\lim_{\alpha \to 1-} T(\alpha) = \infty.$$

It should be remarked that eigenvalues $\hat\lambda_{n,p}$ to (2.20) can be expressed as $\hat\lambda_{n,p} = (nT(0))^p$. These are just the values referred to in the next statement where the solvability of equivalent problems (2.19) and (2.22) is completely described. Its proof reduces to analyze the solutions to (2.23) and is a direct consequence of Proposition 6.

Proposition 7. *Problems* (2.19) *and* (2.22) *admit a nontrivial solution if and only if,*

$$\lambda > \hat\lambda_{1,p}.$$

Moreover, to every

$$\hat{\lambda}_{n,p} < \lambda \le \hat{\lambda}_{n+1,p}$$

there corresponds exactly n *solutions* u *(respectively* v*) to* (2.19) *(r.* (2.22)*) satisfying the normalizing condition* $u_x(0) > 0$ *(r.* $v_t(0) > 0$*).*

The following auxiliary result address the limit behavior as p goes to 1 (see [40, Lemma 2] for a proof). It will be instrumental in the arguments of Sections 4 and 5.

Proposition 8. *Assume that* $1 < p \le 2$*. Then the following properties hold.*

a) *Function* T *introduced in* (2.24) *verifies:*

$$\overline{T}(\alpha) := \lim_{p\to 1} T(\alpha) = \frac{2}{1-\alpha^{q-1}} \qquad \text{for all } 0 < \alpha < 1. \tag{2.28}$$

b) *For* $0 < \alpha < 1$ *the function* v *defined by* (2.27)*, alternatively* $v = \psi_0^{-1}(t)$*, satisfies:*

$$\lim_{p\to 1} v(t) = \alpha,$$

where the convergence holds in $C^1\left(0, \frac{1}{1-\alpha^{q-1}}\right)$*.*

3. Radial solutions

In this section we study (2.6) in a ball $B_R = B(0,R) \subset \mathbb{R}^N$:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u - |u|^{q-2}u & x \in B_R, \\ u = 0 & x \in \partial B_R. \end{cases} \tag{3.29}$$

As was pointed out in Theorem 2 problem (2.6) exhibits a unique positive solution. Thus, it must be radial if $\Omega = B_R$. In fact, uniqueness is in principle necessary since the validity of Gidas–Ni–Nirenberg symmetry for equations $-\Delta_p u = f(u)$ requires suitable conditions on the nonlinear term f ([19]). Nevertheless we are further interested in solutions with both signs and therefore we focus our attention on radial solutions.

Assume that $\tilde{u} \in W_0^{1,p}(B)$ is a radially symmetric solution to (3.29), then $\tilde{u}$ can be a. e. identified with a function $u(r)$, $r = |x|$, such that $u, |u_r|^{p-2}u_r \in C^1[0,R]$, $u_r(0) = u(R) = 0$ and pointwise solves,

$$-(|u_r|^{p-2}u_r)_r - \frac{N-1}{r}|u_r|^{p-2}u_r = \lambda |u|^{p-2}u - |u|^{q-2}u, \qquad 0 < r < R. \tag{3.30}$$

Moreover, we are only concerned with the parameter range $1 < p \le 2$. In this case $u_r = |w|^{p'-2}w$ where $p' = \frac{p}{p-1}$ and $w = |u_r|^{p-2}u_r$. Thus $u \in C^2[0,R]$.

On the other hand, nontrivial solutions u satisfy the estimate $\|u\|_\infty \le \lambda^{\frac{1}{q-p}}$ (Theorem 2). Hence, by introducing the scaling

$$u(r) = \lambda^{\frac{1}{q-p}} v(t), \qquad t = \lambda^{\frac{1}{p}} r, \tag{3.31}$$

nontrivial solutions are sought in the range $\|v\|_\infty \le 1$. In addition, it should be remarked that the decreasing character of the energy $E(v, v_t)$ below (see (3.38) and (3.37)) implies that solutions u to (3.29) satisfying $u(0) > 0$ achieve their maximum at $r = 0$. Accordingly, $\alpha = v(0)$ is a natural parameter to describe normalized solutions (3.31). Observe that unlike the one dimensional case (problems (2.22) and (2.23)), a sift is not necessary now.

So, to handle (3.29) and (3.30), we are led to the initial value problem

$$\begin{cases} -(|v_t|^{p-2}v_t)_t - \frac{N-1}{t}|v_t|^{p-2}v_t = |v|^{p-2}v - |v|^{q-2}v, & t > 0, \\ v(0) = \alpha, \quad v_t(0) = 0. \end{cases} \tag{3.32}$$

with $0 < \alpha < 1$. Notice that when $\alpha = 1$ the solution to (3.32) is given by $v(t) = 1$.

Main features on (3.32) are next depicted. The sequence of *radial* eigenvalues $\lambda = \tilde{\lambda}_{n,p}$ to the Dirichlet problem in the ball B_R (see [2], [9], [44]),

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & x \in B_R, \\ u = 0 & x \in \partial B_R, \end{cases} \tag{3.33}$$

is involved in the next and forthcoming statements. Observe that $\tilde{\lambda}_{n,p} = R^{-p}\tilde{\lambda}_{n,p}(B_1)$ where $\tilde{\lambda}_{n,p}(B_1)$ are the Dirichlet eigenvalues of $-\Delta_p$ in the unit ball B_1.

Due to our purposes here, exponent p is restricted to the range $1 < p \leq 2$.

Lemma 9. *Assume that p, q satisfy (2.7) while $1 < p \leq 2$. Then for every $0 < \alpha < 1$ problem (3.32) satisfies the following properties.*

i) *It admits a unique solution $v = v(\cdot, \alpha)$ which is defined and C^2 in $[0, \infty)$. Moreover,*

$$\lim_{t\to\infty} (v(t), v_t(t)) = (0, 0). \tag{3.34}$$

ii) *Solution v is oscillatory, i. e., it exhibits a sequence of infinitely many simple zeros,*

$$0 < \theta_1 < \theta_2 < \cdots,$$

such that $\theta_n \to \infty$.

iii) *The asymptotic estimate*

$$\lim_{n\to\infty} \Delta\theta_n = t_1(p) \tag{3.35}$$

holds true, where $\Delta\theta_n = \theta_n - \theta_{n-1}$ and $t_1(p)$ is the value introduced in (2.21). In particular,

$$\theta_n \sim n t_1(p) \qquad \text{as } n \to \infty.$$

iv) *Every θ_n defines a continuous function of α and,*

$$\lim_{\alpha\to 0+} \theta_n(\alpha) = \omega_n \qquad \& \qquad \lim_{\alpha\to 1-} \theta_n(\alpha) = \infty, \tag{3.36}$$

where $\omega_n = \tilde{\lambda}_{n,p}(B_1)^{\frac{1}{p}}$ and $\tilde{\lambda}_{n,p}(B_1)$ is the n–th radial eigenvalue of $-\Delta_p$ in B_1. In addition, function θ_1 is increasing in α.

Proof. The existence and uniqueness of a local solution v to this problem have been largely discussed in [37] and [23]. That such a solution can be extended to all $t > 0$ is a consequence of the relation,

$$\frac{dE}{dt} = -\frac{N-1}{t}|v_t|^p, \tag{3.37}$$

which express the decaying along solutions of the total energy E defined by

$$E(v, v_t) = \frac{1}{p'}|v_t|^p + V(v), \quad \text{where } V(v) = \frac{1}{p}|v|^p - \frac{1}{q}|v|^q. \tag{3.38}$$

We next describe the oscillatory character of v. From the equation we get,

$$v' = -\varphi_{p'}\left(\int_0^t \left(\frac{\tau}{t}\right)^{N-1} f(v(\tau))\,d\tau\right).$$

Here $v' = v_t$, $\varphi_r(t) = |t|^{r-2}t$ and $f(v) = |v|^{p-2}v - |v|^{q-2}v$.

Observe that $f(v(t)) > 0$ implies $v'(t) < 0$. Hence, it follows from $0 < \alpha < 1$ that $f(v(t))$ must be positive for t small enough, wherewith $v' < 0$ and v decreases in the same interval. Next, we are showing that v must vanish at finite time. Otherwise $0 < l < v(t) < \alpha$ and so $f(v) \geq \delta > 0$ for all $t > 0$. Then one finds $v'(t) \leq -\varphi_{p'}\left(\frac{\delta}{N}\right)t^{p'-1}$ and consequently

$$v(t) \leq \alpha - \frac{1}{p'}\varphi_{p'}\left(\frac{\delta}{N}\right)t^{p'}, \qquad t > 0,$$

which is not possible. Thus a first zero $t =: \theta_1$ arises. In addition a first value $t =: \tau_1 > \theta_1$ exists such that $v'(\tau_1) = 0$. Otherwise, $v' < 0$ for all $t \geq \theta_1$ and $f(v(t)) \leq -\eta$ for $t \geq t_1 := \theta_1 + \varepsilon$. Then,

$$v'(t) \geq v'(t_1) + \varphi_{p'}\left(\frac{\eta}{N}\left(1 - \frac{t_1^N}{t^N}\right)\right)t^{p'}, \qquad t > t_1.$$

This is again not possible. Finally, by doing $v \to -v$, the conditions on $-v$ for $t \geq \tau_1$ are just the same as those for v at the beginning of the reasoning at $t = 0$. This shows that v exhibits infinitely many simple positive zeros θ_n (notice that $v'(\theta_n) \neq 0$). But v can not accumulate zeros in $(0,\infty)$ since the only solution to (3.32) with initial data $v(t_0) = v'(t_0) = 0$ is $v = 0$. Thus $\theta_n \to \infty$. Moreover, a careful review of the proof permits us concluding the existence of a *unique* critical point $\tau_n \in (\theta_n, \theta_{n+1})$ of v for every n. Additionally, the continuous dependence of v on α ([37], [23]) entails that every θ_n depends continuously on α.

To prove (3.34) assume on the contrary that $\inf_{\mathbb{R}^+} E > 0$. Then

$$\inf_{n\in\mathbb{N}} (-1)^n v(\tau_n) =: \inf_{n\in\mathbb{N}} \alpha_n = \underline{\alpha} > 0.$$

Moreover, $\inf_{\mathbb{R}^+} E = V(\underline{\alpha})$. Define

$$v_n(t) = (-1)^n v(t + \theta_n), \qquad t \geq 0.$$

Sequence $\{v_n\}$ is bounded in $C^1[0,b]$ for all $b > 0$. In addition, $v = v_n(t)$ solves

$$\begin{cases} -(|v_t|^{p-2}v_t)_t - \frac{N-1}{t+\theta_n}|v_t|^{p-2}v_t = |v|^{p-2}v - |v|^{q-2}v, & 0 < t < b, \\ v(0) = 0, \quad v_t(0) = (-1)^n v_t(\theta_n). \end{cases}$$

Let us point out that Ascoli–Arzelà's Theorem implies that

$$v_n(t) \to v_\infty(t), \qquad t \in [0,b],$$

in $C^1[0,b]$, for all $b > 0$. On the other hand, inequalities $E(\theta_n) \geq E(\tau_n) \geq E(\theta_{n+1})$ yield

$$\lim_{n\to\infty} E(\theta_n) = \lim_{n\to\infty} E(\tau_n).$$

Hence,

$$\lim_{n\to\infty} (-1)^n v_t(\theta_n) = \lim_{n\to\infty} (p'V(\alpha_n))^{\frac{1}{p}} = (p'V(\underline{\alpha}))^{\frac{1}{p}} =: v'_\infty > 0.$$

By taking into account both $(-1)^n v_t(\theta_n) \to v'_\infty$ and $\theta_n \to \infty$, together with the uniform convergence of functions v_n and their derivatives, it follows that $v = v_\infty(t)$ solves the problem,

$$\begin{cases} -(|v_t|^{p-2}v_t)_t = |v|^{p-2}v - |v|^{q-2}v, & 0 < t < b, \\ v(0) = 0, \quad v_t(0) = v'_\infty. \end{cases}$$

By choosing $b > T(\underline{\alpha})$, $T(\cdot)$ being the function defined in (2.24), we obtain

$$\lim_{n\to\infty} \Delta\theta_n = \lim_{n\to\infty} (\theta_{n+1} - \theta_n) = T(\underline{\alpha}).$$

We next observe that

$$\int_{\theta_n}^{\theta_{n+1}} \frac{N-1}{\tau} |v_t(\tau)|^p \, d\tau = -\int_{\theta_n}^{\theta_{n+1}} \frac{dE}{dt}(\tau) \, d\tau = E(\theta_n) - E(\theta_{n+1})$$

and, by proceeding as in telescoping series, it leads to

$$\sum_{n=1}^\infty \int_{\theta_n}^{\theta_{n+1}} \frac{N-1}{\tau} |v_t(\tau)|^p \, d\tau \le E(\theta_1) < \infty.$$

Performing a change of variable, we deduce

$$\sum_{n=1}^\infty \int_{\theta_n}^{\theta_{n+1}} \frac{N-1}{\tau} |v_t(\tau)|^p \, d\tau = \sum_{n=1}^\infty \int_0^{\Delta\theta_n} \frac{N-1}{s+\theta_n} |v_t(s+\theta_n)|^p \, ds =: \sum_{n=1}^\infty a_n,$$

and $\sum_{n=1}^\infty a_n$ converges. However,

$$a_n \sim \left\{ (N-1) \int_0^{T(\underline{\alpha})} |v'_\infty(s)|^p \, ds \right\} \frac{1}{\theta_n}, \qquad n \to \infty,$$

while by Cesàro's Theorem

$$\lim_{n\to\infty} \frac{\theta_n}{n} = T(\underline{\alpha}).$$

Thus the series $\sum_{n=1}^\infty a_n$ diverges. The contradiction has arisen from assuming that $\underline{\alpha} > 0$. Therefore, $\inf \alpha_n = 0$.

To show (3.35) set $\beta_n = (-1)^n v_t(\theta_n)$. Then, due to the fact that

$$\max\left\{ \frac{1}{p'} |v_t(t)|^p, V(v(t)) \right\} \le \frac{1}{p'} \beta_n^p, \qquad t \ge \theta_n,$$

together with $\beta_n \to 0$ and $V(v) \sim \frac{1}{p}|v|^p$ as $v \to 0$, we find that the sequence of functions,

$$\tilde v_n(t) = \frac{1}{\beta_n} v(t+\theta_n)$$

is bounded in $C^1[0,b]$ for all $b > 0$. On the other hand, $v = \tilde v_n(t)$ solves the problem,

$$\begin{cases} -(|v_t|^{p-2}v_t)_t - \frac{N-1}{t+\theta_n}|v_t|^{p-2}v_t = |v|^{p-2}v - \beta_n^{q-p}|v|^{q-2}v, & 0 < t < b, \\ v(0) = 0, \quad v_t(0) = 1. \end{cases}$$

A compactness argument again permits us ensuring that

$$\tilde v_n(t) \to \tilde v(t),$$

in $C^1[0,b]$ where $v = \tilde{v}(t)$ is the solution to problem

$$\begin{cases} -(|v_t|^{p-2}v_t)_t = |v|^{p-2}v, & 0 < t < b, \\ \quad v(0) = 0, \quad v_t(0) = 1. \end{cases}$$

This implies that

$$\lim_{n\to\infty} \Delta\theta_n = t_1(p),$$

as desired.

The fact $\theta_1(\alpha) \to \infty$ as $\alpha \to 1-$ follows from the continuous dependence of $v(\cdot,\alpha)$ on the parameter α (see [23]). On the other hand, that θ_1 increases with α is a consequence of the uniqueness of a positive solution to the Dirichlet problem,

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u - |u|^{q-2}u & x \in B, \\ u = c & x \in \partial B, \end{cases}$$

where $c \geq 0$ is constant and B an arbitrary ball (see [13]).

Finally and arguing as above, $v_\alpha(t) := \frac{1}{\alpha}v(t,\alpha)$ solves,

$$\begin{cases} -(|v_t|^{p-2}v_t)_t - \frac{N-1}{t}|v_t|^{p-2}v_t = |v|^{p-2}v - \alpha^{q-p}|v|^{q-2}v, & 0 < t < b, \\ \quad v(0) = 1, \quad v_t(0) = 0, \end{cases}$$

and in the limit as $\alpha \to 0+$, v_α converges in $C^1[0,b]$ for all $b > 0$ to the solution $\phi(t)$ to

$$\begin{cases} -(|v_t|^{p-2}v_t)_t - \frac{N-1}{t}|v_t|^{p-2}v_t = |v|^{p-2}v, & 0 < t < b \\ \quad v(0) = 1, \quad v_t(0) = 0. \end{cases}$$

It is well–known that ϕ exhibits a sequence of positive zeros $\omega_n \to \infty$ and that the sequence $\tilde{\lambda}_{n,p} = \omega_n^p$ just defines the eigenvalues of $-\Delta_p$ in B_1 ([9], [41]). On the other hand, the convergence $v_\alpha \to \phi$ in C^1 together with the simplicity of all of the zeros involved entail that $\theta_n(\alpha) \to \omega_n$ as $\alpha \to 0+$ for all $n \in \mathbb{N}$. □

Theorem 10. *Assume that* $1 < p \leq 2$. *Then, problem* (3.29) *exhibits the following properties.*

i) [Range and amplitude] *Nontrivial solutions u are only possible when $\lambda > \tilde{\lambda}_{1,p}$ while their* normalized amplitude*,*

$$\alpha := \lambda^{-\frac{1}{q-p}}\|u\|_\infty,$$

satisfies $0 < \alpha < 1$.

ii) [Positive solutions] *There exists a* unique *positive (radial) solution $u_{1,\lambda}$ for all $\lambda > \tilde{\lambda}_{1,p}$. Moreover,*

$$\|u_{1,\lambda}\|_\infty \to 0 \quad \text{as } \lambda \to \tilde{\lambda}_{1,p} \qquad \& \qquad \lambda^{-\frac{1}{q-p}}\|u_{1,\lambda}\|_\infty \to 1 \quad \text{as } \lambda \to \infty. \tag{3.39}$$

iii) [Existence of branches] *For every $n \geq 2$ two symmetric families $\pm u_{n,\lambda}(r)$ of nontrivial radial solutions exist which are* exactly *defined for all $\lambda > \tilde{\lambda}_{n,p}$ and satisfy,*

$$\|u_{n,\lambda}\|_\infty \to 0 \quad \text{as } \lambda \to \tilde{\lambda}_{n,p} \qquad \& \qquad \lambda^{-\frac{1}{q-p}}\|u_{n,\lambda}\|_\infty \to 1 \quad \text{as } \lambda \to \infty. \tag{3.40}$$

iv) [Nodal properties] *Every solution $u_{n,\lambda}(r)$ satisfies $u_{n,\lambda}(0) > 0$ and vanishes exactly at $n-1$ values $r_k \in (0,R)$.*

v) [Continuity] *The n–th family $u_{n,\lambda}$ can be globally parameterized, in terms of the normalized amplitude $\alpha \in [0,1)$, as a continuous curve*

$$(\lambda, u) = (\lambda_n(\alpha), u_n(\cdot, \alpha))$$

in $\mathbb{R} \times C^2[0,R]$, that is, $u_{n,\lambda} = u_n(\cdot,\alpha)$ when $\lambda = \lambda_n(\alpha)$. Moreover,

$$\lambda_n(\alpha) > \tilde{\lambda}_{n,p}, \qquad \textit{for all } 0 < \alpha < 1. \tag{3.41}$$

vi) [Uniqueness] *Let u be a nontrivial solution to* (3.29). *Then u belongs to some of the families $\pm u_{n,\lambda}$, $n \in \mathbb{N}$, introduced in ii) and iii).*

Proof. According to the change (3.31) a nontrivial radial solution u to (3.29) is represented as:

$$u(r) = \lambda^{\frac{1}{q-p}} v(\lambda^{\frac{1}{p}} r, \alpha), \tag{3.42}$$

where $0 < \alpha < 1$ and

$$\lambda^{\frac{1}{p}} R = \theta_n(\alpha), \tag{3.43}$$

for some $n \in \mathbb{N}$. Equations (3.42), (3.43) define a continuous curve of solutions $(\lambda_n(\alpha), u_n(\cdot,\alpha))$ parameterized in $\alpha \in (0,1)$. This proves the first assertion in v) while (3.41) is a consequence of inequality (4.53) to be shown in next section. Notice that this curve can be alternatively represented as a (possibly multivalued) family $u_{n,\lambda}$ when λ is regarded as the governing parameter.

From (3.43) one finds that $u_{n,\lambda}$ is defined for $\lambda^{\frac{1}{p}} R > \omega_n$ while the asymptotic behaviors in either (3.39) or (3.40) are a consequence of *iv)* in Lemma 9. In addition, every solution in $u_{n,\lambda}$ vanishes at,

$$r_k = R\frac{\theta_k(\alpha)}{\theta_n(\alpha)}, \qquad 1 \le k \le n-1.$$

The uniqueness of a positive solution to (3.29) was already established in Theorem 2.

The characterization of nontrivial solutions asserted in vi) is achieved when such solutions are observed as solving the initial value problem (3.32). □

Remark 6. First limits in (3.39) and (3.40) assert that the n–th family bifurcates from $u = 0$ at $\lambda = \tilde{\lambda}_{n,p}$. It was stated in [24] (see also [39]) that such a bifurcation locally occurs in the direction $\lambda > \tilde{\lambda}_{n,p}$. However, inequality (3.41) substantially improves this result since it implies that $u_{n,\lambda}$ is only defined when $\lambda > \tilde{\lambda}_{n,p}$.

Remark 7. In the regime $p > 2$, radial solutions u to (3.29) may develop a central core $\{u = \pm\lambda^{\frac{1}{q-p}}\}$ as λ is large.

4. Limit as $p \to 1$: direct approach

It this section the more subtle question of finding the limit profiles as $p \to 1+$ of the branches of solutions $u_{n,\lambda}$ of Theorem 10 is addressed. Our first results provide some partial answers to this problem.

In the forthcoming statements a reference to p is incorporated to the notation whenever it is necessary. For instance $v_p(t,\alpha)$ stands for the solutions to (3.32) while $\theta_{n,p}(\alpha)$ designates its n–th zero. They are just new names for the former $v(t,\alpha)$ and $\theta_n(\alpha)$, respectively.

Lemma 11. *Let $v_p(t,\alpha)$ be the solution to* (3.32) *and let $\theta_{1,p}(\alpha)$ designate its first zero. Then*

$$\frac{1}{1-\alpha^{q-1}} \le \varliminf_{p\to 1+} \theta_{1,p}(\alpha) \le \varlimsup_{p\to 1+} \theta_{1,p}(\alpha) \le \frac{N}{1-\alpha^{q-1}}. \tag{4.44}$$

Moreover,

$$v_p(\cdot,\alpha) \to \alpha \qquad \text{as } p \to 1+, \tag{4.45}$$

the convergence being in the topology of $C^1\left[0, \frac{\overline{T}(\alpha)}{2}\right)$ where $\overline{T}(\alpha)$ is the value introduced in (2.28).

Proof. The energy E in (3.38) is decreasing along $v(t) = v_p(t,\alpha)$ while relation

$$|v'|^{p-2}v' = -\int_0^t \left(\frac{\tau}{t}\right)^{N-1} f(v(\tau))\, d\tau, \tag{4.46}$$

where $f(v) = |v|^{p-2}v - |v|^{q-2}v$, reveals that v decreases up to $t = \theta_{1,p}$ (α is removed to brief). In fact, v decreases until its first critical point $t = \tau_1 \in (\theta_{1,p}, \theta_{2,p})$. Thus,

$$-\frac{1}{(p')^{\frac{1}{p}}}(-v') < (V(\alpha) - V(v))^{\frac{1}{p}},$$

which implies that,

$$\frac{1}{(p')^{\frac{1}{p}}} \int_{v(t)}^{\alpha} \frac{ds}{(V(\alpha)-V(s))^{\frac{1}{p}}} < t, \qquad 0 < t < \tau_1. \tag{4.47}$$

In particular, by setting $t = \theta_{1,p}$ we get

$$\theta_{1,p} > \frac{T(\alpha)}{2}.$$

Hence, the first inequality in (4.44) follows by taking limits and observing that

$$\lim_{p\to 1} \frac{T(\alpha)}{2} = \frac{\overline{T}(\alpha)}{2} = \frac{1}{1-\alpha^{q-1}}.$$

Set now,

$$\psi_0(v) = \frac{1}{(p')^{\frac{1}{p}}} \int_0^v \frac{ds}{(V(\alpha)-V(s))^{\frac{1}{p}}}, \qquad 0 < v < \alpha.$$

Function $v = \psi_0^{-1}(t)$, $t \in \left[0, \frac{T(\alpha)}{2}\right]$, defines the solution to equation in (2.22) having $v'(0) > 0$ and $\max v = \alpha > 0$ (Section 2.4). On the other hand, (4.47) implies that

$$v(t) > \psi_0^{-1}\left(\frac{T(\alpha)}{2} - t\right), \qquad 0 < t < \frac{T(\alpha)}{2},$$

while Proposition 8 asserts $\psi_0^{-1} \to \alpha$ as $p \to 1+$.

In addition, equation (4.46) yields,

$$|v'|^{p-2}v' \to -(1-\alpha^{q-1})\frac{t}{N} \qquad \text{as } p \to 1+,$$

for $0 < t < \frac{\overline{T}(\alpha)}{2}$. All these facts put together entail (4.45).

The complementary upper estimate in (4.44) is a consequence of Theorem 13 below. □

Our next result states the finiteness of the limits,

$$\bar{\theta}_n^-(\alpha) = \underline{\lim}_{p\to 1+} \theta_{n,p}(\alpha), \qquad \bar{\theta}_n^+(\alpha) = \overline{\lim}_{p\to 1+} \theta_{n,p}(\alpha), \tag{4.48}$$

for all $n \in \mathbb{N}$. This is a quite delicate question. Its proof relies upon the following result, one of the featured achievements in [41]. Notice that relevant quantities, e. g. the radial eigenvalues $\tilde{\lambda}_{n,p}$, are labeled with subindex p to stress its dependence on p.

Theorem 12. *Let*

$$\tilde{\lambda}_{n,p} = (\omega_{n,p})^p, \qquad n \in \mathbb{N},$$

be the sequence of Dirichlet radial eigenvalues of the p–Laplacian in the unit ball $B_1 \subset \mathbb{R}^N$. Then, the limits

$$\lim_{p\to 1} \omega_{n,p} = \bar{\omega}_n,$$

exist for all n. Moreover, $\bar{\omega}_n$ is increasing, $\bar{\omega}_n \to \infty$ and,

$$\lim_{n\to\infty} \Delta\bar{\omega}_n = 2, \tag{4.49}$$

where $\Delta\bar{\omega}_n = \bar{\omega}_n - \bar{\omega}_{n-1}$.

We now prove that limits in (4.48) are finite.

Theorem 13. *For all $n \in \mathbb{N}$ and $0 \le \alpha < 1$, limits $\bar{\theta}_n^\pm(\alpha)$ in (4.48) are finite. Moreover,*

$$\bar{\omega}_n \le \bar{\theta}_n^-(\alpha) \le \bar{\theta}_n^+(\alpha) \le \frac{1}{1-\alpha^{q-1}}\bar{\omega}_n. \tag{4.50}$$

In particular $\bar{\theta}_n^\pm(\alpha) \to \infty$ as $n \to \infty$.

Proof. Write again $\theta_n = \theta_{n,p}(\alpha)$ for short. Define,

$$u(r) = v_p(\theta_n r, \alpha) \qquad 0 \le r \le 1.$$

Then, u solves the eigenvalue problem

$$\begin{cases} -\mathcal{L}_p u + q(r)|u|^{p-2}u = \lambda |u|^{p-2}u & 0 < r < 1 \\ \quad u'(0) = 0,\ u(1) = 0, \end{cases} \tag{4.51}$$

where operator $\mathcal{L}_p$ is defined by (the radial p–Laplacian):

$$\mathcal{L}_p u = (|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u',$$

the weight q is defined by

$$q = -\theta_n^p(1 - |u|^{q-p}),$$

and $\lambda = 0$. Notice now that u vanishes exactly at $n-1$ points in the interval $(0,1)$ and that problem (4.51) has a unique eigenvalue exhibiting an eigenfunction with that property ([44]). Namely, the n–th eigenvalue $\lambda_n(q)$. Therefore,

$$\lambda_n(q) = \lambda_n(-\theta_n^p(1 - |u|^{q-p})) = 0.$$

Now,

$$-\theta_n^p \le -\theta_n^p(1-|u|^{q-p}) \le -\theta_n^p(1-\alpha^{q-p}). \tag{4.52}$$

But $\lambda_n(q)$ is increasing in the weight q ([44]). Thus:

$$\lambda_n(-\theta_n^p) < 0 < \lambda_n(-\theta_n^p(1-\alpha^{q-p})).$$

The first inequality implies that

$$\omega_{n,p}^p < \theta_{n,p}(\alpha). \tag{4.53}$$

Thus,

$$\lim_{p\to 1} \omega_{n,p} \leq \underline{\lim}_{p\to 1} \theta_{n,p}(\alpha),$$

and so,

$$\bar{\omega}_n \leq \bar{\theta}_n^-(\alpha).$$

The second inequality entails,

$$\theta_{n,p}(\alpha)^p < \frac{1}{1-\alpha^{q-p}}\omega_{n,p},$$

whence,

$$\bar{\theta}_n^+(\alpha) \leq \frac{1}{1-\alpha^{q-1}}\bar{\omega}_n.$$

To achieve (4.44) in Lemma 11 observe that $\bar{\omega}_1 = N$ ([41]). □

We now analyze the gap between the values $\bar{\theta}_n(\alpha)^\pm$ and its behavior as n becomes large.

Theorem 14. *Limits $\bar{\theta}_n^\pm(\alpha)$ in (4.48) satisfy,*

$$\bar{\theta}_{n-1}^+(\alpha) < \bar{\theta}_n^-(\alpha),$$

for all $\in \mathbb{N}$. Moreover, for n fixed

$$\Delta\underline{\theta}_n(\alpha) := \underline{\lim}_{p\to 1} \Delta\theta_{n,p}(\alpha) \geq 1, \tag{4.54}$$

where $\Delta\theta_{n,p}(\alpha) = \theta_{n,p}(\alpha) - \theta_{n-1,p}(\alpha)$, while

$$\Delta\bar{\theta}_n(\alpha) := \overline{\lim}_{p\to 1} \Delta\theta_{n,p}(\alpha) \leq \frac{2}{(1-\alpha^{q-1})^N}\left(\frac{\bar{\omega}_n}{\bar{\omega}_{n-1}}\right)^{N-1}. \tag{4.55}$$

Furthermore,

$$\overline{\lim}_{n\to\infty} \Delta\bar{\theta}_n(\alpha) \leq \frac{2}{(1-\alpha^{q-1})^N}. \tag{4.56}$$

Proof. A variant of the argument in the proof of Theorem 13 is going to be employed. Define $\lambda_1(m,q)$ the first eigenvalue of the problem,

$$\begin{cases} -(m(t)|u'|^{p-2}u')' + q(t)|u|^{p-2}u = \lambda|u|^{p-2}u & t \in J \\ u_{|\partial J} = 0, \end{cases} \tag{4.57}$$

where $J = (a,b)$ is a finite interval, $m, q \in C(\overline{J})$. It is well–known that $\lambda_{1,p}(m,q)$ is increasing in m and q.

Let $v(t)$ be the solution to (3.32) (subscript p will be omitted whenever possible) and consider the particular case of problem (4.57) where $m = t^{N-1}$, $q = -t^{N-1}(1-|v|^{q-p})$ and $J = J_n := (\theta_{n-1}, \theta_n)$. Then it holds that its main eigenvalue is:

$$\lambda_{1,p}(m,q) = 0,$$

and has $u = v_{|J_n}$ as an associated main eigenfunction. Setting

$$\alpha_{n-1} = \max_{J_n} |v|,$$

the estimates

$$-\theta_n^{N-1} \leq q(t) \leq -\theta_{n-1}^{N-1}(1-\alpha_{n-1}^{q-p}) \leq -\theta_{n-1}^{N-1}(1-\alpha^{q-p}), \tag{4.58}$$

hold true.

The monotonicity of λ_1 in (m,q) then implies,

$$\theta_{n-1}^{N-1}\lambda_{1,p}(J_n)-\theta_n^{N-1}\le 0\le \theta_n^{N-1}\lambda_{1,p}(J_n)-\theta_{n-1}^{N-1}(1-\alpha^{q-p}), \tag{4.59}$$

where $\lambda_{1,p}(J_n)=\lambda_{1,p}(m,q)$ for the choices $m=1$, $q=0$. Thus,

$$\lambda_{1,p}(J_n)=\frac{t_1(p)^p}{(\Delta\theta_n)^p},$$

$t_1(p)$ being the value provided in (2.21).

The second inequality in (4.59) says that

$$\frac{(\Delta\theta_n)^p}{t_1(p)^p}\le\frac{1}{(1-\alpha^{q-p})}\left(\frac{\theta_n}{\theta_{n-1}}\right)^{N-1}.$$

By taking lim sup as $p\to 1$ we find,

$$\overline{\lim_{p\to 1}}\,\Delta\theta_n\le\frac{2}{(1-\alpha^{q-1})}\left(\frac{\bar\theta_n^+(\alpha)}{\bar\theta_{n-1}^-(\alpha)}\right)^{N-1}\le\frac{2}{(1-\alpha^{q-1})^N}\left(\frac{\bar\omega_n}{\bar\omega_{n-1}}\right)^{N-1},$$

which proves (4.55).

Estimate (4.56) follows from (4.55) by noticing (Theorem 12) that $\bar\omega_n-\bar\omega_{n-1}\to 2$ as $n\to\infty$.

As for (4.54) suppose that $v>0$ in J_n (otherwise replace $v\to -v$), set as above $\alpha_{n-1}=\max_{J_n} v$ and τ_{n-1} the critical point in J_n. From the fact that v decreases in $[\tau_{n-1},\theta_n]$ an that

$$\frac{1}{p'}(-v)^p+V(v)<V(\alpha_{n-1}),\qquad t\in(\tau_{n-1},\theta_n],$$

we obtain that

$$\frac{1}{(p')^{\frac1p}}\int_{v(t)}^{\alpha_{n-1}}\frac{ds}{(V(\alpha_{n-1})-V(s))^{\frac1p}}<t-\tau_{n-1},$$

for $\tau_{n-1}<t<\theta_n$. In particular,

$$1=\frac12 T(0)<\frac12 T(\alpha_{n-1})<\theta_n-\tau_{n-1}, \tag{4.60}$$

whence (4.54) follows by taking limits as $p\to 1$. □

Remark 8. Upper estimate in (4.50) and the corresponding ones in (4.54) and (4.55) can slightly be refined. By observing that the upper estimate in (4.58) may be replaced by

$$-\theta_n^p(1-|u|^{q-p})\le -\theta_n^p(1-\alpha_{n-1}^{q-p}),$$

we obtain the sharper one

$$\bar\theta_n^+(\alpha)\le\frac{1}{1-\bar\alpha_{n-1}^{q-1}}\bar\omega_n,$$

where $\bar\alpha_{n-1}=\overline{\lim}_{p\to 1}\alpha_{n-1}$. This in turn implies,

$$\overline{\lim_{p\to 1}}\,\Delta\theta_{n,p}(\alpha)\le\frac{2}{(1-\bar\alpha_{n-1}^{q-1})^{N-1}}\left(\frac{\bar\omega_n}{\bar\omega_{n-1}}\right)^{N-1},$$

a better alternative than (4.55). Moreover, since $\lim_{n\to\infty} \alpha_n = 0$ it should be expected that $\lim_{n\to\infty} \bar{\alpha}_n = 0$. This together with (4.56) would lead to:

$$\overline{\lim_{n\to\infty}} \Delta\bar{\theta}_n(\alpha) \leq 2.$$

Similarly, it follows from (4.60) that for fixed n,

$$\underline{\lim}_{p\to 1} \Delta\theta_{n,p}(\alpha) \geq \frac{1}{1 - \underline{\alpha}_{n-1}^{q-1}},$$

with $\underline{\alpha}_{n-1} = \underline{\lim}_{p\to 1} \alpha_{n-1}$.

We stress that in Section 5 a sharpened version of all of the previous estimates will be stated.

Remark 9. Numerical simulations in Figures 1 and 2 strongly suggest that all of numbers $\theta_{n,p}(\alpha)$ and $\alpha_{n,p}$ stabilize to single values as $p \to 1+$. In addition, solution $v_p(t,\alpha)$ develops flat patterns between consecutive values of the limits of $\theta_{n,p}$. This issue is addressed in detail in the next section.

5. Limit as $p \to 1$: the BV framework

Two main objectives of this work are now to be accomplished. First, to show the existence of the limit as $p \to 1$ of the solutions $u_{n,\lambda}$ to (3.29) obtained in Theorem 10. Second, to prove that these limits define solutions $\bar{u}_{n,\lambda}$ to the limit problem,

$$\begin{cases} -\text{div}\left(\dfrac{Du}{|Du|}\right) u = \lambda \dfrac{u}{|u|} - |u|^{q-2}u, & x \in B_R, \\ u = 0, & x \in \partial B_R. \end{cases} \tag{5.61}$$

Resulting families $\bar{u}_{n,\lambda}$ give rise to continuous curves bifurcating from the eigenvalues to $-\Delta_1$. In fact, radial eigenvalues $\lambda = \bar{\lambda}_n$ to

$$\begin{cases} -\text{div}\left(\dfrac{Du}{|Du|}\right) u = \lambda \dfrac{u}{|u|}, & x \in B_R, \\ u = 0, & x \in \partial B_R. \end{cases} \tag{5.62}$$

have been recently introduced in the form ([41]):

$$\bar{\lambda}_n = \lim_{p\to 1} \tilde{\lambda}_{n,p} = R^{-1}\bar{\omega}_n,$$

$\bar{\omega}_n$ being the values referred to in Theorem 12.

As it is the case when the operator $-\Delta_1$ is involved in the equations, problem (5.61) has a tendency to exhibit an uncontrolled amount of solutions. See for instance [8] dealing with eigenvalue problems, [40] on the one dimensional case (2.19) or Remark 14 below. To identify proper solutions, we handle an energy condition (see (5.67) below) similar to that introduced in [41].

In order to formulate a uniqueness result, we also require suitable symmetry restrictions on the solutions.

Definition 15. *A solution $u \in BV(B_R)$ to* (5.61) *is said to be radial if aside from u, function β and field $\mathbf{z}$ referred to in Definition 3 are also radial. In the latter case this means that,*

$$\mathbf{z} = \tilde{w}(r)\frac{x}{r}, \qquad \lim_{r\to 0} \tilde{w}(r) = 0, \tag{5.63}$$

where $\tilde{w} \in L^\infty(0,R)$.

Remark 10. Condition (5.63) is reminiscent of the fact that for a radial C^1 function $u(x) = v(r)$ one has $\nabla u = v'(r)\dfrac{x}{r}$ where $v'(0) = 0$.

In the next statement, solutions to (5.61) are understood to be radial in the sense of Definition 15. The continuity mentioned in the point iii) below is regarded in the sense of the *strict topology* of the space $BV(B_R)$ ([1]).

Theorem 16. *The problem* (5.61) *exhibits the following properties.*

i) [Range of existence and amplitude estimate] *Nontrivial solutions* $u \in BV(B_R) \cap L^q(B_R)$ *are only possible if* $\lambda > \bar{\lambda}_1$. *Normalized amplitude* $\alpha = \lambda^{-\frac{1}{q-1}}\|u\|_\infty$ *of solutions satisfies* $0 < \alpha < 1$.

ii) [Existence] *To every eigenvalue* $\bar{\lambda}_n$ *there corresponds a symmetric family* $\pm\bar{u}_{n,\lambda}$ *of nontrivial solutions bifurcating from* $u = 0$ *at* $\bar{\lambda}_n$ *which is* exactly *defined for* $\lambda > \bar{\lambda}_n$. *Moreover, the normalized amplitude of* $\bar{u}_{n,\lambda}$ *satisfies*

$$\lim_{\lambda \to \infty} \lambda^{-\frac{1}{q-1}}\|\bar{u}_{n,\lambda}\|_\infty = 1. \tag{5.64}$$

iii) [Continuity of branches] *Family* $\bar{u}_{n,\lambda}$ *can be represented as a* continuous *curve*

$$(\lambda, u) = (\bar{\lambda}_n(\alpha), \bar{u}_n(\cdot, \alpha)) \in \mathbb{R} \times BV(B_R),$$

when parameterized by the normalized amplitude $0 < \alpha < 1$. *This means that* $\bar{u}_{n,\lambda} = \bar{u}_n(\cdot, \alpha)$ *when* $\lambda = \bar{\lambda}_n(\alpha)$. *More precisely,*

$$\bar{\lambda}_n(\alpha) = R^{-1}\bar{\theta}_n(\alpha), \qquad \bar{u}_n(r, \alpha) = \lambda^{\frac{1}{q-1}} \sum_{k=1}^{n} (-1)^k \alpha_{k-1} \chi_{I_k}(\lambda r), \tag{5.65}$$

$\chi_{I_k}(t)$ *being the characteristic function of the interval,*

$$I_k = \left(R\frac{\bar{\theta}_{k-1}(\alpha)}{\bar{\theta}_n(\alpha)}, R\frac{\bar{\theta}_k(\alpha)}{\bar{\theta}_n(\alpha)} \right),$$

and where α_k, $\bar{\theta}_k$ *are* smooth *functions of the amplitude* α.

iv) [Convergence of branches and zeros] *For every* n *and* $0 < \alpha < 1$

$$\lim_{p \to 1} \theta_n(\alpha) = \bar{\theta}_n(\alpha), \tag{5.66}$$

while the family $u_{n,\lambda}$ *converges to the family* $\bar{u}_{n,\lambda}$ *as* $p \to 1+$ *in the sense:*

$$\lim_{p \to 1} (\lambda_n(\alpha), u_n(\cdot, \alpha)) = (\bar{\lambda}_n(\alpha), \bar{u}_n(\cdot, \alpha)),$$

λ_n, u_n *being the functions introduced in v) of Theorem 10.*

v) [Uniqueness] Every *nontrivial solution* u *fulfilling the 'energy' condition,*

$$\frac{d}{dr}\left(\lambda|u| - \frac{|u|^q}{q}\right) = -\frac{N-1}{r}|u_r| \qquad \text{in } \mathcal{D}'(0, R), \tag{5.67}$$

necessarily belongs to some of the previous families $\pm\bar{u}_{n,\lambda}$.

The remaining of this section is devoted to the proof of Theorem 16. Section 5.2 states a compactness result which entails the existence of solutions. The key uniqueness result is presented in Section 5.3.

5.1. **An initial value problem.** We are mimicking the existence analysis in Section 3. Our reference initial value problem (3.32) there:

$$\begin{cases} -(|v_t|^{p-2}v_t)_t - \frac{N-1}{t}|v_t|^{p-2}v_t = |v|^{p-2}v - |v|^{q-2}v, & t>0, \\ v(0)=\alpha, \quad v_t(0)=0. \end{cases}$$

is more conveniently written now in the equivalent form

$$\begin{cases} w = |v'|^{p-2}v' & v(0)=\alpha, \\ w' = -f(v) - \frac{N-1}{t}w & w(0)=0, \end{cases} \qquad t>0, \tag{5.68}$$

where $f(v) = |v|^{p-2}v - |v|^{q-2}v$, $0<\alpha<1$. In addition, notation $' = \dfrac{d}{dt}$ will be often used with the meaning $v' = v_t$.

A formal expression for the limit problem of (5.68) as $p \to 1$ reads as follows,

$$\begin{cases} w \in \text{sign }(v') & v(0)=\alpha, \\ -\left(w' + \frac{N-1}{t}w - |v|^{q-2}v\right) \in \text{sign }(v) & w(0)=0, \end{cases} \qquad t>0, \tag{5.69}$$

where v, w vary in suitable spaces of functions defined in $(0,\infty)$ and equations are understood in distributional sense. Precise details to clarify the meaning of a solution to (5.69) are next explained. Of course, we are keeping in mind Definition 3.

As it turns out from the results below, a convenient space for the solutions (v,w) to (5.69) is

$$BV_{\text{loc}}(0,\infty) \times W^{1,\infty}_{\text{loc}}(0,\infty),$$

where we denote,

$$BV_{\text{loc}}(0,\infty) = \bigcap_{b>0} BV(0,b), \qquad W^{1,\infty}_{\text{loc}}(0,\infty) = \bigcap_{b>0} W^{1,\infty}(0,b).$$

According to Section 2.3, a function u belongs to $BV(I)$ with $I=(0,b)$ if $u \in L^1(I)$ and its distributional derivative u' is a Radon measure with finite total variation $|u'|(I)$. As customary, $W^{1,\infty}(I)$ denotes the space of functions $w \in L^\infty(I)$ with a weak derivative $w' \in L^\infty(I)$.

It can be shown that every function $u \in BV(I)$ can be identified a. e. with a function $\tilde{u}$ which is of bounded variation in the classical sense in I (see [1]). The identification of u with $\tilde{u}$ is henceforth assumed without further comments. In particular $BV(I) \subset L^\infty(I)$.

On the other hand, we point out that the first equation in (5.69) will be satisfied in the sense that the total variation $|v'|$ is equal to the product wv'. When $u \in BV(I)$ and $w \in W^{1,\infty}(I)$ such a product is naturally defined as $\langle wv', \varphi\rangle = \langle v', w\varphi\rangle$, $\varphi \in C_0^\infty(I)$, since w is a Lipschitz function. Moreover, by suitably approximating w, it follows from the definition of v' in the sense of distributions that

$$\int_I \varphi w v' = -\int_I v(w'\varphi + w\varphi') \qquad \text{for all } \varphi \in C_0^\infty(I)\,.$$

Hence, wv' coincides with the definition of the pairing (w,v') introduced in Section 2.3 (see (2.12)). It should be also recalled that (w,v') is a Radon measure in I such that,

$$|(w,v')|(J) \le \|w\|_{\infty,J}|v'|(J),$$

for all open interval $J \subset I$, where $|\cdot|$ means the total variation of the corresponding measure.

Equation (3.37) for the dissipation of the energy

$$E_p(v, v') = \frac{1}{p'}|v'|^p + \frac{1}{p}|v|^p - \frac{1}{q}|v|^q,$$

plays a substantial rôle in the forthcoming considerations. More properly the formal limit equation of (3.37) as $p \to 1$ will play such a rôle. This formal limit is given by

$$\frac{d}{dt}\left(|v| - \frac{1}{q}|v|^q\right) = -\frac{N-1}{t}|v'|. \tag{5.70}$$

For a function $v \in BV_{\rm loc}(0, \infty)$ equation (5.70) is understood in distributional sense. Notice that the power term is well defined as $v \in L^\infty(0, b)$ for each $b > 0$.

The next definition is an adaptation of a corresponding one in [41] where it was proposed for the study of the limit of the eigenvalue problem (3.33) as $p \to 1$.

Definition 17. *A couple of functions $(v, w) \in BV_{loc}(0, \infty) \times W^{1,\infty}_{loc}(0, \infty)$ defines a solution to (5.69) provided that the following conditions hold.*

i) *Function $\|w\|_\infty \le 1$ while,*

$$(w, v') = |v'| \qquad \text{in } \mathcal{D}'(0, \infty). \tag{5.71}$$

ii) *There exists $\beta \in L^\infty(0, \infty)$ satisfying $\|\beta\|_\infty \le 1$, $\beta v = |v|$ and such that v solves the equation*

$$w' + \frac{N-1}{t}w - |v|^{q-2}v = -\beta, \qquad \text{in } \mathcal{D}'(0, \infty)\,. \tag{5.72}$$

iii) *Initial conditions are fulfilled in the following sense,*

$$v(0+) = \alpha, \qquad w(0) = 0.$$

Remark 11. We are showing in Section 5.4 that any radial solution in the sense of Definition 15 gives rise, up to scaling, to a solution of problem (5.69).

5.2. **Existence results.** Our next statement furnishes the existence of a solution to (5.69).

Theorem 18. *Fix $0 < \alpha < 1$ and for $1 < p \le 2$ let $(v_p, w_p) \in C^1([0, \infty))^2$ be the solution to the initial value problem (5.68). Then, up to subsequences,*

$$(v_p, w_p) \to (v, w) \qquad \text{as } p \to 1,$$

and (v, w) solves (5.69). More precisely, the following properties hold true.

i) *For each $b > 0$, $v_p \to v$ in $L^1((0, b), t^{N-1}\,dt)$ where $v \in BV_{loc}(0, \infty)$.*

ii) *There exists $\beta_1 \in L^\infty(0, \infty)$ with $\|\beta_1\|_\infty \le 1$ such that $|v_p|^{p-2}v_p \rightharpoonup \beta_1$ weakly in $L^s((0, b), t^{N-1}\,dt)$ for all $1 \le s < \infty$ and $b > 0$. Moreover,*

$$\beta_1 v = |v|.$$

iii) *$w_p \rightharpoonup w$ weakly in $L^s((0, b), t^{N-1}\,dt)$ for all $1 \le s < \infty$ and all $b > 0$, where $w \in L^\infty(0, \infty) \cap W^{1,\infty}_{loc}(0, \infty)$, $\|w\|_\infty \le 1$ and solves the equation,*

$$-w' - \frac{N-1}{t}w + |v|^{q-2}v = \beta_1, \qquad \text{in } \mathcal{D}'(0, \infty). \tag{5.73}$$

iv) *Identity $(w, v') = |v'|$ is fulfilled in $\mathcal{D}'(0, \infty)$.*

v) *$\|v\|_\infty = \alpha$ while the energy equation (5.70) is satisfied in the sense of $\mathcal{D}'(0, \infty)$.*

Remark 12. Convergence in i) actually holds in $L^s((0,b),t^{N-1}\,dt)$ for all $1\le s<\infty$.

Proof of Theorem 18. We begin by observing that $\bar\theta_n^+(\alpha)\to\infty$ as $n\to\infty$ (see Theorem 13). Thus, if we prove the desired claims on each interval $(0,\bar\theta_n^+(\alpha))$, then it will hold on every $(0,b)$ $(b>0)$.

Fix $n\in\mathbb{N}$ and set:

$$\lambda_p=\frac{\theta_{n,p}(\alpha)^p}{R^p},\qquad u_p(x)=\lambda_p^{\frac{1}{q-p}}v_p(\lambda_p^{\frac{1}{p}}r,\alpha),\tag{5.74}$$

$r=|x|$. Then u_p defines a solution to (3.29) with $\lambda=\lambda_p$. Take a suitable subsequence as $p\to1$ (denoted with the same index) to get $\lim\limits_{p\to1}\theta_{n,p}(\alpha)=\bar\theta_n^+(\alpha)$ and define $\lambda=\lim\limits_{p\to1}\lambda_p$. Now observe that hypotheses of Theorem 5 holds, so that (2.18) implies the estimate

$$\int_0^{\theta_{n,p}}|v_p'|^pt^{N-1}\,dt\le M,\tag{5.75}$$

from where, by Young's inequality, we deduce an estimate of $\{v_p\}$ in $BV(\sigma,\theta_{n,p})$ for all $\sigma>0$. On the other hand, applying Theorem 5, we may choose a further subsequence and find radial functions $u\in BV(B_R)$, $\beta\in L^\infty(B_R)$ and a field $\mathbf{z}\in L^\infty(B_R,\mathbb{R}^N)$ satisfying $\|\mathbf{z}\|_\infty\le1$, $\|\beta\|_\infty\le1$ so that assertions 1) to 4) in the theorem are satisfied. Thus, by extracting again a subsequence if necessary, we infer that $u_p(x)\to u(x)$ a.e. in B_R. Summarizing, a sequence p_m, *no depending* on n, can be found so that all of the previous limits hold true as $p_m\to1+$ (subindex m will be omitted).

In the sequel and by abuse of notation, $u_p(r)$ and $u(r)$ are replacing $u_p(x)$ and $u(x)$ when necessary. The same criterium will be applied to other possible radial functions.

We now set,

$$v(t,\alpha)=\lambda^{-\frac{1}{q-1}}u(\lambda^{-1}t),\qquad t\in(0,\bar\theta_n^+(\alpha))\,.$$

Assertions i) to v) are next to be verified. Explicit reference to α will be avoided whenever possible.

i) The L^1–convergence $u_p\to u$ implies

$$\lim_{p\to1}\int_0^R|\lambda_p^{\frac{1}{q-p}}v_p(\lambda_p^{\frac{1}{p}}r)-\lambda^{\frac{1}{q-1}}v(\lambda r)|r^{N-1}\,dr=0\,,$$

and so,

$$\int_0^{\theta_{n,p}}|\lambda_p^{\frac{1}{q-p}}v_p(t)-\lambda^{\frac{1}{q-1}}v(\sigma_pt)|t^{N-1}\,dt=o(1),\qquad\sigma_p=\lambda\lambda_p^{-\frac{1}{p}},\tag{5.76}$$

as $p\to1$. Since $\sigma_p\to1$ and $v(t)$ is continuous in $(0,\bar\theta_n^+)$ up to a numerable set we observe that,

$$\int_0^{\theta_{n,p}}|v(\sigma_pt)-v(t)|t^{N-1}\,dt=o(1),\qquad\text{as }p\to1.$$

From the estimate,

$$\begin{aligned}\lambda^{\frac{1}{q-1}}|v_p(t)-v(t)| & \\ \leq |\lambda^{\frac{1}{q-1}}-\lambda_p^{\frac{1}{q-p}}||v_p(t)| &+ |\lambda_p^{\frac{1}{q-p}}v_p(t)-\lambda^{\frac{1}{q-1}}v(\sigma_p t)| \\ &+ \lambda^{\frac{1}{q-1}}|v(\sigma_p t)-v(t)|,\end{aligned}$$

and (5.76) we obtain that,

$$\int_0^{\theta_{n,p}} |v_p(t)-v(t)|t^{N-1}\,dt = o(1), \qquad \text{as } p\to 1.$$

As $\theta_{n,p}\to\bar\theta_n^+$ then we find that $v_p\to v$ in $L^1((0,\bar\theta_n^+),t^{N-1}dt)$. This L^1–convergence jointly with our BV–estimate gives $v\in BV(\sigma,\bar\theta_n^+)$ for all $\sigma>0$. We recall that Lemma 11 yields $v(t)=\alpha$ on $\left(0,\frac{1}{1-\alpha^{q-1}}\right)$. Therefore, $v\in BV(0,\bar\theta_n^+(\alpha))$. Finally, as $p_m\to 1$ no depends on n, then v is actually defined on the whole interval $(0,+\infty)$ and $v\in BV_{loc}(0,\infty)$.

ii) By putting $\beta_1(t)=\beta(\lambda^{-1}t)$ one finds that $\beta_1\in L^\infty(0,\bar\theta_n^+(\alpha))$, $\|\beta_1\|_\infty\leq 1$, while the identity $\beta_1 v=|v|$ is a straightforward consequence of the identity $\beta u=|u|$ a.e. in B_R.

We also need to connect test functions on $(0,\bar\theta_n^+(\alpha))$ and test functions on B_R. Given $\psi\in C_0^\infty(0,\bar\theta_n^+(\alpha))$, consider $\varphi(x)=\psi(\lambda|x|)$. Owing to Theorem 5, Property 2), we obtain

$$\lim_{p\to 1+}\int_{B_R}|u_p(x)|^{p-2}u_p(x)\varphi(x)\,dx=\int_{B_R}\beta(x)\varphi(x)\,dx\,.$$

Passing to polar coordinates, we get

$$\lim_{p\to 1+}\int_0^R \lambda_p^{\frac{p-1}{q-p}}|v_p(\lambda_p^{\frac{1}{p}}r)|^{p-2}v_p(\lambda_p^{\frac{1}{p}}r)\psi(\lambda r)r^{N-1}\,dr \;=\; \int_0^R\beta_1(\lambda r)\psi(\lambda r)r^{N-1}\,dr$$

and so, by scaling separately each integral we arrive at,

$$\lim_{p\to 1+}A_p\int_0^{\theta_{n,p}}|v_p(t)|^{p-2}v_p(t)\psi(\sigma_p t)t^{N-1}\,dt=\int_0^{\bar\theta_n^+}\beta_1(t)\psi(t)t^{N-1}\,dt,$$

where $\sigma_p=\lambda\lambda_p^{-\frac{1}{p}}$, $A_p=\lambda_p^{\frac{p-1}{q-p}}\sigma_p^N$. We next observe that both $A_p\to 1$ and $\sigma_p\to 1$ while $\theta_{n,p}\to\bar\theta_n^+$. In addition, $\psi(\sigma_p t)\to\psi(t)$ for each t. Thus,

$$\lim_{p\to 1+}\int_0^{\bar\theta_n^+}|v_p(t)|^{p-2}v_p(t)\psi(t)t^{N-1}\,dt=\int_0^{\bar\theta_n^+}\beta_1(t)\psi(t)t^{N-1}\,dt\,.$$

The desired convergence follows by directly employing $\psi\in L^{s'}(0,\bar\theta_n^+)$ in the previous argument.

iii) First observe that $\mathbf{z}(x)\cdot\frac{x}{|x|}$ is a weak limit of radial functions and so defines a radial function $\tilde w(r)$. If $w(t)=\tilde w(\lambda^{-1}t)$ then $w\in L^\infty(0,\bar\theta_n^+)$ with $\|w\|_\infty\leq 1$. A similar procedure than that developed above gives the weak convergence. To check that equation (5.73) holds, take $\psi\in C_0^\infty(0,\bar\theta_n^+)$ and consider

$$\varphi(x)=\frac{1}{|x|^{N-1}}\,\psi\left(\lambda|x|\right)\quad\text{if } x\neq 0,\qquad \varphi(0)=0. \tag{5.77}$$

As u solves (2.11), we get

$$\int_{B_R} \mathbf{z}\cdot\nabla\varphi = \lambda\int_{B_R}(\beta-|u|^{q-2}u)\varphi.$$

Passing to polar coordinates leads to

$$\int_0^R \lambda\psi'(\lambda r)w(\lambda r)\,dr - \int_0^R \frac{N-1}{r}\psi(\lambda r)w(\lambda r)\,dr = \lambda\int_0^R (\beta_1(\lambda r)-|v(\lambda r)|^{q-2}v(\lambda r))\psi(\lambda r)\,dr.$$

By setting $t=\lambda r$ it is found that w solves (5.73). Observe that then

$$w' = -\frac{N-1}{t}w - \beta_1 + |v|^{q-2}v$$

and the right hand side is bounded on any interval $(a,\bar\theta_n^+)$ with $a>0$. Moreover, we deduce from Lemma 11 that

$$-w' - \frac{N-1}{t}w = 1-\alpha^{q-1} \qquad t\in\left(0,\frac{1}{1-\alpha^{q-1}}\right)$$

whose solution satisfying $w(0)=0$ is given by

$$w(t) = -\frac{1}{N}(1-\alpha^{q-1})t.$$

Thus w' is bounded on $(0,\bar\theta_n^+)$ and so $w\in W^{1,\infty}(0,\bar\theta_n^+)$. Actually, $w\in W^{1,\infty}(0,+\infty)$ since bounds do not depend on the interval.

iv) Choose $\psi\in C_0^\infty(0,\bar\theta_n^+)$ and define $\varphi\in C_0^\infty(B_R)$ as in (5.77). It follows from the identity $|Du|=(\mathbf{z},Du)$ as measures that

$$\int_{B_R}\varphi|Du| = \int_{B_R}\varphi(\mathbf{z},Du) = -\int_{B_R} u\,\varphi\,\mathrm{div}\,\mathbf{z}\;dx - \int_{B_R} u\,\mathbf{z}\cdot\nabla\varphi\;dx\,.$$

Performing the same manipulations as above and employing (2.5) we obtain,

$$\int_0^{\bar\theta_n^+}\psi|v'| = \int_0^{\bar\theta_n^+} v\psi(\beta_1-|v|^{q-2}v)\,dt + \int_0^{\bar\theta_n^+}\left(\frac{N-1}{t}\right)v\psi w\,dt - \int_0^{\bar\theta_n^+} vw\psi'\,dt = -\int_0^{\bar\theta_n^+} v\psi w'\,dt - \int_0^{\bar\theta_n^+} vw(t)\psi'\,dt = \int_0^{\bar\theta_n^+}\psi(w,v')\,,$$

and we are done.

v) For a nonnegative $\psi\in C_0^\infty(0,\bar\theta_n^+(\alpha))$ choose now the variant,

$$\varphi(x) = \frac{1}{|x|^N}\,\psi\Big(\lambda|x|\Big) \quad \text{if } x\neq 0, \qquad \varphi(0)=0,$$

of the test function defined in (5.77). By Theorem 5, Property 4), and taking once again a subsequence,

$$\int_{B_R}\varphi|Du| = \lim_{p\to1}\int_{B_R}\varphi|\nabla u_p|^p dx\,.$$

Passing to polar coordinates, performing separate scalings in the integrals and multiplying by $N-1$ we deduce

$$\int_0^{\bar\theta_n^+} \frac{N-1}{t}\,\psi|v'| = \lim_{p\to 1}\int_0^{\theta_{n,p}} \frac{N-1}{t}\,\psi_p|v_p'|^p dt$$
$$= \lim_{p\to 1}\int_0^{\theta_{n,p}} \psi_p\Big(-\frac{dE_p}{dt}\Big)\,dt = \lim_{p\to 1}\int_0^{\theta_{n,p}} \psi_p' E_p\,dt\,,$$

where:

$$\psi_p(t) = \lambda_p^{\frac{q}{q-p}}\lambda^{-\frac{q}{q-1}}\psi(\sigma_p t), \qquad \sigma_p = \lambda\lambda_p^{-\frac{1}{p}}.$$

Hence,

$$\int_0^{\bar\theta_n^+} \frac{N-1}{t}\,\psi|v'| = \lim_{p\to 1}\frac{1}{p'}\int_0^{\theta_{n,p}} \psi_p'|v_p'|^p\,dt$$
$$+\lim_{p\to 1}\int_0^{\theta_{n,p}} \psi_p'\left[\frac{1}{p}|v_p|^p - \frac{1}{q}|v_p|^q\right]dt\,. \qquad (5.78)$$

Now recalling (5.75) and taking into account that $\psi_p \to \psi$ in $C_0^\infty(0,\bar\theta_n^+)$ as $p\to 1$ and in particular, that its support is bounded away from zero, we find that the first limit in (5.78) vanishes. On the other hand, by Lebesgue's theorem we obtain,

$$\lim_{p\to 1}\int_0^{\theta_{n,p}} \psi_p'\left[\frac{1}{p}|v_p|^p - \frac{1}{q}|v_p|^q\right]dt \;= \int_0^{\bar\theta_n^+} \psi'\left[|v| - \frac{1}{q}|v|^q\right]dt\,.$$

Thus we conclude from (5.78) that,

$$\int_0^{\bar\theta_n^+} \frac{N-1}{t}\,\psi(t)|v'| = \int_0^{\bar\theta_n^+} \psi'(t)\left[|v(t)| - \frac{1}{q}|v(t)|^q\right]dt\,.$$

and the energy identity (5.70) is proved.

Finally, the other assertion of v) follows immediately from the fact that $\|v_p\|_\infty = \alpha$ for all $1 < p \le 2$. □

Figure 1 shows the profiles of $v_p(t,\alpha)$ corresponding to $\alpha = 0.5$, $q = 2.5$, $N = 2$ and decreasing values of $p \in (1,2]$. Flat plateaus arise when p becomes close to one. In strong difference with the 1D case (problem (2.23)) a decaying in the amplitude of the solutions to (3.32) is observed and this feature is transmitted to the limit as $p \to 1+$.

5.3. **A uniqueness result.** The next one is a sort of uniqueness statement for the initial value problem (5.73).

Theorem 19. *Let $0 < \alpha < 1$. Then the initial value problem* (5.69) *admits a unique solution $(v,w) \in BV_{loc}(0,\infty) \times W_{loc}^{1,\infty}(0,\infty)$ satisfying the energy condition* (5.70)*. Moreover, there exist positive monotone sequences α_n and θ_n which verify*

$$\alpha_n \to 0, \qquad\qquad \theta_n \to \infty,$$

such that the following properties are satisfied.

i) *$v(t) = (-1)^{n-1}\alpha_{n-1}$ on each interval (θ_{n-1},θ_n) wherein $\alpha_0 = \alpha$ and $\theta_0 = 0$.*

ii) *$w \in W^{1,\infty}(0,\infty)$, w is strictly monotone on each interval (θ_{n-1},θ_n) while $w(\theta_n) = (-1)^n$ for every $n \ge 1$.*

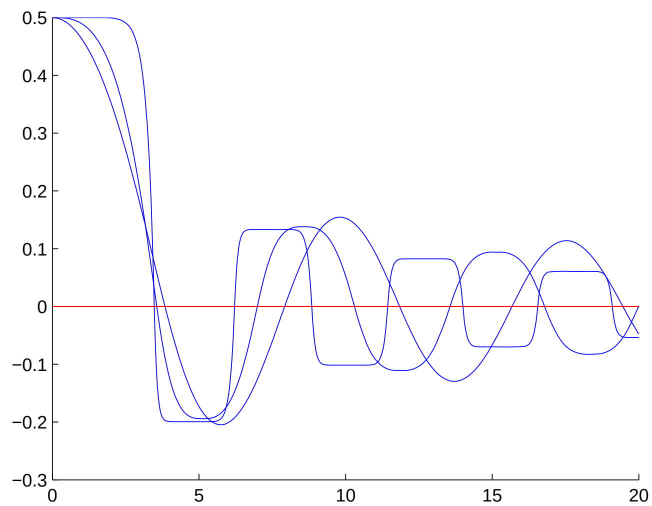

FIGURE 1. Profiles of v_p corresponding to $N = 2$, $q = 2.5$, $\alpha = 0.5$ and $p = 2$, $p = 1.5$ and $p = 1.1$.

iii) *Sequences* α_n *and* θ_n *satisfy the recurrence relations:*

$$\frac{h(\alpha_{n-1})}{N}\theta_n^N - \theta_n^{N-1} = \frac{h(\alpha_{n-1})}{N}\theta_{n-1}^N + \theta_{n-1}^{N-1} \qquad n \geq 1, \tag{5.79}$$

where $\theta_0 = 0$, $\alpha_0 = \alpha$, $h(x) = \operatorname{sign} x - |x|^{q-2}x$ *while*

$$g(\alpha_n) + \frac{N-1}{\theta_n}\alpha_n = g(\alpha_{n-1}) - \frac{N-1}{\theta_n}\alpha_{n-1} \qquad n \geq 1, \tag{5.80}$$

with $g(x) = |x| - \frac{1}{q}|x|^q$.

iv) θ_n *satisfies the following asymptotic estimate,*

$$\lim_{n\to\infty}(\theta_n - \theta_{n-1}) = 2. \tag{5.81}$$

v) *For every* $n \geq 1$ *both* θ_n *and* α_n *are smooth functions of* $\alpha \in [0, 1)$. *Moreover,*

$$\lim_{\alpha\to 0+}\theta_n = \bar{\omega}_n, \tag{5.82}$$

$\bar{\omega}_n$ *being the reference values introduced in Theorem 12.*

Proof of Theorem 19. As a first remark, let (v, w) be any possible solution to (5.69) where $0 < \alpha < 1$. Since v satisfies the energy equation (5.70), it follows that $g(v) = |v| - \frac{1}{q}|v|^q$ is non increasing along the solution. As the function g is increasing in $(0, 1)$ we deduce that $|v|$ is nonincreasing; in particular $|v(t)| \leq \alpha$ for all $t \geq 0$.

We are now following the argument of the proof of [41, Theorem 19].

1) *Function* v *is constant in every component of the set* $\mathcal{C} := \{t : |w(t)| < 1\}$. In fact, let (a, b) be any of such components, $J \subset (a, b)$ an arbitrary open interval. Then,

$$|v'|(J) = (w, v')(J) \leq \|w\|_{\infty,J}|v'|(J).$$

Since $\|w\|_{\infty,J} < 1$ then $|v'|(J) = 0$. Thus, v is constant in J.

2) *Nature of* (v,w) *in the initial component of* $\mathcal{C}$. Since $|w| < 1$ near $t = 0$ there exists a first component $(0,b)$ in $\mathcal{C}$. From $v(0+) = \alpha$ it follows from 1) that $v = \alpha$ in $(0,b)$ while direct integration of (5.72) yields

$$w(t) = -\frac{h(\alpha)}{N}t \qquad\qquad b = \frac{N}{h(\alpha)}, \tag{5.83}$$

for $t \in (0,b)$, the last equality being implied by the relation $w(b) = -1$. Thus we set $\theta_1 = b$. Notice that $b > N$ since $h(\alpha) < 1$. We next use (5.72) to observe that,

$$|w'(t)| \le \frac{1}{N'}h(\alpha) + 1 + \alpha^{q-1} = 1 + \frac{1}{N'} + \frac{1}{N}\alpha^{q-1} \le 2,$$

for $t \ge \theta_1$. This together with (5.83) implies that,

$$\|w'\|_{\infty,(0,\infty)} \le 2. \tag{5.84}$$

3) *Let* (a,b) *be a component of* $\mathcal{C}$ *where* $v(t) = c$, $c \neq 0$, *is a constant. Then we claim the validity of the following facts.*

- a) (a,b) *is finite while* $b - a \ge 1$.
- b) $\operatorname{sign} c = \operatorname{sign} w(a)$, *being* $(\operatorname{sign} c)w(t)$ *decreasing in* (a,b).
- c) *The following relation holds true:*

$$\frac{h(|c|)}{N}b^N - b^{N-1} = \frac{h(|c|)}{N}a^N + a^{N-1}. \tag{5.85}$$

The finiteness of (a,b) is consequence of the representation

$$w(t) = \left(w(a) + a\frac{h(c)}{N}\right)\left(\frac{a}{t}\right)^{N-1} - \frac{h(c)}{N}t, \tag{5.86}$$

which holds in (a,b) and the fact that $|w| < 1$. Furthermore, (5.84) implies the second assertion in a) since

$$2 = |w(b) - w(a)| \le 2(b-a)\,.$$

To check b), observe first that $\operatorname{sign} c = \operatorname{sign} h(c)$ when $|c| < 1$. If $w(a) = 1$ and $c < 0$, then

$$w'(a+) = -\frac{N-1}{a} + h(|c|) > 0,$$

due to $a \ge \theta_1$ and $|c| \le \alpha$. This would imply that $w > 1$ near $t = a$ which is not possible. A similar argument allows to deal with the case $w(a) = -1$ and $c > 0$. Thus, $w(a) = \operatorname{sign} c$. On the other hand,

$$w(a)w(t) = \left(1 + a\frac{h(|c|)}{N}\right)\left(\frac{a}{t}\right)^{N-1} - \frac{h(|c|)}{N}t,$$

which is decreasing. Since $(\operatorname{sign} c)w(t) = w(a)w(t)$, point b) is proven.

Finally, (5.85) follows by direct integration of (5.72).

4) *Solution* v *can not undergo a discontinuity at* $\theta \ge \theta_1$ *such that* $v(\theta-) = c \neq 0$ *and* $v(\theta+) = 0$. In fact, since v only has jump discontinuities, that fact and (5.70) would imply

$$|c| - \frac{|c|^q}{q} = \frac{N-1}{\theta}|c|,$$

and so

$$1 - \frac{|c|^{q-1}}{q} \le \frac{1}{N'}(1 - \alpha^{q-1}).$$

Hence,

$$1 - \frac{1}{N'} \le \left(\frac{1}{q} - \frac{1}{N'}\right)\alpha^{q-1} < \frac{1}{q} - \frac{1}{N'},$$

which is not possible. We stress that condition (5.70) is essential in this step (see Remark 14 below).

5) *If either* $w(t) = 1$ *or* $w(t) = -1$ *in an whole interval* $I = (a, b)$ *then* $v(t) = 0$ *there.* Assume that $w(t) = 1$. Then from $(w, v') = |v'|$ one learns that $v' = |v'|$ and so v is nondecreasing in I. However, (5.72) implies that

$$|v|^{q-2}v \in \text{sign } v + \frac{N-1}{t}, \qquad t \in I.$$

If $v(t_0) \ne 0$ at some $t_0 \in I$, then (near t_0) $|v|^{q-2}v$ would be strictly decreasing and this is not possible. The case $w(t) = -1$ is similarly handled.

6) *Components of* $\mathcal{C}$ *are contiguous in the sense that the upper end of one component coincides with the lower end of another. More precisely, beyond every component* (a, b) *in* $\mathcal{C}$ *where* $v(t) = c \ne 0$ *there exists a further component* (b, d) *where* $v(t) = c'$ *and* $cc' < 0$ *holds.* The assertion is a consequence of 3), 4) and 5) (see [41]).

Proof of i), ii), iii)–(5.79).

Starting at the first interval $(0, \theta_1)$ with $\alpha_0 = \alpha$ and by employing step 6), we are attaching successive components, named $I_n := (\theta_{n-1}, \theta_n)$. Function v attains the constant value $(-1)^n\alpha_n$ in I_n, with $\alpha_n > 0$ since signs on these intervals are alternated. Energy condition (5.70) implies that α_n is not increasing. By (5.85) it is found that θ_n follows the recursive law (5.79). Observe that this law gives $\theta_1 = \frac{N}{h(\alpha)}$ as expected.

Proof of iii)–(5.80)*, dependence* $\theta_n(\alpha)$*,* $\alpha_n(\alpha)$ *and* v*).*

We first discus equation (5.79) to show that every θ_n can actually be computed. Given α_{n-1} and θ_{n-1}, the new term $x = \theta_n$ must be found by solving

$$\frac{h(\alpha_{n-1})}{N}x^N - x^{N-1} = \frac{h(\alpha_{n-1})}{N}\theta_{n-1}^N + \theta_{n-1}^{N-1}.$$

By setting $y = h(\alpha_{n-1})x$, $\tilde{\theta}_{n-1} = h(\alpha_{n-1})\theta_{n-1}$ such an equation is transformed into

$$\frac{1}{N}y^N - y^{N-1} = \frac{1}{N}\tilde{\theta}_{n-1}^N + \tilde{\theta}_{n-1}^{N-1}. \tag{5.87}$$

It is rather clear that this equation possesses a unique root $y = \hat{\theta}_n > N$ which is a smooth function of $\tilde{\theta}_{n-1}$. This implies that

$$\theta_n = \frac{\hat{\theta}_n}{h(\alpha_{n-1})}$$

is the next term in the sequence. Moreover, it also defines a smooth function of both α_{n-1}, θ_{n-1}.

We emphasize that it follows from $\theta_n \ge 1 + \theta_{n-1}$ (see 3) a)) that $\lim_{n\to\infty}\theta_n = \infty$. Thus, v is defined in $(0, \infty)$.

Function v exhibits a jump at every θ_n. Thus, the energy relation (5.70) implies that

$$g(\alpha_n) - g(\alpha_{n-1}) = -\frac{N-1}{\theta_n}(\alpha_{n-1} + \alpha_n),$$

which can be written as (5.80):

$$g(\alpha_n) + \frac{N-1}{\theta_n}\alpha_n = g(\alpha_{n-1}) - \frac{N-1}{\theta_n}\alpha_{n-1}.$$

We now check that this recursive relation certainly produces a decreasing sequence $0 < \alpha_n < \alpha$. Proceeding by induction, assume that both $0 < \alpha_{n-1} < \alpha$ and $\theta_n > \theta_1$ have already been found. Then,

$$g(\alpha_{n-1}) - \frac{N-1}{\theta_n}\alpha_{n-1} > 0.$$

In fact this inequality amounts to

$$1 - \frac{1}{q}\alpha_{n-1}^{q-1} - \frac{N-1}{\theta_n} > 0.$$

But $\theta_n > \frac{N}{h(\alpha_{n-1})}$, so that

$$1 - \frac{1}{q}\alpha_{n-1}^{q-1} - \frac{N-1}{\theta_n} > 1 - \frac{1}{q}\alpha_{n-1}^{q-1} - \frac{1}{N'}(1 - \alpha_{n-1}^{q-1}) > 0,$$

as $q > 1$.

Next, it can be checked that the function $x \mapsto g(x) + \frac{N-1}{\theta_n}x$ is increasing in the interval $0 \le x \le \left(1 + \frac{N-1}{\theta_n}\right)^{\frac{1}{q-1}}$. Thus, equation

$$g(x) + \frac{N-1}{\theta_n}x = g(\alpha_{n-1}) - \frac{N-1}{\theta_n}\alpha_{n-1}$$

has a unique solution x in the range $0 < x < 1$, and such a root must be $x = \alpha_n$. Moreover, we deduce

$$\frac{d}{dx}\left(g(x) + \frac{N-1}{\theta_n}x\right)_{x=\alpha_n} > 0.$$

This means that α_n is a smooth function of both θ_n, α_{n-1}. In addition, since α_n lies in the range where $g(\cdot) + \frac{N-1}{\theta_n}\cdot$ is increasing, it follows from (5.80) that $0 < \alpha_n < \alpha_{n-1}$.

We now proceed recursively and use the dependence of θ_n on θ_{n-1} and α_{n-1} shown above, to conclude that α_n and θ_n are smooth functions of α. Moreover, in the particular case $n = 1$ both functions are increasing in α.

Estimate (5.82) in assertion v) is shown by direct substitution and the help of [41, Theorem 19].

Proof of $\alpha_n \to 0$ and estimate (5.81). By setting,

$$a_n = \frac{\theta_n}{\theta_{n-1}} > 1,$$

then (5.79) leads to

$$a_n^N\left(1 - \frac{N}{h(\alpha_{n-1})\theta_n}\right) = 1 + \frac{N}{h(\alpha_{n-1})\theta_{n-1}},$$

whence $\lim a_n = 1$. On the other hand,

$$\theta_n - \theta_{n-1} = \frac{1}{h(\alpha_{n-1})} \frac{N(a_n^{N-1}+1)}{\sum_{k=0}^{N-1} a_n^{N-1-k}} \to \frac{2}{h(\bar{\alpha})}, \tag{5.88}$$

as $n \to \infty$ where $\bar{\alpha} = \lim \alpha_n = \inf \alpha_n$. We are next showing that $\bar{\alpha} = 0$ so the proof of estimate (5.81) is attained.

Accordingly, let us verify that $\bar{\alpha} = 0$. For n fixed choose $0 < a < b$ so that,

$$a < \theta_1 < \cdots < \theta_n < b < \theta_{n+1}.$$

The decaying character of the energy E:

$$E = |v| - \frac{|v|^q}{q},$$

and equation (5.70) imply that,

$$\int_a^b \frac{N-1}{t}|v'| \le E(a) - E(b) < E(0) < \alpha,$$

and so,

$$(N-1)\sum_{k=1}^{n} \frac{\alpha_k + \alpha_{k-1}}{\theta_k} = \int_a^b \frac{N-1}{t}|v'| \le \alpha.$$

Thus the series

$$\sum_{n=1}^{\infty} \frac{\alpha_n + \alpha_{n-1}}{\theta_n}$$

converges. On the other hand, it follows from (5.88) and Cesàro's Theorem that $\lim_{n\to\infty} \frac{\theta_n}{n} = \frac{2}{h(\bar{\alpha})}$. Hence,

$$\frac{\alpha_n + \alpha_{n-1}}{\theta_n} \sim C\frac{\bar{\alpha}}{n}$$

for a certain constant $C > 0$. Therefore $\bar{\alpha}$ must be zero.

□

Remark 13. For $0 < \alpha < 1$ the sequence θ_n of values obtained in Theorem 19 are denoted in the sequel as $\bar{\theta}_n(\alpha)$. This is done to highlight on the one hand their dependence on α, and on the other its rôle as a limit when $p \to 1$. Next statement clarifies this last remark. Notations $v_p(\cdot, \alpha)$ and $\theta_{n,p}(\alpha)$ (beginning of Section 4) are going to be employed.

Corollary 20. *Fixed* $0 < \alpha < 1$, *let* $v_p(t, \alpha)$ *be the solution* (5.68) *while* $v(t, \alpha)$ *designates the solution to* (5.69) *computed in Theorem 19. Then the whole family* v_p, *not merely a subsequence, satisfies*

$$v_p \to v \qquad \text{as } p \to 1, \tag{5.89}$$

in $L^1((0,b), t^{N-1}\,dt)$ *for every* $b > 0$. *Moreover,*

$$\lim_{p\to 1} \theta_{n,p}(\alpha) = \bar{\theta}_n^-(\alpha) = \bar{\theta}_n^+(\alpha) = \bar{\theta}_n(\alpha), \tag{5.90}$$

for all $n \in \mathbb{N}$, *where* $\theta_{n,p}(\alpha)$ *denotes the sequence of zeros of* v_p.

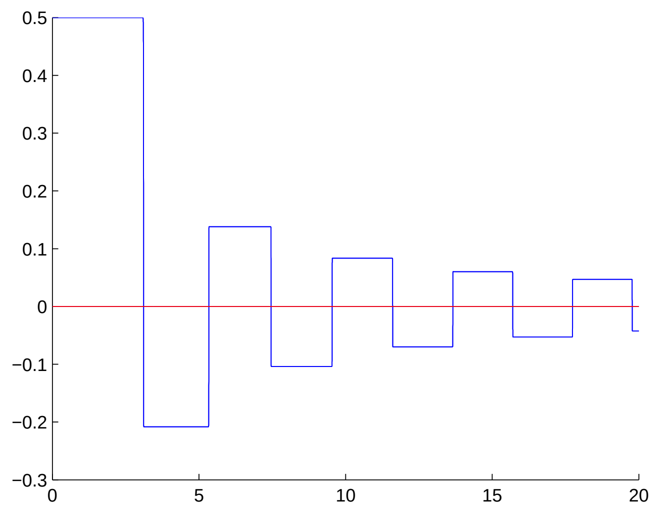

FIGURE 2. Drawing of $v_p(t,\alpha)$ for $N = 2$, $q = 2.5$, $\alpha = 0.5$ and $p = 1.001$. Profile is dramatically steepened.

Proof. Convergence assertion (5.89) is a consequence of the uniqueness shown in Theorem 19.

To prove (5.90) we proceed by induction and firstly check that $\bar{\theta}_1^{\pm} = \bar{\theta}_1$ (α is omitted for simplicity). Thus, choose a subfamily $v_{p'}$ so that $\bar{\theta}_{1,p'} \to \bar{\theta}_1^+$ while $v_{p'} \to \tilde{v}$ a. e. in $(0,\infty)$ (Theorem 18). Thanks to Theorem 13 (finiteness of limits) and Theorem 14 (gaps between the limits), $\tilde{v} \geq 0$ in an interval $(\bar{\theta}_1^+ - \delta, \bar{\theta}_1^+)$ while $\tilde{v} \leq 0$ in $(\bar{\theta}_1^+, \bar{\theta}_1^+ + \delta)$ for certain $\delta > 0$. But uniqueness entails that $\tilde{v} = v$ and so $\bar{\theta}_1^+$ must coincide with $\bar{\theta}_1$. Otherwise a discrepancy in signs should arise. By the same token, $\bar{\theta}_1^-$ must be $\bar{\theta}_1$.

Assume now that $\bar{\theta}_k^{\pm} = \bar{\theta}_k$ for $1 \leq k \leq n$. We are proving that $\bar{\theta}_{n+1}^+ = \bar{\theta}_{n+1}$. In fact, choose again a subfamily $v_{p''}$ so that $v_{p''} \to \hat{v}$ and satisfying $\theta_{n+1,p''} \to \bar{\theta}_{n+1}^+$ (Theorem 13). Then, Theorem 14 provides some $\eta > 0$ such that $(-1)^{n+1}\hat{v} \geq 0$ in $(\bar{\theta}_{n+1}^+, \bar{\theta}_{n+1}^+ + \eta)$ and $(-1)^{n+1}\hat{v} \leq 0$ in $(\bar{\theta}_{n+1}^+ - \eta, \bar{\theta}_{n+1}^+)$ (in fact this is just the sign in the whole interval $(\bar{\theta}_n, \bar{\theta}_{n+1}^+)$). But again $\hat{v} = v$ and necessarily $\bar{\theta}_{n+1}^+ = \bar{\theta}_{n+1}$, to avoid inconsistency in the signs. For $\bar{\theta}_{n+1}^- = \bar{\theta}_{n+1}$ the argument is the same. □

Figures 2 and 3 show plottings of $v_p(t,\alpha)$ and $w_p(t,\alpha)$ with the parameters of Figure 1 but p reduced to $p = 1.001$.

5.4. **Proof of Theorem 16.** Let $(v(t), w(t))$ be the solution to (5.69) introduced in Theorem 19. By setting,

$$u = \bar{u}_{n,\lambda}(r) = \lambda^{\frac{1}{q-1}} v(\lambda r, \alpha), \quad \text{with } \lambda = \bar{\lambda}_n(\alpha) = R^{-1}\bar{\theta}_n(\alpha), \tag{5.91}$$

we are checking that the assertions in Theorem 16 hold true.

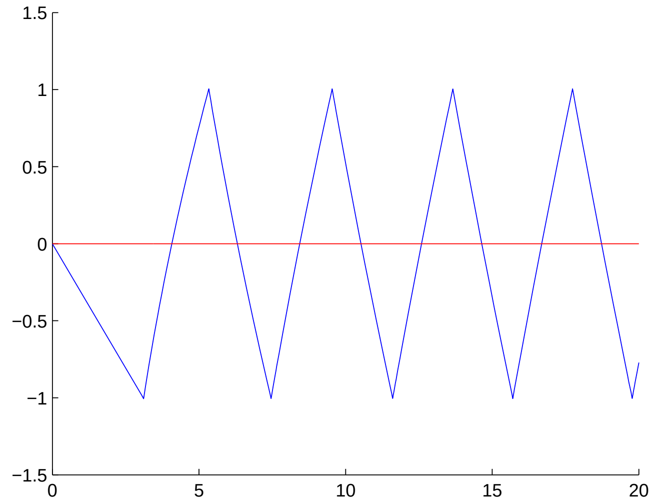

FIGURE 3. Drawing of $w_p(t,\alpha)$ for $N = 2$, $q = 2.5$, $\alpha = 0.5$ and $p = 1.001$. The Lipschitz nature of w_p is clearly reflected.

Regarding the property of being a solution to (5.61) we choose

$$\mathbf{z} = \tilde{w}(r)\frac{x}{r}, \qquad \beta(x) = \beta_1(\lambda r),$$

where $\tilde{w}(r) = w(\lambda r)$ and $\beta_1(t)$ is just the function,

$$\beta_1 = \sum_{n=1}^{\infty}(-1)^{n-1}\chi_{(\bar{\theta}_{n-1}(\alpha),\bar{\theta}_n(\alpha))}.$$

It is clear then that $\beta u = |u|$.

On the other hand, distributions $\operatorname{div}\mathbf{z}$, $(\mathbf{z}, Du)$ and $|Du|$ are invariant under rotations in $\mathbb{R}^N$ (a detailed checking of this and forthcoming similar assertions is omitted to brief). Accordingly, they are equal provided that take the same values when acting on radial test functions $\varphi \in C_0^\infty(B_R)$, $\varphi(x) = \psi(|x|)$. Thus, to check that (5.61) holds we observe that both v and w are smooth enough up to $t = 0$ and that equality

$$-(t^{N-1}w)' = t^{N-1}(\beta_1 - |v|^{q-2}v),$$

is satisfied. It is equivalent to,

$$-(r^{N-1}\tilde{w})' = r^{N-1}(\lambda\beta - |u|^{q-2}u).$$

Multiplying by a test function $\psi \in C^1[0,R]$ which vanishes near $r = R$ and integrating by parts we obtain,

$$\int_0^R \tilde{w}\psi' r^{N-1}\,dr = \int_0^R (\lambda\beta - |u|^{q-2}u)\psi r^{N-1}\,dr,$$

which is the weak version of $-\operatorname{div}\mathbf{z} = \lambda\beta - |u|^{q-2}u$ in polar coordinates.

Regarding the identity $(\mathbf{z}, Du) = |Du|$ it suffices with checking it in $D(\sigma) := B_R \setminus \overline{B_\sigma}$ for $0 < \sigma < R$ small since it is clearly true near zero. Thus, define φ as in

(5.77) where $\psi \in C_0^\infty(\lambda\sigma, \bar{\theta}_n(\alpha))$. Then $\varphi \in C_0^\infty(D(\sigma))$ while some computations show that

$$\begin{aligned}
\langle(\mathbf{z}, Du), \varphi\rangle &= -\int_{D(\sigma)} u(x)\varphi(x)\operatorname{div}\mathbf{z}(x)\,dx - \int_{D(\sigma)} u(x)\mathbf{z}(x)\cdot\nabla\varphi(x)\,dx \\
&= -N\omega_N\lambda^{\frac{1}{q-1}}\int_{\lambda\sigma}^{\lambda R} v(t)\psi(t)\left(w_t(t) + \frac{N-1}{t}w(t)\right)dt \\
&\quad - N\omega_N\lambda^{\frac{1}{q-1}}\int_{\lambda\sigma}^{\lambda R} v(t)w(t)\left(\psi_t(t) - \frac{N-1}{t}\psi(t)\right)dt \\
&= N\omega_N\lambda^{\frac{1}{q-1}}\left[-\int_{\lambda\sigma}^{\lambda R} v(t)\psi(t)w_t(t)\,dt - \int_{\lambda\sigma}^{\lambda R} v(t)w(t)\psi_t(t)\,dt\right] \\
&= N\omega_N\lambda^{\frac{1}{q-1}}\langle(w, v_t), \psi\rangle.
\end{aligned}$$

Taking the same test functions ψ and φ, it can be seen that

$$\langle|Du|, \varphi\rangle = N\omega_N\lambda^{\frac{1}{q-1}}\langle|v_t|, \psi\rangle .$$

Since (v, w) is the solution to (5.69), it follows that $(w, v_t) = |v_t|$ and so $(\mathbf{z}, Du) = |Du|$ as measures in $D(\sigma)$. It yields $(\mathbf{z}, Du) = |Du|$ as measures in B_R, so that the required coupling between $\mathbf{z}$ and $|Du|$ is verified.

The validity of the energy relation (5.67) is proven by a direct scaling argument based on (5.70).

Regarding the boundary condition, it follows from [1, Theorem 3.87] that the trace of u at R is given by:

$$u_{|r=R} = \lim_{r\to R-} u(r) = \lambda^{\frac{1}{q-1}} \lim_{t\to\bar{\theta}_n(\alpha)} v(t,\alpha) = (-1)^{n-1}\alpha_{n-1}\lambda^{\frac{1}{q-1}}.$$

Hence,

$$\operatorname{sign} u_{|r=R} = (-1)^{n-1} = -[\mathbf{z}, \nu],$$

since $[\mathbf{z}, \nu] = \tilde{w}(R) = w(\bar{\theta}_n(\alpha)) = (-1)^n$.

Parametrization (5.65) for $\bar{u}_{n,\lambda}$ together with its continuity in α are provided by the expression for $v(t,\alpha)$ and the smoothness of $\bar{\theta}_n$ and α_n with respect to α stated in Theorem 19. In addition, crucial relation (5.66) was the objective of Corollary 20.

The fact that $\bar{u}_{n,\lambda}$ bifurcates from zero at $\lambda = \bar{\lambda}_n$ follows from (5.91) and the convergence $\bar{\lambda}_n(\alpha) \to 0$ as $\alpha \to 0+$. Similarly, that $\bar{\lambda}_n(\alpha) \to \infty$ as $\alpha \to 1-$ proves (5.64).

We next address the uniqueness issue in v). So, let $u \in BV(B_R)$ be a radial solution in the sense of Definition 15, with associated function $\beta(r)$ and field $\mathbf{z} = \tilde{w}(r)\frac{x}{r}$. It is also supposed that u satisfies the energy relation (5.67).

We start with the equation,

$$-\operatorname{div}\mathbf{z} = \lambda\beta - |u|^{q-2}u.$$

By testing with radial functions $\psi(|x|) \in C_0^\infty(B_R)$ we obtain,

$$\int_0^R \tilde{w}(r)\psi'(r)r^{N-1}\,dr = \int_0^R (\lambda\beta - |u|^{q-2}u)\psi(r)r^{N-1}\,dr.$$

By using the limit condition in (5.63) we arrive at:

$$\tilde{w}(r) = -\int_0^r \left(\frac{s}{r}\right)^{N-1} (\lambda\beta - |u|^{q-2}u)\ ds.$$

Since $u \in L^\infty(B_R)$ (Theorem 4) it follows that $\tilde{w} \in W^{1,\infty}(0,R)$. Moreover, equation

$$-\tilde{w}_r - \frac{N-1}{r}\tilde{w} = \lambda\beta - |u|^{q-2}u, \qquad 0 < r < R,$$

is satisfied in weak sense.

Define now,

$$\alpha = \lim_{r\to 0+} \lambda^{-\frac{1}{q-1}} u(r).$$

Such a limit exists because u is chosen to be of bounded variation in classical sense. In addition $|\alpha| < 1$ due to (2.16) (Theorem 4) and no generality is lost if we assume that $\alpha \geq 0$. We now observe that (5.67) implies that the group $\lambda|u| - \frac{|u|^q}{q}$ is non increasing. Therefore,

$$\lambda^{-\frac{1}{q-1}}|u(r)| \leq \alpha, \qquad r > 0.$$

This in particular rules out the option $\alpha = 0$.

Let us introduce now the scalings,

$$v(t) = \lambda^{-\frac{1}{q-1}} u(\lambda^{-1}t), \qquad w(t) = \tilde{w}(\lambda^{-1}t), \qquad \beta_1(t) = \beta(\lambda^{-1}t).$$

Then it is found that the pair (v,w) fulfills the properties i), ii) and iii) in Definition 17, where β_1 assumes the rôle of β in iii), being $(0,R\lambda)$ the reference interval. In addition, a scaling computation ensures us that the energy relation (5.70) holds. Finally, the boundary condition:

$$-\,w(b)v(b-) = |v(b-)|, \tag{5.92}$$

is satisfied at the endpoint $b = R\lambda$. We now come back to the proof of Theorem 19 and observe that dispose of enough conditions to conclude that $v(t)$ exactly matches, in the interval $(0,R\lambda)$, the solution obtained in this theorem. Since the boundary condition (5.92) is only fulfilled at the points $\bar{\theta}_k(\alpha)$ there must exist some n so that,

$$R\lambda = \bar{\theta}_n(\alpha).$$

Thus, we have shown that solution $u = \bar{u}_{n,\lambda}$ with $\lambda = R^{-1}\bar{\theta}_n(\alpha)$. This finishes the proof of Theorem 16.

Remark 14. If we drop condition (5.67) then further families of solutions than those in Theorem 16 can be found. The most simple example is extracted from the solution (v,w) to problem (5.69) defined by

$$v(t) = \alpha\,\chi_{(0,\bar{\theta}_1(\alpha))}(t), \qquad \bar{\theta}_1(\alpha) = \frac{N}{1-\alpha^{q-1}}, \qquad \alpha > 0,$$

together with

$$w(t) = \begin{cases} -\dfrac{t}{\bar{\theta}_1(\alpha)} & 0 \leq t \leq \bar{\theta}_1(\alpha) \\ -1 & t > \bar{\theta}_1(\alpha), \end{cases}$$

$$\beta(t) = \begin{cases} 1 & 0 \leq t \leq \bar{\theta}_1(\alpha) \\ \dfrac{N-1}{t} & t > \bar{\theta}_1(\alpha). \end{cases}$$

Then,

$$\hat{u}_\lambda(r) = \lambda^{\frac{1}{q-1}} v(\lambda r),$$

defines a solution to (5.61) in every ball whose radius is greater than R provided that

$$\lambda \geq \lambda_c := R^{-1}\bar{\theta}_1(\alpha) > \bar{\lambda}_1.$$

Observe that a dead core $\{\hat{u}_\lambda = 0\}$ propagates towards $x = 0$ as $\lambda \to \infty$. Many other families of solutions can be obtained. Of course, none of them satisfying the energy condition (5.67).

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