HIGHER ROBIN EIGENVALUES FOR THE p-LAPLACIAN OPERATOR AS p APPROACHES 1

JOSÉ C. SABINA DE LIS AND SERGIO SEGURA DE LEÓN

To the memory of Prof. Manuel Elgueta Dedes, a dear friend and wonderful person.

Abstract. This work addresses several aspects of the dependence on p of the higher eigenvalues λ_n to the Robin problem,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & x \in \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} + b|u|^{p-2} u = 0 & x \in \partial\Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^N$ is a C^1 bounded domain, ν is the outer unit normal, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p–Laplacian operator and $b \in L^\infty(\partial\Omega)$. Main results concern: a) the existence of the limits of λ_n as $p \to 1$, b) the 'limit problems' satisfied by the 'limit eigenpairs', c) the continuous dependence of λ_n on p when $1 < p < \infty$ and d) the limit profile of the eigenfunctions as $p \to 1$. The latter study is performed in the one dimensional and radially symmetric cases. Corresponding properties on the Dirichlet and Neumann eigenvalues are also studied in these two special scenarios.

Contents

Date: June 28, 2024.

2010 *Mathematics Subject Classification.* 35J92,35P30,35B40.

Key words and phrases. p–Laplacian operator, Ljusternik–Schnirelmann eigenvalues, Robin conditions, Γ- convergence.

1. Introduction

The analysis of the eigenvalues of the p–Laplacian operator under different sets of boundary conditions is a challenging topic in nonlinear analysis. Just a distinguished example of such problems correspond to the so–called Robin conditions,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & x \in \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + b|u|^{p-2}u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^N$ a bounded C^1 domain, ν being the outer unit normal and $b \in L^\infty(\Omega)$. In most parts of this paper, and unless otherwise stated, coefficient b is nonnegative.

Weak eigenpairs $(\lambda, u) \in \mathbb{R} \times W^{1,p}(\Omega)$ to (1.1) are defined as follows.

Definition 1. *A function $u \in W^{1,p}(\Omega) \setminus \{0\}$ is an eigenfunction associated to an eigenvalue $\lambda \in \mathbb{R}$ if,*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx + \int_{\partial\Omega} b|u|^{p-2}uv \, d\mathcal{H}^{N-1} = \lambda \int_\Omega |u|^{p-2}uv \, dx,$$

for every $v \in W^{1,p}(\Omega)$.

Problem (1.1) has been exhaustively studied in the linear diffusion case $p = 2$ ([10]). However, the regime $p \neq 2$ is substantially more complicated, specially when dealing with higher eigenvalues. It shares many basic features with a broad class of eigenvalue problems. The most important examples are the Dirichlet problem:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

and the Neumann problem,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Some of these common properties are,

a) All possible eigenvalues λ are nonnegative while there exists a unique principal eigenvalue λ_1. This means an eigenvalue with an associated eigenfunction which does not change its sign. It is the minimum eigenvalue and is characterized as,

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + \int_{\partial\Omega} b|u|^p}{\int_\Omega |u|^p}. \tag{1.4}$$

b) λ_1 is a simple and *isolated* eigenvalue ([3], [32], [30]).

c) There exists a further nondecreasing family λ_n of eigenvalues, $\lambda_n \to \infty$, which are variationally defined. Corresponding eigenvalues to the Dirchlet and Neumann problems are denoted $\lambda_n^{\mathcal{D}}$ and $\lambda_n^{\mathcal{N}}$, respectively. These are the so–called Ljusternik–Schnirelmann eigenvalues ([2], [24], [23], [30] and Section 2.4 below). However it is still not known if they are the only possible eigenvalues.

d) Second eigenvalue λ_2 in the sequence (2.13) is actually the second eigenvalue. In other words, no other eigenvalues between λ_1 and λ_2 exist ([4]).

A discussion of the different aspects of the dependence on p of the eigenvalues of $-\Delta_p$ has been the subject of recent interest in the literature. It goes back at least to [17] where the continuity on p of the first Dirichlet eigenvalue $\lambda_1^{\mathcal{D}}$ is shown. A possible failure of the left continuity of this eigenvalue in nonsmooth domains Ω was discovered in [33], while optimum conditions to achieve such continuity have been recently given in [15]. As for the higher Dirichlet eigenvalues, continuity of $\lambda_2^{\mathcal{D}}$ is stated in [25], being thoroughly discussed in [11] the continuity of the full family of eigenvalues $\lambda_n^{\mathcal{D}}$ on p (see also [40] and [14]).

The present work is mainly concerned with the behavior of the eigenvalues to (1.1) as $p \to 1$. The interest on this kind of results has its origin in the analysis of the limit $\lambda_1^{\mathcal{D}} \to \bar{\lambda}_1$ of the first Dirichlet eigenvalue as $p \to 1$ and, more importantly, the rôle of $\bar{\lambda}_1$ as the first eigenvalue of the 1–Laplacian operator $-\Delta_1$ ([19, 20], [28], [29] and rough preliminary approximations in [25], [31]). As for the higher Dirichlet eigenvalues $\lambda_n^{\mathcal{D}}$, the existence of the limits $\bar{\lambda}_n^{\mathcal{D}} = \lim_{p\to 1} \lambda_n^{\mathcal{D}}$ together with a variational expression for such limits were obtained in [38, 39], [34]. Nevertheless, the mere existence of these limits is deduced in [41] following a direct approach inspired in [33]. In addition, the identification of $\bar{\lambda}_n^{\mathcal{D}}$ as a critical value of the total variation functional $\mathcal{D}(u)$ (see (6.80)), in the sense of the so–called 'weak slope', was also shown in [34] (see [36] and [12] for related results). In the one–dimensional case, a complete characterization of the variational Dirichlet eigenvalues was accomplished in [12], while the limits of eigenvalues and eigenfunctions profiles for a convective perturbation of $-\Delta_1$ are addressed in [13].

A further field of research on the p–dependence of eigenvalues consists in determining the possible limit eigenfunctions u_n associated to a limit eigenvalue $\bar{\lambda}_n^{\mathcal{D}} = \lim_{p\to 1} \lambda_n^{\mathcal{D}}$. It is shown in [41] the existence of such eigenfunctions which solve a suitable limit problem governed by the 1–Laplacian operator $-\Delta_1$ (see also [12], [36]). The natural framework to manage these problems is the space $BV(\Omega)$ of functions of bounded variation (Section 2). A detailed computation of both the limits $\bar{\lambda}_n^{\mathcal{D}}$ and the profiles of the eigenfunctions in a ball is performed in [41], while the case of annuli has been recently addressed in [27]. Natural nonlinear perturbations of the Dirichlet eigenvalue problem for $-\Delta_1$ have also been recently considered in [43, 42].

Although this kind of results will not be considered in this paper, we would like to mention that the monotonicity with respect to p of the eigenvalues either to (1.2) or (1.3), is another area of recent study (see [26], [35], [27]).

The contents of this work deals with the topics described above. One of our main results, Theorem 20, both states the existence of the limits $\bar{\lambda}_n = \lim_{p\to 1} \lambda_{n,p}$ of the Robin eigenvalues together with the variational expression (6.79) for such limits. As shown in Theorem 18, Corollary 23 and it is going to be observed in the course of Sections 3 and 4, the relative values of b with respect to unity exert a strong influence on the expression and behavior of $\bar{\lambda}_n$. In addition, it turns out when proving Theorem 20 that eigenvalues $\lambda_{n,p}$ are right continuous as functions of $p > 1$ (Theorem 26). Analysis of the full continuity of $\lambda_{n,p}$ is delayed to Theorem 31 in Section 7 and this is the only part of the work dealing with Γ-convergence.

The connection between the limits $\bar{\lambda}_n$ and the 1–Laplacian is confined to Section 5. We do not need there the existence of $\lim_{p\to 1} \lambda_{n,p}$. Rather, the weaker assumption that $\bar{\lambda} = \lim_{m\to\infty} \lambda_{n,p_m}$ where n is fixed and $p_m \to 1$. Thus, the existence of such limit values $\bar{\lambda}$ is simply ensured through uniform estimates of $\lambda_{n,p}$ as $p \to 1$ (Lemma 6). Main result of the section, Theorem 18, states that an eigenfunction $u \in BV(\Omega)$ can be found so that $(\bar{\lambda}, u)$ defines an eigenpair of the 'natural' boundary value problem associated to $-\Delta_1$ (see (5.76) and (5.78)). In this problem, the boundary conditions depend significantly on the relative values of b with respect to unity. More precisely, if $0 < b < 1$ on Γ_1 and $b \geq 1$ on Γ, Γ_1, Γ being open parts of $\partial\Omega$ with $\overline{\Gamma_1} \cup \overline{\Gamma} = \partial\Omega$, then $(\bar{\lambda}, u)$ solves,

$$\begin{cases} -\Delta_1 u = \lambda \dfrac{u}{|u|} & x \in \Omega, \\ u = 0, & \text{on } \Gamma, \\ \dfrac{Du}{|Du|}\nu + b\dfrac{u}{|u|} = 0, & \text{on } \Gamma_1. \end{cases}$$

Here Δ_1 stands for the 1-Laplacian operator which is formally defined as $\operatorname{div}\left(\frac{Du}{|Du|}\right)$. A precise definition of the concept of solution to this problem is presented in Section 5.

To attain a deeper insight on the profiles of the limit eigenfunctions, the one dimensional and the radial versions of (1.1) are considered in Sections 3 and 4. In dimension $N = 1$, Theorems 8 and 9 describe the exact values of $\bar{\lambda}_n$, the profile and the exact position of the zeros of the limit eigenfunctions u. On the other hand, analysis of the *radial* Robin eigenvalues is much harder to carry out. It is clear from the start that a fine knowledge of the asymptotic behavior of both the Dirichlet and Neumann eigenvalues is also required. Accordingly, we develop a unified approach to study all these three problems. By using previous results in [41], which clarify the Dirichlet problem, a very precise computation of the limit values $\bar{\lambda}_n^{\mathcal{N}} = \lim_{p\to 1} \lambda_{n,p}{}^{\mathcal{N}}$ and $\bar{\lambda}_n = \lim_{p\to 1} \lambda_{n,p}$ is provided in Theorems 14 and 15, respectively. Results also include recurrence relations to determine the exact limit positions of the eigenfunction zeros. In addition, Theorem 16 accounts for the limit profile of the eigenfunctions. It should be stressed that both in Sections 3 and 4, coefficient b is not restricted to be positive.

This paper is organized so that special cases are presented before the more general statements. In addition, most of the sections can be independently read. Section 2 is devoted to auxiliary results. Sections 3 and 4 deal with the one dimensional and radial cases, respectively. It must be remarked that Section 4 also contains a detailed analysis of the Neumann eigenvalue problem. The limit version as $p \to 1$ of problem (1.1) and its connection with $-\Delta_1$ are studied in Section 5. The existence of the limit of $\lambda_{n,p}$ as $p \to 1$ is presented in Section 6. Finally, the continuity of $\lambda_{n,p}$ with respect to p is addressed in Section 7.

2. Estimates and technical results

As a matter of notation $L^p(\Omega)$ and $W^{1,p}(\Omega)$ will designate the standard Lebesgue and Sobolev spaces in a bounded domain $\Omega \subset \mathbb{R}^N$ with exponent $1 \le p \le \infty$. Their norms are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, respectively. If μ is a Radon measure in Ω, $|\mu|$ stands for its total variation. For an open set $G \subset \mathbb{R}^k$ convergence of a sequence $u_n \to u$ in $C^m(G)$ means that u_n together with their derivatives up to the order m converge uniformly in compact sets of G to the corresponding ones of u. Finally, integrals of functions on the boundary $\partial\Omega$ will be understood in the sense of the Hausdorff measure $\mathcal{H}^{N-1}$. An explicit reference to this fact will be generally omitted to brief the notation.

2.1. **Dependence on p of a well–known inequality.** We begin with a basic well–known estimate. Very attention is put on the dependence on p.

Lemma 2. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. Then there exists a constant C_Ω, only depending on Ω such that,*

$$\int_{\partial\Omega} |u|^p \le 2^{p-1} C_\Omega \left(\int_\Omega |\nabla u|^p + (2p-1) \int_\Omega |u|^p \right), \tag{2.5}$$

for all $u \in W^{1,p}(\Omega)$, $p > 1$.

Proof. Choose $\{(U_i, H_i)\}_{i=1}^m$, $U_i \subset \mathbb{R}^N$, $H_i : U_i \to B$, $y = H_i(x)$, a C^1 diffeomorphism, a coordinate system for $\partial\Omega$ so that $\partial\Omega \subset \cup U_i$ and $H_i(U_i \cap \Omega) = B^+$ while $H_i(U_i \cap \partial\Omega) = B \cap \{y_N = 0\}$ for all $1 \le i \le m$. Here B stands for the unit ball of $\mathbb{R}^N$. Let also (U_i, φ_i) be a partition of unity associated to $\{U_i\}$. Then,

$$\int_{\partial\Omega} |u|^p \; dx \le 2^{p-1} \sum \int_{\partial\Omega \cap U_i} |u_i|^p \; dx = 2^{p-1} \sum \int_{\{y_N=0\}} |\tilde{u}_i|^p J_i(y') \; dy',$$

where $u_i = u\varphi_i$, $\tilde{u}_i = u_i \circ H_i^{-1}$, $y' = (y_1, \dots, y_{N-1})$ and the J_i's are positive functions associated to the transformation of integrals rule. By estimating the functions J_i from above and taking into account that,

$$\begin{aligned}\int_{\{y_N=0\}} |\tilde{u}_i|^p \; dy' &\le p \int_{\mathbb{R}^N_+} |\tilde{u}_i|^{p-1} |\partial_n \tilde{u}_i| \; dy \le pC_i \int_{U_i \cap \Omega} |u_i|^{p-1} |\nabla u_i| \\ &\le pC_i \int_{U_i \cap \Omega} \left(|u|^p + |u|^{p-1} |\nabla u| \right) \\ &\le C_i \int_\Omega |\nabla u|^p + C_i (2p-1) \int_\Omega |u|^p,\end{aligned}$$

where value of C_i may possible change in the course of the computation, we finally find that,

$$\int_{\partial\Omega} |u|^p \le 2^{p-1} C \left(\int_\Omega |\nabla u|^p + (2p-1) \int_\Omega |u|^p \right),$$

where C comprises both m and an estimate of the C_i's. □

2.2. **BV–functions and pairings.** The space of all functions of finite variation, that is the space of those $u \in L^1(\Omega)$ whose distributional gradient is a Radon measure with finite total variation, is denoted by $BV(\Omega)$. It is the natural energy space to deal with problems involving the 1–Laplacian operator. As it is going to be shown in Section 5, the limit of (1.1) as p goes to 1 constitutes an important example of such problems. We recall that the notion of trace can be extended to any $u \in BV(\Omega)$ and this fact allows us to interpret it as the boundary values of u and to write $u\big|_{\partial\Omega}$. By means of the trace, the expression,

$$\|u\|_{BV(\Omega)} = \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \int_{\Omega} |Du|\,,$$

furnishes a norm for $BV(\Omega)$. Here $\int_\Omega |Du|$ stands for the total variation of the vector Radon measure Du and $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

Instead of the norm convergence, it is more convenient to deal with the strict convergence in $BV(\Omega)$: we say that a sequence u_n strictly converges to u if

$$u_n \to u \qquad \text{strongly in } L^1(\Omega)$$

and

$$\lim_{n\to\infty} \int_\Omega |Du_n| = \int_\Omega |Du|\,.$$

For further information on functions of bounded variation, we refer to [1] and [22].

In addition, the concept of solution to problems as (5.76) in Section 5 relies on Anzellotti's theory (see [6]), which we next recall. For $u \in BV(\Omega)$ and $z \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\operatorname{div} z \in L^N(\Omega)$, a pairing (z, Du) was defined: it is a Radon measure over Ω which is nothing else than the dot product $z\cdot\nabla u$ when $u \in W^{1,1}(\Omega)$. This pairing satisfies the inequality $|(z, Du)| \le \|z\|_{L^\infty(\Omega,\mathbb{R}^N)}|Du|$ as measures.

A further notion of the theory in [6] is that of the weak trace on $\partial\Omega$ of the normal component of a bounded vector field z whose divergence belongs to $L^N(\Omega)$. It is a bounded function, denoted by $[z,\nu]$, which satisfies $\|\,[z,\nu]\,\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega,\mathbb{R}^N)}$. Most importantly, a Green formula connecting the measure (z, Du) and the weak trace $[z,\nu]$ is established in [6, Th. 1.9]. Namely:

$$\int_\Omega (z, Du) + \int_\Omega u \operatorname{div} z = \int_{\partial\Omega} u\,[z,\nu]\, d\mathcal{H}^{N-1}. \tag{2.6}$$

2.3. **Approximation lemmas.** Our next result is a corrected version of [34, Lemma 4.1], [40, Prop. 3.1] and [38, Ths. 2.12 and 2.17] where defective proofs are provided (specifically, the one corresponding to the 'limsup inequality'). Observe that no sign restriction is imposed to the coefficient b.

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain which satisfies the graph property while $b \in L^\infty(\partial\Omega)$. Consider $p \ge 1$ and a fixed sequence $p_n \to p+$. Then,*

a) *For every $u \in W^{1,p}(\Omega)$ there exists $u_n \in C^1(\overline{\Omega})$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ and*

$$\int_\Omega |\nabla u|^p = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n} \;\;\&\;\; \int_{\partial\Omega} b|u|^p = \lim_{n\to\infty} \int_{\partial\Omega} b|u_n|^{p_n}. \tag{2.7}$$

b) *If $u \in BV(\Omega)$ and $p = 1$, then there also exists a sequence $u_n \in C^1(\overline{\Omega})$ such that $u_n \to u$ in the* strict *topology of $BV(\Omega)$ together with*

$$\int_\Omega |Du| = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n} \;\;\&\;\; \int_{\partial\Omega} b|u| = \lim_{n\to\infty} \int_{\partial\Omega} b|u_n|^{p_n}, \tag{2.8}$$

where $\int_\Omega |Du|$ stands for the total variation of Du.

Proof. Case a). Choose $v_n \in C^1(\overline{\Omega})$ satisfying $v_n \to u$ in $W^{1,p}(\Omega)$. Let $\varepsilon_n > 0$ be a decreasing sequence so that $\varepsilon_n \to 0$. Fixed n there exists a member v_{k_n} such that,

$$\int_\Omega |\nabla u|^p \, dx + \varepsilon_n > \int_\Omega |\nabla v_{k_n}|^p,$$

$$\int_{\partial\Omega} b|u|^p \; + \varepsilon_n > \int_{\partial\Omega} b|v_{k_n}|^p,$$

together with $\|v_{k_n} - u\|_{1,p} < \frac{1}{n}$. Notice that traces theorem has been employed in the last assertion. Observe also that,

$$\int_\Omega |\nabla v_{k_n}|^p = \lim_{l\to\infty} \int_\Omega |\nabla v_{k_n}|^{p_l}, \quad \int_{\partial\Omega} b|v_{k_n}|^p = \lim_{l\to\infty} \int_{\partial\Omega} b|v_{k_n}|^{p_l}.$$

as shown in Remark 1 below. Hence, associated to n there corresponds $\varphi(n) \in \mathbb{N}$ such that

$$\int_\Omega |\nabla u|^p \, dx + \varepsilon_n > \int_\Omega |\nabla v_{k_n}|^{p_l},$$

and,

$$\int_{\partial\Omega} b|u|^p \; + \varepsilon_n > \int_{\partial\Omega} b|v_{k_n}|^{p_l},$$

in both cases for every $l \geq \varphi(n)$.

In the next step we take $v_{k_{n+1}}$ such that $\|v_{k_{n+1}} - u\|_{1,p} < \frac{1}{n+1}$ and

$$\int_\Omega |\nabla u|^p \; + \varepsilon_{n+1} > \int_\Omega |\nabla v_{k_{n+1}}|^p,$$

$$\int_{\partial\Omega} b|u|^p \, dx + \varepsilon_{n+1} > \int_{\partial\Omega} b|v_{k_{n+1}}|^p,$$

are satisfied while,

$$\int_\Omega |\nabla v_{k_{n+1}}|^p = \lim_{s\to\infty} \int_\Omega |\nabla v_{k_{n+1}}|^{p_s}, \quad \int_{\partial\Omega} b|v_{k_{n+1}}|^p = \lim_{s\to\infty} \int_{\partial\Omega} b|v_{k_{n+1}}|^{p_s}.$$

Thus, $\varphi(n+1) > \varphi(n)$ exists such that

$$\int_\Omega |\nabla u|^p \, dx + \varepsilon_{n+1} > \int_\Omega |\nabla v_{k_{n+1}}|^{p_l},$$

and

$$\int_{\partial\Omega} b|u|^p \; + \varepsilon_{n+1} > \int_{\partial\Omega} b|v_{k_{n+1}}|^{p_l} \quad \text{for all } l \geq \varphi(n+1).$$

An increasing sequence:

$$\varphi(n) < \varphi(n+1) < \cdots \varphi(n+h) < \cdots, \qquad \varphi(n) \to \infty,$$

is found such that $\|u - v_{k_{n+h}}\|_{1,p} < \frac{1}{n+h}$ jointly with,

$$\int_\Omega |\nabla u|^p + \varepsilon_{n+h} > \int_\Omega |\nabla v_{k_{n+h}}|^{p_s},$$

$$\int_{\partial\Omega} b|u|^p + \varepsilon_{n+h} > \int_{\partial\Omega} b|v_{k_{n+h}}|^{p_s},$$

for every $s \geq \varphi(n+h)$. The desired sequence is defined as

$$u_m = v_{k_{n+h}} \qquad \text{for} \quad \varphi(n+h) \leq m < \varphi(n+h+1).$$

It is quite clear that

$$\int_\Omega |\nabla u|^p \geq \varlimsup_{m\to\infty} \int_\Omega |\nabla u_m|^{p_m}, \quad \int_{\partial\Omega} b|u|^p \geq \varlimsup_{m\to\infty} \int_{\partial\Omega} b|u_m|^{p_m},$$

On the other hand, the converse inequalities,

$$\int_\Omega |\nabla u|^p \leq \varliminf_{m\to\infty} \int_\Omega |\nabla u_m|^{p_m}, \quad \int_{\partial\Omega} b|u|^p \leq \varliminf_{m\to\infty} \int_{\partial\Omega} b|u_m|^{p_m},$$

are checked without difficulty.

Case b). If $u \in BV(\Omega)$ there exists $v_n \in C^\infty(\Omega) \cap BV(\Omega)$ approaching u in the strict topology ([1], [22]). In addition every such v_n can be also arbitrarily approached in $W^{1,1}(\Omega)$ by a function in $C^1(\overline{\Omega})$. This follows from the fact that Ω fulfils the graph property ([45]). Hence it can be assumed from the start that $v_n \in C^1(\overline{\Omega})$. Now, by replacing the norm $\|\cdot\|_{1,p}$ in $W^{1,p}(\Omega)$ by the strict distance (see [1]),

$$d(u,v) = \|u-v\|_1 + \left| \int_\Omega |Du| - \int_\Omega |Dv| \right|, \qquad u,v \in BV(\Omega),$$

in the argument of case a), one achieves as well the existence of $u_n \in C^1(\overline{\Omega})$ such that

$$\int_\Omega |Du| \geq \varlimsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n}, \tag{2.9}$$

together with,

$$\int_{\partial\Omega} b|u| \ \geq \varlimsup_{n\to\infty} \int_{\partial\Omega} b|u_n|^{p_n}. \tag{2.10}$$

In fact, to check the last assertion one employs the continuity of the trace mapping with respect the strict topology (see [1]) to conclude that

$$\int_{\partial\Omega} b|u| \ = \lim_{n\to\infty} \int_{\partial\Omega} b|v_n|\,.$$

Then the first election of v_{n_k} in the proof of case a) should now further include the requirement:

$$\int_{\partial\Omega} b|u| \ + \varepsilon_n \geq \int_{\partial\Omega} b|v_{n_k}| \ = \lim_{s\to\infty} \int_{\partial\Omega} b|v_{n_k}|^{p_s}.$$

The remaining steps keep the same.

Converse estimates to (2.9) and (2.10) are straightforward. □

Remark 1. If $v \in C^1(\overline{\Omega})$ then,

$$\|\nabla v\|_\infty^{p-p_n} \int_\Omega |\nabla v|^{p_n} \leq \int_\Omega |\nabla v|^p \leq |\Omega|^{1-\frac{p}{p_n}} |\nabla v|_{p_n}^p, \qquad p_n > p.$$

That is why,

$$\int_\Omega |Dv|^p = \lim_{n\to\infty} \int_\Omega |Dv|^{p_n}.$$

By employing a more direct argument than the one in Lemma 3 the following improved statement can be shown.

Lemma 4. *Suppose $p_n \to p-$ and so $p>1$ while $u_n \in C^1(\overline{\Omega})$ is* any *sequence so that $u_n \to u$ in $W^{1,p}(\Omega)$. Then u_n satisfies the identities* (2.7).

Proof. By observing that,

$$\int_\Omega |\nabla u_n|^{p_n} \le |\Omega|^{1-\frac{p_n}{p}} \left(\int_\Omega |\nabla u_n|^p\right)^{\frac{p_n}{p}},$$

together with,

$$\int_{\partial\Omega} b^\pm |u_n|^{p_n} \le \left(\int_{\partial\Omega} b^\pm\right)^{1-\frac{p_n}{p}} \left(\int_{\partial\Omega} b^\pm |u_n|^p\right)^{\frac{p_n}{p}},$$

it follows that,

$$\varlimsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n} \le \int_\Omega |\nabla u|^p, \qquad \varlimsup_{n\to\infty} \int_{\partial\Omega} b^\pm |u_n|^{p_n} \le \int_{\partial\Omega} b^\pm |u|^p.$$

In the previous relations $b^\pm$ stand for $b^\pm = \max\{\pm b, 0\}$. To complete the proof leading to (2.7) we take $\varepsilon > 0$ small enough and observe that,

$$\int_\Omega |\nabla u_n|^{p-\varepsilon} \le |\Omega|^{1-\frac{p-\varepsilon}{p_n}} \left(\int_\Omega |\nabla u_n|^{p_n}\right)^{\frac{p-\varepsilon}{p_n}}.$$

Since $u_n \to u$ in $W^{1,p-\varepsilon}(\Omega)$ then,

$$\int_\Omega |\nabla u|^{p-\varepsilon} \le |\Omega|^{\frac{\varepsilon}{p}} \left(\varliminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n}\right)^{\frac{p-\varepsilon}{p}}.$$

By taking limits as $\varepsilon \to 0+$ we achieve,

$$\int_\Omega |\nabla u|^p = \lim_{\varepsilon\to 0+} \int_\Omega |\nabla u|^{p-\varepsilon} \le \varliminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n}.$$

The same argument shows that,

$$\int_{\partial\Omega} b^\pm |u|^p = \lim_{\varepsilon\to 0+} \int_{\partial\Omega} b^\pm |u|^{p-\varepsilon} \le \varliminf_{n\to\infty} \int_{\partial\Omega} b^\pm |u_n|^{p_n}.$$

□

In some steps of our forthcoming results we need to know wether the convergence $u_n \to u$ in $L^{p_0}(\Omega)$ implies,

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p_n} = \int_\Omega |u|^{p_0}, \qquad \text{provided } p_n \to p_0. \tag{2.11}$$

Next statement is an improved version of [39, Lemma 5.9].

Lemma 5.

a) *Assume that $p_n \to p_0-$ together with $u_n \to u$ in $L^{p_0}(\Omega)$. Then the equality* (2.11) *holds.*

b) *Suppose that $u_n \to u$ in $L^{p_0}(\Omega)$ when $p_n \to p_0+$, $p_0 \ge 1$. If the extra condition*

$$\int_\Omega |u_n|^{p_0+\delta} \le M, \qquad n \in \mathbb{N}, \tag{2.12}$$

is satisfied for some small $\delta > 0$, then (2.11) *also holds.*

Proof. In case a) and after extraction of a subsequence, there exists $h \in L^{p_0}(\Omega)$ such that $|u_n| \le h$ a.e. in Ω ([9, Theorem 4.9]). In addition, we may assume $u_n(x) \to u(x)$ for almost all $x \in \Omega$. Since,

$$|u_n|^{p_n} \le \chi_{\{h\le 1\}} + \chi_{\{h>1\}} h^{p_0} \in L^1(\Omega),$$

then the stated equality holds for this particular sequence. The uniqueness of the limit implies that the whole sequence u_n (not merely a subsequence) satisfies (2.11).

For b), (2.12) implies the estimate,

$$\int_A |u_n|^{p_n} \quad \leq \quad |A|^{\frac{p_0+\delta-p_n}{p_0+\delta}} \left(\int_A |u_n|^{p_0+\delta}\right)^{\frac{p_n}{p_0+\delta}} \quad \leq \quad |A|^{\frac{\delta/2}{p_0+\delta}} \max\{1, M\},$$

for large n provided $|A| < 1$. Hence, the family $|u_n|^{p_n}$ is equi integrable. This entails (2.11). □

Remark 2. A relaxed version of (2.12) as

$$\int_\Omega |u_n|^{p_n} = O(1), \qquad \text{as } n \to \infty,$$

does not suffice for the validity of (2.11).

2.4. **Eigenvalue estimates.** Our main objective is to describe the limit of eigenvalues to problem (1.1) as $p \to 1$. Of course, we focus on those eigenvalues λ_n variationally defined using the Ljusternik–Schnirelmann theory. In this subsection, we begin by introducing these eigenvalues and stating rough estimates for the behavior of λ_n as $p \to 1$.

For $p > 1$, the sequence of the Ljusternik–Schnirelmann eigenvalues is defined as

$$\lambda_n = \inf_{A\in\mathcal{A}_n} \max_{u\in A} \frac{\int_\Omega |\nabla u|^p + \int_{\partial\Omega} b|u|^p}{\int_\Omega |u|^p}, \tag{2.13}$$

where $\mathcal{A}_n = \{A \subset W^{1,p}(\Omega) : A = -A,\ A \text{ compact},\ \gamma(A) \geq n\}$, and $\gamma(A)$ stands for the Krasnoselskii genus of A ([44]). It should be observed that Neumann eigenvalues $\lambda_n^{\mathcal{N}}$ are obtained from (2.13) by choosing $b = 0$. Moreover, Dirichlet eigenvalues $\lambda_n^{\mathcal{D}}$ are obtained by taking $b = 0$ in (2.13) and changing the class $\mathcal{A}_n$ by

$$\mathcal{A}_n^{\mathcal{D}} = \{A \subset W_0^{1,p}(\Omega) : A = -A,\ A \text{ compact},\ \gamma(A) \geq n\},$$

that is,

$$\lambda_n^{\mathcal{D}} = \inf_{A\in\mathcal{A}_n^{\mathcal{D}}} \max_{u\in A} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}. \tag{2.14}$$

When b is nonnegative on $\partial\Omega$, it can be checked by direct inspection that:

$$\lambda_n^{\mathcal{N}} \leq \lambda_n \leq \lambda_n^{\mathcal{D}}, \qquad n \in \mathbb{N}. \tag{2.15}$$

In the sequel, generic dependence on p will be denoted by adding a subindex p when it is necessary to stress this fact. It should be remarked that the existence and finiteness of both limits $\lim_{p\to 1} \lambda_{n,p}^{\mathcal{D}}$ and $\lim_{p\to 1} \lambda_{n,p}^{\mathcal{N}}$ have been stated in [41, Cor. 3, Rem. 3] (see also [34] and [38] for the Dirichlet case). An immediate consequence of (2.15) is the following observation.

Lemma 6. *Estimate,*

$$0 < \bar{\lambda}_n^{\mathcal{N}} := \lim_{p\to 1} \lambda_{n,p}^{\mathcal{N}} \leq \varliminf_{p\to 1} \lambda_{n,p} \leq \varlimsup_{p\to 1} \lambda_{n,p} \leq \bar{\lambda}_n^{\mathcal{D}} := \lim_{p\to 1} \lambda_{n,p}^{\mathcal{D}},$$

holds true for every $n \in \mathbb{N}$.

Remark 3. Owing to the lower estimate of the second eigenvalue ${\lambda_{2,p}}^{\mathcal{N}}$ in [8, Th. 1.1] it can be assured that the limits $\lim_{p\to 1} \lambda_{n,p}$ are bounded away from zero for $n \geq 2$. In addition, that $\lim_{p\to 1} \lambda_{1,p} > 0$ follows Corollary 27 if $b \gneq 0$ (see Remark 12 for a finer estimate when $b(x) \geq b_- > 0$).

3. The one–dimensional setting

As a first stage in the analysis of (1.1) as $p \to 1$ we are addressing the symmetric version of the Robin problem (1.1) in the unit interval. That is,

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda |u|^{p-2}u & 0 < x < 1, \\ u'(0) = 0, \\ |u'(1)|^{p-2}u'(1) = -b|u(1)|^{p-2}u(1), \end{cases} \tag{3.16}$$

where for the moment b is a positive constant and $' = \frac{d}{dx}$. We will later suppress this restriction on b (see Section 3.1). It can be shown that a weak eigenfunction $u \in W^{1,p}(0,1)$ satisfies $u \in C^1[0,1]$ with $|u'|^{p-2}u' \in C^1[0,1]$ (see [16]).

To find out the eigenpairs (λ, u) to (3.16) it is convenient introducing the normalized equation,

$$(|\dot v|^{p-2}\dot v)^{\cdot} + |v|^{p-2}v = 0, \qquad (^{\cdot} = \frac{d}{dt}), \tag{3.17}$$

subject to initial data,

$$v(0) = 1, \qquad \dot v(0) = 0. \tag{3.18}$$

It follows from phase portrait analysis (see [16], [13]) the existence of a unique solution $v(t)$ to (3.17), (3.18) which satisfies the following properties.

i) $v(-t) = v(t)$ and $|v(t)| \le 1$ for $t \in \mathbb{R}$.

ii) v is $2\pi_p$ periodic with,

$$\frac{\pi_p}{2} = (p-1)^{\frac{1}{p}} \int_0^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}}.$$

iii) v is decreasing in $0 \le t \le \pi_p$ and it is implicitly defined as,

$$(p-1)^{\frac{1}{p}} \int_v^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} = t, \qquad 0 \le t \le \pi_p.$$

Moreover, zeros of v and $\dot v$ are,

$$t = \frac{\pi_p}{2} + k\pi_p, \qquad \text{and} \qquad t = k\pi_p, \qquad k \in \mathbb{Z},$$

respectively.

Solutions $w(t)$ to (3.17) keep constant the energy $E(w, \dot w) = \frac{1}{p'}|\dot w|^p + \frac{1}{p}|w|^p$. Thus v satisfies,

$$(p-1)|\dot v(t)|^p + |v(t)|^p = 1, \qquad t \in \mathbb{R}. \tag{3.19}$$

For next use it is appropriate to introduce the functions,

$$v(t) = \cos_p t, \qquad -(p-1)^{\frac{1}{p}}\dot v(t) = \sin_p t, \qquad \tan_p t = \frac{\sin_p t}{\cos_p t}.$$

It should be noted that this is a simple matter of 'ad hoc' notation. See for instance [21, Chapter 1] where a slight modified version are termed as the 'half–linear' trigonometrical functions. It is then clear that $\sin_p t$ is an odd, $2\pi_p$ periodic function such that the fundamental relation,

$$|\cos_p t|^p + |\sin_p t|^p = 1, \qquad t \in \mathbb{R}, \tag{3.20}$$

holds.

Now, to solve (3.16) we first observe that all eigenvalues are positive if $b > 0$. Moreover,

$$u(x) = \cos_p(\lambda^{\frac{1}{p}} x),$$

both satisfies the equation and the initial condition $u'(0) = 0$. The boundary condition at $x = 1$ reads as,

$$\lambda^{\frac{1}{p}} \tan_p(\lambda^{\frac{1}{p}}) = (p-1)^{\frac{1}{p}} b^{\frac{1}{p-1}}. \tag{3.21}$$

Some few features of the function $\tan_p t$ should be highlighted. As in the 'circular' case, $\tan_p t$ turns out to be a π_p periodic function. In fact, it follows from assertion iii) that,

$$\cos_p(\pi_p - t) = -\cos_p t, \qquad \sin_p(\pi_p - t) = \sin_p t.$$

Thus,

$$\tan_p(t - \pi_p) = -\tan_p(\pi_p - t) = \tan_p t, \tag{3.22}$$

and $\tan_p t$ is π_p periodic. In addition, relation (3.20) entails that both $\sin_p t$ and $\tan_p t$ are increasing in $0 \le t < \frac{\pi_p}{2}$. Hence $\tan_p t$ is increasing in $|t| < \dfrac{\pi_p}{2}$ and it follows from $\cos_p(\pi_p/2) = 0$ that

$$\lim_{t \to \frac{\pi_p}{2} \pm} \tan_p t = \pm\infty.$$

On the other hand, a few of calculus leads to the formulas,

$$(p-1)^{\frac{1}{p}} \frac{d}{dt} (\sin_p t) = |\tan_p t|^{2-p} \cos_p t,$$

and

$$(p-1)^{\frac{1}{p}} \frac{d}{dt} (\tan_p t) = |\tan_p t|^{2-p} + \tan_p^2 t,$$

which agree with the circular case as $p = 2$.

By taking into account the periodicity of $\tan_p t$ we conclude that equation (3.21) exhibits an increasing sequence λ_m of positive solutions such that,

$$(m-1)\pi_p < \lambda_m{}^{\frac{1}{p}} < \left(m - \frac{1}{2}\right)\pi_p, \qquad m \in \mathbb{N}. \tag{3.23}$$

Moreover, there only exists a unique root in this interval, as illustrated in Figure 2.

Theorem 7. *Problem* (3.16) *admits an increasing sequence of positive eigenvalues $\lambda = \lambda_m$ which are defined by the solutions to equation* (3.21)*. An associated eigenfunction u to $\lambda = \lambda_m$, normalized as $u(0) = 1$, is defined by,*

$$u_m = \cos_p(\lambda_m{}^{\frac{1}{p}} x), \qquad m \in \mathbb{N}.$$

Eigenfunction u_m vanishes $m-1$ times in the interval $0 < x < 1$ at the points,

$$x_k = \frac{\left(k - \frac{1}{2}\right)\pi_p}{\lambda_m{}^{\frac{1}{p}}}, \qquad k = 1, \dots, m-1.$$

Moreover,

$$0 < (-1)^{m-1} u_m(1) < 1, \qquad m \in \mathbb{N}. \tag{3.24}$$

Remark 4. In the Neumann problem $u'(1) = 0$ replaces the boundary condition in (3.16). The eigenvalues are,

$$\lambda_m^{\mathcal{N}} = \{(m-1)\pi_p\}^p, \qquad m \in \mathbb{N},$$

while the corresponding Dirichlet eigenvalues (in this case $u(1) = 0$ in (3.16)) are,

$$\lambda_m^{\mathcal{D}} = \left[\left(m - \frac{1}{2}\right)\pi_p\right]^p, \qquad m \in \mathbb{N}.$$

Thus, relation (3.23) is the well–known relative distribution (2.15) of Neumann, Dirichlet and Robin eigenvalues.

We are now going to elucidate the limit behavior of the eigenvalues to (3.16) as $p \to 1+$. To emphasize their dependence on p we are writing $\lambda_{m,p}$ instead of λ_m.

Theorem 8. *Let $\lambda_{m,p}$ be the m–th eigenvalue to problem* (3.16)*. Then,*

i) *For $b > 1$,*

$$\lim_{p\to 1} \lambda_{m,p} = \lim_{p\to 1} \lambda_{m,p}{}^{\mathcal{D}} = 2m - 1, \qquad m \in \mathbb{N}.$$

ii) *For $0 < b \le 1$,*

$$\lim_{p\to 1} \lambda_{m,p} = \lim_{p\to 1} \lambda_{m,p}{}^{\mathcal{N}} + b = 2(m-1) + b, \qquad m \in \mathbb{N}. \tag{3.25}$$

Proof. Our first goal is expressing equation (3.21) in a more manageable way. We observe that roots $\lambda_{m,p}{}^{\frac{1}{p}}$ fall on an interval:

$$I_m = \{t : (m-1)\pi_p \le t \le (m - \frac{1}{2})\pi_p\},$$

where $\cos_p t$ is decreasing when $m = \dot{2} + 1$ while $\cos_p t$ is increasing in this interval provided $m = \dot{2}$. By employing the boundary condition we conclude that,

$$0 < (-1)^{m-1} \cos_p(\lambda_{m,p}{}^{\frac{1}{p}}) < 1, \qquad m \in \mathbb{N}.$$

The $2\pi_p$ periodicity of $v(t) = \cos_p t$, condition iii) and the symmetry relation $\cos_p(\pi_p - t) = -\cos_p t$ altogether imply,

$$(p-1)^{\frac{1}{p}} \int_v^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} = t - (m-1)\pi_p, \qquad t \in I_m,$$

provided $m = \dot{2} + 1$ while if m is even then,

$$(p-1)^{\frac{1}{p}} \int_{-1}^v \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} = t - (m-1)\pi_p, \qquad t \in I_m.$$

Coming back to the boundary condition and employing the energy conservation (3.19) we find that,

$$\cos_p(\lambda_{m,p}{}^{\frac{1}{p}}) = (-1)^{m-1} \left[\frac{\lambda_{m,p}}{(p-1)b^{p'} + \lambda_{m,p}}\right]^{\frac{1}{p}},$$

where the remark on the sign of $\cos_p(\lambda_{m,p}{}^{\frac{1}{p}})$ has been used. By resorting to a symmetry argument in the integral we achieve the following alternative version of equation (3.21),

$$(p-1)^{\frac{1}{p}} \int_{(-1)^{m-1}\cos_p(\lambda_{m,p}{}^{\frac{1}{p}})}^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} = \lambda_{m,p}{}^{\frac{1}{p}} - (m-1)\pi_p. \tag{3.26}$$

At this point observe that,

$$\lim_{p\to 1} \cos_p({\lambda_{m,p}}^{\frac{1}{p}}) = 0,$$

provided that $b > 1$. Hence, on account of $\lim_{p\to 1} \pi_p = 2$ we arrive at,

$$\bar{\lambda}_m := \lim_{p\to 1} \lambda_{m,p} = \lim_{p\to 1} \{(m-1)\pi_p + \frac{\pi_p}{2}\} = 2m-1, \qquad m \in \mathbb{N}.$$

According to [12], [41], values $\bar{\lambda}_m$ coincide with the Dirichlet eigenvalues of the 1–Laplacian in the unit interval. This proves part i).

The remaining cases are more delicate. We are first assuming $m \geq 2$ since this implies that ${\lambda_{m,p}}^{-1}$ remains bounded as $p \to 1$. By setting,

$$A = \left[\frac{\lambda_{m,p}}{(p-1)b^{p'} + \lambda_{m,p}}\right]^{\frac{1}{p}}.$$

we then notice that $\lim_{p\to 1} A = 1$ for every $m \geq 2$ as $0 < b \leq 1$. By an elementary estimate we get,

$$\overline{\lim_{p\to 1}} (p-1)^{\frac{1}{p}} \int_A^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} \leq \overline{\lim_{p\to 1}} (p-1) \int_A^1 \frac{ds}{(1-s)^{\frac{1}{p}}} = \overline{\lim_{p\to 1}}\, p(1-A)^{\frac{1}{p'}}.$$

We now write,

$$1 - A = 1 - \frac{1}{[(p-1)\mu_p b^{p'} + 1]^{\frac{1}{p}}},$$

where $\mu_p = \lambda_{m,p}^{-1} < \left[\lambda_{m,p}^{\mathcal{N}}\right]^{-1}$ and so $\mu_p = O(1)$ as $p \to 1$. Thus,

$$\ln(1-A) = \ln\left([(p-1)\mu_p b^{p'} + 1]^{\frac{1}{p}} - 1\right) + o(1), \qquad \text{as } p \to 1,$$

since $(p-1)\mu_p b^{p'} = o(1)$ as $p \to 1$.

On the other hand,

$$[(p-1)\mu_p b^{p'} + 1]^{\frac{1}{p}} = 1 + \frac{p-1}{p}\mu_p b^{p'} + o\left((p-1)\mu_p b^{p'}\right),$$

and so,

$$\ln(1-A) = \ln\left(\frac{p-1}{p}\mu_p b^{p'}\right) + o(1) = p' \ln b + \ln\left(\frac{p-1}{p}\mu_p\right) + o(1).$$

Gathering together the previous estimates,

$$(1-A)^{\frac{1}{p'}} = e^{\frac{1}{p'}\ln(1-A)} = e^{\ln b + \frac{1}{p'}\ln\left(\frac{\mu_p}{p'}\right) + o(1)}.$$

This implies that,

$$\lim_{p\to 1} p(1-A)^{\frac{1}{p'}} = b.$$

Therefore,

$$\overline{\lim_{p\to 1}} (p-1)^{\frac{1}{p}} \int_A^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} \leq b.$$

To check the complementary estimate we first observe that,

$$1 - s^p = p\int_s^1 x^{p-1} = p\int_0^{(1-s)} (t+s)^{p-1} < p(1+\delta)^{p-1}(1-s),$$

provided that $1-\delta < s$. Thus, for $\delta > 0$ and p close enough to unity we achieve that,

$$\varliminf_{p\to 1}(p-1)^{\frac{1}{p}}\int_A^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} \geq \varliminf_{p\to 1}\frac{p-1}{p^{\frac{1}{p}}(1+\delta)^{\frac{1}{p'}}}\int_A^1 \frac{ds}{(1-s)^{\frac{1}{p}}} = b.$$

This completes the checking of ii) for $m \geq 2$.

Only remains studying $\lambda_{1,p}$ when $0 < b \leq 1$. From the equation (3.17),

$$|\dot v_p(t)|^{p-2}\dot v_p(t) = -\int_0^t |v_p|^{p-2}v_p > -t,$$

where $v_p(t) = \cos_p t$. This implies $\dot v_p(t) > -t^{\frac{1}{p-1}}$ and so we get that the inequalities,

$$1 > v_p(t) > 1 - \frac{p-1}{p}t^{p'}$$

hold. Hence, both $v_p \to 1$ and $|\dot v_p(t)|^{p-2}\dot v_p(t) \to -t$ uniformly in $[0,1]$ as $p \to 1$. Since,

$$-t^{p-1}\frac{|\dot v_p|^{p-2}\dot v_p}{|v_p|^{p-2}v_p} \to t, \qquad \text{as } p \to 1,$$

uniformly in $[0,1]$, it follows that $\lambda_{1,p} \to b$ as $p \to 1$ when $0 < b \leq 1$. $\square$

Let us review next the limit profiles of the eigenfunctions as $p \to 1$.

Theorem 9. *Let $u_{m,p}$ be the eigenfunction to* (3.16) *associated to λ_m and normalized as $u_{m,p}(0) = 1$. Then,*

$$\lim_{p\to 1} u_{m,p} = \sum_{k=1}^{m}(-1)^{k-1}\chi_{D_k}, \tag{3.27}$$

χ_{D_k} being the the characteristic function of the interval $D_k = (\bar x_{k-1}, \bar x_k)$, where

$$\bar x_k = \begin{cases} \dfrac{2k-1}{2m-1} & b > 1, \\ \dfrac{2k-1}{2(m-1)+b} & 0 < b \leq 1, \end{cases} \qquad 1 \leq k \leq m. \tag{3.28}$$

Moreover, the convergence (3.27) *holds in $C^1(D_k)$ for $1 \leq k \leq m$.*

Proof. It is consequence of the fact,

$$\lim_{p\to 1}\cos_p t = \sum_{k\in\mathbb{Z}}(-1)^{k-1}\chi_{(\xi_{k-1},\xi_k)}(t),$$

$\xi_k = 2k-1$, where the convergence holds in $C^1(\xi_{k-1}, \xi_k)$ for all $k \in \mathbb{Z}$ (see [41, Prop. 9] and Figure 1). $\square$

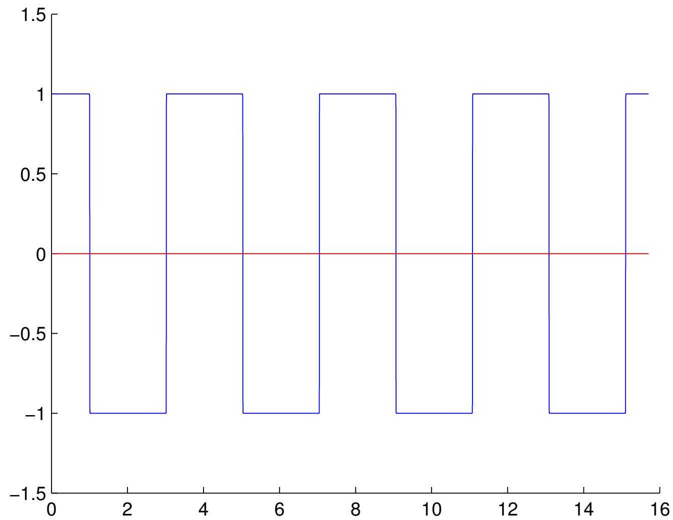

FIGURE 1. Function $v(t) = \cos_p t$ for $p = 1.001$. In this case, $\pi_p = 2.0138$.

3.1. **The case when b is negative.** If $b < 0$ all eigenvalues $\lambda = \lambda_m$ to (3.16) for $m \geq 2$ are positive, solve

$$\lambda^{\frac{1}{p}} \tan_p \lambda^{\frac{1}{p}} = -|b|^{\frac{1}{p-1}} (p-1)^{\frac{1}{p}},$$

and satisfy (see Figure 2),

$$\lambda_{m-1}^{\mathcal{D}} < \lambda_m < \lambda_m^{\mathcal{N}}, \qquad m \geq 2.$$

Conclusions of Theorem 8 remain true if conditions $b > 1$ and $0 < b \leq 1$ in items i) and ii) change to $b < -1$ and $-1 \leq b < 0$, respectively. Notice that in the latter case, estimate (3.25) retains its meaning. The same happens to Theorem 9 if inequalities $b > 1$ and $0 < b \leq 1$ in (3.28) are replaced by $b < -1$ and $-1 \leq b < 0$, respectively.

However, as $b < 0$ a further negative eigenvalue λ_1 exists. In fact, to find out negative eigenvalues λ we try,

$$u(x) = v(t), \qquad t = |\lambda|^{\frac{1}{p}} x,$$

as eigenfunctions, where v solves,

$$\begin{cases} (|\dot{v}|^{p-2}\dot{v})^{\cdot} - |v|^{p-2}v = 0, & t > 0, \\ v(0) = 1, \qquad \dot{v}(0) = 0. \end{cases} \tag{3.29}$$

This hyperbolic version of (3.17), (3.18) has a unique solution $v(t)$ globally defined in $t \in \mathbb{R}$. Since $|\dot{v}|^{p-2}\dot{v}$ is increasing and $\dot{v}(0) = 0$ then v is increasing in $t \geq 0$. In addition,

$$\ddot{v} = \frac{1}{p-1} \left|\frac{\dot{v}}{v}\right|^{2-p} v > 0,$$

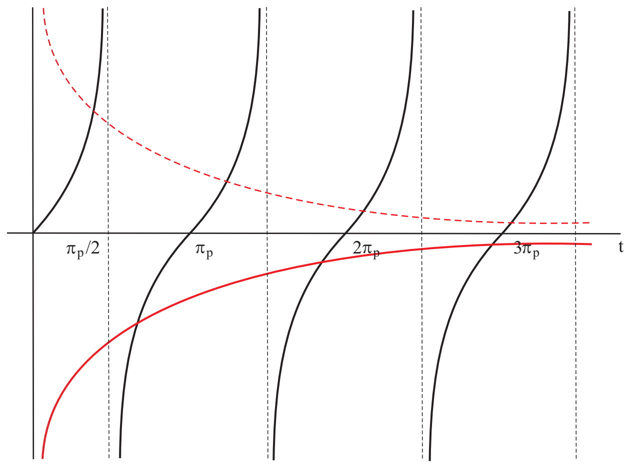

FIGURE 2. Function $\tan_p t$ vs $\operatorname{sign} b \dfrac{|b|^{\frac{1}{p-1}}(p-1)^{\frac{1}{p}}}{t}$ as b changes sign.

and so v is convex and diverges to ∞ as $|t| \to \infty$. Moreover, v is implicitly defined through,

$$(p-1)^{\frac{1}{p}} \int_1^v \frac{ds}{(s^p-1)^{\frac{1}{p}}} = t, \qquad t \in \mathbb{R}.$$

By similarity with the previous section we set,

$$v(t) = \cosh_p t, \qquad (p-1)^{\frac{1}{p}} \dot{v}(t) = \sinh_p t, \qquad \tanh_p t = \frac{\sinh_p t}{\cosh_p t}.$$

Then, the equivalent to the fundamental relation, that is,

$$|\cosh_p t|^p - |\sinh_p t|^p = 1,$$

holds true. Setting $u(x) = \cosh_p(|\lambda|^{\frac{1}{p}} x)$, u satisfies the equations and the condition $u'(0) = 0$. The other boundary condition for the negative eigenvalues λ becomes,

$$|\lambda|^{\frac{1}{p}} \tanh_p |\lambda|^{\frac{1}{p}} = |b|^{\frac{1}{p-1}} (p-1)^{\frac{1}{p}}. \tag{3.30}$$

Our next result discuses the existence of negative eigenvalues.

Theorem 10. *Assume $b < 0$. Then, problem (3.16) admits a unique negative eigenvalue $\lambda_{1,p}$ with corresponding normalized and positive eigenfunction,*

$$u_{1,p}(x) = \cosh_p(|\lambda_{1,p}|^{\frac{1}{p}} x).$$

Moreover,

i) $\lim_{p\to1} \lambda_{1,p} = b$ *if* $-1 \le b < 0$.

ii) $\lim_{p\to1} \lambda_{1,p} = -\infty$ *as* $b < -1$.

iii) $\lim_{p\to1} u_{1,p} = 1$ *in* $C^1[0,1]$ *provided* $-1 < b < 0$ *while such convergence holds in* $C^1[0,1)$ *when* $b = -1$.

iv) *On the contrary,*

$$\lim_{p\to1} u_{1,p}(x) = \infty, \qquad \textit{for each } 0 < x \le 1,$$

when $b < -1$.

Proof. Owing to,

$$(p-1)^{\frac{1}{p}}\frac{d}{dt}(\tanh_p t)=\tanh_p^2 t\frac{1}{|\sinh_p t|^p},$$

together with,

$$\tanh_p t=\left(1-\frac{1}{\cosh_p^p t}\right)^{\frac{1}{p}},\qquad t>0, \tag{3.31}$$

it follows that equation,

$$t\tanh_p t=|b|^{\frac{1}{p-1}}(p-1)^{\frac{1}{p}}, \tag{3.32}$$

has a unique positive root $t=t_b$ (which depend on p). Indeed, the left hand side is increasing while $t\tanh_p t\sim t$ as $t\to\infty$. Thus,

$$\lambda_{1,p}=-t_b^p,$$

is the unique negative eigenvalue.

On the other hand, we deduce from (3.31) and (3.32) that $t=t_b$ satisfies,

$$\frac{t^p-|b|^{p'}(p-1)}{t^p}=\frac{1}{\cosh_p^p t}. \tag{3.33}$$

In particular,

$$t_b^p>|b|^{p'}(p-1),$$

and $t_b\to\infty$ as $p\to 1$ when $b<-1$. This proves ii).

To show i) we notice that,

$$t_b=(p-1)^{\frac{1}{p}}\int_1^B\frac{ds}{(s^p-1)^{\frac{1}{p}}},$$

with $B=\cosh_p t_b$. We now claim that,

$$\lim_{p\to1}B=\lim_{p\to1}\left[\frac{t_b^p}{t_b^p-|b|^{p'}(p-1)}\right]^{\frac{1}{p}}=1\qquad\text{provided } -1\le b<0.$$

By assuming this fact we first observe that,

$$t_b\sim(p-1)\int_1^B\frac{ds}{(s^p-1)^{\frac{1}{p}}}\le(p-1)\int_1^B\frac{ds}{(s-1)^{\frac{1}{p}}}=p(B-1)^{\frac{1}{p'}},$$

as $p\to1$. Now, an argument similar to the one leading to (3.25) in Theorem 8 shows that,

$$\frac{1}{p'}\ln(B-1)=\ln|b|+\frac{p-1}{p}\ln\left(\frac{(p-1)}{p}\frac{1}{t_b}\right)+o(1),$$

as $p\to1$. This implies that,

$$\varlimsup_{p\to1}t_b\le|b|.$$

For the complementary limit one uses the inequality,

$$x^p-1\le p(1+\delta)^{p-1}(x-1),\qquad 1\le x\le1+\delta,$$

to achieve the lower estimate,

$$t_b\sim(p-1)\int_1^B\frac{ds}{(s^p-1)^{\frac{1}{p}}}\ge\frac{(p-1)}{p^{\frac{1}{p}}(1+\delta)^{\frac{p-1}{p}}}\int_1^{1+\delta}\frac{ds}{(s-1)^{\frac{1}{p}}},$$

where we identify $B = 1 + \delta$. Then the previous computation shows that,

$$\underline{\lim}_{p\to 1} t_b \geq |b|.$$

The proof of i) is thus completed.

To show the claim we notice that both the increasing character of $v = \cosh_p t$ and the equation imply that,

$$(\dot{v}(t))^{p-1} = \int_0^t v(s)^{p-1} < tv(t)^{p-1}, \qquad t > 0.$$

Hence,

$$\left(t\frac{\dot{v}}{v}\right)^{p-1} < t^p, \qquad t > 0. \tag{3.34}$$

By setting $t = t_b$ in this inequality and resorting to (3.32) we find the estimate,

$$|b| < t_b^p,$$

which implies that,

$$\underline{\lim}_{p\to 1} t_b \geq |b|.$$

Then,

$$\lim_{p\to 1} B = \lim_{p\to 1} \frac{1}{\left[1 - |b|^{p'}(p-1)t_b^{-p}\right]^{\frac{1}{p}}} = 1,$$

and the claim is proved.

To show the remaining assertions we first deduce from (3.34) that,

$$1 < v(t) < e^{\frac{t^{p'}}{p'}}, \qquad t > 0, \tag{3.35}$$

and so,

$$\lim_{p\to 1} v = 1, \qquad \lim_{p\to 1} \dot{v} = 0,$$

in both cases in $C[0,1]$. Since $v = \cosh_p t$ one concludes iii).

As for iv) we are showing that $v = \cosh_p t \to \infty$ uniformly on compacta of $t > 1$ as $p \to 1$. In fact, choose $A > 1$ and set $t_A > 0$ such that,

$$\cosh_p t_A = A.$$

Because of (3.35) we find that,

$$\underline{\lim}_{p\to 1} t_A \geq 1.$$

On the other hand,

$$t_A = (p-1)^{\frac{1}{p}} \int_1^A \frac{ds}{(s^p - 1)^{\frac{1}{p}}} < (p-1)^{\frac{1}{p}} p'(A-1)^{\frac{1}{p'}}.$$

Hence,

$$\overline{\lim}_{p\to 1} t_A \leq 1.$$

Since $\lim_{p\to 1} t_A = 1$ we achieve the desired divergence to ∞ of $\cosh_p t$ in the interval $(1, \infty)$. See Figure 3. $\square$

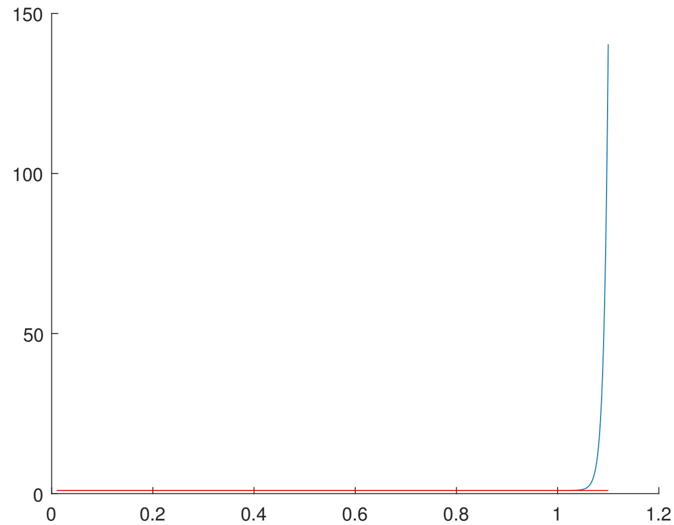

FIGURE 3. Function $v = \cosh_p t$ corresponding to $p = 1.008$. Notice the difference between the scales in the axis together with the graph of the function $v = 1$.

4. Radially symmetric eigenvalues

When searching for radial eigen pairs (λ, u), we denote $u = u(r)$, $r = |x|$ in $\Omega = B$ a ball (assume it is the unit ball). Then (1.1) is expressed in the form,

$$\begin{cases} -(|u'|^{p-2}u')' - \dfrac{N-1}{r}|u'|^{p-2}u' = \lambda |u|^{p-2}u, & 0 < r < 1, \\ u'(0) = 0, & \\ |u'|^{p-2}u' = -b|u|^{p-2}u\,, \quad \text{at } r = 1. & \end{cases} \tag{4.36}$$

As in the previous section the influence of the sign of b on the eigenvalues will be closely analyzed. Weak radial eigenfunctions $u \in W^{1,p}(B)$ exhibit the smoothness $u \in C^1[0,1]$, $|u'|^{p-2}u' \in C^1[0,1]$. Thus, solvability of (4.36) can be understood in a classical sense.

By patterning the argument in Section 3 we consider the initial value problem,

$$\begin{cases} -(|\dot v|^{p-2}\dot v)^{\cdot} - \dfrac{N-1}{r}|\dot v|^{p-2}\dot v = |v|^{p-2}v, & t > 0, \\ v(0) = 1, \qquad \dot v(0) = 0, & \end{cases} \tag{4.37}$$

where $v = v(t)$. Then, eigenfunctions u to (4.36) corresponding to *positive* eigenvalues λ, when properly normalized, can be expressed as,

$$u(r) = v(t), \qquad t = \lambda^{\frac{1}{p}} r, \qquad \lambda > 0.$$

It is warned that all eigenvalues are positive if $b > 0$. When $b < 0$ positive and negative eigenvalues must be separately studied.

On the other hand, existence of a unique global solution v to (4.37) is stated in [41, Lemma 10] (see also [17]). Among other features, such a solution satisfies $|v(t)| \le 1$ in $t \ge 0$ and possesses a sequence of simple zeros,

$$0 < \theta_1 < \theta_2 < \cdots, \qquad \theta_n \to \infty.$$

We now claim the existence of a unique zero $t = \sigma_i$ of $\dot v$ in the interval (θ_{i-1}, θ_i) for every $i \in \mathbb{N}$. In fact, equation (4.37) can be written as,

$$-(t^{N-1}|\dot v|^{p-2}\dot v)^{\cdot} = t^{N-1}|v|^{p-2}v.$$

Beginning with the interval $(0, \theta_1)$ we find that $\dot v < 0$ there, even up to $t = \theta_1$. Since $v(\theta_1) = v(\theta_2) = 0$ a zero of $\dot v$ exists in (θ_1, θ_2). Assume $t = \sigma_2$ is the first one, then $\dot v > 0$ in (σ_2, θ_2) up to $t = \theta_2$. Hence σ_2 is the unique zero of $\dot v$ in $\theta_1 \le t \le \theta_2$. The claim is proved by repeating the argument in the other intervals. Thus, the following distribution of zeros of v and $\dot v$ is deduced,

$$0 = \sigma_1 < \theta_1 < \sigma_2 < \theta_2 < \cdots, \qquad \sigma_n \to \infty.$$

Note that the σ_i is 'simple' in the sense $(t^{N-1}|\dot v|^{p-2}\dot v)^{\cdot}_{|t=\sigma_i} \ne 0$ for all i. Moreover, relations,

$$(-1)^{i-1}v(t) > 0, \qquad \theta_{i-1} < t < \theta_i,$$

and

$$(-1)^{i-1}\dot v(t) > 0, \qquad \sigma_{i-1} < t < \sigma_i,$$

hold true for every i, where $\theta_0 = 0$ in the first inequality. It should be remarked that in the case $\Omega = B$, Dirichlet and Neumann 'radial' eigenvalues correspond to the values,

$$\lambda_m^{\mathcal{D}} = \theta_m^p, \qquad \lambda_m^{\mathcal{N}} = \sigma_m^p, \qquad m \in \mathbb{N}. \tag{4.38}$$

Remark 5. Radial eigenvalues in B could be computed by means of the Ljusternik–Schnirelmann approach (Section 2.4). In this case classes $\mathcal{A}_n^{\mathcal{D}}$ and $\mathcal{A}_n$ should be further subject to radial symmetry. This procedure gives rise to radial Dirichlet or Neumann eigenvalues that exactly coincide with (4.38). See [41, Th. 12] for a detailed account on this assertion.

We now observe that $u(r) = v(\lambda^{\frac{1}{p}} r)$ satisfies the Robin condition if $t = \lambda^{\frac{1}{p}}$ solves the equation,

$$-\, t^{p-1}\frac{|\dot v|^{p-2}\dot v}{|v|^{p-2}v}_{|t=\lambda^{\frac{1}{p}}} = b \tag{4.39}$$

wherewith

$$-\, t\frac{\dot v(t)}{v(t)}_{|t=\lambda^{\frac{1}{p}}} = \operatorname{sign} b|b|^{\frac{1}{p-1}}. \tag{4.40}$$

As a consequence of the nodal behavior of v and $\dot v$ described before, when $b > 0$ we conclude the existence of at least a solution $\lambda_m^{\frac{1}{p}}$ to (4.40) satisfying,

$$\sigma_m < {\lambda_m}^{\frac{1}{p}} < \theta_m, \qquad m \in \mathbb{N}. \tag{4.41}$$

Bearing in mind (4.38), this fact fits with (2.15). On the contrary, when $b < 0$ there exists a infinite sequence of positive roots $t = {\lambda_m}^{\frac{1}{p}}$, $m \ge 2$, to (4.40) satisfying,

$$\theta_{m-1} < {\lambda_m}^{\frac{1}{p}} < \sigma_m, \qquad m \ge 2. \tag{4.42}$$

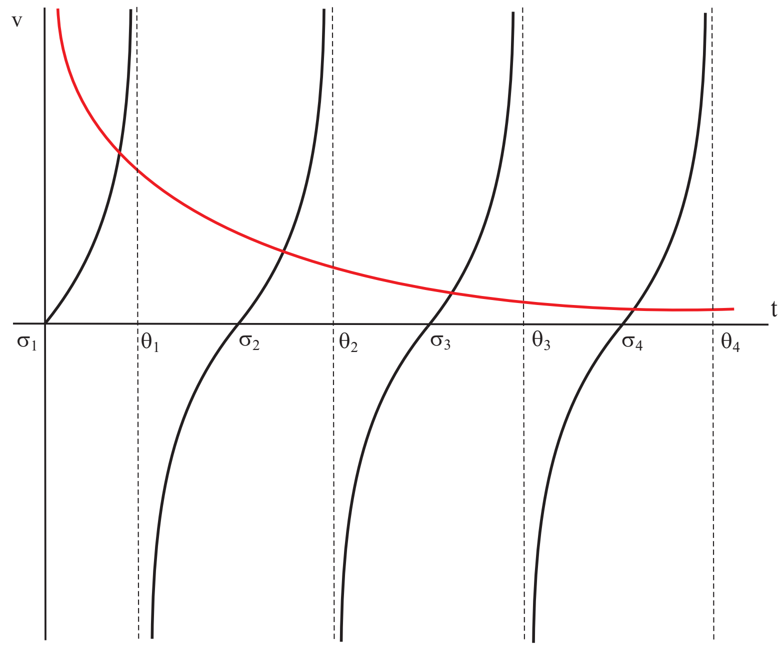

FIGURE 4. A draft of the graph of the function $-\frac{\dot{v}}{v}$ versus $|b|^{\frac{1}{p-1}}/t$.

On the other hand, by employing the approach in [46] it can be shown that $\lambda_m^{\frac{1}{p}}$ is the unique root of (4.40) in the intervals referred to in (4.41) and (4.42). Anyway, some computations lead to the following result.

Lemma 11. *Let $t > 0$ be a solution to* (4.40). *Then,*

$$\frac{d}{dt}\left(t\frac{\dot{v}}{v}\right) = \frac{|b|^{\frac{1}{p-1}}}{(p-1)t}\left[\operatorname{sign} b(N-p) - (p-1)|b|^{\frac{1}{p-1}} - \frac{t^p}{|b|}\right]. \tag{4.43}$$

It follows from (4.43) that,

$$\frac{d}{dt}\left(t\frac{\dot{v}}{v}\right) < 0,$$

at every solution t to (4.40) when $b < 0$, while this also holds for all roots such that $t \geq t_0$,

$$t_0 = \left(N - p - (p-1)b^{\frac{1}{p-1}}\right)^{\frac{1}{p}} b^{\frac{1}{p}},$$

when $b > 0$. In either case, this ensures us the uniqueness of the roots $\lambda_m{}^{\frac{1}{p}}$ in the intervals considered in (4.42), and the uniqueness of the corresponding ones in (4.41) for large m.

We are now in position to present a unified statement on the three standard eigenvalue problems in the ball.

Theorem 12. *Let v be the solution to problem* (4.37). *If $b > 0$, radial Neumann, Robin and Dirichlet problems in the ball B exhibit a family of eigenvalues,*

$$\lambda_m^{\mathcal{N}} < \lambda_m < \lambda_m^{\mathcal{D}}, \qquad \lambda_m \to \infty.$$

On the contrary, if $b<0$, then there exists a family λ_m, $m \geq 2$, of positive eigenvalues satisfying,

$$\lambda_{m-1}^{\mathcal{D}} < \lambda_m < \lambda_m^{\mathcal{N}}, \qquad \lambda_m \to \infty.$$

In all of these cases, corresponding normalized eigenfunctions have the form,

$$u_m(r) = v(\lambda^{\frac{1}{p}} r), \qquad \lambda \in \{\lambda_m^{\mathcal{N}}, \lambda_m, \lambda_m^{\mathcal{D}}\}.$$

Every eigenfunction u_m vanishes exactly $m-1$ times in $0<r<1$ at values,

$$r_k = \frac{\theta_k}{\lambda^{\frac{1}{p}}}, \qquad 1 \leq k \leq m-1.$$

Moreover, in the Neumann and Robin cases,

$$\operatorname{sign} u_m(1) = (-1)^{m-1}.$$

The study of the limit behavior of the radial eigenvalues $\lambda_{m,p}$ and associated eigenfunctions u_m as $p \to 1$ require the knowledge of the corresponding behavior for the solution v to (4.37). This is a most delicate issue that involves the response of both the Dirichlet and Neumann eigenvalues as $p \to 1$. Next statement extracts some of the features of [41, Th. 19, Cor. 21] which are relevant to the forthcoming arguments. Notice that subindex 'p' means 'dependence on p'.

Lemma 13. *Let $v_p(t)$ be the solution to (4.37) whose zeros are,*

$$0 < \theta_{1,p} < \theta_{2,p} < \cdots .$$

Then, there exist sequences,

$$0 < \bar{\theta}_{1,p} < \bar{\theta}_{2,p} < \cdots , \quad \& \quad \alpha_0 < \alpha_1 < \cdots ,$$

such that,

$$\lim_{p \to 1} \lambda_{m,p}{}^{\mathcal{D}} = \lim_{p \to 1} \theta_{m,p} = \bar{\theta}_m,$$

$$\bar{v} = \lim_{p \to 1} v_p = \sum_{k=1}^{\infty} \alpha_{k-1} \chi_{(\bar{\theta}_{k-1}, \bar{\theta}_k)} \qquad \text{in } L^1(0,a) \text{ for all } a>0, \tag{4.44}$$

where $(-1)^k \alpha_k > 0$. Moreover, the following recursive relations are satisfied for all $m \in \mathbb{N}$,

i)

$$\frac{\bar{\theta}_m^N - \bar{\theta}_{m-1}^N}{\bar{\theta}_m^{N-1} + \bar{\theta}_{m-1}^{N-1}} = N, \qquad \text{with } \theta_0 = 0. \tag{4.45}$$

ii)

$$|\alpha_{m+1}| = \frac{\bar{\theta}_{m+1} - (N-1)}{\bar{\theta}_{m+1} + (N-1)} |\alpha_m|, \qquad \text{where } \alpha_0 = 1.$$

iii) *Finally,*

$$\lim_{m \to \infty} \alpha_m = 0, \qquad \text{and} \qquad \lim_{m \to \infty} (\bar{\theta}_m - \bar{\theta}_{m-1}) = 2. \tag{4.46}$$

Next result determines the limit values of the radial Neumann eigenvalues as $p \to 1$.

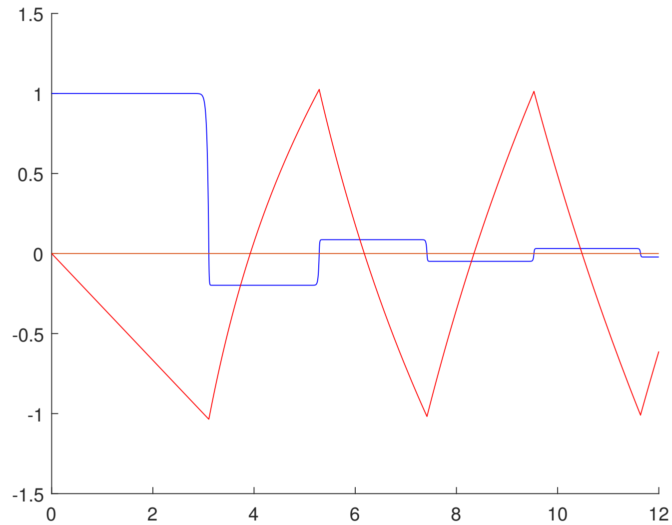

FIGURE 5. Plots of v_p and $|\dot{v}_p|^{p-2}\dot{v}_p$ corresponding to $N = 3$ and $p = 1.01$.

Theorem 14. *Let v_p be the solution to* (4.37) *while*

$$0 = \sigma_{1,p} < \sigma_{2,p} < \cdots ,$$

designates the family of zeros of $\dot{v}_p$. Then, limits

$$\lim_{p\to 1} \lambda_{m,p}{}^{\mathcal{N}} = \lim_{p\to 1} \sigma_{m,p} = \bar{\sigma}_m,$$

exist for all $m \in \mathbb{N}$. In addition,

i) $\bar{\sigma}_{m+1}^N = 2\bar{\theta}_m^N - \bar{\sigma}_m^N$, *for each $m \geq 2$.*

ii) $\bar{\sigma}_m^N = \bar{\theta}_{m-1}^N + N\bar{\theta}_{m-1}^{N-1} = \bar{\theta}_m^N - N\bar{\theta}_m^{N-1}$ *for $m \geq 2$.*

iii) *Relation $\bar{\theta}_{m-1} < \bar{\sigma}_m < \bar{\theta}_m$ holds for every $m \geq 2$. Moreover,*

$$\bar{\theta}_m - \bar{\sigma}_m \geq 1 \quad \textit{for all } m \in \mathbb{N} \qquad \textit{and} \qquad \lim_{m\to\infty} (\bar{\theta}_m - \bar{\sigma}_m) = 1.$$

Proof. The mere existence of the limits $\lim_{p\to 1} \lambda_{m,p}^{\mathcal{N}}$ of the Neumann eigenvalues in a general domain Ω is shown by the same argument of [41, Th. 2] (see Remark 3 there). Unfortunately this direct approach does not work for the Robin problem (1.1). Moreover, such existence assertion is valid when restricting classes $\mathcal{A}_{m,p}$ to the radial case and thus the limits $\bar{\sigma}_m := \lim_{p\to 1} \sigma_{m,p}$ exist for all m.

We next use the relation,

$$\int_{\sigma_{m-1,p}}^{\sigma_{m,p}} s^{N-1} |v_p|^{p-2} v_p \, ds = 0,$$

which is a consequence of our equation. Observe that, as proven in [41, Prop. 16],

$$|v_p|^{p-2} v_p \rightharpoonup \beta, \qquad \text{weakly in } L^q(0,a) \text{ for all } q \geq 1, \tag{4.47}$$

in every interval $(0, a)$. Here function $\beta \in L^\infty(0, +\infty)$ satisfies

$$\beta = (-1)^{k-1} = \operatorname{sign} \alpha_{k-1} \qquad \text{in } (\bar\theta_{k-1}, \bar\theta_k),$$

for each k. Thus we deduce,

$$-\operatorname{sign} \alpha_{m-1} \int_{\bar\sigma_{m-1}}^{\bar\theta_{m-1}} s^{N-1}\, ds = \operatorname{sign} \alpha_m \int_{\bar\theta_{m-1}}^{\bar\sigma_m} s^{N-1}\, ds,$$

which is the relation i).

The equality in ii) is shown in the course of the proof of Theorem 15.

Now, from (4.45) one obtains the inequalities $\bar\theta_{m-1} < \bar\sigma_m < \bar\theta_m$ and

$$\bar\theta_m - \bar\sigma_m = \frac{N\bar\theta_m^{N-1}}{\bar\sigma_m^{N-1} + \bar\theta_m \bar\sigma_m^{N-2} + \cdots + \bar\theta_m^{N-1}} \geq 1. \tag{4.48}$$

Since it is also clear that,

$$\lim_{m\to\infty} \frac{\bar\sigma_m}{\bar\theta_m} = 1,$$

the equality in (4.48) implies that $\bar\theta_m - \bar\sigma_m \to 1$ as $m \to \infty$. □

We are already in position to determine the limit value of the radial Robin eigenvalues as $p \to 1$. Notice that we focus our analysis on positive eigenvalues. The study of the negative eigenvalues when $b < 0$ is delayed to Section 4.1.

Theorem 15. *Let $\lambda_{m,p}$ be the sequence of radial Robin eigenvalues in the unit ball B. Then,*

a) *For $b > 0$ (respectively, $b < 0$) limit $\bar\lambda_m = \lim_{p\to 1} \lambda_{m,p}$ exists for all $m \in \mathbb{N}$ ($m \geq 2$).*

b) *$\bar\lambda_m = \bar\theta_m$ for all m if $b \geq 1$ while $\bar\lambda_m = \bar\theta_{m-1}$ if $b \leq -1$ and $m \geq 2$.*

c) *For $0 < b < 1$ the first limit eigenvalue is just,*

$$\bar\lambda_1 = bN.$$

d) *For $m \geq 2$, $t = \bar\lambda_m$ is the unique root to the equation,*

$$\frac{1}{t^{N-1}} \left[\frac{1}{N}(t^N - \bar\theta_{m-1}^N) - \bar\theta_{m-1}^{N-1} \right] = b, \tag{4.49}$$

either in the interval $\bar\sigma_m < t < \bar\theta_m$ if $b > 0$, or in the interval $\bar\theta_{m-1} < t < \bar\sigma_m$ if $b < 0$.

e) *For $|b| < 1$,*

$$\lim_{m\to\infty} \bar\lambda_m - \bar\sigma_m = b.$$

Proof. We begin by recalling some convergence features included in [41, Prop. 16]. Namely,

$$w_p := |\dot v_p|^{p-2} \dot v_p \rightharpoonup w \qquad \text{weakly in } L^q(0, a) \text{ for all } 1 \leq q < \infty, \tag{4.50}$$

and each $a > 0$. Moreover, since $|v_p| \leq 1$, it follows from the differential equation in (4.37) the equicontinuity of the family $\{t^{N-1} w_p(t)\}$ on every interval $[c, a]$ with $0 < c < a$. Combined with [41, Lem.15] it implies that $\{w_p\}$ is equicontinuous on $[0, a]$ for all $a > 0$. The uniqueness of the limit w in (4.50) ([41, Th.19]) then yields that such a convergence is uniform in every interval $0 \leq t \leq a$.

It is further shown ([41, Prop. 16]) that $w \in W^{1,\infty}(0,+\infty)$ solves the equation,

$$\dot{w} + \frac{N-1}{t} w = -\beta, \tag{4.51}$$

where $\beta \in L^\infty(0,+\infty)$ is the function in (4.47).

On the other hand, it turns out in the course of the proof of [41, Th.19] that $|w| < 1$ in every interval $(\bar{\theta}_{m-1}, \bar{\theta}_m)$ while $w(\bar{\theta}_m) = (-1)^m$. Two consequences can be drawn from this fact. The first one is that the limit,

$$v_p \to \alpha_{m-1}, \tag{4.52}$$

holds in the topology of $C^1(\bar{\theta}_{m-1}, \bar{\theta}_m)$, so that the convergence in (4.44) is considerably up–graded. The second one is the explicit expression for w,

$$w(t) = \frac{(-1)^{m-1}}{t^{N-1}} \left(\bar{\theta}_{m-1}^{N-1} - \frac{1}{N}(t^N - \bar{\theta}_{m-1}^N) \right), \qquad \bar{\theta}_{m-1} \le t \le \bar{\theta}_m,$$

which is deduced from (4.51) by integration. At this point one observes that w_p vanishes at $t = \sigma_{m,p}$. Therefore $\bar{\sigma}_m = \lim_{p\to 1} \sigma_{m,p}$ coincides with the unique zero of w in $\bar{\theta}_{m-1} < t < \bar{\theta}_m$. This proves ii) in Theorem 14.

We next observe that (4.52) entails that $|v_p|^{p-2} v_p \to \operatorname{sign} \alpha_{m-1} = (-1)^{m-1}$ in the topology of $C(\bar{\theta}_{m-1}, \bar{\theta}_m)$. That is why, roots $t = \lambda_{m,p}{}^{\frac{1}{p}}$ to (4.39), that is,

$$-t^{p-1} \frac{|\dot{v}_p|^{p-2} \dot{v}_p}{|v|^{p-2} v_p} = b,$$

accumulate as $p \to 1$ to the possible roots of the limit equation,

$$-\frac{w(t)}{(-1)^{m-1}} = b, \tag{4.53}$$

in the interval $\bar{\theta}_{m-1} \le t \le \bar{\theta}_m$.

Now one observes that the function,

$$g(t) = -\frac{w(t)}{(-1)^{m-1}} = \frac{1}{t^{N-1}} \left[\frac{1}{N}(t^N - \bar{\theta}_{m-1}^N) - \bar{\theta}_{m-1}^{N-1} \right],$$

is increasing in $\bar{\theta}_{m-1} \le t \le \bar{\theta}_m$, satisfies $g(\bar{\theta}_m) = -g(\bar{\theta}_{m-1}) = 1$ while vanishes at $t = \bar{\sigma}_m$.

When $|b| < 1$, there just exists a solution t to (4.49) either in the interval $\bar{\sigma}_m < t < \bar{\theta}_m$ if $0 < b < 1$ or in $\bar{\theta}_{m-1} < t < \bar{\sigma}_m$ if $-1 < b < 0$.

For the special case $m = 1$ and $b > 0$, we have $\beta(t) = 1$ on $(0, \bar{\theta}_1)$ and $w(0) = 0$. By integrating (4.51) we deduce

$$t^{N-1} w(t) = -\int_0^t \tau^{N-1} d\tau = -\frac{t^N}{N},$$

and hence equation (4.53) reduces to,

$$\frac{t}{N} = b.$$

This shows c), d) and the existence of the limits in a) when $|b| < 1$.

On the other hand, as $|b| \ge 1$, $\lambda_{m,p}$ can not accumulate to a value belonging to the interval $(\bar{\theta}_{m-1}, \bar{\theta}_m)$ since $|g(t)| < 1$ there. Thus, either $\bar{\lambda}_m = \bar{\theta}_m$ if $b \ge 1$ or either $\bar{\lambda}_m = \bar{\theta}_{m-1}$ provided $b \le -1$. This completes the proofs of b) and the existence of the limits in a).

As for assertion e) first notice that (4.49) admits the alternative expression,

$$\frac{1}{Nt^{N-1}}\left[t^N - \bar{\sigma}_m^N\right] = b.$$

Hence,

$$(t-\bar{\sigma}_m)\frac{t^{N-1}+\bar{\sigma}_m t^{N-2}+\cdots+\bar{\sigma}_m^{N-1}}{Nt^{N-1}} = b.$$

For $0 < b < 1$ one has $\bar{\sigma}_m < t < \bar{\theta}_m$ and so:

$$1 < \frac{t}{\bar{\sigma}_m} < \frac{\bar{\theta}_m}{\bar{\sigma}_m} = \frac{\bar{\theta}_{m-1}}{\bar{\sigma}_m}\frac{\bar{\theta}_m}{\bar{\theta}_{m-1}} \to 1,$$

as $m \to \infty$. In fact, observe that second limit in (4.46) implies that $\dfrac{\bar{\theta}_m}{\bar{\theta}_{m-1}} \to 1$ as $m \to \infty$. An identical reasoning applies to the case $-1 < b < 0$.

This ends the proof. □

Remark 6. Existence of the limits in a) when $b > 0$ is independently assured by Theorem 20. To apply properly this result, classes $\mathcal{A}_{m,p}$ and $\mathcal{A}_{m,1}$ in Section 6 might be subject to radial symmetry. It should be also checked, as in [41, Th. 12], that Ljusternik–Schnirelmann eigenvalues coincide with the radial ones. In the same vein, first assertion in b) is a consequence of Corollary 23. Accordingly, an alternative proof of these facts has been given in the course of the previous proof.

Next result summarizes the asymptotic behavior as $p \to 1$ of the eigenfunctions to Neumann, Robin and Dirichlet problems.

Theorem 16. *Let,*

$$u_{m,p}(r) = v(\lambda^{\frac{1}{p}} r), \qquad \lambda \in \{{\lambda_{m,p}}^{\mathcal{N}}, \lambda_{m,p}, {\lambda_{m,p}}^{\mathcal{D}}\},$$

be the eigenfunction to either of the three boundary value problems, normalized under the condition $u_m(0) = 1$, where $m \in \mathbb{N}$ if $b > 0$, $m \geq 2$ as $b < 0$.

Then,

$$\bar{u}_m = \lim_{p\to 1} u_{m,p} = \sum_{k=1}^{m} \alpha_{k-1}\chi_{(\rho_{k-1},\rho_k)},$$

where the convergence is in $C^1\left((0,1)\setminus\{\rho_1,\dots,\rho_{m-1}\}\right)$ and the limit values ρ_k are given in each case by,

$$\rho_k \in \left\{\frac{\bar{\theta}_k}{\bar{\sigma}_m}, \frac{\bar{\theta}_k}{\bar{\lambda}_m}, \frac{\bar{\theta}_k}{\bar{\theta}_m}\right\}, \qquad 1 \leq k \leq m.$$

4.1. **Negative eigenvalues for $b < 0$.** We are next discussing the possible existence of negative eigenvalues to (4.36) when $b < 0$. A number $\lambda < 0$ is an eigenvalue provided that its normalized eigenfunction $u(r) = v(t)$, $t = |\lambda|^{\frac{1}{p}} r$, $u(0) = 1$, solves,

$$\begin{cases} (|\dot{v}|^{p-2}\dot{v})^{\cdot} + \dfrac{N-1}{t}|\dot{v}|^{p-2}\dot{v} - |v|^{p-2}v = 0, & t > 0, \\ v(0) = 1, \qquad \dot{v}(0) = 0. \end{cases} \tag{4.54}$$

This problem exhibits a unique maximal solution $v(t)$ defined for $0 \leq t < \omega \leq \infty$ which is a sort of modified Bessel function (see Figure 6). From the equation one gets that $v(t)$ is increasing while the energy inequality,

$$(p-1)|\dot{v}|^p - |v|^p < -1, \tag{4.55}$$

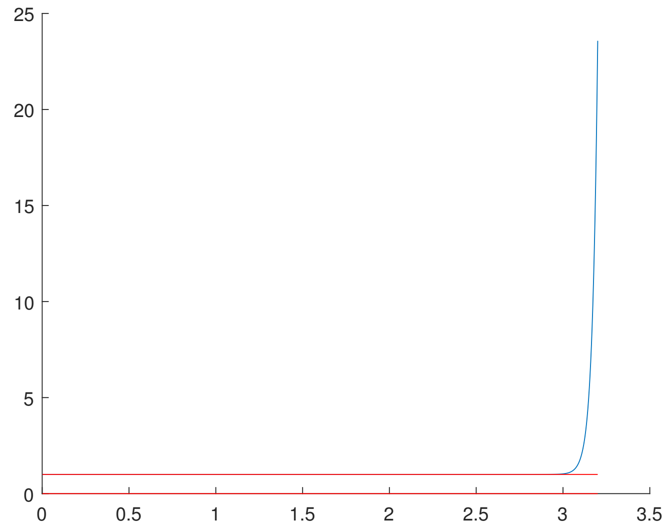

FIGURE 6. Modified Bessel function $v(t)$, $p = 1.01$, $N = 3$. In red $v = 1$.

holds true for all t. In fact, the group $\frac{1}{p'}|\dot{v}|^p - \frac{1}{p}|v|^p$ decreases in $0 \le t < \omega$. Hence,

$$1 < v(t) < \cosh_p t, \qquad 0 \le t < \omega.$$

This entails that v is indeed a global solution, i. e. $\omega = \infty$.

As $b < 0$, equation for the eigenvalues becomes,

$$t^{p-1}\frac{|\dot{v}|^{p-2}\dot{v}}{|v|^{p-2}v} = |b|, \qquad t > 0. \tag{4.56}$$

The relevant features on these eigenvalues are next stated.

Theorem 17. *Suppose $b < 0$. Then problem (4.36) admits a unique negative eigenvalue $\lambda_{1,p}$. Moreover,*

i) *$\lambda_{1,p}$ is a principal eigenvalue with associated eigenfunction,*

$$u_{1,p}(r) = v_p(|\lambda_{1,p}|^{\frac{1}{p}} r),$$

where $v_p(t)$ stands for the solution to (4.54).

ii) *The limit behavior of the eigenvalue is,*

$$\lim_{p\to 1} \lambda_{1,p} = \begin{cases} bN & \text{if } -1 < b < 0, \\ -\infty & \text{as } b < -1. \end{cases} \tag{4.57}$$

iii) *If $-1 < b < 0$ the limit profile of the eigenfunction is,*

$$\lim_{p\to 1} u_{1,p} = 1, \qquad \text{in } C^1[0,1],$$

while,

$$\lim_{p\to 1} |u_{1,p}'(r)|^{p-2}u_{1,p}'(r) = |b|r, \qquad \text{in } C[0,1).$$

iv) $\lim_{p\to 1} u_{1,p} = \infty$ *uniformly on compacts of* $(0,1]$ *as* $b < -1$.

Proof. It is known that problem (1.1) possesses a unique principal eigenvalue $\lambda_{1,p}$ in a general smooth domain Ω, that is, an eigenvalue with an associated eigenfunction which does not change its sign. See [30, Sect.5] whose analysis can be extended to the case $b < 0$. Such an eigenvalue is simple so it must be radial in the case $\Omega = B$. Taking into account that eigenfunctions to (4.36) associated to eigenvalues $\lambda_{n,p}$, $n \geq 2$, change their sign, then the principal eigenvalue must be necessarily negative. This is coherent with the positivity of the solution $v = v_p(t)$ to (4.54). Furthermore, as a consequence of these facts equation (4.56) actually exhibits a *unique* positive solution $t = t_b$ for every $b > 0$ so that the first eigenvalue is,

$$\lambda_{1,p} = -t_b^p.$$

Thus i) is proven.

We next observe from the differential equation that $v = v_p(t)$ satisfies,

$$|\dot v(t)|^{p-2}\dot v(t) = \int_0^t \left(\frac{s}{t}\right)^{N-1} |v(s)|^{p-2} v(s)\, ds < \frac{t}{N}|v(t)|^{p-2}v(t),$$

for all $t > 0$, and so,

$$\frac{\dot v}{v} < \left(\frac{t}{N}\right)^{\frac{1}{p-1}}, \qquad t > 0.$$

We get by integration,

$$v(t) < \exp\left(\frac{N}{p'}\left(\frac{t}{N}\right)^{p'}\right).$$

In particular,

$$v_p \to 1, \qquad \text{and} \qquad |\dot v_p|^{p-2}\dot v_p \to \frac{t}{N}, \tag{4.58}$$

uniformly on compacta of $[0, N)$ as $p \to 1$. From the latter we also get,

$$\dot v_p \to 0, \qquad p \to 1, \tag{4.59}$$

uniformly on compacta of $[0, N)$.

In addition, the inequality $v_p(t) > 1$ in $t > 0$ leads to

$$|\dot v_p(t)|^{p-2}\dot v_p(t) = \int_0^t \left(\frac{s}{t}\right)^{N-1} |v(s)|^{p-2} v(s)\, ds > \frac{t}{N}, \qquad t > 0.$$

Thus,

$$v_p(t) > 1 + \frac{N}{p'}\left(\frac{t}{N}\right)^{p'}, \qquad t > 0,$$

which implies that,

$$v_p \to \infty, \qquad p \to 1, \tag{4.60}$$

holds uniformly on compacta of (N, ∞).

Let us discuss now the response of the root t_b (which depends on p) as $p \to 1$. We first use the inequality (4.55) to obtain,

$$1 < \left(1 - \frac{p-1}{t_b^p}|b|^{p'}\right) v_p(t_b)^p,$$

and the positiveness of $v_p(t_b)$ yields,

$$(p-1)|b|^{p'} < t_b^p.$$

This is enough to conclude that,

$$t_b \to \infty, \qquad \text{as } p \to 1,$$

when $b < -1$ and hence the second option in (4.57) is proved.

We now assume $-1 < b < 0$. As shown in (4.58),

$$t^{p-1} \left(\frac{\dot{v}}{v}\right)^{p-1} \to \frac{t}{N},$$

uniformly on compacta of $[0, N)$ as $p \to 1$. This implies that equation (4.56) becomes

$$\frac{t}{N} = |b|,$$

when $p \to 1$. Therefore, provided that $|b| < 1$ equation (4.56) admits at least a root $t \in [0, N)$ close to $N|b|$ as $p \to 1$. By uniqueness this root must be necessarily t_b and the first option of (4.57) is shown.

Assertions iii) and iv) on the eigenfunctions are now a consequence of the convergence features in (4.58), (4.59) and (4.60). □

Remark 7. The fact that $\lim_{p\to 1} \lambda_{1,p} = bN$ for the ball B as $|b| < 1$ has been independently shown in [18].

5. An eigenvalue problem for the limit eigen pairs

We are next going to analyze the behavior of a family of eigen pairs $(\lambda_{n,p_m}, u_{n,p_m})$ where n is fixed, $p_m \to 1$ while $\lambda_m := \lambda_{n,p_m}$ accumulates to a limit value $\bar{\lambda}$. We will also write $u_m = u_{n,p_m}$ to brief.

Existence of such families and corresponding limits $\bar{\lambda}$ is ensured by Lemma 6. A complete analysis of the whole limit of $\lambda_{n,p}$ as $p \to 1$ is delayed to Section 6. Please note that possible omitted details in the forthcoming reasonings can be checked at [41].

By normalizing eigenfunctions as $\|u_m\|_{p_m} = 1$ and observing that,

$$\int_\Omega |\nabla u_m|^{p_m} + \int_{\partial\Omega} b|u_m|^{p_m} = \lambda_m, \tag{5.61}$$

one gets $\|\nabla u_m\|_{p_m} = O(1)$ which in turn implies –by means of (2.5)– that $\|u_m\|_{p_m,\partial\Omega} = O(1)$. Therefore

$$\int_\Omega |\nabla u_m|^{p_m} + \int_{\partial\Omega} |u_m|^{p_m} = O(1),$$

so Young's inequality gives the existence of a positive constant M such that

$$\|u_m\|_{BV(\Omega)} \leq M.$$

We find out $u \in BV(\Omega)$ and a subsequence (not relabeled) such that $u_m \to u$ in $L^1(\Omega)$, so by Lemma 5,

$$\int_\Omega |u| = \lim_{m\to\infty} \int_\Omega |u_m|^{p_m} = 1.$$

At a second step it is remarked that $|u_m|^{p_m-2}u_m \rightharpoonup \gamma \in L^\infty(\Omega)$ weakly in $L^q(\Omega)$, for every $q \geq 1$, (this might involve an additional subsequence extraction) where $\|\gamma\|_\infty \leq 1$ and,

$$\gamma \in \operatorname{sign}(u). \tag{5.62}$$

A third step allows us concluding the existence of a field

$$z \in L^\infty(\Omega, \mathbb{R}^N), \qquad \|z\|_\infty \le 1,$$

such that $|\nabla u_m|^{p_m-2}\nabla u_m \rightharpoonup z$ weakly in every $L^q(\Omega, \mathbb{R}^N)$ for all $q \ge 1$.

Some information on the boundary behavior of u_m is now required. Set $v_m = |u_m|^{p_m-2}u_m$. Then $v_m \in L^{p'_m}(\partial\Omega)$ while,

$$\|v_m\|_{q,\partial\Omega} \le \mathcal{H}^{N-1}(\partial\Omega)^{\frac{1}{q}-\frac{1}{p'_m}}\|v_m\|_{p'_m,\partial\Omega} = \mathcal{H}^{N-1}(\partial\Omega)^{\frac{1}{q}-\frac{1}{p'_m}}\|u_m\|^{p_m-1}_{p_m,\partial\Omega},$$

where q has been chosen so that $q < p'_m = \frac{p_m}{p_m-1}$. By a new extraction of a subsequence one concludes that v_m is bounded in every $L^q(\partial\Omega)$:

$$\|v_m\|_{q,\partial\Omega} \le \mathcal{H}^{N-1}(\partial\Omega)^{\frac{1}{q}-\frac{1}{p'_m}}K^{p_m-1},$$

for some $K > 0$. A diagonal procedure leads to a new extraction of a subsequence and the existence of $\gamma_1 \in L^q(\partial\Omega)$ such that $v_m \rightharpoonup \gamma_1$ weakly in $L^q(\partial\Omega)$ for all $q \ge 1$. Moreover,

$$\|\gamma_1\|_{q,\partial\Omega} \le \mathcal{H}^{N-1}(\partial\Omega)^{\frac{1}{q}}.$$

holds for all $q \ge 1$. Hence, $\gamma_1 \in L^\infty(\partial\Omega)$ with $\|\gamma_1\|_\infty \le 1$. However, notice that the lack of a better knowledge of the pointwise convergence of u_m on $\partial\Omega$ doesn't permit us getting a better connection between u and γ_1 as in (5.62).

By gathering all the previous results together we can conclude now from the weak equation of eigen pairs (λ_m, u_m) that,

$$\int_\Omega z\nabla v + \int_{\partial\Omega} b\gamma_1 v = \bar\lambda \int_\Omega \gamma v, \qquad v \in C^1(\overline{\Omega}). \tag{5.63}$$

As a first consequence,

$$-\operatorname{div} z = \bar\lambda\gamma, \qquad \text{in } \mathcal{D}'(\Omega). \tag{5.64}$$

Since $\operatorname{div} z \in L^\infty(\Omega)$, Anzellotti's theory in [6] applies (see Section 2), so that,

$$\int_\Omega z\nabla v + v \operatorname{div} z = \int_{\partial\Omega} [z,\nu]v, \qquad v \in C^1(\overline{\Omega}), \tag{5.65}$$

holds true. Therefore,

$$\int_{\partial\Omega} [z,\nu]v = -\int_{\partial\Omega} b\gamma_1 v,$$

for all $v \in C^1(\overline{\Omega})$ and consequently,

$$[z,\nu] + b\gamma_1 = 0 \qquad \mathcal{H}^{N-1} \text{ a. e. in } L^\infty(\partial\Omega). \tag{5.66}$$

The following step of this analysis consists in proving the identity,

$$|Du| = (z, Du), \tag{5.67}$$

where the pairing in the right hand side is understood in the sense of [6]. The approach to achieve (5.67) is well–known. It goes back to [5] (see further referring in [41]) and is next described. By testing with φu_m, $\varphi \in C_0^1(\Omega)^+$, in the weak equation for the eigenfunctions to (1.1), $\lambda = \lambda_m$, we observe that,

$$\int_\Omega \varphi|\nabla u_m|^{p_m} = \lambda_m \int_\Omega |u_m|^{p_m}\varphi - \int_\Omega u_m|\nabla u_m|^{p_m-2}\nabla u_m\nabla\varphi.$$

By taking limits,

$$\begin{aligned}\lim_{m\to\infty}\int_\Omega \varphi|\nabla u_m|^{p_m} &= \bar\lambda\int_\Omega |u|\varphi - \int_\Omega uz\nabla\varphi \\ &= -\int_\Omega u\varphi\operatorname{div} z - \int_\Omega uz\nabla\varphi = \langle (z,Du),\varphi\rangle,\end{aligned}\tag{5.68}$$

where (z,Du) is understood in distributional sense according to [6]. Limit in the left hand side can be estimated from below due to the lower semicontinuity of the functional $u\to\int_\Omega \varphi|Du|$ in $BV(\Omega)$, thus leading to the estimate:

$$\int_\Omega \varphi|Du| \le \langle (Du,z),\varphi\rangle, \qquad \varphi\in C_0^1(\Omega)^+.$$

Equality (5.67) is a consequence from this estimate and the complementary one,

$$\langle (z,Du),\varphi\rangle \le \|z\|_\infty\int_\Omega \varphi|Du|,$$

which is valid since $(z,Du)\le \|z\|_\infty|Du|$ as measures.

We are going to analyze now the boundary condition (5.66) in a deeper way. While the volumetric term γ is connected to u through the explicit relation (5.62), the corresponding linking between u and the surface coefficient γ_1 still remains undetermined in the expression (5.66).

Three cases will be considered in turn. In what follows, inequalities between functions in the boundary will be understood in the $\mathcal{H}^{N-1}$ sense.

a) $0<b(x)\le 1$ on $\partial\Omega$. Due to the fact that $u_m\to u$ in $L^1(\Omega)$ together with [37, Prop. 1.2] it holds that,

$$\int_\Omega |Du| + \int_{\partial\Omega} b|u| \;\le\; \varliminf_{m\to\infty}\left\{\int_\Omega |Du_m| + \int_{\partial\Omega} b|u_m|\right\}.$$

On the other hand,

$$\begin{aligned}&\int_\Omega |Du_m| + \int_{\partial\Omega} b|u_m| \\ &\qquad\le \frac{1}{p_m'}\left\{\int_{\partial\Omega} b + |\Omega|\right\} + \frac{1}{p_m}\left\{\int_\Omega |Du_m|^{p_m} + \int_{\partial\Omega} b|u_m|^{p_m}\right\},\end{aligned}$$

hence,

$$\int_\Omega |Du| + \int_{\partial\Omega} b|u| \;\le \bar\lambda\int_\Omega |u|.$$

In addition (5.64), (2.6) and (5.66) imply,

$$\int_\Omega (z,Dv) + \int_{\partial\Omega} b\gamma_1 v \;=\bar\lambda\int_\Omega \gamma v,$$

for all $v\in BV(\Omega)$. Thus

$$\int_\Omega |Du| + \int_{\partial\Omega} b|u| \;\le \int_\Omega |Du| + \int_{\partial\Omega} b\gamma_1 u$$

and so

$$\int_{\partial\Omega} b(|u|-\gamma_1 u) \;\le 0.$$

Therefore,

$$|u| = \gamma_1 u \qquad \text{on } \partial\Omega. \tag{5.69}$$

In other words $\gamma_1 \in \operatorname{sign}(u)$ on $\partial\Omega$.

b) $b \geq 1$ on $\partial\Omega$. Again [37, Prop. 1.2] implies that,

$$\int_\Omega |Du| + \int_{\partial\Omega} |u| \\ \leq \varliminf_{m\to\infty} \left\{ \int_\Omega |Du_m| + \int_{\partial\Omega} |u_m| \right\} \leq \varliminf_{m\to\infty} \left\{ \int_\Omega |Du_m| + \int_{\partial\Omega} b|u_m| \right\}.$$

Accordingly we conclude,

$$\int_\Omega |Du| + \int_{\partial\Omega} |u| \leq \int_\Omega |Du| - \int_{\partial\Omega} [z,\nu]u\,.$$

Thus,

$$|u| = -[z,\nu]u \qquad \text{on } \partial\Omega. \tag{5.70}$$

c) $b \geq 1$ in Γ while $0 < b < 1$ in Γ_1 where Γ, Γ_1 are disjoint open sets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma}_1$. By the same reasons as before,

$$\int_\Omega |Du| + \int_{\partial\Omega} \min\{b,1\}|u| \\ \leq \varliminf_{m\to\infty} \left\{ \int_\Omega |Du_m| + \int_{\partial\Omega} \min\{b,1\}|u_m| \right\} \\ \leq \varliminf_{m\to\infty} \left\{ \int_\Omega |Du_m| + \int_{\partial\Omega} b|u_m| \right\} \leq \int_\Omega |Du| - \int_{\partial\Omega} [z,\nu]u\,.$$

Thus,

$$|u| = \gamma_1 u, \quad \text{on } \Gamma_1, \qquad \text{and} \qquad |u| = -[z,\nu]u \qquad \text{on } \Gamma. \tag{5.71}$$

These three options for the boundary conditions for u can be then presented in an unified way as,

$$-[z,\nu] \in \begin{cases} b\operatorname{sign}(u), & \text{on } \Gamma_1, \\ \operatorname{sign}(u), & \text{on } \Gamma, \end{cases} \tag{5.72}$$

where it is understood that a) correspond to $\Gamma_1 = \partial\Omega$, b) to $\Gamma = \partial\Omega$ and c) to the intermediate case stated above.

Remark 8. The fact that the limit eigenpairs $(\bar{\lambda}, u)$ satisfy the Robin condition (5.66) – (5.69) in $\partial\Omega$ relies upon the property that the functional,

$$u \to \int_\Omega |Du| + \int_{\partial\Omega} b|u|, \qquad u \in BV(\Omega),$$

is lower semicontinuous with respect to $L^1(\Omega)$ only when $|b| \leq 1$ ([37]). Moreover, in the threesold value $b = 1$, this functional defines the Dirichlet problem for the 1–Laplacian ([5]). In this case, (5.70) is regarded as the Dirichlet boundary condition and, as shown in b), it is satisfied by the limit eigenpairs when $b \geq 1$ in $\partial\Omega$, regardless its size. These features somehow explain why the regimes $0 \leq b \leq 1$ and $b \geq 1$ affect the boundary conditions in the forthcoming limit problem. On the other hand, the above discussion agrees with the conclusions of our previous Theorems 8 and 15, also with the general scenario considered in Corollary 23 below.

The contents of the present section can be summarized in the next statement. For simplicity we are assuming that $b > 0$ on $\partial\Omega$ (see Remark 9 below) while, as in c), there exist disjoint open sets $\Gamma, \Gamma_1 \subset \partial\Omega$, $\overline{\Gamma} \cup \overline{\Gamma}_1 = \partial\Omega$ such that,

$$0 < b < 1, \qquad \text{on } \Gamma_1, \qquad \text{and} \qquad b \geq 1 \qquad \text{on } \Gamma, \tag{5.73}$$

and all inequalities are understood in the $\mathcal{H}^{N-1}$- sense. In addition, extreme possibilities $\Gamma = \emptyset$ or $\Gamma_1 = \emptyset$ are also considered.

Theorem 18. *Fix $n \in \mathbb{N}$ and let $\{\lambda_{n,p_m}\}$ be a sequence of eigenvalues to* (1.1) *satisfying $\bar{\lambda} = \lim_{m\to\infty} \lambda_{n,p_m}$ where $p_m \to 1$. Let $u_m = u_{n,p_m} \in W^{1,p_m}(\Omega)$ be a corresponding family of eigenfunctions normalized as,*

$$\int_\Omega |u_m|^{p_m} = 1.$$

Then, there exist $u \in BV(\Omega)$, $z \in L^\infty(\Omega, \mathbb{R}^N)$, $\gamma \in L^\infty(\Omega)$ and $\gamma_1 \in L^\infty(\partial\Omega)$ such that, up to subsequences,

i) $|u_m|^{p_m-2}u_m \rightharpoonup \gamma$ *weakly in $L^q(\Omega)$ while $|u_m|^{p_m-2}u_m \rightharpoonup \gamma_1$ weakly in $L^q(\partial\Omega)$, in both cases for all $q \geq 1$. Moreover,*

$$\|\gamma\|_\infty \leq 1, \qquad \|\gamma_1\|_{\infty,\partial\Omega} \leq 1.$$

ii) $|\nabla u_m|^{p_m-2}\nabla u_m \rightharpoonup z$ *weakly in $L^q(\Omega, \mathbb{R}^N)$ for all $q \geq 1$ where,*

$$\|z\|_\infty \leq 1.$$

iii) $-\operatorname{div} z = \bar{\lambda}\gamma$ *in $\mathcal{D}'(\Omega)$.*

iv) $|Du| = (z, Du)$ *as measures, where $|Du|$ stands for the total variation of Du.*

v) $|u| = \gamma u$ *a. e. in Ω.*

vi) $|u| = \gamma_1 u$ $\mathcal{H}^{N-1}$*- a. e. on Γ_1 and*

$$-[z,\nu] = b\gamma_1 \qquad \mathcal{H}^{N-1}\text{-a. e. on } \Gamma_1. \tag{5.74}$$

vii) $-[z,\nu]u = |u|$ $\mathcal{H}^{N-1}$*-a. e. on Γ, or in an alternative expression,*

$$-[z,\nu] \in \operatorname{sign}(u) \qquad \mathcal{H}^{N-1}\text{-a. e. on } \Gamma. \tag{5.75}$$

Next definition is based upon the assertions of the previous theorem. Introduces the problem satisfied by the 'limit' eigenfunctions to (1.1) as $p \to 1$.

Definition 19. *Assume that $b > 0$ on $\partial\Omega$ and fulfills* (5.73). *A couple $(\bar{\lambda}, u) \in \mathbb{R} \times BV(\Omega)$ is said to be a weak eigen pair to problem*

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \lambda\dfrac{u}{|u|} & x \in \Omega, \\ u = 0, & \text{on } \Gamma, \\ \dfrac{Du}{|Du|}\nu + b\dfrac{u}{|u|} = 0, & \text{on } \Gamma_1, \end{cases} \tag{5.76}$$

if there are a field z and coefficients γ, γ_1 as in the statement of Theorem 18 so that properties i) to vii) are satisfied. Boundary conditions are understood according to (5.75) *and* (5.74) *and one refers to* (5.76) *as the Robin eigenvalue problem for $-\Delta_1$ if $\Gamma = \emptyset$, while* (5.76) *is the Dirichlet eigenvalue problem when $\Gamma_1 = \emptyset$.*

Remark 9. A further interesting case arises when $b \geq 0$ satisfies,

$$b = 0 \qquad \text{on } \Gamma_0, \qquad \text{and} \qquad 0 < b \leq 1, \qquad \text{on } \Gamma_1, \tag{5.77}$$

where, as in c), $\Gamma_0, \Gamma_1 \subset \partial\Omega$ are disjoint open sets, $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$. Under these conditions, our previous analysis permits us concluding that a limit eigenpair $(\bar{\lambda}, u)$ solves the problem,

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \lambda \dfrac{u}{|u|} & x \in \Omega, \\ \dfrac{Du}{|Du|}\nu = 0, & \text{on } \Gamma_0, \\ \dfrac{Du}{|Du|}\nu + b\dfrac{u}{|u|} = 0, & \text{on } \Gamma_1. \end{cases} \tag{5.78}$$

In this case all properties of Theorem 18 remain true but replacing (5.75) with the restriction $[z, \nu] = 0$ on Γ_0. The latter is regarded as a Neumann condition for the 1–Laplacian operator.

6. Existence of the limits as $p \to 1$

In this section a finer description of the behavior of the eigenvalues $\lambda_{n,p}$ as $p \to 1+$ is pursued. In particular, we focuss the interest in the own existence of $\lim_{p\to 1} \lambda_{n,p}$.

Consider the classes,

$$\mathcal{A}_{n,p} = \{A \subset W^{1,p}(\Omega) \ : \ A \in \mathcal{K}_p(\Omega) : A \subset \mathcal{M}_p,\, \gamma(A) \geq n\},$$

where $p > 1$, $\mathcal{K}_p(\Omega)$ designates the family of compact symmetric sets in the space $W^{1,p}(\Omega)$ and $\mathcal{M}_p = \{u \in W^{1,p}(\Omega) : \int_\Omega |u|^p = 1\}$. We are also dealing with the extra family,

$$\mathcal{A}_{n,1} = \{A \subset BV(\Omega) : A \in \mathcal{K}_1(\Omega),\, A \subset \mathcal{M},\, \gamma(A) \geq n\},$$

where $\mathcal{M} = \{u \in BV(\Omega) : \int_\Omega |u| = 1\}$ and $\mathcal{K}_1(\Omega)$ is the corresponding class of closed and bounded symmetric sets in $BV(\Omega)$ which are compact, in this case with respect to the L^1 topology (please, notice this slight change). We introduce in addition the functionals,

$$J_p(u) = \int_\Omega |\nabla u|^p + \int_{\partial\Omega} b|u|^p, \qquad J(u) = \int_\Omega |Du| + \int_{\partial\Omega} b|u|.$$

According to (2.13), the variational expression for $\lambda_{n,p}$ can be reformulated as:

$$\lambda_{n,p} = \inf_{A\in\mathcal{A}_{n,p}} \sup_{u\in A} J_p(u).$$

Our goal here is relating these eigenvalues with the 'characteristic numbers',

$$\bar{\lambda}_n = \inf_{A\in\mathcal{A}_{n,1}} \sup_{u\in A} J(u). \tag{6.79}$$

In fact the main result can be stated as follows.

Theorem 20. *Assume that $b \geq 0$ and let $\lambda_{m,p}$ be the m–th eigenvalue to the Robin problem* (1.1)*. Then,*

$$\lim_{p\to 1} \lambda_{m,p} = \bar{\lambda}_m, \qquad \textit{for every } m.$$

Let us explain why the values $\bar{\lambda}_n$ are connected to the limits of $\lambda_{n,p}$ as $p \to 1$ and what the rôle of the weight b is in the forthcoming computations. To this purpose define,

$$\mathcal{D}(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u|, \qquad u \in BV(\Omega). \tag{6.80}$$

This is not only the functional J computed at $b = 1$ but also the total variation of $u \in BV(\Omega)$ when extended as zero outside Ω. The following result was formerly sketched at the end of the proof of [11, Th. 3.3] and shown with more detail in [38] (see a further proof in [34]). It not only provides the existence of the limit of the Dirichlet eigenvalues $\lambda_{n,p}^{\mathcal{D}}$ as $p \to 1+$, but also its precise value. However, it should be remarked that the mere *existence* of the limits may be obtained in a more direct way ([41]).

Proposition 21. *Let $\lambda_{n,p}^{\mathcal{D}}$ be the family of variational eigenvalues to the Dirichlet problem* (1.2). *Then:*

$$\lim_{p\to 1} \lambda_{n,p}^{\mathcal{D}} =: \bar{\lambda}_n^{\mathcal{D}} = \inf_{A\in\mathcal{A}_{n,1}} \sup_{u\in A} \mathcal{D}(u).$$

In other words, asymptotic values $\bar{\lambda}_n^{\mathcal{D}}$ of the Dirichlet eigenvalues correspond to $\bar{\lambda}_n$ when $b = 1$. This somehow indicates that the $\bar{\lambda}_n$'s are natural candidates for the limits of the Robin eigenvalues as $p \to 1$.

Proof of Theorem 20. It is a consequence of Theorems 22 and 25 below. □

The first ingredient in the proof of Theorem 20 is the following result.

Theorem 22. *The lower estimate,*

$$\varliminf_{p\to 1} \lambda_{n,p} \ge \bar{\lambda}_n, \tag{6.81}$$

holds true for every $n \in \mathbb{N}$.

It is implicit in Theorem 22 that the asymptotic values $\bar{\lambda}_n$ of the Robin problem 'can't see' weight $b \in L^\infty(\partial\Omega)$ when $b(x) \ge 1$ on $\partial\Omega$ (see Remark 8). This is the assertion we are next stating.

Corollary 23. *Assume $b \ge 1$. Then following limits exist and satisfy the stated equalities:*

$$\bar{\lambda}_n = \lim_{p\to 1} \lambda_{n,p} = \bar{\lambda}_n^{\mathcal{D}}, \qquad n \in \mathbb{N}. \tag{6.82}$$

Proof. Since $b \ge 1$ implies that $\bar{\lambda}_n^{\mathcal{D}} \le \bar{\lambda}_n$ then it is clear that,

$$\varliminf_{p\to 1} \lambda_{n,p} \ge \bar{\lambda}_n \ge \bar{\lambda}_n^{\mathcal{D}}.$$

On the other hand, from (2.15) we have $\lambda_{n,p} \le \lambda_{n,p}^{\mathcal{D}}$ for all $n \in \mathbb{N}$. Thus,

$$\varlimsup_{p\to 1} \lambda_{n,p} \le \varlimsup_{p\to 1} \lambda_{n,p}^{\mathcal{D}} = \bar{\lambda}_n^{\mathcal{D}}.$$

□

Remark 10. It is well–known that the eigenvalues of $-\Delta$ in a smooth bounded domain Ω under the Robin condition $\frac{\partial u}{\partial \nu} + \beta u = 0$, converge to the Dirichlet eigenvalues as $\beta \to \infty$ ([10]). Our previous result is somehow reminiscent of this fact if one thinks of $b^{\frac{1}{p-1}}$ as β and observes that $b^{\frac{1}{p-1}} \to \infty$ as $p \to 1$ when $b > 1$ (see Sections 3 and 4).

Proof of Theorem 22. We are following the ideas in [11] and [34]. As a first step a subsequence $p_m \to 1$ is chosen so as,

$$\lim_{m\to\infty} \lambda_{n,p_m} = \varliminf_{p\to 1} \lambda_{n,p}. \tag{6.83}$$

Denote $\lambda_m = \lambda_{n,p_m}$ to brief. After a refining process it can be assumed that

$$1 < p_m < N' = \frac{N}{N-1}, \qquad m \in \mathbb{N},$$

and

$$\sup_{A_m} J_{p_m} < \lambda_{p_m} + \frac{1}{m}, \qquad m \in \mathbb{N},$$

for a certain A_m such that $A_m \in \mathcal{K}_{p_m}(\Omega)$, $A_m \subset \mathcal{M}_{p_m}$ and $\gamma(A_m) \geq n$. We now observe that due to the continuous embedding $W^{1,p_m}(\Omega) \subset L^{p_m{}^*}(\Omega)$, $p_m{}^* = \frac{Np_m}{N-p_m}$, one finds:

$$\gamma^{W^{1,p_m,}(\Omega)}(A_m) = \gamma^{L^q(\Omega)}(A_m), \qquad 1 \leq q \leq N',$$

where 'super index' space refers to the topology with respect to which the genus γ is computed. In fact, that A_m is compact in $W^{1,p}(\Omega)$ entails that A_m is isomorphic to itself when endowed with the $L^q(\Omega)$ topology.

To complete the 'approaching' to the definition of $\bar{\lambda}_n$ we introduce next the projection $\mathcal{P} : L^1(\Omega) \setminus \{0\} \to L^1(\Omega)$, $\mathcal{P}u = \dfrac{u}{\|u\|_1}$ onto the unit L^1–sphere. Basic genus properties allow us asserting that ([44]),

$$\gamma^{L^1(\Omega)}(B_m) \geq \gamma^{L^1(\Omega)}(A_m) \geq n, \qquad B_m := \mathcal{P}(A_m).$$

Of course, $B_m \in \mathcal{K}_1(\Omega)$ for all $m \in \mathbb{N}$. Thus $\{B_m\} \subset \mathcal{A}_{n,1}$ and we have the estimate,

$$\bar{\lambda}_n = \inf_{A\in\mathcal{A}_{n,1}} \sup_A J \leq \varliminf_{m\to\infty} \sup_{B_m} J. \tag{6.84}$$

By writing $v \in B_m$ as $v = \dfrac{u}{\|u\|_1}$ with $u \in A_m$ we find,

$$J(v) \leq \frac{1}{\|u\|_1} \left\{ \frac{1}{p_m{}'} \left(|\Omega| + \int_{\partial\Omega} b \right) + \frac{1}{p_m} J_{p_m}(u) \right\},$$

and hence,

$$J(v) \leq \frac{1}{\|u\|_1} \left\{ \frac{1}{p_m{}'} \left(|\Omega| + \int_{\partial\Omega} b \right) + \frac{1}{p_m} \left(\lambda_m + \frac{1}{m} \right) \right\}. \tag{6.85}$$

So, in order to measure how large $\sup_{B_m} J$ is, an estimate of $\|u\|_1^{-1}$ on A_m is required. Thus we start with $u \in A_m$ and observe that,

$$\|u\|_{p_m} \leq \|u\|_1^{\theta_m} \|u\|_{N'}^{(1-\theta_m)}, \qquad \theta_m = \frac{\frac{1}{p_m} - \frac{1}{N'}}{1 - \frac{1}{N'}}.$$

By taking into account that $A_m \subset \mathcal{M}_{p_m}$ and so $\|u\|_{p_m} = 1$ we get,

$$\|u\|_1^{-1} \leq \|u\|_{N'}^{\frac{1-\theta_m}{\theta_m}}.$$

Now observe that,

$$\|u\|_{N'} \leq |\Omega|^{\frac{1}{N'} - \frac{1}{p_m{}^*}} \|u\|_{p_m{}^*}.$$

In addition,

$$\|u\|_{p_m{}^*} \le C_{p_m}\left(\int_\Omega |\nabla u|^{p_m} + \int_\Omega |u|^{p_m}\right) \le C_{p_m}\left(\lambda_m + \frac{1}{m} + 1\right).$$

On the other hand, constant $C_{p_m} = O(1)$ as $m \to \infty$ (see [9]) therefore,

$$\|u\|_{N'} \le M,$$

for a certain positive constant M. Coming back to the estimate of $\|u\|_1^{-1}$ we conclude,

$$\|u\|_1^{-1} \le M^{\frac{1-\theta_m}{\theta_m}}, \qquad u \in A_m. \tag{6.86}$$

Finally by gathering together estimates (6.84), (6.85) y (6.86) we obtain,

$$\bar\lambda_n \le \lim_{m\to\infty} M^{\frac{1-\theta_m}{\theta_m}}\left(\lambda_m + \frac{1}{m} + o(1)\right) = \varliminf_{p\to 1} \lambda_{n,p},$$

as desired. □

As a byproduct of the above proof, the same statement as Theorem 22 holds true when $p = 1$ is replaced with $p > 1$.

Theorem 24. *Suppose $p \ge 1$. Then,*

$$\varliminf_{q\to p+} \lambda_{n,q} \ge \lambda_{n,p}.$$

We are now addressing the complementary estimate to (6.81).

Theorem 25. *The upper estimate,*

$$\varlimsup_{p\to 1} \lambda_{n,p} \le \bar\lambda_n, \tag{6.87}$$

also holds for every $n \in \mathbb{N}$.

Proof. The approach in [11] and [34] is also followed. For the readers benefit we are describing computations in detail.

By picking $\delta > 0$ some compact $A \in \mathcal{A}_{n,1}$ exists such that,

$$\sup_A J \le \bar\lambda_n + \delta,$$

while a sequence $1 < p_m < N'$, $p_m \to 1$, can be found satisfying,

$$\varlimsup_{p\to 1} \lambda_{n,p} = \lim_{m\to\infty} \lambda_{n,p_m} := \lim_{m\to\infty} \lambda_m.$$

The strategy of the proof consists in obtaining a family of compact sets $B_m \in \mathcal{A}_{n,p_m}$ such that

$$\varlimsup_{m\to\infty} \sup_{B_m} J_{p_m},$$

can be properly estimated in terms of $\sup_A J$. Since the construction is slightly involved we are going to proceed step by step.

First step. Consider $k > 0$ as a parameter which is going to be suitably defined near the end of the process. Now take $1 < q < N'$ so close to 1 as to achieve the uniform estimate:

$$\|u\|_r \le \left(1 + \frac{\delta}{k}\right)\|u\|_s \qquad \text{for all } 1 \le r < s \le q, \tag{6.88}$$

and every $u \in L^q(\Omega)$. From now on, p_m is assumed to satisfy $1 < p_m < q$.

At this early stage we observe that $A \subset BV(\Omega)$ is relative compact in $L^q(\Omega)$ and compact in $L^1(\Omega)$ (check the definition of $\mathcal{A}_{n,1}$ at the beginning of the section). Thus, A is also compact in $L^q(\Omega)$ and satisfies,

$$\gamma^{L^q(\Omega)}(A) = \gamma^{L^1(\Omega)}(A).$$

To conclude this preliminary part of the proof we select l functions, $u_i \in A \subset BV(\Omega)$ so that,

$$A \subset \left\{ \cup_{i=1}^{l} B_{L^q}\left(u_i, \frac{\delta}{k}\right) \right\},$$

B_{L^q} standing for the open ball in L^q.

Second step. Functions u_i are approximated by a sequence $u_{i,m} \in C^1(\overline{\Omega})$ in the strict topology of $BV(\Omega)$. By Lemma 3 it can be assumed that relations,

$$\int_\Omega |Du_i| = \lim_{m\to\infty} \int_\Omega |\nabla u_{i,m}|^{p_m}, \qquad \int_{\partial\Omega} b|u_i| = \lim_{m\to\infty} \int_{\partial\Omega} b|u_i|^{p_m},$$

$1 \le i \le l$, are satisfied. This implies that,

$$J(u_i) = \lim_{m\to\infty} J_{p_m}(u_{i,m}), \qquad u_i = L^q\text{-} \lim_{m\to\infty} u_{i,m}. \tag{6.89}$$

Third step. We are introducing the family B_m according to the following instructions. A compact convex set $C_m \subset C^1(\overline{\Omega}) \subset W^{1,p_m}(\Omega)$ is defined as,

$$C_m = \text{co}\, \{\pm u_{i,m} : i = 1, \dots, l\},$$

'co' standing for convex hull. Notice that C_m is compact since it is the convex envelope of a finite set. Let $\pi_m : L^{p_m}(\Omega) \to L^{p_m}(\Omega)$ be the projection onto C_m, i. e., for $u \in L^{p_m}$, $\pi_m(u) \in C_m$ is uniquely defined through the relation,

$$\|u - \pi_m(u)\|_{p_m} \le \|u - v\|_{p_m}, \qquad \text{for all } v \in C_m.$$

Set in addition $\mathcal{P}_m : L^{p_m}(\Omega) \setminus \{0\} \to L^{p_m}(\Omega)$ the projection $\mathcal{P}_m u = \dfrac{u}{\|u\|_{p_m}}$ onto the unit sphere in L^{p_m}. Estimate (6.92) below show that $0 \notin \pi_m(A)$. Therefore, family:

$$B_m = \mathcal{P}_m(\pi_m(A)), \qquad m \in \mathbb{N},$$

is properly defined. In addition, the set $B_m \subset C^1(\overline{\Omega})$ is not only compact in $L^{p_m}(\Omega)$ $(1 < p_m < q)$ but also in $W^{1,p_m}(\Omega)$ due to the finite dimensional character of C_m. By the standard genus properties,

$$\gamma^{W^{1,p_m}}(B_m) = \gamma^{L^{p_m}}(B_m) \ge \gamma^{L^{p_m}}(\pi_m(A)) \ge \gamma^{L^q}(A) = \gamma^{L^1}(A) \ge n.$$

Fourth step. Since,

$$\lim_{m\to\infty} \lambda_m \le \overline{\lim_{m\to\infty}} \sup_{B_m} J_{p_m},$$

and for $u \in B_m$ it holds,

$$J_{p_m}(u) = \frac{1}{\|v\|_{p_m}^{p_m}} J_{p_m}(v) \le \frac{1}{\|v\|_{p_m}^{p_m}} \max_{1\le i\le l} J_{p_m}(u_{i,m}), \tag{6.90}$$

where $u = \dfrac{v}{\|v\|_{p_m}}$ and $v \in \pi_m(A)$, we require a lower estimate of $\|v\|_{p_m}^{p_m}$ in $\pi_m(A)$ to measure the size of $\sup_{B_m} J_{p_m}$. To this purpose we start with the conditions (valid for large m),

$$\|u_{i,m}\|_1 > 1 - \frac{\delta}{k}, \qquad \|u_{i,m}\|_{p_m} > 1 - \frac{2\delta}{k}, \qquad \|u_i - u_{i,m}\|_q < \frac{\delta}{k}, \tag{6.91}$$

where uniform inequalities (6.88) have been employed to deduce the second estimate. On the other hand, every $v \in \pi_m(A)$ has the form $v = \pi_m(a)$ where $a \in A$ with $\|a - u_i\|_q < \frac{\delta}{k}$ for some i. Accordingly,

$$\|v\|_{p_m} > \|u_{i,m}\|_{p_m} - \|\pi_m(a) - a\|_{p_m} - \|a - u_i\|_{p_m} - \|u_i - u_{i,m}\|_{p_m},$$

where $u_{i,m}$ satisfies (6.91). Second term is estimated as,

$$\|\pi_m(a) - a\|_{p_m} \leq \|u_{i,m} - a\|_{p_m} \leq \|u_{i,m} - u_i\|_{p_m} + \|u_i - a\|_{p_m} \leq \frac{2\delta}{k}\left(1 + \frac{\delta}{k}\right).$$

The remaining terms are handled by means of (6.91) and (6.88) thus leading to,

$$\|v\|_{p_m} \geq \frac{k - 10\delta}{k} > 0, \qquad v \in \pi_m(A), \tag{6.92}$$

provided that $k > 0$ is chosen suitably large. Now, relation (6.90) implies,

$$\overline{\lim_{m\to\infty}} \sup_{B_m} J_{p_m} \leq \frac{k}{k - 10\delta} \max_{1\leq i\leq l} J(u_i) \leq \frac{k}{k - 10\delta} \sup_A J \leq \bar{\lambda}_n + o(1),$$

as $\delta \to 0+$. This finishes the proof. □

The asymptotic estimate in Theorem 25 can be also proved when $q \to p+$ with $p > 1$:

$$\overline{\lim_{q\to p+}} \lambda_{n,q} \leq \lambda_{n,p}.$$

In combination with Theorem 24 we can state the following.

Theorem 26. *The sequence of higher eigenvalues $\lambda_{n,p}$ to the Robin problem are* right *continuous when regarded as a function of $p \in [1, \infty)$.*

As a special case of Theorem 20 we single out the first eigenvalue (see [18] for a further account when $b > -1$ is constant). Dependence of $\bar{\lambda}_1$ on b is stressed for technical pursues.

Corollary 27. *Assume that $b \geq 0$, $b \neq 0$ while $\lambda_{1,p}(b)$ designates the principal eigenvalue to* (1.1)*. Then,*

$$\bar{\lambda}_1(b) = \lim_{p\to 1} \lambda_{1,p}(b) = \inf_{u\in BV(\Omega)} \frac{\int_\Omega |Du| + \int_{\partial\Omega} b|u|}{\int_\Omega |u|}, \tag{6.93}$$

where it can be set $b = 1$ if $b \geq 1$. Moreover, $\bar{\lambda}_1(b) > 0$.

Proof. The variational expression follows from (6.79) when $n = 1$.

As for the positivity, it is clear when $b \geq 1$ since in this case $\bar{\lambda}_1(b) = \bar{\lambda}_1^{\mathcal{D}} > 0$. Otherwise, the set $\Gamma_1 = \{b \leq 1\}$ must have $\mathcal{H}^{N-1}(\Gamma_1) > 0$. Define $b_1 = b\chi_{\Gamma_1} \neq 0$. We claim that $\bar{\lambda} = \bar{\lambda}_1(b_1) > 0$. In fact, its corresponding associated eigenfunction u satisfies, according to Section 5, the equality,

$$\int_\Omega |Du| + \int_{\partial\Omega} b_1\gamma_1 u = \bar{\lambda}\int_\Omega |u| = \bar{\lambda}.$$

If it were $\bar{\lambda} = 0$, then u would be a positive constant and $\int_{\partial\Omega} b_1 = 0$ which is impossible. Thus,

$$\bar{\lambda}_1(b) \geq \bar{\lambda}_1(b_1) > 0,$$

and we are done. □

Remark 11. It can be shown that function $b(x)$ in (6.93) may be replaced with $\min\{1, b(x)\}$. Details are omitted for brevity and will be given in a future work.

Remark 12. From Theorem 15 the explicit value for $\bar{\lambda}_1(B(0,R))$ in the ball $B(0,R)$ is,

$$\bar{\lambda}_1(B(0,R)) = b\frac{N}{R}.$$

Assume that $b_- = \min_{\partial\Omega} b > 0$ in $\partial\Omega$. It is a consequence of the Faber–Krahn inequality (see [18]) that,

$$\bar{\lambda}_1 \geq b_- \frac{N}{R^{\#}},$$

where the value $R^{\#}$ is chosen so that $|\Omega| = |B(0,R^{\#})|$.

7. Continuity of the eigenvalues with respect to $p \in (1,\infty)$

For the sake of completeness we are finishing the analysis of the dependence of $\lambda_{n,p}$ with respect to p by showing its left continuity in every $p > 1$ (see Theorem 26). Indeed, the experience with the first Dirichlet eigenvalue ${\lambda_{1,p}}^{\mathcal{D}}$ reveals that this is the hardest part of the analysis ([33], [15]).

For technical reasons we begin with a direct proof of the continuity of $\lambda_{1,p}$. It can be obtained in the same spirit as in the Dirichlet case (see [17], [33]).

Theorem 28. *The principal Robin eigenvalue $\lambda_{1,p}$ is continuous in $p > 1$.*

Proof. On view of Theorem 26 only the left continuity must be shown. So, let $p_n < p_0$ satisfy $p_n \to p_0$ and choose $u \in W^{1,p_0}(\Omega)$ a normalized eigenfunction to λ_{1,p_0} with $\|u\|_{p_0} = 1$. Then,

$$\lambda_{1,p_n} \leq \frac{J_{p_n}(u)}{\int_\Omega |u|^{p_n}}.$$

Since,

$$J_{p_n}(u) \leq \frac{p_0 - p_n}{p_n}\left\{|\Omega| + \int_{\partial\Omega} b\right\} + \frac{p_0}{p_n} J_{p_0}(u) =: \frac{p_0 - p_n}{p_n} A + \frac{p_0}{p_n} J_{p_0}(u),$$

and $\lim_{n\to\infty} \int_\Omega |u|^{p_n} = \int_\Omega |u|^{p_0} = 1$, we find,

$$\underline{\lambda} := \varliminf_{n\to\infty} \lambda_{1,p_n} \leq \varlimsup_{n\to\infty} \lambda_{1,p_n} \leq \lambda_{1,p_0}.$$

To estimate $\underline{\lambda}$ from below we fix $\delta > 0$ so that $(p_0 - \delta)^* > p_0$ where $(p_0 - \delta)^*$ means the Sobolev conjugate, and select eigenfunctions u_n to λ_{1,p_n} with $\|u_n\|_{p_n} = 1$, $J_{p_n}(u_n)$ bounded and $p_0 - \delta < p_n$. In addition and up to a subsequence we assume that $\underline{\lambda} = \lim_{n\to\infty} \lambda_{1,p_n}$.

From,

$$J_{p_0-\delta}(u_n) \leq \frac{p_n - p_0 + \delta}{p_n} A + \frac{p_0 - \delta}{p_n} J_{p_n}(u_n), \tag{7.94}$$

we conclude that $u_n \rightharpoonup u$ weakly in $W^{1,p_0-\delta}(\Omega)$ and strongly in $L^{p_0}(\Omega)$ since $\|u_n\|_{1,p_0-\delta} = O(1)$. Moreover, u can be chosen no depending on δ. By taking limits we obtain,

$$J_{p_0-\delta}(u) \leq \frac{\delta}{p_0} A + \frac{p_0 - \delta}{p_0} \underline{\lambda}.$$

This inequality together with [9, Prop. 9.3] and letting $\delta \to 0$ yields $u \in W^{1,p_0}(\Omega)$. In addition, it also implies that,

$$J_{p_0}(u) \leq \underline{\lambda}, \qquad \& \qquad \int_\Omega |u|^{p_0} = 1.$$

Thus, $\underline{\lambda} \geq \lambda_{1,p_0}$ and the proof is complete. □

For the remain of this section we follow the approach in [11] which involves the notion of Γ–convergence of functionals (the account in [7] on the subject is enough for our aim). This is a suitable notion when studying the infimum of a 'limit' f of functionals f_n.

Definition 29. *Let X be a metric space and $f, f_n : X \to \bar{\mathbb{R}} := [-\infty, \infty]$ a family of functionals in X. It is said that f_n Γ–converges to f, written $f = \Gamma\text{-}\lim_{n\to\infty} f_n$, if*

$$f = \Gamma\text{-}\varliminf_{n\to\infty} f_n = \Gamma\text{-}\varlimsup_{n\to\infty} f_n, \tag{7.95}$$

where:

$$\Gamma\text{-}\varliminf_{n\to\infty} f_n = \inf\{\varliminf_{n\to\infty} f_n(x_n)\}, \qquad \Gamma\text{-}\varlimsup_{n\to\infty} f_n = \inf\{\varlimsup_{n\to\infty} f_n(x_n)\},$$

where the 'inf' and 'sup' are extended to the class of all sequences $x_n \to x$. In that case, $\Gamma\text{-}\lim_{n\to\infty} f_n$ is regarded as the common value in (7.95).

Remark 13.

a) Condition $f(x) \leq \Gamma\text{-}\varliminf_{n\to\infty} f_n$ is equivalent to,

$$f(x) \leq \varliminf_{n\to\infty} f_n(x_n), \qquad \text{for all } x_n \to x.$$

b) Inequality $\Gamma\text{-}\varlimsup_{n\to\infty} f_n(x) \leq f(x)$ is equivalent to,

$$\text{There exists } x_n \to x \text{ such that } \qquad \varlimsup_{n\to\infty} f_n(x_n) \leq f(x).$$

Thus, the Γ–convergence of f_n to f can expressed in terms of a), b) ([7, Chap. 2]).

To put the continuity of $\lambda_{n,p}$ in the framework of [11] we introduce the family of functionals $F_p : L^1(\Omega) \to [0, \infty]$ defined as,

$$F_p(u) = \begin{cases} J_p(u)^{\frac{1}{p}}, & u \in W^{1,p}(\Omega), \\ \infty, & \text{otherwise.} \end{cases}$$

It should be remarked that the analysis in [11] is specialized on the Dirichlet eigenvalue problem. However, results can be adapted with minor changes to the Robin problem (1.1). In this layout, main hypotheses in [11] take the form of the properties i) and ii) stated next.

Proposition 30. *The following properties are satisfied.*

i) *For every compact interval $I = [a, b] \subset (1, \infty)$ there exist constants C_1, C_2 such that,*

$$C_1\|u\|_{1,p} \leq F_p(u) \leq C_2\|u\|_{1,p}, \qquad \textit{for all } u \in W^{1,p}(\Omega),\ p \in I.$$

ii) *For each $p_0 > 1$, $F_{p_0} = \Gamma\text{-}\lim_{p\to p_0} F_p$.*

A slightly larger class than $\mathcal{A}^{\mathcal{D}}_{n,p}$ for Dirichlet conditions is also introduced in [11]. The adaptation to the Robin problem (1.1) is $\widetilde{\mathcal{A}}_{n,p} \supset \mathcal{A}_{n,p}$ defined as:

$$\begin{aligned}\widetilde{\mathcal{A}}_{n,p} = \{A \subset W^{1,p}(\Omega) : &A \text{ closed and bounded},\\ &A = -A, A \text{ compact in } L^p(\Omega),\ \gamma^{L^p(\Omega)}(A) \geq n\}.\end{aligned}$$

A characteristic number associated to the class $\widetilde{\mathcal{A}}_{n,p}$ is defined as,

$$\tilde{\lambda}_{n,p} = \inf_{A \in \widetilde{\mathcal{A}}_{n,p}} \sup_A J_p \leq \inf_{A \in \mathcal{A}_{n,p}} \sup_A J_p = \lambda_{n,p}.$$

Main result [11, Th. 3.3], conveniently adapted to our framework, allows us concluding that,

$$\lim_{p \to p_0} \tilde{\lambda}_{n,p} = \tilde{\lambda}_{n,p_0}. \tag{7.96}$$

A further consequence of its proof is the following assertion (see [11, Cor. 3.6]):

$$\tilde{\lambda}_{n,p} = \lambda_{n,p}, \qquad n \in \mathbb{N}, \quad p > 1. \tag{7.97}$$

Therefore, both (7.96) and (7.97) lead to the the main statement of the present section. It comprises the left continuity of $\lambda_{n,p}$ on p as a special case.

Theorem 31. *Eigenvalue $\lambda_{n,p}$ is a continuous function of p for $p > 1$.*

To complete our analysis it only remains to prove Proposition 30.

Proof of Proposition 30. To check the validity of i) observe that an estimate of the form,

$$J_p(u) \leq C_2^p \|u\|_{1,p}^p, \qquad u \in W^{1,p}(\Omega),$$

holds true in the basis of $b \in L^\infty(\Omega)$ and Lemma 2. In fact, C_2 can be chosen non depending on $p \in I$. In addition, inequality,

$$\int_\Omega |u|^p \leq \frac{1}{\lambda_{1,p}} J_p(u),$$

together with the continuity of $\lambda_{1,p}$ (Theorem 28) permit us finding a constant C_1, non depending on $p \in I$, such that $C_1^p \|u\|_{1,p}^p \leq J_p(u)$. Thus assertion i) is proved.

As for the Γ–convergence statement we are showing in turn properties a), b) of Remark 13. Beginning with b), for any fixed $u \in W^{1,p_0}(\Omega)$ and $p_n \to p_0$ one can use either Lemma 3-a) if $p_n \to p_0+$, or either Lemma 4 when $p_n \to p_0-$, to conclude the existence of $u_n \in W^{1,p_n}(\Omega)$ such that $J_{p_n}(u_n) \to J_{p_0}(u)$. Observe in addition that

$$\Gamma\text{-} \overline{\lim_{n \to \infty}} F_{p_n}(u_n) \leq F_{p_0}(u_0),$$

holds regardless the sequence u_n whenever $u \in L^{p_0}(\Omega)$.

In order to proceed with a) assume $u_n \to u$ in $L^{p_0}(\Omega)$. We only need to check,

$$J_{p_0}(u) \leq \underline{\lim_{n \to \infty}} J_{p_n}(u_n),$$

when the right hand side limit is finite. First consider the case $p_n \to p_0$ with $p_n > p_0$ to have,

$$J_{p_0}(u_n) \leq \frac{p_n - p_0}{p_n} A + \frac{p_n}{p_0} J_{p_n}(u_n), \qquad A = |\Omega| + \int_{\partial\Omega} b\,.$$

Since $J_{p_0}(u_n) = O(1)$ and $u_n \to u$ in $L^{p_0}(\Omega)$ then $u_n \rightharpoonup u$ weakly in $W^{1,p_0}(\Omega)$. Hence,

$$J_{p_0}(u) \le \varliminf_{n\to\infty} J_{p_0}(u_n) \le \varliminf_{n\to\infty} J_{p_n}(u_n),$$

and we are donde.

Let us deal now with the case $p_n \to p_0-$ and as before assume the finiteness of $\varliminf_{n\to\infty} J_{p_n}(u_n)$. As in the proof of Theorem 28 pick $\delta > 0$ so small as $p_0 < (p_0-\delta)^*$. From $J_{p_n}(u_n) = O(1)$ we deduce that $u_n \rightharpoonup u$ weakly in $W^{1,p_0-\delta}(\Omega)$, strongly in $L^{p_0}(\Omega)$ where u can be chosen non depending on δ. From (7.94) we deduce again that,

$$\int_\Omega |\nabla u|^{p_0} + \int_{\partial\Omega} b|u|^{p_0} \le \frac{\delta}{p_0} A + \frac{p_0-\delta}{p_0} \varliminf_{n\to\infty} J_{p_n}(u_n).$$

This entails $u \in W^{1,p_0}(\Omega)$ together with $\int_\Omega |u|^p_0 = 1$ (see Lemma 5). The desired estimate follows from letting $\delta \to 0+$. $\square$

Acknowledgements. The authors would like to thank the anonymous referee for the interesting comments. This research has been partially supported by CIUCSD (Generalitat Valenciana) under project AICO/2021/223 and by Grant PID2022-136589NB-I00 founded by MCIN/AEI/10.13039/501100011033.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[2] A. Anane. *Etude des valeurs propres et de la résonance pour l'opérateur p–Laplacien.* Thése de doctorat. Université Libre de Bruxelles, 1987.

[3] A. Anane. Simplicité et isolation de la première valeur propre du p-laplacien avec poids. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(16):725–728, 1987.

[4] A. Anane and N. Tsouli. On the second eigenvalue of the p-Laplacian. In *Nonlinear partial differential equations (Fès, 1994)*, volume 343 of *Pitman Res. Notes Math. Ser.*, pages 1–9. Longman, Harlow, 1996.

[5] F. Andreu-Vaillo, V. Caselles, and J. M. Mazón. *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals*, volume 223 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2004.

[6] G. Anzellotti. Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl. (4)*, 135:293–318 (1984), 1983.

[7] A. Braides. *Γ-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2002.

[8] B. Brandolini, F. Chiacchio, and C. Trombetti. Optimal lower bounds for eigenvalues of linear and nonlinear Neumann problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 145(1):31–45, 2015.

[9] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Universitext. Springer, New York, 2011.

[10] D. Bucur, P. Freitas, and J. Kennedy. The Robin problem. In *Shape optimization and spectral theory*, pages 78–119. De Gruyter Open, Warsaw, 2017.

[11] T. Champion and L. De Pascale. Asymptotic behaviour of nonlinear eigenvalue problems involving p-Laplacian-type operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(6):1179–1195, 2007.

[12] K. C. Chang. The spectrum of the 1-Laplace operator. *Commun. Contemp. Math.*, 11(5):865–894, 2009.

[13] B. de la Calle Ysern, J. C. Sabina de Lis, and S. Segura de León. The convective eigenvalues of the one-dimensional p-Laplacian as $p \to 1$. *J. Math. Anal. Appl.*, 484(1):123738, 28, 2020.

[14] M. Degiovanni and M. Marzocchi. Limit of minimax values under Γ-convergence. *Electron. J. Differential Equations*, pages No. 266, 19, 2014.

[15] M. Degiovanni and M. Marzocchi. On the dependence on p of the variational eigenvalues of the p-Laplace operator. *Potential Anal.*, 43(4):593–609, 2015.
[16] M. del Pino, M. Elgueta, and R. Manásevich. A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0,\ u(0) = u(T) = 0,\ p > 1$. *J. Differential Equations*, 80(1):1–13, 1989.
[17] M. A. del Pino and R. F. Manásevich. Global bifurcation from the eigenvalues of the p-Laplacian. *J. Differential Equations*, 92(2):226–251, 1991.
[18] F. Della Pietra, C. Nitsch, F. Oliva, and C. Trombetti. On the behavior of the first eigenvalue of the p-Laplacian with Robin boundary conditions as p goes to 1. *Adv. Calc. Var.*, 16(4):1123–1135, 2023.
[19] F. Demengel. Théorèmes d'existence pour des équations avec l'opérateur "1-laplacien", première valeur propre pour $-\Delta_1$. *C. R. Math. Acad. Sci. Paris*, 334(12):1071–1076, 2002.
[20] F. Demengel. Functions locally almost 1-harmonic. *Appl. Anal.*, 83(9):865–896, 2004.
[21] O. Došlý and P. Řehák. *Half-linear differential equations*, volume 202 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2005.
[22] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.
[23] L. Friedlander. Asymptotic behavior of the eigenvalues of the p-Laplacian. *Comm. Partial Differential Equations*, 14(8-9):1059–1069, 1989.
[24] J. P. García Azorero and I. Peral Alonso. Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues. *Comm. Partial Differential Equations*, 12(12):1389–1430, 1987.
[25] Y. X. Huang. On the eigenvalues of the p-Laplacian with varying p. *Proc. Amer. Math. Soc.*, 125(11):3347–3354, 1997.
[26] R. Kajikiya, M. Tanaka, and S. Tanaka. Bifurcation of positive solutions for the one-dimensional (p, q)-Laplace equation. *Electron. J. Differential Equations*, pages Paper No. 107, 37, 2017.
[27] R. Kajikiya, M. Tanaka, and S. Tanaka. Asymptotic behavior and monotonicity of radial eigenvalues for the p-Laplacian. *J. Differential Equations*, 387:496–531, 2024.
[28] B. Kawohl and V. Fridman. Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant. *Commentat. Math. Univ. Carol.*, 44(4):659–667, 2003.
[29] M. Kraïem. On some nonlinear partial differential equations involving the 1-Laplacian. *Ann. Fac. Sci. Toulouse Math. (6)*, 16(4):905–921, 2007.
[30] A. Lê. Eigenvalue problems for the p-Laplacian. *Nonlinear Anal.*, 64(5):1057–1099, 2006.
[31] L. Lefton and D. Wei. Numerical approximation of the first eigenpair of the p-Laplacian using finite elements and the penalty method. *Numer. Funct. Anal. Optim.*, 18(3-4):389–399, 1997.
[32] P. Lindqvist. On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.*, 109(1):157–164, 1990.
[33] P. Lindqvist. On nonlinear Rayleigh quotients. *Potential Anal.*, 2(3):199–218, 1993.
[34] S. Littig and F. Schuricht. Convergence of the eigenvalues of the p-Laplace operator as p goes to 1. *Calc. Var. Partial Differential Equations*, 49(1-2):707–727, 2014.
[35] M. Mihăilescu. Monotonicity properties for the variational Dirichlet eigenvalues of the p-Laplace operator. *J. Differential Equations*, 335:103–119, 2022.
[36] Z. Milbers and F. Schuricht. Existence of a sequence of eigensolutions for the 1-Laplace operator. *J. Lond. Math. Soc. (2)*, 82(1):74–88, 2010.
[37] L. Modica. Gradient theory of phase transitions with boundary contact energy. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 4(5):487–512, 1987.
[38] E. Parini. *Asymptotic behavior of higher eigenfunctions of the p–Laplacian as p goes to one*. 2009. Thesis (Ph.D.)–Universität zu Köln.
[39] E. Parini. The second eigenvalue of the p-Laplacian as p goes to 1. *Int. J. Differ. Equ.*, pages 23, Art. ID 984671, 2010.
[40] E. Parini. Continuity of the variational eigenvalues of the p-Laplacian with respect to p. *Bull. Aust. Math. Soc.*, 83(3):376–381, 2011.
[41] J. C. Sabina de Lis and S. Segura de León. The limit as $p \to 1$ of the higher eigenvalues of the p-Laplacian operator $-\Delta_p$. *Indiana Univ. Math. J.*, 70(4):1395–1439, 2021.
[42] J. C. Sabina de Lis and S. Segura de León. p-Laplacian diffusion coupled to logistic reaction: asymptotic behavior as p goes to 1. *Ann. Mat. Pura Appl. (4)*, 201(5):2197–2240, 2022.
[43] J. Sabina Lis and S. Segura de León. 1D logistic reaction and p-Laplacian diffusion as p goes to one. *Ric. Mat.*, 71(2):529–547, 2022.

[44] M. Struwe. *Variational methods*. Modern Surveys in Mathematics. Springer-Verlag, Berlin, fourth edition, 2008.
[45] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Berlin; UMI, Bologna, 2007.
[46] W. Walter. Sturm-Liouville theory for the radial Δ_p-operator. *Math. Z.*, 227(1):175–185, 1998.

José Sabina de Lis
Departamento de Análisis Matemático and IUEA,
Universidad de La Laguna,
P. O. Box 456, 38200 – La Laguna, SPAIN.
Email address: josabina@ull.edu.es

Sergio Segura de León
Departament d'Anàlisi Matemàtica, Universitat de València,
Dr. Moliner 50, 46100 Burjassot, València, SPAIN.
Email address: sergio.segura@uv.es