RADIAL SPECTRA OF THE 1–LAPLACIAN IN THE ANNULUS

JOSÉ C. SABINA DE LIS AND SERGIO SEGURA DE LEÓN

Abstract. This paper is concerned with the behaviour, as p goes to 1, of eigenvalues of the p–Laplace operator associated to radial eigenfunctions. The Dirichlet, Neumann and Robin conditions are analyzed in an annulus. In each case we prove that there exist the limits of both eigenvalues and eigenfunctions and the limits define in a proper way an eigenpair of the 1–Laplacian.

Contents

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1. Introduction

Our aim in this work is to analyze the behavior, as p goes to 1, of the solutions to the family of classical eigenvalue problems,

$$\begin{cases} -\operatorname{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ \mathcal{B}_p(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{B}_p$ stands for either of the following boundary operators,

$$\mathcal{B}_p(u) = u, \quad \mathcal{B}_p(u) = |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}, \quad \mathcal{B}_p(u) = |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \beta u. \tag{1.2}$$

Subject to these conditions (1.1) becomes the Dirichlet, Neumann or Robin eigenvalue problem, respectively, for $-\Delta_p$ in Ω. In these expressions, ν designates the outer unit normal at $\partial\Omega$ while in the latter one β is a nonnegative function defined on $\partial\Omega$.

Our interest here is focussed on radial solutions and so Ω is supposed to be a radially symmetric bounded domain. The case where Ω is a ball has been treated in detail by the authors in both [26] and [28]. Thus we are now assuming that Ω is the annulus $\mathcal{A} = \{x \in \mathbb{R}^N : 0 < a < |x| < b\}$. In particular, when dealing with Robin conditions β is taken a positive constant on each of the components of the boundary $\partial\Omega$.

We are going to prove that eigenpairs (λ_p, u_p) to (1.1) converge as $p \to 1$ to some (λ, u) which is a solution of the limit problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \lambda \dfrac{u}{|u|} & \text{in } \Omega, \\ \mathcal{B}_1(u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

with $\Omega = \mathcal{A}$. Of course, we are giving a proper meaning to both the equation and the corresponding boundary conditions $\mathcal{B}_1$ for each of the classical cases (1.2) (Section 6). Divergence expression in (1.3) defines the so–called 1–Laplacian operator Δ_1 (see [7], [8], [13]).

Eigenvalue problems (1.1) to $-\Delta_p$ is a relevant subject in nonlinear analysis since the early eighties where most part of the literature concerns the case $p > 1$. Naturally, it also encompasses radially symmetric problems as an special case. However, the behavior of (1.1) as $p \to 1$ and the study of the corresponding limit problems is a more recent and quite less treated issue.

This job may be regarded as a continuation of [26], [28] concerned with the ball. It should be remarked that in this case all eigenpairs to (1.1) are deduced, after scaling, from the global solution to a single specific initial value problem defined in $[0, \infty)$. In fact, its solution is a sort of generalized Bessel type function (see Remark 3). On the contrary and as a main difference, this is not longer possible for annuli. Each one of the eigenvalues corresponding to the boundary conditions (1.2) requires a separate analysis since their possible values perturb the equation.

The present paper has some overlap with [18] which studies the limit behavior as $p \to 1$ of radial eigenvalues to (1.1) only under Dirichlet conditions. One of its main results ([18, Theorem 2.9]) proves that the n–th eigenvalue of (1.1) converges as $p \to 1$ to the n–th Cheeger constant of Ω, which may be regarded as the n–th eigenvalue of (1.3). Nevertheless our approach is quite different since we are also interested in identifying the eigenfunctions of problem (1.3). In addition, the present analysis involves the remaining conditions in (1.2).

It is worth mentioning that the definition of the spectrum of $-\Delta_1$ is by no means straightforward. Finding out eigenpairs does not reduces to solving (1.3). In fact, being a solution to (1.3) is a necessary but not sufficient condition for (λ, u) to be an eigenpair. Actually, the notion of spectrum of $-\Delta_1$ relies upon an appropriate extension of the critical point theory for nonsmooth functionals. Such functionals depend on the boundary conditions. For instance, in the case of the Dirichlet problem theory is associated to the total variation,

$$\mathcal{T}(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u|,$$

defined in the space $BV(\Omega)$ of functions bounded variation. To get a deeper insight on the eigenvalues to $-\Delta_1$ under the three boundary conditions, we are next reviewing some well–known features on the eigenvalues to (1.1) and their limits as $p \to 1$. Most of them hold true in a general bounded smooth domain $\Omega \subset \mathbb{R}^N$.

1) For every $p > 1$ there exists an increasing sequence $\{\lambda_{n,p}\}_n$ of variational eigenvalues to each one of the problems (1.1), (1.2). They are defined by the Ljusternik–Schnirelman theory as ([1], [31], [3], [16], [15], [23], [5], [20]),

$$\lambda_{n,p} = \inf_{C \in \mathcal{C}_n} \sup_{u \in C} J_p(u), \qquad n \in \mathbb{N}, \tag{1.4}$$

where,

$$J_p(u) = \int_\Omega |\nabla u|^p,$$

for Dirichlet or Neumann conditions in (1.2), while

$$J_p(u) = \int_\Omega |\nabla u|^p + \int_{\partial\Omega} \beta |u|^p,$$

in case of the Robin condition where $\beta \in L^\infty(\partial\Omega)$. As for the class $\mathcal{C}_n$,

$$\mathcal{C}_n = \{C \subset \mathcal{M}_p : C \text{ compact in } W^{1,p}(\Omega),\ C = -C,\ \gamma(C) \geq n\}, \quad (1.5)$$

being $\mathcal{M}_p = \{u \in W^{1,p}(\Omega) : \int_\Omega |u|^p\ dx = 1\}$ and supposing we are dealing with Neumann or Robin conditions. In the Dirichlet case, space $W^{1,p}(\Omega)$ must be replaced by $W_0^{1,p}(\Omega)$. As a matter of notation, $\gamma(C)$ designates the Krasnosel'skii genus of C ([29]).

In addition, eigenvalues $\lambda_{n,p}$ are positive, with the sole exception of the Neumann problem where $\lambda_{1,p} = 0$. Furthermore, $\lim_{n\to\infty} \lambda_{n,p} = +\infty$ (see [3], [16]).

2) Limits,

$$\bar{\lambda}_n := \lim_{p\to 1} \lambda_{n,p}, \quad (1.6)$$

exists for every $n \in \mathbb{N}$. Concerning the Dirichlet problem and the principal eigenvalue this was first shown in [19] and [14] (see preliminary results in [17], [21]), while higher eigenvalues were addressed in [12] (for the 1–dimensional case), [25], [24] and [26]. Moreover, variational expressions as (1.4) for the limits $\bar{\lambda}_n$ were proposed in [25], [24] in the Dirichlet case and in [28] for either Neumann or Robin conditions.

3) A sequence of variational Dirichlet eigenvalues $\lambda_n > 0$, $\lambda_n \to \infty$ as $n \to \infty$, to (1.3) was introduced and studied in [12]. Such eigenvalues λ_n are proved to agree with the limits $\bar{\lambda}_n$ in the one–dimensional case. This reference also exhibits examples of solutions (λ, u) to (1.3) that do not define variational eigenpairs (λ_n, u_n). Accordingly, problem (1.3) does not characterizes by itself the Dirichlet spectrum of $-\Delta_1$ meanwhile the family of limits $\lambda_n = \lim_{p\to 1} \lambda_{n,p}$ may be regarded as the true Dirichlet eigenvalues of this operator.

4) It is further shown in [24] that the Dirichlet eigenvalues λ_n introduced in [12] coincide, in a general Lipschitz domain $\Omega \subset \mathbb{R}^N$, with the limits $\bar{\lambda}_n$ referred to in (1.6). In other words, $\bar{\lambda}_n = \lambda_n$ for every n.

5) As for the behavior of eigenfunctions as $p \to 1$, a pioneering result in [12] states that Dirichlet eigenfunctions to (1.1) pointwise converge to eigenfunctions of $-\Delta_1$ in dimension $N = 1$. Such a result was somehow extended in [26] to N–dimensional domains Ω. Namely, that normalized eigenpairs $(\lambda_{n,p}, u_{n,p}) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ weakly–$*$ converges in $\mathbb{R} \times BV(\Omega)$, and modulus a subsequence, to a solution $(\bar{\lambda}_n, \bar{u}_n)$ of (1.3). In the framework of $-\Delta_1$, $\bar{u}_n$ may be regarded as an eigenfunction associated to the eigenvalue $\bar{\lambda}_n$.

6) The limit profile of eigenfunctions to either the Neumann or Robin problem as $p \to 1$ in the ball was studied in [28].

Our main concern in the present work is to analyze the behavior of the radial eigenpairs (λ_p, u_p) to (1.1) as p goes to 1 on annuli. We are proposing the proper definition of eigenvalue to problem (1.3), which involves the 1–Laplacian together with the three boundary conditions. It should be remarked in this regard that extra energy relation (6.80), involved in the definition, is instrumental to discard spurious solutions. Moreover, we are showing that the limit (λ, u) of the eigenpairs (λ_p, u_p) as $p \to 1$ constitutes the *unique* solutions to (1.3). Accordingly, such values λ may be considered as the radial spectrum of $-\Delta_1$.

This paper is organized as follows. Section 2 introduces the functional setting and settles the problem that radial eigenpairs to (1.1), (1.2) satisfy in an annulus. The existence of these eigenpairs is proved in Section 3. An analysis of the limit as $p \to 1$ of the radial eigenvalues is contained in Section 4. Section 5 addresses a detailed study of the limit of the eigenfunctions as $p \to 1$. Theorem 15 there summarizes our main findings. As a further achievement, Corollary 14 shows the explicit expression of the main Robin eigenvalue λ_1 to the 1–Laplacian in an annulus $\mathcal{A} = \{x \in \mathbb{R}^N : a < |x| < b\}$. Section 6 introduces the classical eigenvalue problems (1.3) for the 1–Laplacian (Definition 17). Main result of the section, Theorem 20, states among other features that the radial eigenpairs to (1.3) in $\mathcal{A}$ are just the limit as $p \to 1$ of the corresponding ones to (1.1).

2. Preliminaries

2.1. **Sobolev spaces.** The natural framework to deal with (1.1) is $W^{1,p}(\Omega)$, the Sobolev space of functions $u \in L^p(\Omega)$ whose gradient, in weak sense, satisfies $\nabla u \in (L^p(\Omega))^N$. Subspace $W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ denotes all those functions vanishing on $\partial\Omega$. When Ω is the annulus $\mathcal{A} = \{0 < a < |x| < b\}$, $\widetilde{W}^{1,p}(\mathcal{A})$ and $\widetilde{W}_0^{1,p}(\mathcal{A})$ denote the subspaces of $W^{1,p}(\mathcal{A})$ and $W_0^{1,p}(\mathcal{A})$, respectively, consisting of their radially symmetric elements (radial functions to short). Namely, functions u such that $u(x) = u(Tx)$ a. e. in $\mathcal{A}$, for every orthogonal transformation of $T : \mathbb{R}^N \to \mathbb{R}^N$. For $u \in L^1(\mathcal{A})$ this is equivalent to the existence of $v \in L^1(a, b)$ such that $u(x) = v(r)$ a. e. in $\mathcal{A}$ with $r = |x|$. In this way, the associated function v to any $u \in \widetilde{W}^{1,p}(\mathcal{A})$ belongs to the one dimensional space,

$$W^{1,p}(a, b, r^{N-1}\, dt) = \{v \in L^p(a, b, r^{N-1}\, dt) : v' \in L^p(a, b, r^{N-1}\, dt)\}, \tag{2.7}$$

where the derivative v' is computed in weak sense of $\mathcal{D}'(a,b)$. Moreover,

$$\nabla u(x) = \frac{x}{r} v'(r), \tag{2.8}$$

and so $v' = \frac{x}{r} \cdot \nabla u$. Accordingly, for a function $u \in \widetilde{W}^{1,p}(\mathcal{A})$,

$$\|u\|^p_{\widetilde{W}^{1,p}(\mathcal{A})} = N\omega_N \left\{ \int_a^b (|v|^p + |v'|^p)\, t^{N-1}\, dt \right\},$$

where $\omega_N = |\{|x| < 1\}|$. We remark that space (2.7) is equivalent to $W^{1,p}(a,b)$ since r is bounded away from zero, so the latter will replace the former one in future references.

Anticipating the case where v' is a measure, relations between v, v' and $u, \nabla u$ are more conveniently expressed in distributional language as,

$$\begin{aligned} \langle u, \psi(|\cdot|)\rangle_{\mathcal{D}'(\mathcal{A})} &= \left\langle N\omega_N r^{N-1} v, \psi \right\rangle_{\mathcal{D}'(a,b)}, \\ \left\langle \left(\frac{x}{r} \cdot \nabla u\right), \psi(|\cdot|) \right\rangle_{\mathcal{D}'(\mathcal{A})} &= \left\langle N\omega_N r^{N-1} v', \psi \right\rangle_{\mathcal{D}'(a,b)}, \end{aligned} \tag{2.9}$$

where $\psi \in C_0^\infty(a,b)$ and $\langle \cdot, \cdot \rangle_{\mathcal{D}'}$ stands for the corresponding duality pairings. In the second expression one recognizes the radial derivative

$$u_r = \frac{x}{r} \cdot \nabla u,$$

of u which can be checked to define a radial function in $L^p(\mathcal{A})$. In fact, according the second equality in (2.9) it may identified with v'. Thus, elements in $\widetilde{W}^{1,p}(\mathcal{A})$ consist in absolutely continuous functions v in the interval $[a,b]$ ([10]). In particular, functions $u \in \widetilde{W}^{1,p}_0(\mathcal{A})$ vanish at $|x| = a, b$ in the standard way, rather than in traces sense.

2.2. BV–**functions.** When dealing with problems involving the 1–Laplacian Δ_1 as (1.3), the proper space to work with is $BV(\Omega)$, constituted by all functions of bounded variation in Ω ([2]). It is made up of those $u \in L^1(\Omega)$ such that its distributional gradient Du is a vectorial Radon measure with $|Du|(\Omega) < \infty$, $|Du|$ standing for the 'total variation measure' associated to Du.

In an annulus $\Omega = \mathcal{A}$, $\widetilde{BV}(\mathcal{A})$ comprises the radial functions of $BV(\mathcal{A})$. Previous notion of radial symmetry is upgraded to Radon measures $\mu \in \mathcal{D}'(\mathcal{A})$ as follows,

$$\langle \mu, \varphi \circ T\rangle_{\mathcal{D}'(\mathcal{A})} = \langle \mu, \varphi\rangle_{\mathcal{D}'(\mathcal{A})},$$

for all test function $\varphi \in C_0^\infty(\mathcal{A})$ and every orthogonal transformation T, being $\varphi \circ T(\cdot) = \varphi(T\cdot)$. Since $\frac{x}{r} \in C^\infty(\mathcal{A}, \mathbb{R}^N)$, the scalar distribution $u_r := \frac{x}{r} \cdot Du$ is well defined. It can be checked that $\mu = u_r$ is radial provided u is. Then, previous relations (2.9) are extended to functions $u \in \widetilde{BV}(\mathcal{A})$. In this case, $v \in L^1(a,b)$, its distributional derivative v' defines a finite Radon measure in (a,b) and so $v \in BV(a,b)$. In addition, total variations $|Du|$ and $|v'|$ are connected through,

$$\langle |Du|, \psi(|\cdot|)\rangle_{\mathcal{D}'(\mathcal{A})} = \left\langle N\omega_N r^{N-1}\,|v'|, \psi\right\rangle_{\mathcal{D}'(a,b)}, \qquad \psi \in \mathcal{D}'(a,b). \tag{2.10}$$

It should be remarked that every function $v \in BV(a,b)$ agrees a. e. in (a,b) with the function,

$$\bar{v}(r) = c + v'(a,r) = c + \int_{(a,r)} v',$$

c being a constant ([2]). Moreover, $\bar{v}$ turns out to be left–continuous and of bounded variation in $[a,b]$ in the classical sense. In future computations it will be understood that v has been replaced by $\bar{v}$. In particular, this fact is entailing the existence of side limits $v(r\pm)$ at every $r \in [a,b]$.

The concept of solution to problems subject to the Δ_1 operator as (1.3) requires the notion of pairing (z, Du) between a field $z \in L^\infty(\Omega, \mathbb{R}^N)$ and the gradient of a function $u \in BV(\Omega)$, $\Omega \subset \mathbb{R}^N$ being a Lipschitz domain. It is a distribution defined, according to [9], as,

$$\langle (z, Du), \varphi\rangle = -\int_\Omega u(\nabla\varphi z + \varphi \operatorname{div} z), \qquad \varphi \in C_0^\infty(\Omega),$$

where the restriction $\operatorname{div} z \in L^\infty(\Omega)$ is assumed. In fact, (z, Du) is a Radon Measure such that,

$$|(z, Du)|(U) \le \|z\|_\infty |Du|(U),$$

for every Borel set $U \subset \Omega$ ([9]). Once the pairing is defined, the action of the operator Δ_1 in (1.3) is properly defined as ([8], [13]),

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = \operatorname{div} z,$$

where the field z is required fulfilling $\|z\|_\infty \le 1$ and the coupling condition with u,

$$(z, Du) = |Du|, \qquad \text{in } \mathcal{D}'(\Omega). \tag{2.11}$$

Of course we are more interested here in the radial case where $u \in \widetilde{BV}(\mathcal{A})$ and fields z exhibit the form,

$$z = \frac{x}{r} w(r),$$

with $w \in L^\infty(a,b)$. Now condition $\operatorname{div} z \in L^\infty(\mathcal{A})$ is equivalent to $w \in W^{1,\infty}(a,b)$ and the corresponding pairing satisfies,

$$\langle (z, Du), \psi(|x|) \rangle_{\mathcal{D}'(\mathcal{A})} = \left\langle N\omega_N r^{N-1} w v', \psi(r) \right\rangle_{\mathcal{D}'(a,b)} .$$

We remark that the distribution (w, v') satisfies $(w, v') = wv'$. In fact, the latter is well defined since $w \in W^{1,\infty}(a,b)$ and so identity (2.11) can be expressed as,

$$wv' = |v'| \qquad \text{in } \mathcal{D}'(a,b). \tag{2.12}$$

For a couple $(v,w) \in BV(a,b) \times W^{1,\infty}(a,b)$ the integration by parts formula,

$$\int_a^b wv' + \int_a^b w'v = w(b)v(b-) - w(a)v(a+), \tag{2.13}$$

holds true, where function v is assumed to be of bounded variation in classical sense and w is absolutely continuos. First integral stands for the measure $wv'((a,b))$ of the interval (a,b). Notice that identity (2.13) is nothing else than a particular case of the more general Green formula in [9, Th. 1.9], however it can be shown by a direct approximation argument under the previous hypotheses.

2.3. **Radial eigenfunctions.** We begin with a definition of solution to (1.1).

Definition 1. *A function $u \in W^{1,p}(\Omega) \setminus \{0\}$ is a weak eigenfunction to the Robin problem* (1.1) *with the choice $\mathcal{B}_p(u) = |\nabla u|^{p-2}\nabla u\, \nu + \beta |u|^{p-2}u$ in* (1.2)) *if the equality,*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \psi + \int_{\partial\Omega} \beta |u|^{p-2} u\psi = \lambda \int_\Omega |u|^{p-2} u\psi, \tag{2.14}$$

holds for all test function $\psi \in W^{1,p}(\Omega)$ and in this case, λ becomes a Robin eigenvalue. Definition of an eigenpair (λ, u) for the Neumann problem ((1.1), *$\mathcal{B}_p(u) = \frac{\partial u}{\partial \nu}$) reduces to set $\beta = 0$ in* (2.14) *while a corresponding eigenpair $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ for the Dirichlet problem* ((1.1), *$\mathcal{B}_p(u) = u$) is obtained by setting $\beta = 0$ and testing* (2.14) *with functions $\psi \in W_0^{1,p}(\Omega)$.*

Our main interest here is focussed on radial eigenvalues λ to (1.1) when Ω is an annulus $\mathcal{A}$. These are just those ones associated to radial eigenfunctions $u(x) = v(r)$, $r = |x|$, $v \in W^{1,p}(a, b)$. Since $\Delta_p u$ is a radial distribution whenever u is radial, then checking equation (2.14) reduces, in this symmetric scenario, to test with radial functions $\psi(r)$, $\psi \in C^1[a, b]$, what means,

$$\int_a^b \varphi_p(v')\psi' \, r^{N-1} \, dr + (\beta r^{N-1}\varphi_p(v)\psi)_{r=b} + (\beta r^{N-1}\varphi_p(v)\psi)_{r=a} = \lambda \int_a^b \varphi_p(v)\psi \, r^{N-1} \, dr, \quad (2.15)$$

where $' = \dfrac{d}{dr}$ and the notation $\varphi_p(t) = |t|^{p-2}t$ is going to be employed henceforth to shorten. In this equation, we set $\beta = 0$ in the Neumann problem while both v and ψ must vanish at $r = a, b$ in the Dirichlet case.

In addition, (2.15) implies that both $v, \varphi_p(v')$ are C^1 in $[a, b]$, up to a possible modification in a null set, and equation,

$$-\left(r^{N-1}\varphi_p(v')\right)' = \lambda r^{N-1}\varphi_p(v), \quad (2.16)$$

is satisfied in $a \leq r \leq b$. Integration by parts in (2.15) together with (2.16) lead to the relation,

$$\left(r^{N-1}[\varphi_p(v') + \beta\varphi_p(v)]\psi\right)_{r=b} + \left(r^{N-1}[-\varphi_p(v') + \beta\varphi_p(v)]\psi\right)_{r=a} = 0, \quad (2.17)$$

for all $\psi \in C^1[a, b]$. Thus, both brackets must vanish when dealing with either the Neumann or Robin problem.

Summarizing these previous features, the radial version of (1.1) in the annulus $\mathcal{A} = \{a < r < b\}$ consists in finding nontrivial pairs $(\lambda, v) \in \mathbb{R} \times C^1[a, b]$ solving,

$$\begin{cases} -(r^{N-1}\varphi_p(v'))' = \lambda r^{N-1}\varphi_p(v), & r \in (a, b), \\ \mathcal{B}_p(v)_{r=a} = 0, \quad \mathcal{B}_p(v)_{r=b} = 0, & \end{cases} \quad (2.18)$$

where $\mathcal{B}_p(v) = v$ in the Dirichlet problem, $\mathcal{B}_p(v) = v'$ in the Neumann one, while,

$$\mathcal{B}_p(v)_{r=a} = \{-\varphi_p(v') + \beta_1\varphi_p(v)\}_{|r=a}, \quad (2.19a)$$

$$\mathcal{B}_p(v)_{r=b} = \{\varphi_p(v') + \beta_2\varphi_p(v)\}_{|r=b}, \quad (2.19b)$$

in the Robin case, β_i being positive for $i = 1, 2$.

Remark 1. When $1 < p \leq 2$ radial eigenfunctions $v \in C^2[a,b]$. This is not the case as $p > 2$ since they cease to be C^2 near their critical points.

3. Existence of radial eigenvalues

We already know that existence of radial eigenvalues to (1.1) is an issue of ordinary differential equations (odes), as reflected through the equivalent formulation (2.18). A complete discussion of this problem is contained in the next statement. In the vein of Sturm–Liouville theory, it also provides us a complete picture of the radial eigenfunctions.

Theorem 2. *Problem* (2.18) *possesses an infinite sequence of eigenvalues,*

$$0 \leq \lambda_1 < \cdots < \lambda_n < \cdots , \qquad \lambda_n \to \infty,$$

where $\lambda_1 = 0$ in the Neumann case, $\lambda_1 > 0$ in the remaining ones. Moreover, the following properties hold true.

i) *Every eigenvalue λ_n is simple which means that any eigenfunction v associated to λ_n is a scalar multiple of a fixed one v_n.*

ii) *To every λ_n there exists $n-1$ points $\theta_i \in (a,b)$ such that an arbitrary eigenfunction v to λ_n exactly vanishes at these points. Moreover, points associated to the following eigenvalue λ_{n+1} separate the points θ_i in the interval (a,b).*

iii) *Any eigenfunction v associated to λ_n possesses n critical points $a \leq \sigma_i \leq b$, $1 \leq i \leq n$. Moreover,*

$$\theta_{i-1} < \sigma_i < \theta_i, \tag{3.20}$$

for $1 \leq i \leq n$ in either the Dirichlet or the Robin problem where $\theta_0 = a$, $\theta_n = b$. In the Neumann one, $\sigma_1 = a$, $\sigma_n = b$ for $n \geq 2$ while (3.20) *holds for $i \notin \{1,n\}$ and $n \geq 3$.*

Remark 2. Observe that assertion iii) holds in the case $n = 1$ leading to the existence of a unique critical point σ_1 of the first eigenfunctions in both the Dirichlet and Robin problems. As for the Neumann case notice that the first eigenfunction is constant and so $v' = 0$ in $[a,b]$. In this case, second eigenfunction does not admit *inner* critical points.

Proof of Theorem 2. The proof is addressed in an 'odes' framework. Assertions i), ii) are consequence of [30, Th. 5, Cor. 5], which are just shown in the more adverse scenario of the ball. In fact, notice that r keeps away the singular value $r = 0$ in the case of the annulus. On the other hand, once an eigenfunction v to λ_n vanishes at θ then all other ones corresponding to this eigenvalue also vanishes at this point. Thus, both families of zeros $\{\theta_i\}$ and critical points $\{\sigma_i\}$ in ii) and iii) only depend on λ_n.

We next show the assertions regarding the critical points set of any eigenfunction v to λ_n. We first observe that $-\operatorname{sign}(v)(r^{N-1}\varphi_p(v'))'$ is positive in all of the intervals $I_i := (\theta_{i-1}, \theta_i)$, $1 \le i \le n$, where $\theta_0 = a$, $\theta_n = b$ (all cases are covered with this convention). Hence, v' just exhibits a unique zero σ_i in the 'inner' intervals I_i, $2 \le i \le n-1$.

The previous assertion also holds for the Dirichlet problem in the initial (a, θ_1) and final (θ_{n-1}, b) intervals, respectively. As for the Robin one, notice that neither v nor v' vanish at $r = a$. Since v keeps sign in (a, θ_1), it follows that v' has opposite signs at the points $r = a$ and $r = \theta_1$. Thus v' has an intermediate zero that must be unique in (a, θ_1). A similar conclusion is achieved in the final interval (θ_{n-1}, b).

Regarding the Neumann problem observe that v' can not vanish in neither of the intervals $(a, \theta_1]$ and $[\theta_{n-1}, b)$ and the proof of iii) is completed. ☐

Remark 3. When Ω is the ball $B(0, b)$, problem (2.18) is reduced to study the initial value problem,

$$\begin{cases} -(t^{N-1}\varphi_p(u'))' = t^{N-1}\varphi_p(u), & t > 0, \\ u(0) = 1\,, \quad u'(0) = 0, \end{cases} \tag{3.21}$$

$' = \dfrac{d}{dt}$, after scaling λ in the form $t = \lambda^{\frac{1}{p}} r$. In this layout, the zeros of u determine the eigenvalues of (1.1). Observe that $u(t)$ is a kind of Bessel-type function. See a further account in both [26], [28]. Unfortunately, this approach is not so well–behaved for dealing neither with annuli nor with variable coefficients problems.

Let $v \in W^{1,p}(a, b)$ be a solution of the equation in (2.18). If $w(r) = \varphi_p(v'(r))$ then it follows from the discussion in Section 2.3 that v defines a classical solution $(v, w) \in C^1[a, b] \times C^1[a, b]$ to the first order system,

$$\begin{cases} v' = \varphi_{p'}(w), & \\ w' = -\dfrac{N-1}{r} w - \lambda \varphi_p(v), & a < r < b, \end{cases} \tag{3.22}$$

which satisfies the boundary conditions:

$$\mathcal{B}_p(v, w)_{r=a,b} = 0,$$

where $\mathcal{B}_p(v, w) = v$ in the Dirichlet case, $\mathcal{B}_p(v, w) = w$ in the Neumann one while $\mathcal{B}_p(v, w)_{r=a} = \{-w' + \beta_1 \varphi_p(v)\}_{r=a}$ and $\mathcal{B}_p(v, w)_{r=b} = \{w' + \beta_2 \varphi_p(v)\}_{r=b}$ in the Robin problem.

The key point in writing the equation in (2.18) as (3.22) is just introducing the Lyapunov function,

$$E(v,w) = \frac{1}{p'}|w|^{p'} + \frac{\lambda}{p}|v|^p, \tag{3.23}$$

which is relevant for dynamical purposes. In fact, a direct computation leads to,

$$\frac{d}{dr}E(v,w) = -\frac{N-1}{r}|w|^{p'}, \tag{3.24}$$

and so E decreases on trajectories. A first consequence is that solutions (v,w) to (3.22), initially only defined in $[a,b]$, can be extended to the whole interval $[a,\infty)$ and constitute global solutions. Moreover ([26, Lem. 10–iv)]),

$$\lim_{r\to\infty}(v(r),w(r)) = (0,0). \tag{3.25}$$

Further applications of E are going to be presented later.

On the other hand, an eigenfunction v_n regarded as a solution (v_n,w_n) to (3.22) gives raise to an orbit Γ which turns clockwise infinitely many times around $(0,0)$. In fact, it meets the v–axis at $\{(v_n(\sigma_i),0)\}_i$ and crosses the w–axis at the points $\{(0,w_n(\theta_i))\}_i$, $\theta_0 = a$, $\theta_n = b$. Furthermore, (3.25) implies that the orbit amplitude progressively decays to zero beyond $r=b$ as $r\to\infty$.

Another consequence of (3.24) is the next result.

Lemma 3. *Let* (λ_n, v_n) *be an eigenpair to* (2.18) *and* $w_n = \varphi_p(v_n')$. *Then,*

i)

$$|w_n(\theta_1)| > \cdots > |w_n(\theta_{n-1})|,$$

where in the Dirichlet case the sequence further involves $|w_n(a)|$ *and* $|w_n(b)|$ *as the first and last terms, respectively.*

ii) *For every* n*, the inequalities,*

$$|v_n(\sigma_1)| > |v_n(\sigma_2)| > \cdots > |v_n(\sigma_n)|,$$

hold true, where $\sigma_1 = a$*,* $\sigma_n = b$ *in the Neumann case corresponding to* $n \geq 2$*.*

iii) *Between consecutive inner zeros* $\theta_{i-1},\theta_i \in (a,b)$ *of a* n*–th eigenfunction* v_n*, relations*

$$(p-1)|w_n(\theta_i)|^{p'} < \lambda|v_n(\sigma_i)|^p < (p-1)|w_n(\theta_{i-1})|^{p'},$$

are satisfied. They also include $\theta_0 = a$ *and* $\theta_n = b$ *in the Dirichlet case.*

Our next objective is to relate the radial eigenvalues to (2.18), obtained in Theorem 2 by means of odes methods, with the variational eigenvalues derived from critical points theory in (1.4). In fact, a possibly distinct family of radial eigenvalues to (2.18) could be obtained from the variational expressions,

$$\tilde{\lambda}_{n,p} = \inf_{C \in \widetilde{\mathcal{C}}_n} \sup_{u \in C} \int_{\mathcal{A}} |\nabla u|^p \, dx + \int_{\partial \mathcal{A}} \beta |u|^p, \qquad n \in \mathbb{N}, \tag{3.26}$$

where $\widetilde{\mathcal{C}}_n = \{C \in \mathcal{C}_n : C \subset \widetilde{W}^{1,p}(\mathcal{A})\}$. In other words, $\widetilde{\mathcal{C}}_n$ is the restriction of the class $\mathcal{C}_n$ to the space of radial functions $\widetilde{W}^{1,p}(\mathcal{A})$. Of course, the latter space is replaced by $\widetilde{W}_0^{1,p}(\mathcal{A})$ in the Dirichlet problem.

As it is going to be observed in Section 4 the variational approach (3.26) is crucial to study the existence of the limits of the eigenvalues $\lambda_{n,p}$ as $p \to 1$. Thus, a result as the next one is required. Subindex p is drop both in λ_n and $\tilde{\lambda}_n$ to brief.

Theorem 4. *The family of 'ode' eigenvalues* $\{\lambda_n\}$ *to* (2.18) *and the corresponding one* $\{\tilde{\lambda}_n\}$ *deduced from Ljusternik–Schnirelman theory agree,*

$$\lambda_n = \tilde{\lambda}_n, \qquad \textit{for every } n \in \mathbb{N}.$$

Proof. We are adapting the corresponding proof in the case of the ball and Dirichlet conditions (see [26, Th. 12]). Firstly and according to the discussion in Section 2.3, eigenvalues in the variational family $\{\tilde{\lambda}_n\}$ gives rise to radial eigenvalues in the family $\{\lambda_n\}$. In addition, there can not exist repetitions in the $\tilde{\lambda}_n$'s. In fact, if it were,

$$\tilde{\lambda}_n = \cdots = \tilde{\lambda}_{n+k-1}, \qquad k \geq 2,$$

for some n, then there would exists a compact $K \subset \mathcal{M}_p \cap \widetilde{W}^{1,p}(\mathcal{A})$ of eigenfunctions to $\tilde{\lambda}_n$ with genus $\gamma(K) \geq k$ ([29, Lem. 5.6]). So, if a repetition occurs then K becomes infinite. This is not posible since the simplicity of $\tilde{\lambda}_n$ stated in Theorem 2–i) implies that $K = \{\tilde{u}_n, -\tilde{u}_n\}$. Thus, repetitions in the sequence $\{\tilde{\lambda}_n\}$ are not allowed.

We next observe that $\tilde{\lambda}_1 = \lambda_1$ since $\tilde{\lambda}_1$ always defines a principal eigenfunction, therefore without inner zeros in (a, b). Hence,

$$\lambda_n \leq \tilde{\lambda}_n, \qquad \text{for all } n \in \mathbb{N}.$$

On the other hand, [6, Prop. 1] states, in the general framework of a bounded smooth domain $\Omega \subset \mathbb{R}^N$, that for an arbitrary Dirichlet eigenvalue λ to (1.1), regardless its nature (variational or not), the following estimate is satisfied,

$$\lambda_N^{LS} \leq \lambda. \tag{3.27}$$

Here λ_N^{LS} denote the N–th term in the sequence given by (1.4), while $N = N(\lambda)$ is the maximal number of nodal regions that an eigenfunction u to λ can exhibit. The nodal regions of u are the family of components in the set $\{x \in \Omega : u(x) \neq 0\}$. Such a number is shown to be bounded above by a maximum value $N(\lambda) < \infty$ only depending on λ (see [4, Prop. 2] for Dirichlet conditions, [20, Th. 5.11] for the remaining two). Moreover, estimate (3.27) has been upgraded in [20, Prop. 5.17] to include both Neumann and Robin conditions.

Now observe that Theorem 2–ii) implies that the n–th radial eigenvalue λ_n satisfies,

$$N(\lambda_n) = n.$$

Thus, (3.27) implies,

$$\tilde{\lambda}_n = \lambda_{N(\lambda_n)}^{LS} \leq \lambda_n,$$

and we are done. □

4. Behavior of the radial eigenvalues as $p \to 1$

Our first result states the existence of the limits of the radial eigenvalues as $p \to 1$. As a matter of notation, whenever necessary a subindex p is added to the relevant objects (numbers, functions) in order to indicate their dependence on p. For instance $(\lambda_{n,p}, v_{n,p})$ will be used instead the former (λ_n, v_n) to stress the influence of p on this eigenpair to (2.18). On the other hand, it will be assumed in the sequel that the domain Ω in (1.1) is the annulus $\mathcal{A} = \{0 < a < |x| < b\}$.

Theorem 5. *Limits,*

$$\bar{\lambda}_n = \lim_{p \to 1} \lambda_{n,p}, \tag{4.28}$$

exist for every $n \in \mathbb{N}$.

Proof. It is shown in [26, Cor. 3] that for Dirichlet conditions the Ljusternik–Schnirelman eigenvalues $\lambda_{n,p}$ to (1.1) admits a limit,

$$\bar{\lambda}_n := \lim_{p \to 1} \lambda_{n,p}, \tag{4.29}$$

in a general Lipschitz bounded domain Ω. The argument makes use in a crucial way of the Rayleigh quotients (1.4) to show the inequality, ([22] for the case $n = 1$ and [26, Th. 2] for an arbitrary n),

$$p {\lambda_{n,p}}^{\frac{1}{p}} \leq s {\lambda_{n,s}}^{\frac{1}{s}}, \qquad \text{for } 1 < p < s,$$

from which (4.29) follows. The same reasoning proves the latter inequality for the Neumann eigenvalues in Ω and thus the existence of the limits (4.29) is also stated in this case. Finally, to achieve (4.29) subject to Robin conditions requires a more elaborate reasoning and it is answered in positive in [28, Th. 20].

In the case of the annulus $\mathcal{A}$ previous arguments remain valid when applied to the radial variational eigenvalues $\tilde{\lambda}_{n,p}$ defined in (3.26). Thus we deduce the existence of the limits,

$$\bar{\lambda}_n = \lim_{p\to 1} \tilde{\lambda}_{n,p}.$$

Then (4.28) follows from Theorem 4. □

Next statement is a further consequence of Theorem 4. It is the specialization to radial solutions of theorems [26, Th. 6] and [28, Th. 18], the first one concerning the Dirichlet problem, the latter dealing with either Neumann or Robin boundary conditions. We recall the notation $\varphi_p(t) = |t|^{p-2}t$ to be next used.

Theorem 6. *For a fixed n let $(\lambda_m, v_m) = (\lambda_{n,p_m}, v_{n,p_m}(r))$, $a < r < b$, be a family of* radial *eigenpairs to* (2.18) *corresponding to a sequence of exponents $p_m \to 1$ and thus satisfying,*

$$\lambda_m \to \bar{\lambda} = \bar{\lambda}_n, \qquad \textit{as } m \to \infty.$$

Assume also that eigenfunctions are normalized so that,

$$\|v_m\|_\infty = 1, \quad \textit{for each } m \in \mathbb{N}. \tag{4.30}$$

Then, up to subsequences, the following properties hold.

i) *$\{v_m\}$ converges strongly in $L^1(a,b)$ to a function $v \in BV(a,b)$.*

ii) *$\{\varphi_{p_m}(v_m)\}$ converges weakly in $L^s(a,b)$ to a function $\gamma \in L^\infty(a,b)$, for every $1 < s < \infty$. Furthermore, $\|\gamma\|_\infty \le 1$ and,*

$$\gamma v = |v|, \qquad \textit{a. e. in } (a,b). \tag{4.31}$$

iii) *Family $w_m = \varphi_{p_m}(v_m')$ converges uniformly in $[a,b]$ to a Lipschitz function $w \in W^{1,\infty}(a,b)$ with $\|w\|_\infty \le 1$.*

iv) *Function w solves in the sense of distributions the equation,*

$$-w' - \frac{N-1}{r} w = \overline{\lambda}\gamma. \tag{4.32}$$

v) *Identity $|v'| = (w, v')$ holds in (a,b) as measures.*

vi) *Boundary conditions,*

$$w(a)v(a+) = \min\{1, \beta_1\}|v(a+)|, \tag{4.33a}$$

$$w(b)v(b-) = -\min\{1, \beta_2\}|v(b-)| \tag{4.33b}$$

are satisfied, with values $\beta_i = 1$ in the Dirichlet case, $\beta_i = 0$, in the Neumann problem and $\beta_i > 0$, $i = 1, 2$, in the Robin one.

vii) *Identity,*

$$\overline{\lambda}\,\frac{d|v|}{dr} = -\frac{N-1}{r}\left|\frac{dv}{dr}\right|, \tag{4.34}$$

holds in (a,b) *in a distributional sense.*

Remark 4. Relations (4.33) assert that Robin eigenfunctions v_n can only 'see' the genuine Robin conditions as $p \to 1$ if $0 < \beta_i \le 1$. Otherwise, both Dirichlet and Robin eigenfunctions exhibits the same limit behavior at $r = a, b$. Notice in addition that the existence of the side limits is ensured after a possible modification of v in a null set (see Section 2.2).

Proof. Starting assumption in both [26, Th. 6] and [28, Th. 18] is the control,

$$0 < k_1 \le \int_\Omega |u|^p \le k_2, \tag{4.35}$$

on the L^p norm of the eigenfunctions u to (1.1). Thus we are checking that provided a family of radial eigenpairs $(\lambda_m, \tilde{v}_m)$ satisfies,

$$\int_\Omega |\tilde{v}_m|^{p_m} = 1, \tag{4.36}$$

with $\lambda_m \to \bar{\lambda}$, then $\|\tilde{v}_m\|_\infty = O(1)$ as $m \to \infty$. Thus, $v_m = \dfrac{\tilde{v}_m}{\|\tilde{v}_m\|_\infty}$ both fulfills (4.30) and (4.35). In fact, by setting $v = \tilde{v}_m$, $p = p_m$,

$$\begin{aligned} |v(r)| \le |v(a)| + \int_a^b |v'| &\le |v(a)| + (b-a)^{1-\frac{1}{p}}\left(\int_a^b |v'|^p\right)^{\frac{1}{p}} \\ &\le |v(a)| + (b-a)^{1-\frac{1}{p}} a^{-\frac{N-1}{p}}\left(\int_a^b |v'|^p r^{N-1}\right)^{\frac{1}{p}} \\ &\le |v(a)| + (b-a)^{1-\frac{1}{p}} a^{-\frac{N-1}{p}} \lambda^{\frac{1}{p}}. \end{aligned}$$

Hence, family $\tilde{v}_m$ satisfies the estimate,

$$\|\tilde{v}_m\|_\infty \le \sup|\tilde{v}_m(a)| + (b-a)\left(\frac{\lambda_m}{(b-a)a^{(N-1)}}\right)^{\frac{1}{p_m}},$$

and a uniform bound is attained provided that the first term in the right hand side is finite. This is obvious in the Dirichlet case. As for Robin conditions one has,

$$\int_a^b |\tilde{v}_m'|^{p_m} r^{N-1}\,dr + \beta_1 a^{N-1}|\tilde{v}_m(a)|^{p_m} + \beta_2 b^{N-1}|\tilde{v}_m(b)|^{p_m} = \lambda_m,$$

what shows the finiteness of $\sup |\tilde{v}_m(a)|$. Finally, under Neumann conditions Barrow's rule implies

$$|\tilde{v}_m(a)| \leq \int_a^b \eta |\tilde{v}_m'| \, dr + \int_a^b |\eta' \tilde{v}_m| \, dr,$$

where $\eta \in C^1[a,b]$ is any fixed nonnegative function satisfying $\eta(a) = 1$, $\eta(b) = 0$. Both integrals are uniformly bounded because of (4.36) and the equality,

$$\int_a^b |\tilde{v}_m'|^{p_m} r^{N-1} \, dr = \lambda_m.$$

Therefore, $\|\tilde{v}_m\|_\infty = O(1)$ as $m \to \infty$.

As a consequence of (4.35), all features of the statement, except for iii), are essentially developed in [26, Th. 6], [26, Prop. 16] and [28, Th. 18].

Regarding iii), results [26, Th. 6] and [28, Th. 18] supply $w \in L^\infty(a,b)$ such that

$$w_m = \varphi_{p_m}(v_m') \rightharpoonup w,$$

weakly in $L^q(a,b)$ for all $1 < q < \infty$. We are now upgrading both the quality of the latter convergence and the regularity of w. First observe that by integrating the radial equation in (2.18) with initial data $w_m(\sigma_{1,m}) = 0$, $r = \sigma_{1,m}$ standing for the first zero of v_m' in (a,b), we deduce,

$$w_m(r) = -\lambda_m \int_{\sigma_{1,m}}^r \left(\frac{s}{r}\right)^{N-1} \varphi_{p_m}(v_m(s)) \, ds,$$

what implies that $\{w_m\}$ is uniformly bounded in $[a,b]$. In addition, both the normalization (4.35) and the equation,

$$(r^{N-1} w_m)' = -\lambda_m r^{N-1} \varphi_{p_m}(v_m),$$

imply that the family $\{r^{N-1} w_m\}$ and hence $\{w_m\}$ are equicontinuous in $[a,b]$ and, after continuation, in any interval $[a,b_1]$ with $b_1 \geq b$. Thus, modulus a subsequence, $w_m \to w$ in $C[a,b]$. After a possibly redefinition of w in a null set we arrive at,

$$w(r) = \left(\frac{t_0}{r}\right)^{N-1} w(t_0) - \bar{\lambda} \int_{t_0}^r \left(\frac{s}{r}\right)^{N-1} \gamma(s) \, ds,$$

for some $t_0 \in [a,b]$ and conclude that $w \in W^{1,\infty}(a,b)$. □

Remark 5. As to be explained in Section 6, assertions iv), v), (4.31), (4.34) together with (4.33) constitute the requirements for $(\bar{\lambda}, v) \in \mathbb{R} \times BV(\Omega)$ to define a radial eigenpair to (1.3). In particular, relations (4.33) encode the boundary conditions (see Definitions 16 and 17).

To ascertain the limit profile of eigenfunctions $v_{n,p}$ to (2.18) as $p \to 1$ is essentially equivalent to master the corresponding behavior of their zeros $\theta_{i,p}$ and critical points $\sigma_{i,p}$. It should be remarked that the own existence of the limits of $\theta_{i,p}$ and $\sigma_{i,p}$ as $p \to 1$ is by no means evident. This fact strongly contrasts with the case when Ω is a ball where such an existence is a direct consequence of the one for the eigenvalues $\lambda_{n,p}$ as $p \to 1$ ([26, Th. 13]).

Nevertheless, to start on a firm ground, next result ensuring the existence of positive gaps between the limit values of these quantities is quite convenient for our purposes.

A first step in this direction is the next statement.

Lemma 7. *Let $\{\theta_{i,p}\}$, $\{\sigma_{i,p}\}$ be the family of zeros and critical points associated to n–th eigenvalue $\lambda_{n,p}$. Then, the estimates*

$$\varliminf_{p\to 1}(\theta_{i,p} - \sigma_{i,p}) > 0, \qquad \varliminf_{p\to 1}(\sigma_{i,p} - \theta_{i-1,p}) > 0, \tag{4.37}$$

are satisfied for all possible choices of $\{\sigma_{i,p}, \theta_{i,p}\}$, $\{\theta_{i-1,p}, \sigma_{i,p}\}$, including the extreme cases $\theta_{0,p} = a$ and $\theta_{n,p} = b$. These facts hold for the three boundary conditions (1.2)*, with the sole possible exceptions of the couples $\{a, \sigma_{1,p}\}$ and $\{\sigma_{n,p}, b\}$ in the Robin problem.*

Latter missing estimates in the previous lemma are separately achieved in the following one.

Lemma 8. *For the Robin problem and the n–th eigenvalue $\lambda_{n,p}$, estimates*

$$\varliminf_{p\to 1}(\sigma_{1,p} - a) > 0, \qquad \varliminf_{p\to 1}(b - \sigma_{n,p}) > 0, \tag{4.38}$$

are also satisfied.

In the course of the following proofs we must resort to the function $\cos_p t$. It is implicitly defined through the integral,

$$(p-1)^{\frac{1}{p}} \int_{\cos_p t}^{1} \frac{ds}{(1-|s|^p)^{\frac{1}{p}}} = t, \qquad t \in I_p := \{0 \le t \le \pi_p\}, \tag{4.39}$$

where,

$$\pi_p = 2(p-1)^{\frac{1}{p}} \int_0^1 \frac{ds}{(1-|s|^p)^{\frac{1}{p}}}. \tag{4.40}$$

We remark that $\lim_{p\to 1} \pi_p = 2$ ([26]) while function $\cos_p t$ decreases from 1 to -1 in I_p and vanishes at $t = \dfrac{\pi_p}{2}$ (see an 'ad hoc' account in [28, Sec. 3]).

Proof of Lemma 7. Define to brief $c = \sigma_{i,p}$, $d = \theta_{i,p}$ and $\lambda_p = \lambda_{n,p}$. Then $\mu = \lambda_p$ is the principal eigenvalue to the weighted problem,

$$\begin{cases} -(t^{N-1}\varphi_p(u'))' = \mu t^{N-1}\varphi_p(u), & c < t < d, \\ u'(c) = 0, \quad u(d) = 0, \end{cases} \tag{4.41}$$

which is expressed as,

$$\mu_1 = \inf \frac{\int_c^d t^{N-1}|u'|^p dt}{\int_c^d t^{N-1}|u|^p dt}.$$

Here, the infimum involves all those functions $u \in W^{1,p}(c,d)$ satisfying $u(d) = 0$. Then,

$$\mu_1 \le \left(\frac{d}{c}\right)^{N-1} \left(\frac{\pi_p}{2(d-c)}\right)^p, \tag{4.42}$$

where π_p is defined by (4.40). To achieve this result notice that the p power factor in the inequality corresponds to the principal eigenvalue to (4.41) when factors t^{N-1} are replaced by unity. In fact, an associated eigenfunction is defined by,

$$u = \cos_p \left(\frac{\pi_p(t-c)}{2(d-c)}\right).$$

A symmetric reasoning leads to the complementary estimate to (4.42) and so we arrive at,

$$\left(\frac{c}{d}\right)^{N-1} \left(\frac{\pi_p}{2(d-c)}\right)^p \le \lambda_p \le \left(\frac{d}{c}\right)^{N-1} \left(\frac{\pi_p}{2(d-c)}\right)^p,$$

from which we deduce,

$$\varliminf_{p\to 1}(d-c) = \varliminf_{p\to 1}(\theta_{i,p}^n - \sigma_{i,p}^n) > 0.$$

The second estimate in (4.37) is obtained by arguing in the same way but interchanging the boundary conditions in (4.41). □

Proof of Lemma 8. As in the previous lemma, by setting $\lambda_p = \lambda_{n,p}$ and $c = \sigma_{1,p}$ then $\mu = \lambda_p$ becomes the principal eigenvalue to the problem,

$$\begin{cases} -(t^{N-1}\varphi_p(u'))' = \mu t^{N-1}\varphi_p(u), & a < t < c, \\ \varphi_p(u'(a)) = \beta_1\varphi_p(u(a)), \\ u'(c) = 0, \end{cases} \tag{4.43}$$

whose variational expression is,

$$\mu_1 = \inf \frac{\int_a^c |u'|^p t^{N-1}\, dt + a^{N-1}\beta_1|u(a)|^p}{\int_a^c |u|^p t^{N-1}\, dt},$$

and the infimum is extended to all functions $u \in W^{1,p}(a,c)$. This implies that,

$$\left(\frac{a}{c}\right)^{N-1}\lambda_{p,l} \le \mu \le \left(\frac{c}{a}\right)^{N-1}\lambda_{p,l}, \tag{4.44}$$

where $l = \sigma_{i,p} - a$ and $\lambda = \lambda_{p,l}$ is the principal eigenvalue to:

$$\begin{cases} -(\varphi_p(u'))' = \lambda\varphi_p(u), & 0 < t < l, \\ u'(0) = 0, & \\ \varphi_p(u'(l)) = -\beta_1\varphi_p(u(l)). & \end{cases} \tag{4.45}$$

Notice that interval endpoints have been interchanged in (4.45) to normalize. Our goal is to show that the behavior $l \to 0$ as $p \to 1$ can not occur. In fact, assume on the contrary that $l \to 0$. Since $\mu_1 \to \bar{\lambda}_n$ as $p \to 1$ with $\bar{\lambda}_n = \lim_{p\to 1}\lambda_{n,p}$, we firstly get from (4.44) a uniform estimate,

$$0 < k_1 \le \lambda_{p,l} \le k_2, \tag{4.46}$$

both as $l \to 0$ and $p \to 1$.

By assuming the normalizing condition $u(0) = 1$ it follows from the equation in (4.45) that,

$$(p-1)|u'|^p + \lambda|u|^p = \lambda, \qquad 0 \le t \le l. \tag{4.47}$$

From the second boundary condition we get,

$$u(l) = \frac{1}{\left[1 + (p-1)\lambda_{p,l}^{-1}\beta_1^{p'}\right]^{\frac{1}{p}}}.$$

Then, by integrating the equation (4.47) in $0 \le t \le l$ we arrive at the equality,

$$(p-1)^{\frac{1}{p}}\int_{u(l)}^{1}\frac{ds}{(1-s^p)^{\frac{1}{p}}} = \lambda_{p,l}^{\frac{1}{p}}l, \tag{4.48}$$

where the normalization $u(0) = 1$ has been employed. By resorting to the computations in [28, Th. 8] it is shown that,

$$\lim_{p\to 1,\, l\to 0}(p-1)^{\frac{1}{p}}\int_{u(l)}^{1}\frac{ds}{(1-s^p)^{\frac{1}{p}}} = \min\{1,\beta_1\}. \tag{4.49}$$

To this purpose remark that the dependence on l is linked to $\lambda_{p,l}$ through the expression for $u(l)$. Estimates (4.46) are enough to keep $\lambda_{p,l}^{-1}$ bounded away from zero as $p \to 1$. This is the only requirement for the limit above to be valid.

Since (4.49) is not compatible with (4.48) provided $l \to 0$ it means that l must keep bounded away from zero as $p \to 1$. The proof of (4.38) has been completed. □

5. Eigenfunctions profile as $p \to 1$

In the next results we are assuming that limit eigenvalue $\bar{\lambda}$ in Theorem 6 is given by,

$$\bar{\lambda} = \bar{\lambda}_n,$$

where n is a fixed index. When dealing with either Dirichlet or Robin conditions, it is going to be assumed that the normalizing condition (4.30) is attained in the following way,

$$\|v_m\|_\infty = v_m(\sigma_{1,p_m}) = 1, \quad \text{for every } m \in \mathbb{N}, \tag{5.50}$$

where $a < \sigma_{1,p_m} < b$ stands for the first critical point of the eigenfunction $v_m = v_{n,p_m}$ in the interval (a, b) (Theorem 2). We stress that then $v_m'(a) > 0$. For the Neumann problem, (5.50) should be replaced with,

$$\|v_m\|_\infty = v_m(a) = 1, \quad \text{for every } m \in \mathbb{N}. \tag{5.51}$$

In fact, notice that, according to Lemma 3, the maximum value of $|v_m(r)|$ is attained at $r = a$ provided $v_m'(a) = 0$.

Our immediate objective is describing the limit profile of the normalized eigenfunction $v_m = v_{n,p_m}$ as $p_m \to 1$ in the *initial* interval $(a, \theta_{1,m})$, where $\theta_{1,m} = \theta_{1,p_m}$. Full analysis in the interval (a, b) is delayed until the final part of the section (Theorem 15). To this aim, the following technical result will be useful.

Lemma 9. *Let $w \in W^{1,\infty}(a, b)$ be the function introduced in Theorem 6. Then v becomes constant in every component of the set $\{t : |w(t)| < 1\}$.*

Proof. See item 1) in the proof of [26, Th. 7.7] (also [27, Th. 10]). □

We are proceeding separately in each of the boundary conditions.

Theorem 10 (Dirichlet Problem)**.** *Let $(\lambda_m, v_m) = (\lambda_{n,p_m}, v_{n,p_m})$ be a sequence of radial Dirichlet eigenpairs asymptotics to $(\bar{\lambda}, v) = (\bar{\lambda}_n, v)$ as $p_m \to 1$, $v \in BV(a, b)$ being the function introduced in Theorem 6. Then the following properties are satisfied.*

i) *There exists $a < \sigma_1 < \theta_1$ such that,*

$$\lim_{m\to\infty} \theta_{1,p_m} = \theta_1, \qquad \textit{and} \qquad \lim_{m\to\infty} \sigma_{1,p_m} = \sigma_1. \tag{5.52}$$

ii) Numbers σ_1 and θ_1 satisfy the relations,

$$\frac{N a^{N-1}}{\sigma_1^N - a^N} = \frac{N({\theta_1}^{N-1} + a^{N-1})}{\theta_1^N - a^N} = \bar{\lambda}. \tag{5.53}$$

iii) Limit $v_m \to 1$ holds in the topology of $C^1(a, \theta_1)$.

Moreover, all of the previous limits do not depend on the sequence $p_m \to 1$.

Proof. In order to brief, we are using $\theta_{1,m}$ and $\sigma_{1,m}$ instead θ_{1,p_m} and σ_{1,p_m}, respectively.

Firstly, observe that, due to (3.24), the estimate

$$(p_m - 1)|w_m(a)|^{p'_m} \geq \lambda_m |v_m(\sigma_{1,m})|^{p_m} = \lambda_m,$$

holds for every m. So, passing to the limit we obtain,

$$w(a) = 1,$$

since $v'_m(a) > 0$ while $\|w\|_\infty \leq 1$. We also have,

$$w_m(r) = \frac{a^{N-1}}{r^{N-1}} w_m(a) - \lambda_m \int_a^r \left(\frac{s}{r}\right)^{N-1} \varphi_{p_m}(v_m(s))\, ds, \tag{5.54}$$

and by letting $p_m \to 1$,

$$w(r) = \frac{a^{N-1}}{r^{N-1}} - \bar{\lambda} \int_a^r \left(\frac{s}{r}\right)^{N-1} \gamma(s)\, ds, \tag{5.55}$$

where $\gamma \in L^\infty(a,b)$ is the weak limit of functions $\varphi_{p_m}(v_m)$ as $m \to \infty$ (Theorem 6–ii)). This is equivalent to integrating (4.32) with datum $w(a) = 1$.

If we take $r = \sigma_{1,m}$ in (5.54) and set $\sigma_1 = \underline{\lim}_{m\to\infty} \sigma_{1,m}$ then,

$$a^{N-1} = \bar{\lambda} \int_a^{\sigma_1} s^{N-1} \gamma(s)\, ds, \tag{5.56}$$

and so $a^{N-1} \leq \frac{\bar{\lambda}}{N}(\sigma_1^N - a^N)$, wherewith we get,

$$\sigma_1 > a.$$

This relation is also ensured by the second estimate in (4.37) for $i = 1$ (see Lemma 7).

From (3.24) we also deduce,

$$-(p_m - 1)^{\frac{1}{p_m}} v'_m(r) \leq \lambda_m^{\frac{1}{p_m}} (1 - |v_m(r)|^{p_m})^{\frac{1}{p_m}},$$

for all $\sigma_{1,m} < r < \theta_{1,m}$ owing to $v_m(\sigma_{1,m}) = 1$. This fact also implies

$$(p_m - 1)^{\frac{1}{p_m}} \int_{v_m(r)}^1 \frac{ds}{(1 - |v_m(s)|^{p_m})^{\frac{1}{p_m}}} \leq \lambda_m^{\frac{1}{p_m}} (r - \sigma_{1,m}),$$

and, recalling that $\cos_p t$ decreases in I_p, we obtain

$$v_m(r) \geq \cos_{p_m}\left(\lambda_m^{\frac{1}{p_m}} (r - \sigma_{1,m})\right), \tag{5.57}$$

as $\sigma_{1,m} < r < \theta_{1,m}$. In addition, it is well–known that,

$$\cos_p t \to 1, \qquad \text{as } p \to 1,$$

in the topology of $C^1(-1,1)$ while $\lim_{p\to 1}\pi_p = 2$ (see [26]). So, it follows from (5.57) that,

$$v_m \to 1, \qquad \text{as } m \to \infty, \tag{5.58}$$

uniformly on compacta of $\big(\sigma_1, \sigma_1 + \bar\lambda^{-1}\big)$.

On account of equation (2.18) we observe that $(r^{N-1}w_m(r))' < 0$ for all $a < r < \sigma_{1,m}$, so that

$$a^{N-1} = a^{N-1}w_m(a) > r^{N-1}w_m(r) > \sigma_{1,m}^{N-1}w_m(\sigma_{1,m}) = 0\,.$$

Letting p_m go to 1, we get

$$a^{N-1} \geq r^{N-1}w(r) \geq 0 \qquad a < r < \sigma_1 - \epsilon\ ,$$

for all $\epsilon > 0$ which implies $1 > w(r) \geq 0$ for all $a < r < \sigma_1$. Therefore, we deduce

$$-1 < w(r) < 1, \qquad \text{as } a < r \leq \sigma_1 + \delta, \tag{5.59}$$

for some small $\delta > 0$. According to Lemma 9, limit function v achieves the value 1 in the whole component $(a, \hat\theta_1)$ of the set $\{|w(t)| < 1\}$ containing the interval $(a, \sigma_1 + \delta)$.

We are now extracting some consequences from this fact. By inserting $\gamma = 1$ in (5.56) we get,

$$a^{N-1} = \frac{\bar\lambda}{N}(\sigma_1^N - a^N),$$

a representation of σ_1 which does not depend on the sequence $p_m \to 1$ and that shows in particular that the limit,

$$\lim_{m\to\infty} \sigma_{1,m} = \sigma_1,$$

holds. By setting now $\gamma = 1$ in (5.55) it follows that,

$$w(r) = \frac{a^{N-1}}{r^{N-1}} - \frac{\bar\lambda}{N}\frac{(r^N - a^N)}{r^{N-1}} = \left(1 + \frac{\bar\lambda a}{N}\right)\left(\frac{a}{r}\right)^{N-1} - \frac{\bar\lambda}{N} r,$$

whenever $a < r < \hat\theta_1$. Thus w is decreasing in this interval, satisfies $|w| < 1$ and reach the value $w(\hat\theta_1) = -1$. So, $\hat\theta_1$ is defined through the relation,

$$\frac{N(\hat\theta_1^{N-1} + a^{N-1})}{\hat\theta_1^N - a^N} = \bar\lambda. \tag{5.60}$$

In fact, such an equation is uniquely solvable in $\hat\theta_1$ (see Lemma 22 in Section 6) and this is also a further expression which does not depend on the particular sequence $p_m \to 1$.

We next observe that,

$$v_m' = \varphi_{p_m'}(w_m) = |w_m|^{p_m'-2} w_m \,, \qquad p_m' = \frac{p_m}{p_m - 1}.$$

Since $w_m \to w$ uniformly on compacta of $(a, \hat{\theta}_1)$ then $v_m' \to 0$ in that precise way. In addition, this fact entails that $v_m \to 1$ in $C(a, \hat{\theta}_1)$.

Set now $\theta_1 = \underline{\lim}_{m\to\infty} \theta_{1,m}$. Convergence (5.58) in $C(a, \hat{\theta}_1)$ necessarily implies that,

$$\theta_1 \geq \hat{\theta}_1.$$

We are showing that,

$$\theta_1 = \hat{\theta}_1. \tag{5.61}$$

This would imply from (5.60) that,

$$\frac{N({\theta_1}^{N-1} + a^{N-1})}{\theta_1^N - a^N} = \bar{\lambda},$$

and so the limit,

$$\lim \theta_{1,m} = \theta_1,$$

does not depend on $p_m \to 1$. Hence, the proof of (5.53) would be accomplished.

To show (5.61) suppose $\theta_1 > \hat{\theta}_1$. This means that for each $\hat{\theta}_1 < r < \theta_1$, $v_m(r) > 0$ for p_m close to 1 (m large). First assume that $\hat{\theta}_1 < r_2 < \theta_1$ exists such that $\underline{\lim}_{m\to\infty} v_m(r_2) > 0$. Since v_m decreases in $J := [r_1, r_2]$ with $\sigma_1 < r_1 < \hat{\theta}_1 < r_2$ then $v > 0$ in this interval. On the other hand,

$$w' = -\frac{N-1}{r} w - \bar{\lambda} \leq \frac{N-1}{\hat{\theta}_1} - \bar{\lambda}, \qquad r \in [\hat{\theta}_1, r_2].$$

Now, it is clear from (5.60) that $N\hat{\theta}_1^{N-1} - \bar{\lambda}\hat{\theta}_1^N < 0$ and so,

$$\frac{N-1}{\hat{\theta}_1} < \bar{\lambda}. \tag{5.62}$$

Hence, $w(r) < -1$ in $(\hat{\theta}_1, r_2]$, which is not possible. Thus, $v = 0$ almost everywhere in $\hat{\theta}_1 \leq r \leq \theta_1$. But in this case v undergoes a unit jump at the point $r = \hat{\theta}_1$. However, (4.34) implies that,

$$\frac{N-1}{\hat{\theta}_1} = \bar{\lambda}. \tag{5.63}$$

Since this can not be true, we finally conclude (5.61). □

Corollary 11. *In the Dirichlet problem,*

$$\bar{\lambda}_1 = \lim_{p\to 1} \lambda_{1,p} = \frac{N(a^{N-1} + b^{N-1})}{b^N - a^N}, \tag{5.64}$$

while $v_{1,p} \to 1$ as $p \to 1$ in the topology of $C^1(a,b)$.

Remark 6. Value $\bar{\lambda}_1$ in (5.64) is just the Cheeger constant for the annulus $\mathcal{A}$ (see for instance [11]). In fact, it is the first eigenvalue to the 1–Laplacian subject to Dirichlet conditions in the annulus (Section 6).

Theorem 12 (Neumann Problem)**.** *Let now $(\lambda_m, v_m) = (\lambda_{n,p_m}, v_{n,p_m})$ be a family of radial Neumann eigenpairs in the conditions of Theorem 6, satisfying in particular,*

$$\lambda_m \to \bar{\lambda}, \qquad v_m \to v \qquad \text{in } L^1(a,b),$$

as $p_m \to 1$. Then $v_m \to 1$ in $C^1(a,\theta_1)$, where

$$\theta_1 = \lim_{m\to\infty} \theta_{1,m},$$

and is given by means of the relation,

$$\frac{N{\theta_1}^{N-1}}{\theta_1^N - a^N} = \bar{\lambda}. \tag{5.65}$$

Moreover, all these features do not depend on the sequence $p_m \to 1$ in Theorem 6.

Proof. The value $r = a$ plays here the rôle of variable critical point $r = \sigma_{1,m}$ in the proof of Theorem 10, whose argument is going to be closely followed. In accordance, $v_m \to 1$ uniformly on compacta of $[a, a + \bar{\lambda}^{-1})$. In addition, it holds that limit function $v = 1$ in a maximal interval $[a, \tilde{\theta}_1)$ where $w(\tilde{\theta}_1) = -1$ (Lemma 9). By noticing that $w(a) = 0$, (5.54) implies in the limit,

$$r^{N-1} w(r) = -\frac{\bar{\lambda}}{N}(r^N - a^N), \qquad a < r < \tilde{\theta}_1,$$

and so,

$$\frac{N\tilde{\theta}_1^{N-1}}{\tilde{\theta}_1^N - a^N} = \bar{\lambda}. \tag{5.66}$$

According Lemma 22 this equation has a unique solution $\tilde{\theta}_1$. Thus, such a value does not depend on the choice of $p_m \to 1$. Setting $\theta_1 = \underline{\lim}_{m\to\infty} \theta_{1,m}$ it follows that $\tilde{\theta}_1 \le \theta_1$ and the strict inequality $\tilde{\theta}_1 < \theta_1$ is discarded as in the discussion of Theorem 10 since the inequality,

$$\frac{N-1}{\tilde{\theta}_1} < \bar{\lambda},$$

is fulfilled. Indeed it is a direct consequence of (5.66). Hence $\tilde{\theta}_1 = \theta_1$ and the proof is concluded. □

Theorem 13 (Robin Problem). *Assume* $(\lambda_m, v_m) = (\lambda_{n,p_m}, v_{n.p_m})$ *is a family of radial eigenpairs to the Robin problem corresponding to exponents* $p_m \to 1$ *and satisfying (Theorem 6),*

$$\lambda_m \to \bar{\lambda}, \qquad v_m \to v \qquad \text{in } L^1(a,b),$$

for a certain $(\bar{\lambda}, v) \in \mathbb{R} \times BV(a,b)$. *Then,*

i) *Limits* $\lim_{m\to\infty} \sigma_{1,m} = \sigma_1$ *and* $\lim_{m\to\infty} \theta_{1,m} = \theta_1$ *hold true for numbers* $a < \sigma_1 < \theta_1$ *which are defined through the relations,*

$$\bar{\lambda} = \frac{N \min\{1,\beta_1\} a^{N-1}}{\sigma_1^N - a^N} = \frac{N({\theta_1}^{N-1} + \min\{1,\beta_1\} a^{N-1})}{\theta_1^N - a^N}. \tag{5.67}$$

ii) *Sequence* v_m *converges to* 1 *in* $C^1(a, \theta_1)$.

Moreover, previous limits do not depend on the choice of exponents $p_m \to 1$.

Proof. We are repeating the pattern of Theorem 10. As a first step notice that $0 \le w(a) \le 1$. Moreover, from (3.24) we deduce that,

$$(p_m - 1) w_m(a)^{p'_m} + \lambda_m v_m(a)^{p_m} \ge \lambda_m,$$

and so,

$$1 \ge v_m(a)^{p_m} \ge \frac{\lambda_m}{\lambda_m + (p_m - 1)\beta_1^{p'_m}}.$$

Hence,

$$\lim_{m\to\infty} v_m(a) = 1, \qquad \text{and} \qquad w(a) = \lim_{m\to\infty} w_m(a) = \beta_1, \tag{5.68}$$

provided that $0 < \beta_1 \le 1$. The same argument as in Theorem 10 permits us asserting both the estimates (5.58) and (5.59). In fact, the validity of the latter only requires $w(a) \le 1$ so the case $\beta_1 > 1$ is included. From Lemma 9 we deduce again that

$$v = 1 \qquad \text{in } a < r < \tilde{\theta}_1, \tag{5.69}$$

$(a, \tilde{\theta}_1)$ being the component of $\{|w| < 1\}$ containing $(a, \sigma_1 + \bar{\lambda}^{-1})$, where $\sigma_1 = \underline{\lim}_{m\to\infty} \sigma_{1,m}$. Accordingly, $a < \sigma_1 < \tilde{\theta}_1 \le \theta_1$ with $\theta_1 = \underline{\lim}_{m\to\infty} \theta_{1,m}$. Now, both (5.69) and the boundary conditions (4.33) imply $w(a) = 1$ if $\beta_1 > 1$, what combined with (5.68) leads to,

$$w(a) = \min\{1, \beta_1\}.$$

By means of the expression,

$$r^{N-1} w(r) = \min\{1,\beta_1\} a^{N-1} - \frac{\bar{\lambda}}{N}(r^N - a^N),$$

and noticing that $w(\sigma_1) = 0$ we attain the first equality in (5.67). Then, after setting $w(\tilde{\theta}_1) = -1$ we get the second inequality but with $\tilde{\theta}_1$ replacing θ_1. Finally, it is shown that $\theta_1 = \tilde{\theta}_1$ by the same reasons as in Theorem 10 and so we are done. □

Next one is a quite interesting achievement of this work. It states for the first time the explicit expression of the principal Robin eigenvalue for $-\Delta_1$ in the annulus (see Section 6).

Corollary 14. *In the Robin problem,*

$$\bar{\lambda}_1 = \lim_{p\to 1} \lambda_{1,p} = \frac{N(\min\{1,\beta_1\}a^{N-1} + \min\{1,\beta_2\}b^{N-1})}{b^N - a^N}, \tag{5.70}$$

while the first normalized eigenfunction $v_{1,p}$ to $\lambda_{1,p}$ satisfies $v_{1,p} \to 1$ as $p \to 1$ in $C^1(a,b)$.

Proof. Set $v_m = v_{1,p_m}$ the first eigenfunction to λ_{1,p_m} which is normalized according to (5.50). As a solution to (2.18) it can be continued for all $r \geq b$ (Section 2.3) and is oscillatory in (a,∞) (see [26, Lem. 10–ii]), so it exhibits a first zero $t_m > b$ with a limit $t_1 = \underline{\lim}\, t_m \geq b$. The arguments in Theorem 10 permit us asserting that $v_m \to 1$ in $C^1(a,t_1)$. With $\sigma_1 = \lim_{m\to\infty} \sigma_{1,p_m}$ and $w = \lim_{m\to\infty} w_m$, successive integrations of (4.32) furnish the identities,

$$a^{N-1}w(a) = \frac{\bar{\lambda}}{N}(\sigma_1^N - a^N), \qquad -b^{N-1}w(b) = \frac{\bar{\lambda}}{N}(b^N - \sigma_1^N).$$

Adding them and setting $w(a) = \min\{1,\beta_1\}$ and $w(b) = -\min\{1,\beta_2\}$, which are the boundary conditions (4.33), lead to (5.70). □

The main result of the section is next introduced. It extends to the whole interval (a,b) the analysis of the eigenfunctions to (2.18) initiated in Theorems 10, 12 and 13.

Theorem 15. *Let $\{(\lambda_{n,p}, v_{n,p}) : p > 1\}$ be the family of n–th radial eigenpairs to* (2.18)*, normalized according either to* (5.50) *in the Dirichlet and Robin problems or to* (5.51) *in the Neumann one. Assume that,*

$$\lim_{p\to 1} \lambda_{n,p} = \bar{\lambda}_n.$$

Then, the following properties are satisfied.

i) *There exist the limits,*

$$\lim_{p\to 1} \theta_{i,p} = \theta_i\,, \qquad 1 \leq i \leq n-1, \tag{5.71a}$$

$$\lim_{p\to 1} \sigma_{i,p} = \sigma_i\,, \qquad 1 \leq i \leq n, \tag{5.71b}$$

where the latter expression assumes $n \geq 2$ in the Neumann case.

ii) *Relations,*

$$\frac{N({\theta_i}^{N-1} + {\theta_{i-1}}^{N-1})}{\theta_i^N - \theta_{i-1}^N} = \bar{\lambda}_n, \qquad 1 \le i \le n, \tag{5.72}$$

hold true with $\theta_0 = a$*,* $\theta_n = b$*, where in the Robin problem the cases* $i = 1$ *and* $i = n$ *are replaced with,*

$$\frac{N({\theta_1}^{N-1} + \min\{1, \beta_1\} a^{N-1})}{\theta_1^N - a^N} = \frac{N(\min\{1, \beta_2\} b^{N-1} + {\theta_{n-1}}^{N-1})}{b^N - {\theta_{n-1}}^N} = \bar{\lambda}_n. \tag{5.73}$$

Neumann conditions for $n \ge 2$ *are included in* (5.73) *by choosing* $\beta_i = 0$*,* $i = 1, 2$*.*

iii) *Inner values* $\{\sigma_i\}$ *satisfy,*

$$\frac{N{\theta_{i-1}}^{N-1}}{\sigma_i^N - \theta_{i-1}^N} = \bar{\lambda}_n, \qquad 1 \le i \le n, \tag{5.74}$$

while $2 \le i \le n-1$ *in the Neumann problem as* $n \ge 3$*.*

iv) *There exist numbers* $\{\alpha_i\}_{0 \le i \le n-1}$*,*

$$1 = \alpha_0 = |\alpha_0| > |\alpha_1| > \cdots > |\alpha_{n-1}| > 0,$$

with alternating signs, i. e. $\alpha_{i-1}\alpha_i < 0$ *for all* i*, and such that,*

$$v_{n,p}(r) \to \alpha_{i-1} \qquad \text{as } p \to 1, \tag{5.75}$$

in the topology of $C^1(\theta_{i-1}, \theta_i)$*, for* $1 \le i \le n$*.*

v) *Families* $\{\alpha_i\}$ *and* $\{\theta_i\}$ *are related through,*

$$|\alpha_i| = \frac{\bar{\lambda}_n \theta_i - (N-1)}{\bar{\lambda}_n \theta_i + (N-1)} |\alpha_{i-1}|, \qquad 1 \le i \le n-1. \tag{5.76}$$

vi) $w(\theta_i) = \operatorname{sign}(\alpha_i) = -\operatorname{sign}(\alpha_{i-1})$, $1 \le i \le n-1$ *where* w *is the uniform limit of* $\varphi_p(v'_{n,p})$ *as* $p \to 1$ *in* $[a, b]$*.*

Proof. We are proceeding separately according to the boundary conditions.

A) *Dirichlet problem.* From Lemma 7 families $\{\bar{\theta}_i\}_{0 \le i \le n}$ and $\{\bar{\sigma}_i\}_{1 \le i \le n}$ defined as,

$$\bar{\theta}_i = \varliminf_{p \to 1} \theta_{i,p}, \qquad \bar{\sigma}_i = \varliminf_{p \to 1} \sigma_{i,p},$$

satisfy the strict inequalities,

$$\bar{\theta}_{i-1} < \bar{\sigma}_i < \bar{\theta}_i, \qquad 1 \le i \le n, \qquad \bar{\theta}_0 = a\,, \quad \bar{\theta}_n = b,$$

where we have set $\theta_{0,p} = a$, $\theta_{n,p} = b$. On the other hand, values $\bar{\theta}_1 = \theta_1$, $\bar{\sigma}_1 = \sigma_1$ have been already addressed in Theorem 10.

We are next proceeding by induction supposing that the assertions have been achieved in the intervals (a,θ_1),. . . ,$(\theta_{i-2},\theta_{i-1})$ to show them in the subsequent one (θ_{i-1},θ_i).

Firstly, no generality is lost by assuming that $v>0$ in $(\theta_{i-2},\theta_{i-1})$, that is $\alpha_{i-2}>0$ together with $w(\theta_{i-1})=-1$. Notice that,

$$\theta_{i-1}<\bar\sigma_i<\bar\theta_i,$$

and we are showing that,

$$v<0 \qquad \text{in } (\theta_{i-1},\bar\sigma_i). \tag{5.77}$$

In fact, take

$$\theta_{i-1}<x<y<\bar\sigma_i,$$

so that $x,y\in(\theta_{i-1,p},\sigma_{i,p})$ for p close to 1. Then

$$0>v_p(x)>v_p(y),$$

what implies,

$$0\ge v(x)\ge v(y), \qquad \text{a. e. in } (\theta_{i-1},\bar\sigma_i),$$

where v_p has been employed instead $v_{n,p}$ to brief. In addition,

$$\lim_{x\to\theta_{i-1}+} v(x)<0,$$

since otherwise the limit must be zero, v jumps at θ_{i-1} and then (4.34) implies,

$$\bar\lambda_n=\frac{N-1}{\theta_{i-1}}<\frac{N-1}{\theta_1},$$

what is not possible. Thus we conclude (5.77).

We next use the differential equation (4.32) for w together with $w=-1$ at $r=\theta_{i-1}$, $w=0$ at $r=\bar\sigma_i$ and $\gamma=-1$ to deduce that,

$$\frac{N{\theta_{i-1}}^{N-1}}{\bar\sigma_i^N-\theta_{i-1}^N}=\bar\lambda_n.$$

So, (5.74) holds and the limit $\bar\sigma_i$ does not depend of the way of convergence $p\to1$. Accordingly we set $\sigma_i=\bar\sigma_i$.

As a further step we are taking into account (5.77), $w(\theta_{i-1})=-1$ and the differential equation (4.32) to express w in the form,

$$w(r)=-\left(1+\frac{\bar\lambda_n\theta_{i-1}}{N}\right)\left(\frac{\theta_{i-1}}{r}\right)^{N-1}+\frac{\bar\lambda_n}{N}r,$$

for $r\in(\theta_{i-1},\sigma_i)$. Since w is increasing in that interval and $-1<w<0$ there, a component $(\theta_{i-1},\hat\theta_i)\supsetneq(\theta_{i-1},\sigma_i)$ of the set $\{|w|<1\}$ exists. As

$\hat{\theta}_i > \sigma_i$ and $w(\hat{\theta}_i) > 0$ we get the relation $w(\hat{\theta}_i) = 1$ what in turn says that,

$$\frac{N(\hat{\theta}_i^{N-1} + \theta_{i-1}{}^{N-1})}{\hat{\theta}_i^N - \theta_{i-1}^N} = \bar{\lambda}_n, \tag{5.78}$$

an equation with a unique solution $\hat{\theta}_i$ (Lemma 22 below). According to Lemma 9, v becomes constant in the interval $(\theta_{i-1}, \hat{\theta}_i)$ and we set,

$$v(r) = \alpha_{i-1},$$

in this interval. Notice that $\alpha_{i-2}\alpha_{i-1} < 0$. On the other hand, since $v_{n,p} \to \alpha_{i-1}$ almost everywhere in $(\theta_{i-1}, \hat{\theta}_i)$, then necessarily,

$$\hat{\theta}_i \leq \bar{\theta}_i.$$

By employing $w(\hat{\theta}_i) = 1$ and arguing just as in the proof of $\hat{\theta}_1 = \theta_1$ in Theorem 6 we obtain $\hat{\theta}_i = \bar{\theta}_i$. Again, this expression of $\bar{\theta}_i$ is independent of the way in which we make $p \to 1$ and so $\lim_{p\to 1} \theta_{i,p} = \bar{\theta}_i$, we directly use θ_i instead and relation (5.72) holds true.

On the other hand, v exhibits a jump at $r = \theta_{i-1}$ and evaluating (4.34) at this point yields,

$$\bar{\lambda}_n(|\alpha_{i-1}| - \alpha_{i-2}) = -\frac{N-1}{\theta_{i-1}}(\alpha_{i-2} + |\alpha_{i-1}|),$$

equivalently,

$$|\alpha_{i-1}| = \frac{\left(\bar{\lambda}_n - \dfrac{N-1}{\theta_{i-1}}\right)}{\left(\bar{\lambda}_n + \dfrac{N-1}{\theta_{i-1}}\right)}\,\alpha_{i-2},$$

which proves (5.76).

As for the convergence assertion (5.75) we first recall that,

$$w_p = \varphi_p(v_p') = |v_p'|^{p-2}v_p' \to w, \qquad \text{as } p \to 1,$$

uniformly on compacta of (a, b) while $-1 < w < 1$ in (θ_{i-1}, θ_i). This implies that,

$$v_p' \to 0,$$

as $p \to 1$ and also in the topology of $C(\theta_{i-1}, \theta_i)$ from which the convergence (5.75) is uniform on compacta of (θ_{i-1}, θ_i).

Thus, the proof of Theorem 15 in the Dirichlet case is complete.

B) *Robin problem.* The reasoning keeps the steps of the case A) but starting at the initial interval (a, θ_1) under the conditions of Theorem 13, and then proceeding by induction until the interval $(\theta_{n-2}, \theta_{n-1})$. Latter equality in (5.73) follows by integrating (4.29) in $\theta_{n-1} \leq r \leq$

b with values $\gamma = \operatorname{sign}(\alpha_{n-1}) = \operatorname{sign}(w(\theta_{n-1}))$ and employing the boundary condition $w(b) = -\min\{1, \beta_2\} \operatorname{sign}(\alpha_{n-1})$ at $r = b$.

C) *Neumann problem.* In this case it suffices with launching the argument in A) at the first interval (a, θ_1) (Theorem 12) and proceeding by induction until the final one (θ_{n-1}, b). □

6. The limit problem

The concepts of weak eigenvalue λ to (1.3) with an associated eigenfunction $u \in BV(\Omega)$, and subject to either Dirichlet, Neumann or Robin conditions are currently available in the literature. See for instance [26], [28] and references therein. Of course, they rely upon the 1–Laplacian theory launched in [7], [8] and [13].

When the spatial domain Ω is an annulus $\mathcal{A} = \{a < |x| < b\}$ and the search for eigenvalues is restricted to radial eigenfunctions, such notions lead to Definition 17 below which is suggested by the following considerations.

We begin with the equation in (1.3). Just like radial solutions to (1.1) are furnished by equations (3.22), corresponding ones to (1.3) are provided by their formal limit as $p \to 1$,

$$\begin{cases} w = \operatorname{sign}(v'), \\ w' = -\dfrac{N-1}{r} w - \lambda \operatorname{sign}(v), \end{cases} \qquad a < r < b. \tag{6.79}$$

Solving (6.79) must be understood as follows.

Definition 16. *A couple $(v, w) \in BV(a, b) \times W^{1,\infty}(a, b)$ is a solution to (6.79) if $\|w\|_\infty \le 1$ and there exists $\gamma \in L^\infty(a, b)$, with $\|\gamma\|_\infty \le 1$, such that,*

a) *$\gamma v = |v|$ in (a, b).*
b) *$|v'| = (w, v')$ as measures in (a, b).*
c) *w is a weak solution to the equation,*

$$-w' - \frac{N-1}{r} w = \lambda \gamma, \qquad \text{in } (a, b).$$

Remark 7. The pairing (w, v') is equivalent to wv' (Section 2.2).

Existence of solutions to (6.79) can be obtained by taking the limit as $p \to 1$ of solutions to (3.22) with fixed initial data. In fact, it is enough with imitating the argument in [26, Prop. 16] (see Remark 9 below). Since limits of solutions to (3.22) satisfy (4.34) and eigenfunctions to (1.3) are expected to be the limit as $p \to 1$ of corresponding ones to (1.1), it is natural to constraint radial solutions to (1.3) with the additional requirement,

$$\lambda \frac{d|v|}{dr} = -\frac{N-1}{r}\left|\frac{dv}{dr}\right| . \tag{6.80}$$

As for the different boundary conditions we are assuming that both w is Lipschitz continuous in $[a,b]$ and that v defines a function of bounded variation in the classical sense ([2] and Section 2.2). Of course, this could involve a modification in a null set.

Classical eigenvalue problems for the 1–Laplacian are now introduced.

Definition 17. *It is said that* $(\lambda, u) \in \mathbb{R} \times BV(\mathcal{A})$ *defines a radial eigenpair to* (1.3) *under Dirichlet conditions if there exists* $(v,w) \in BV(a,b) \times W^{1,\infty}(a,b)$*,* $v \neq 0$*, such that* $u(x) = v(r)$ *a. e. in* $\mathcal{A}$ *with* $r = |x|$*,* (v,w) *both solves* (6.79) *and fulfills* (6.80) *and satisfies the following* Dirichlet *conditions,*

$$w(a) = \operatorname{sign} v(a+), \qquad w(b) = -\operatorname{sign} v(b-).$$

A Neumann eigepair (λ, u) *is defined by changing the latter to* Neumann *conditions,*

$$w(a) = w(b) = 0.$$

Corresponding Robin eigenpairs are defined by employing instead Robin *conditions,*

$$w(a) = \min\{1,\beta_1\} \operatorname{sign} v(a+), \qquad w(b) = -\min\{1,\beta_2\} \operatorname{sign} v(b-),$$

where $\beta_i > 0$*,* $i = 1,2$*.*

Remark 8. All these boundary conditions could be formulated in a single unified form by taking Robin conditions with coefficients $\beta_i \geq 0$, $i = 1,2$, where both $\beta_i \geq 1$ correspond to Dirichlet and $\beta_1 = \beta_2 = 0$ to Neumann. Of course, a mixed combination of them can also be considered.

As a first feature, negative eigenvalues to (1.3) can be discarded.

Proposition 18. *Eigenvalues to either of the three boundary value problems* (1.3) *are positive with the sole exception of* $\lambda = 0$ *which is the first Neumann eigenvalue.*

Proof. Let (v,w) be a solution to (6.79). It follows from c),

$$-(r^{N-1}w(r))' = \lambda r^{N-1}\gamma(r) \qquad r \in (a,b)\,.$$

Multiplying by v and integrating over (a,b), we get

$$-\int_a^b (r^{N-1}w(r))'v(r)\,dr = \lambda \int_a^b r^{N-1}|v(r)|\,dr,$$

where condition a) has been used. Integrating by parts (Section 2.2) and taking into account condition b) lead to,

$$-\int_a^b (r^{N-1}w(r))'v(r)\,dr$$
$$= \int_a^b r^{N-1}|v'(r)| + b^{N-1}|w(b)||v(b)| + a^{N-1}|w(a)||v(a)|.$$

Hence,

$$\int_a^b r^{N-1}|v'(r)|\,dr + b^{N-1}|w(b)||v(b-)| + a^{N-1}|w(a)||v(a+)|$$
$$= \lambda \int_a^b r^{N-1}|v(r)|\,dr\,.$$

Since the left hand side is nonnegative, $\lambda < 0$ is excluded. If $\lambda = 0$, v must be zero excepting the Neumann problem where constants can be regarded as the only associated eigenfunctions. □

Next result allow us normalizing eigefuntions v with the condition

$$v(a+) = 1. \tag{6.81}$$

Lemma 19. *Let* (v,w) *be a non trivial solution to* (6.79) *satisfying condition* (6.80). *Then* $v(a+) \neq 0$. *In particular, this holds true for the radial eigenfunctions* $u(x) = v(|x|)$, $v \in BV(a,b)$, *to* (1.3).

Proof. By assuming that v is of bounded variation in classical sense then limit $v(a+)$ exists. In addition, function $|v|$ is non increasing due to (6.80) where we suppose $\lambda > 0$, otherwise v should be constant. Thus $|v(a+)|$ must be positive. □

Remark 9. Let us illustrate the kind of solutions that equation (6.79) may exhibit. Consider the family $(v_p, w_p) = (v_p, \varphi_p(v_p'))$, $p > 1$, of solutions to (3.22) corresponding to initial Robin type data,

$$v_p(a) = 1, \qquad w(a) = \beta_1 \varphi_p(v_p(a)),$$

with $a = 2$ and $\beta_1 = 0.5$. Dimension and frequency parameters N and λ respectively, have been both set to 2. In Figure 1 a numerical integration reveals the typical behavior of v_p evolving to a *step* function profile v as $p \to 1$. Value of p is decreased from 2 to 1.1 in steps of size $h = 0.1$. The closer to 1 is p, the steepest the slope of v_p near certain

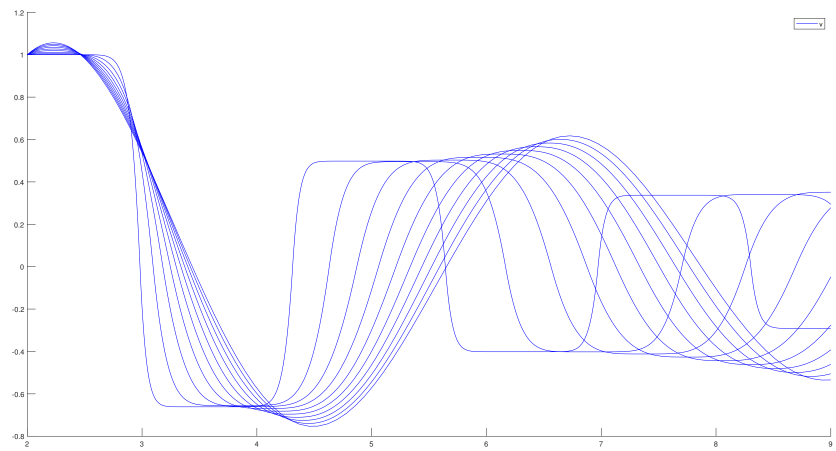

FIGURE 1. Profile of solutions v_p with p ranging from $p = 2$ to $p = 1.1$

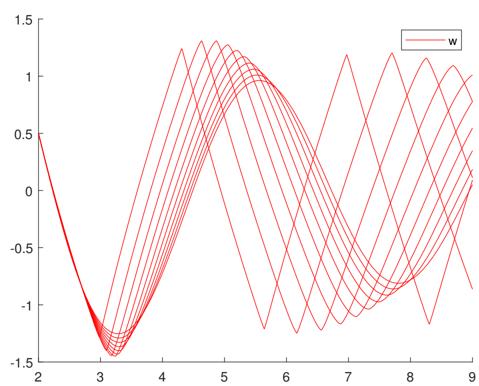

FIGURE 2. Profile of the component $w_p = \varphi_p(v_p')$ where p varies from $p = 2$ to $p = 1.1$

points becomes. The corresponding response of w_p is evolving to a more smooth sawtooth wave w (Figure 2). Observe that the singularities developed by the derivatives v_p' are balanced by means of the power $p - 1$ involved in w_p. As it is going to be shown in Theorem 20 the suggested limit pair (v, w) shows the characteristic form of a solution to (6.79) when solutions are subject to condition (6.80). See Figure 3 below.

A full description of the radial eigenpairs (λ, u) to problem (1.3) is now stated. It is warned that values θ_i, σ_i, α_i referred to as in the next statement are in principle *different* from the corresponding ones introduced in Theorem 15. However, it will soon become clear that the values coincide (Remark 10).

Theorem 20. *There exists an increasing sequence of numbers,*

$$0 \leq \lambda_1 < \cdots < \lambda_n < \cdots , \qquad \lambda_n \to \infty,$$

such that problem (1.3) *admits a radial eigenvalue λ only when $\lambda = \lambda_n$ for some $n \in \mathbb{N}$. In addition, to each λ_n there corresponds a unique radial eigenfunction $u_n(x) = v_n(r)$ whose associated pair (v_n, w_n) fulfills the normalization $v_n(a+) = 1$.*

Furthermore, the following properties are satisfied.

ia) *The first eigenvalue is,*

$$\lambda_1 = N\,\frac{b^{N-1}\min\{1,\beta_2\} + a^{N-1}\min\{1,\beta_1\}}{b^N - a^N},$$

where values β_i, $i = 1, 2$, are assumed to be nonnegative parameters.

ib) *For each $n \geq 2$ eigenvalue λ_n is characterized as the* unique *number $\lambda > 0$ such that there exist $n-1$ points θ_i,*

$$a < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < b,$$

satisfying,

$$\lambda = N\frac{\theta_i^{N-1} + \theta_{i-1}^{N-1}}{\theta_i^N - \theta_{i-1}^N}, \qquad i = 2, \ldots, n-1\,, \tag{6.82}$$

together with,

$$\lambda = N\frac{\theta_1^{N-1} + a^{N-1}\min\{1,\beta_1\}}{\theta_1^N - a^N} = N\frac{b^{N-1}\min\{1,\beta_2\} + \theta_{n-1}^{N-1}}{b^N - \theta_{n-1}^N},$$

the values of β_i, $i = 1, 2$, being given as in ia).

ii) *Function $w_n(r)$ is strictly monotone in each interval (θ_{i-1}, θ_i), $i = 2, \ldots, n-1$, oscillates between -1 and 1, fulfills $w_n(\theta_{i-1})w_n(\theta_i) = -1$ and vanishes at $r = \sigma_i \in (\theta_{i-1}, \theta_i)$ defined by,*

$$\lambda_n = \frac{N\theta_{i-1}^{N-1}}{\sigma_i^N - \theta_{i-1}^N}, \qquad i = 2, \ldots, n-1\,. \tag{6.83}$$

Function $w_n(r)$ is also strictly monotone in the intervals (a, θ_1) and (θ_{n-1}, b). Under either Dirichlet or Robin conditions it vanishes at $\sigma_1 \in (a, \theta_1)$ and $\sigma_n \in (\theta_{n-1}, b)$ satisfying,

$$\lambda_n = \frac{Na^{N-1}\min\{1,\beta_1\}}{\sigma_1^N - a^N}, \qquad \lambda_n = \frac{Nb^{N-1}\min\{1,\beta_2\}}{b^N - \sigma_n^N}\,. \tag{6.84}$$

iii) *There exist n alternating values $\{\alpha_i\}_{0\le i\le n-1}$,*

$$1 = \alpha_0 > -\alpha_1 > \alpha_2 > \cdots > (-1)^{n-1}\alpha_{n-1} > 0,$$

such that $v_n(r) = \alpha_{i-1}$ for all $r \in (\theta_{i-1}, \theta_i)$, $i = 1, \cdots, n$, where it is assumed that $\theta_0 = a$, $\theta_n = b$. They fulfill the recursive formula,

$$\alpha_i = -\frac{\lambda_n\theta_i - (N-1)}{\lambda_n\theta_i + (N-1)}\alpha_{i-1}\,, \qquad \alpha_0 = 1. \tag{6.85}$$

As a consequence, the eigenfunction v_n possesses exactly n nodal regions.

iv) *Values $w(\theta_i)$ and α_{i-1} have opposite signs for $i = 1, \ldots, n-1$ while $-\operatorname{sign}(\alpha_{n-1})w(b) = \min\{1, \beta_2\}$.*

v) *The family of eigenvalues satisfies the asymptotic estimate,*

$$\lambda_n = \frac{2(n-1) + \min\{1,\beta_1\} + \min\{1,\beta_2\}}{b-a} + o(1)\,, \tag{6.86}$$

as $n \to \infty$, where the three boundary conditions are parameterized by $\beta_i \ge 0$, $i = 1, 2$.

Remark 10. It was shown in Theorem 5 that, for every $n \in \mathbb{N}$, there exists $\bar\lambda_n = \lim_{p\to 1}\lambda_{n,p}$, where $(\lambda_{n,p}, v_{n,p})$ is the n–th normalized radial eigenpair to (1.1). Moreover, Theorem 15 describes the limits of the zeros $\theta_{i,p}$ of $v_{n,p}$ as $p \to 1$. Since $\bar\lambda_n$ and the family $\{\lim_{p\to 1}\theta_{i,p}\}_{1\le i\le n-1}$ are linked through (5.72) and (5.73), the uniqueness assertion in *ib*) of Theorem 20 permits us asserting that $\bar\lambda_n = \lambda_n$ for every n. Moreover, that limits $\lim_{p\to 1}\theta_{i,p}$ coincide with the values θ_i introduced in Theorem 20. It is also a conclusion of Theorem 15 that the limit profile $\lim_{p\to 1}(v_{n,p}, \varphi_p(v_{n,p}'))$ coincides with the normalized eigenfunction (v_n, w_n) associated to λ_n as it is described in Theorem 20.

Corollary 21. *For each $n \in \mathbb{N}$, the n–th normalized radial eigenpair (λ_n, v_n) to* (1.3) *coincides with the limit $\lim_{p\to 1}(\lambda_{n,p}, v_{n,p})$ of the n–th radial eigenpair to* (1.1)*, where the precise form of the convergence $v_{n,p} \to v_n$ is specified in Theorem 15.*

Before addressing the full proof of Theorem 20 we are first achieving some partial results. The first one shows the existence and uniqueness of the family of numbers λ_n referred to as in points *ia*)–*ib*) of the statement.

Lemma 22. *For each $n \ge 2$, there exists a* unique *positive number $\lambda = \lambda_n$ and exactly $n-1$ points θ_i,*

$$a < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < b,$$

satisfying (6.82) *together with,*

$$\lambda = N\frac{\theta_1^{N-1} + a^{N-1}\min\{1,\beta_1\}}{\theta_1^N - a^N} = N\frac{b^{N-1}\min\{1,\beta_2\} + \theta_{n-1}^{N-1}}{b^N - \theta_{n-1}^N},$$

where the three boundary conditions in Definition 17 are implicit in the different possible values of $\beta_i \geq 0$, $i = 1,2$.

Proof. We begin by analyzing the Dirichlet case. Define the auxiliary function ψ whose inverse is,

$$\psi^{-1}(t) = \frac{t^{N-1}+1}{t^N - 1}, \qquad t > 1. \tag{6.87}$$

Then $\psi \in C^1(0,\infty)$, $\psi'(s) < 0$ for all $s > 0$ and satisfies,

$$\lim_{s\to 0}\psi(s) = \infty, \qquad \lim_{s\to\infty}\psi(s) = 1. \tag{6.88}$$

Consider next the function $\tilde{\psi} : (0,\infty)\times(0,\infty) \to \mathbb{R}$ defined as, $\tilde{\psi}(s,\lambda) = s\psi\left(\frac{\lambda}{N}s\right)$. A careful checking shows that its derivative satisfies,

$$\tilde{\psi}'_s(s,\lambda) = \psi\left(\frac{\lambda}{N}s\right) + \left(\frac{\lambda}{N}s\right)\psi'\left(\frac{\lambda}{N}s\right) > 0,$$

and so $\tilde{\psi}$ is increasing with respect to s. In addition

$$\lim_{s\to 0}\tilde{\psi}(s,\lambda) = \frac{N}{\lambda}, \qquad \lim_{s\to\infty}\tilde{\psi}(s,\lambda) = \infty, \tag{6.89}$$

and,

$$\tilde{\psi}(s,\lambda) > s, \qquad \text{for all } s > 0. \tag{6.90}$$

On the other hand, the derivative with respect to λ is $\tilde{\psi}'_\lambda(s,\lambda) = \left(\frac{s^2}{N}\right)\psi'\left(\frac{\lambda}{N}s\right) < 0$, and so $\tilde{\psi}$ is decreasing in λ while,

$$\lim_{\lambda\to 0}\tilde{\psi}(s,\lambda) = \infty, \qquad \lim_{\lambda\to\infty}\tilde{\psi}(s,\lambda) = s. \tag{6.91}$$

Set now $\tilde{\psi}^{(k)}$ the k–th iterate of $\tilde{\psi}$ with respect to the first argument,

$$\tilde{\psi}^{(k+1)}(s,\lambda) = \tilde{\psi}\circ\tilde{\psi}^{(k)}(s,\lambda) = \tilde{\psi}\big(\tilde{\psi}^{(k)}(s,\lambda),\lambda\big), \qquad k\in\mathbb{N}, \tag{6.92}$$

where $\tilde{\psi}^{(1)} = \tilde{\psi}$ and "$\circ$" means composition. By computing the derivative with respect to λ it follows that $\tilde{\psi}^{(k)}$ is decreasing in λ. Moreover, a combination of (6.89) and (6.91) yields,

$$\lim_{\lambda\to 0}\tilde{\psi}^{(k)}(s,\lambda) = \infty, \qquad \lim_{\lambda\to\infty}\tilde{\psi}^{(k)}(s,\lambda) = s. \tag{6.93}$$

Therefore, for every $n\in\mathbb{N}$, we can find a *unique* solution $\lambda = \lambda_n > 0$ to the equation,

$$\tilde{\psi}^{(n)}(a,\lambda) = b. \tag{6.94}$$

Setting $\theta_0 = a$ and

$$\theta_i = \tilde{\psi}(\theta_{i-1}, \lambda_n) = \tilde{\psi}^{(i)}(\theta_0, \lambda_n), \qquad i = 1, \dots, n\,, \tag{6.95}$$

it follows that $\{\theta_i\}_{0\le i\le n}$ satisfy (6.82) and the Dirichlet conditions. In fact,

$$\theta_i = \theta_{i-1}\psi\Big(\frac{\lambda}{N}\theta_{i-1}\Big) \qquad \Leftrightarrow \qquad \frac{\lambda}{N} = \frac{\theta_i^{N-1} + \theta_{i-1}^{N-1}}{\theta_i^N - \theta_{i-1}^N}\,.$$

To deal with the Robin case, we consider the functions,

$$\psi_1^{-1}(t) = \frac{t^{N-1} + \min\{1, \beta_1\}}{t^N - 1}\,, \quad \psi_2^{-1}(t) = \frac{\min\{1, \beta_2\}t^{N-1} + 1}{t^N - 1}\,, \tag{6.96}$$

where $t > 1$. Their inverses $\psi_i \in C^1(0,\infty)$ and satisfy $\psi_i'(s) < 0$ together with (6.88), $i = 1, 2$. Corresponding functions,

$$\tilde{\psi}_i(s, \lambda) = s\psi_i\left(\frac{\lambda}{N}s\right), \qquad i = 1, 2,$$

fulfill properties (6.89) to (6.91) with the exception that in the case of $\tilde{\psi}_2$, first limit in (6.89) reads,

$$\lim_{s\to 0} \tilde{\psi}_2(s, \lambda) = \min\{1, \beta_2\}\frac{N}{\lambda}.$$

Now, instead of $\tilde{\psi}^{(n)}$ we consider the family of iterations $\tilde{\psi}_{\mathcal{R}}^{(n)}$, defined as,

$$\tilde{\psi}_{\mathcal{R}}^{(n)}(s, \lambda) = \tilde{\psi}_2 \circ \tilde{\psi}^{(n-1)} \circ \tilde{\psi}_1(s, \lambda),$$

where, for instance $\tilde{\psi}^{(n-1)} \circ \tilde{\psi}_1(s, \lambda) = \tilde{\psi}^{(n-1)}(\tilde{\psi}_1(s, \lambda), \lambda)$. For completeness we fix the special cases $\tilde{\psi}_{\mathcal{R}}^{(1)} = \tilde{\psi}_1$, $\tilde{\psi}_{\mathcal{R}}^{(2)} = \tilde{\psi}_2 \circ \tilde{\psi}_1$.

Then, the function $\tilde{\psi}_{\mathcal{R}}^{(n)}$ also verifies the limit conditions (6.93). Therefore, the unique solution $\lambda = \lambda_n$ to

$$\tilde{\psi}_{\mathcal{R}}^{(n)}(a, \lambda) = b,$$

together with the values $\theta_1 = \tilde{\psi}_1(a, \lambda_n)$, $\theta_i = \tilde{\psi}_{\mathcal{R}}^{(i)}(s, \lambda_n)$, $2 \le i \le n$ provide us the desired solution in the Robin option.

Finally, observe that the Neumann case is included in the Robin one. In fact it is enough with employing just the functions ψ_i in (6.96) where the coefficients β_i are taken zero. This finishes the proof.

□

Monotonicity of the sequence λ_n and its diverging character is the next step.

Lemma 23. *The sequence $\{\lambda_n\}$ obtained in Lemma 22 is increasing and tends to ∞.*

Proof. To show the increasing character of λ_n first consider the Dirichlet case and observe that (6.90) implies that,

$$\tilde{\psi}^{(n+1)}(s,\lambda) = \tilde{\psi}(\tilde{\psi}^{(n)}(s,\lambda),\lambda) > \tilde{\psi}^{(n)}(s,\lambda).$$

Since,

$$\tilde{\psi}^{(n)}(a,\lambda_{n+1}) < \tilde{\psi}^{(n+1)}(a,\lambda_{n+1}) = b,$$

and $\tilde{\psi}^{(n)}(a,\lambda)$ decreases in λ then the equality $\tilde{\psi}^{(n)}(a,\lambda_n) = b$ entails $\lambda_n < \lambda_{n+1}$ as desired. For both the Robin and Neumann cases it is enough with replacing $\tilde{\psi}^{(n)}$ with $\tilde{\psi}^{(n)}_{\mathcal{R}}$ (see the proof of Lemma 22).

Regarding the limit $\lambda_n \to \infty$, we take $n \geq 3$ and go back to (6.82) to get,

$$\theta_i^N - \theta_{i-1}^N = \frac{N}{\lambda_n}[\theta_i^{N-1} + \theta_{i-1}^{N-1}],$$

and deduce,

$$\theta_i - \theta_{i-1} = \frac{N}{\lambda_n} g(t), \qquad t = \frac{\theta_i}{\theta_{i-1}}, \tag{6.97}$$

for $i = 2, \ldots, n-1$, where function g is defined as,

$$g(t) = \frac{t^{N-1} + 1}{t^{N-1} + t^{N-2} + \cdots + 1}, \qquad t \geq 1\,.$$

Notice now that g is increasing and so $g(t) \geq g(1) = \frac{2}{N}$. Thus, it follows that

$$\theta_i - \theta_{i-1} \geq \frac{2}{\lambda_n}, \qquad i = 2, \ldots, n-1\,.$$

Adding all these inequalities, we deduce the estimate

$$b - a \geq \frac{2}{\lambda_n}(n-2), \tag{6.98}$$

from where the result follows. □

Our final partial achievement is assertion $v)$ of Theorem 20.

Lemma 24. *Let λ_n be the sequence of values introduced in Lemma 22. Then, the following asymptotic estimates hold true as $n \to \infty$,*

$$\lambda_n = \frac{2n}{b-a} + o(1), \qquad \text{(Dirichlet)}, \tag{6.99}$$

$$\lambda_n = \frac{2(n-1)}{b-a} + o(1), \qquad \text{(Neumann)}, \tag{6.100}$$

and,

$$\lambda_n = \frac{2(n-1) + \beta_1 + \beta_2}{b-a} + o(1), \qquad \text{(Robin)}. \tag{6.101}$$

Proof. For our proposals in this proof we are writing $\theta_i = \theta_{n,i}$ to stress the dependence of θ_i on n. By setting $\Delta\theta_{n,i} = \theta_{n,i} - \theta_{n,i-1}$ equation (6.97) is expressed as,

$$\Delta\theta_{n,i} = \frac{N}{\lambda_n} g\left(t_{n,i}\right), \qquad t_{n,i} = \frac{\theta_{n,i}}{\theta_{n,i-1}}, \qquad 2 \leq i \leq n-1.$$

Since $\lambda_n \to \infty$ then $\Delta\theta_{n,i} \to 0$, $t_{n,i} \to 1$ and $g\left(t_{n,i}\right) \to \frac{2}{N}$ as $n \to \infty$.

We now write,

$$\sum_{i=2}^{n-1} \Delta\theta_{n,i} = \frac{2(n-2)}{\lambda_n} + \frac{N}{\lambda_n} \sum_{i=2}^{n-1} \left(g\left(t_{n,i}\right) - \frac{2}{N} \right),$$

and estimate the reminder as,

$$\begin{aligned} \left| \frac{N}{\lambda_n} \sum_{i=2}^{n-1} \left(g\left(t_{n,i}\right) - \frac{2}{N} \right) \right| &\leq N \frac{(n-2)}{\lambda_n} \max \left| g\left(t_{n,i}\right) - \frac{2}{N} \right| \\ &\leq \frac{N(b-a)}{2} \max \left| g\left(t_{n,i}\right) - \frac{2}{N} \right| \to 0, \qquad \text{as } n \to \infty, \end{aligned} \tag{6.102}$$

where inequality (6.98) has been used. Thus,

$$\sum_{i=2}^{n-1} \Delta\theta_{n,i} = \frac{2(n-2)}{\lambda_n} + o(1), \qquad \text{as } n \to \infty. \tag{6.103}$$

Put now $\theta_{n,0} = a$, $\theta_{n,n} = b$. By using the same argument we find in the Dirichlet case,

$$\Delta\theta_{n,1} + \Delta\theta_{n,n} = \frac{4}{\lambda_n} + o\left(\frac{1}{\lambda_n}\right) = \frac{4}{\lambda_n} + o\left(1\right), \tag{6.104}$$

as $n \to \infty$. Such computations applied to the Robin case yield,

$$\Delta\theta_{n,1} + \Delta\theta_{n,n} = \frac{2 + \beta_1 + \beta_2}{\lambda_n} + o\left(\frac{1}{\lambda_n}\right) = \frac{2 + \beta_1 + \beta_2}{\lambda_n} + o\left(1\right). \tag{6.105}$$

The Dirichlet estimate (6.99) is obtained from,

$$b - a = \sum_{i=1}^{n} \Delta\theta_{n,i},$$

together with (6.103) and (6.104). The Robin estimate (6.101) follows by replacing the later estimate by (6.105). Finally, Neumann estimate (6.100) is just the Robin one with $\beta_i = 0$, $i = 1, 2$. □

We are now in a position to address a full proof of Theorem 20.

Proof of Theorem 20. We analyze each boundary condition separately.

A) *Dirichlet and Robin problems.* By regarding β_i, $i = 1, 2$, as *positive* parameters in the Robin condition we are handling both problems at the same time.

Let (v, w) be an arbitrary nontrivial solution to system (6.79) corresponding to a certain $\lambda > 0$ such that v satisfies condition (6.80). Then $v(a+) \neq 0$ (see Lemma 19) and v can be normalized so as $v(a+) = 1$.

Assume that (v, w) satisfies the Robin condition at $r = a$. Then $w(a) = \min\{1, \beta_1\}$ while the differential equation for w (Definition 17) and the sign of v both entail that $w(r) < \min\{1, \beta_1\} \leq 1$ near $r = a$. Keeping in mind Lemma 9, it follows that a is the endpoint of a connected component (a, θ_1) of the set $\{|w| < 1\}$ where $v(r) = \alpha_0$ with $\alpha_0 = 1$. In particular w solves,

$$w' = -\frac{N-1}{r}w - \lambda, \qquad w(a) = \min\{1, \beta_1\}, \tag{6.106}$$

and so,

$$w(r) = \frac{\lambda a^N + N a^{N-1} \min\{1, \beta_1\}}{N r^{N-1}} - \frac{\lambda}{N} r\,.$$

in this interval (a, θ_1).

Assuming now that $\theta_1 = b$, two options are possible. The first one is $w(b) \neq -\min\{1, \beta_2\}$, then the boundary condition does not hold and so λ can not be an eigenvalue. On the contrary, if $w(b) = -\min\{1, \beta_2\}$, then λ defines an eigenvalue since (v, w) with $v(r) = 1$ fulfills all the required conditions to be a solution. In this case, $w(b) = -\min\{1, \beta_2\}$ implies that,

$$\lambda_1 = N \frac{b^{N-1} \min\{1, \beta_2\} + a^{N-1} \min\{1, \beta_1\}}{b^N - a^N} > 0\,.$$

This is the expression for the first eigenvalue which coincides with the Cheeger constant of the annulus $\mathcal{A}$ as $\min\{\beta_1, \beta_2\} \geq 1$ (Dirichlet problem).

As a further remark, from the decreasing character of $w(r)$ in (a, θ_1) and $w(\theta_1) = -\min\{1, \beta_2\} < 0$, function w vanishes at a point $\sigma_1 \in (a, \theta_1)$ which verifies,

$$\lambda = \frac{N a^{N-1} \min\{1, \beta_1\}}{\sigma_1^N - a^N}\,.$$

Suppose next that $\theta_1 < b$. Then $w(\theta_1) = -1$ and so $w(\theta_1)$ and α_0 have different signs. Moreover, it holds

$$\lambda \theta_1^N - N \theta_1^{N-1} = \lambda a^N + N a^{N-1} \min\{1, \beta_1\}. \tag{6.107}$$

Since the right hand side is positive, it follows that $\theta_1 > \dfrac{N}{\lambda}$. As a consequence, we are showing that v can not jump to zero at any $\theta_1 \le r < b$. In fact, assume on the contrary that $r \ge \theta_1$ exists such that $v(r-) \neq 0$ and $v(r+) = 0$. Condition (6.80) evaluated at r implies,

$$\lambda|v(r+)| - \lambda|v(r-)| = -\frac{N-1}{r}|v(r+) - v(r-)|,$$

so that $r = \frac{N-1}{\lambda} < \theta_1$ which is not possible.

As a further implication of $\theta_1 > \dfrac{N}{\lambda}$, functions $w'(r)$ and $v(\theta_1+)$ have opposite signs for $r > \theta_1$ near θ_1. Since $w(\theta_1) = -1$ then necessarily $v(\theta_1+) < 0$ and then there must exist a component (θ_1, θ_2) of the set $\{|w| < 1\}$ where $v = \alpha_1$, a negative constant. Its value can be computed by means of (6.80) which implies

$$\lambda|\alpha_1| - \lambda|\alpha_0| = -\frac{N-1}{\theta_1}|\alpha_1 - \alpha_0|\,.$$

To continue the reasoning we consider the problem (6.106) with initial value $w(\theta_1) = -1$ and $-\lambda$ replaced by λ in the equation for w. Its solution in the interval (θ_1, θ_2) is,

$$w(r) = -\frac{\lambda\theta_1^N + N\theta_1^{N-1}}{Nr^{N-1}} + \frac{\lambda}{N}r\,.$$

Suppose $\theta_2 = b$, then λ is an eigenvalue only when $w(b) = \min\{1, \beta_2\}$. Since in this case $w(\theta_1) = -1$ also holds then,

$$\lambda = N\frac{b^{N-1}\min\{1,\beta_2\} + \theta_1^{N-1}}{b^N - \theta_1^N} = N\frac{\theta_1^{N-1} + a^{N-1}\min\{1,\beta_1\}}{\theta_1^N - a^N}\,.$$

Thus $\lambda = \lambda_2 > \lambda_1$ (Lemmas 22 and 23). Actually, $\lambda = \lambda_2$ is an eigenvalue. To see this, by setting,

$$v(r) = \begin{cases} \alpha_0, & r \in (a, \theta_1], \\ \alpha_1, & r \in (\theta_1, b)\,, \end{cases}$$

then (v, w) solves (6.79). In fact, we are checking that all the conditions of Definition 17 hold true. Function $v \in BV(a, b)$ and by hypothesis $w \in W^{1,\infty}(a, b)$ with $|w| \le 1$. Since v' is trivial except at $r = \theta_1$, the identity $|v'| = (w, v')$ only must be checked at this point. Notice that v' is negative owing to be v decreasing at $r = \theta_1$. Hence, $w(\theta_1) = -1$ implies

$$(w, v')\llcorner\theta_1 = -v'\llcorner\theta_1 = |v'|\llcorner\theta_1\,,$$

where '$\llcorner$' means restriction. On the other hand, it is enough to define

$$\gamma(r) = \begin{cases} 1, & r \in (a, \theta_1), \\ -1, & r \in (\theta_1, b), \end{cases}$$

to get $\gamma v = |v|$ and that equation $-w' - \frac{N-1}{r} w = \lambda \gamma$ holds. Regarding the boundary conditions, $w(a) = \min\{1, \beta_1\} \operatorname{sign}(\alpha_0) = \min\{1, \beta_1\}$ and $w(b) = -\min\{1, \beta_2\} \operatorname{sign}(\alpha_1) = \min\{1, \beta_2\}$. Finally, condition (6.80) must only be checked at θ_1 and at this point it follows from the definition of α_1.

Observe that w vanishes at the point $r = \sigma_2 \in (\theta_1, \theta_2)$ given by the identity,

$$\lambda = \frac{N\theta_1^{N-1}}{\sigma_2^N - \theta_1^N}.$$

Assume next that $\theta_2 < b$. Our previous argument can be recursively applied. A sequence of components (θ_i, θ_{i+1}) of the set $\{|w| < 1\}$ is successively adjoined to (θ_1, θ_2) in the interval (a, b). Function $v = \alpha_i$, $\alpha_i \neq 0$, in every interval (θ_i, θ_{i+1}) with $\alpha_i w(\theta_{i+1}) < 0$, $w(\theta_i)w(\theta_{i+1}) = -1$ whenever $\theta_{i+1} < b$. On the other hand, w is a Lipschitz function from the start with a constant L provided by the estimate,

$$|w'| \leq \frac{N-1}{r} + \lambda < \frac{N-1}{a} + \lambda = L.$$

Thus,

$$2 = |w(\theta_{i+1}) - w(\theta_i)| \leq L(\theta_{i+1} - \theta_i),$$

which implies $\theta_{i+1} - \theta_i \geq 2/L$. Therefore, our recursive process ends at some $\theta_n = b$ after a finite number of steps and we face a dichotomy. Either,

$$w(b) \neq -\min\{1, \beta_2\} \operatorname{sign}(v(b-)),$$

and λ is not an eigenvalue, or either $w(b) = -\min\{1, \beta_2\} \operatorname{sign}(v(b-))$. This latter case entails the existence of $n-1$ values $a < \theta_1 < \cdots < \theta_{n-1} < b = \theta_n$ such that,

$$\lambda = N\frac{b^{N-1}\min\{1, \beta_2\} + \theta_{n-1}^{N-1}}{b^N - \theta_{n-1}^N} = N\frac{\theta_1^{N-1} + a^{N-1}\min\{1, \beta_1\}}{\theta_1^N - a^N}$$
$$= N\frac{\theta_i^{N-1} + \theta_{i-1}^{N-1}}{\theta_i^N - \theta_{i-1}^N}, \qquad 2 \leq i \leq n-1, \tag{6.108}$$

where the fact that $\alpha_{n-1} w(\theta_{n-1}) > 0$ has been employed to manage the boundary condition. According to Lemma 22, $\lambda = \lambda_n$ and the couple

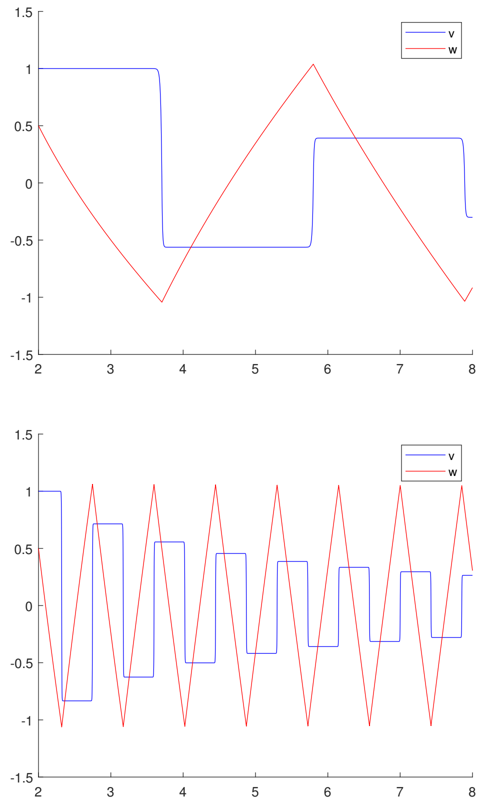

FIGURE 3. Profiles of the solution (v_p, w_p) corresponding to $p = 1.01$ and frequency values $\lambda = 1$ and $\lambda = 5$.

(v, w) provides us an associated Robin eigenfunction by setting,

$$v = \sum_{i=1}^{n} \alpha_{i-1} \chi_{(\theta_{i-1}, \theta_i]}, \qquad 1 \leq i \leq n,$$

where $\chi_{(\theta_{i-1}, \theta_i]}$ stands for the characteristic function of the interval $(\theta_{i-1}, \theta_i]$. In fact, a careful checking in the line of the previous arguments shows that (v, w) is an eigenfunction to λ_n and that assertions in points *ii*) to *iv*) of the statement of Theorem 20 are satisfied. We omite the details.

B) *Neumann problem.* As pointed out in Proposition 18, $\lambda = 0$ is the first eigenvalue whose normalized eigenfunction is $v = 1$. The analysis of the other positive eigenvalues λ is performed as in the previous cases by setting values $\beta_i = 0$, $i = 1, 2$ in the argument. Actually, this only affects to the initial and final step of the iterative process.

This completes the proof. □

Remark 11. It is worth noting that parameter λ somehow plays the role of a "frequency" in equation (6.79). The higher its value is, the greater the number of oscillations exhibited in the interval (a, b) by both components of a solution (v, w) which satisfies condition (6.80). Figure 3 shows a numerical simulation of the solution (v_p, w_p) of (3.22) with the initial conditions and values for N and β_1 chosen in Remark 9. Value of p has been taken as 1.01 while those for λ were set 1 and 5 respectively.

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José Sabina de Lis
Departamento de Análisis Matemático and IUEA,
Universidad de La Laguna,
P. O. Box 456, 38200 – La Laguna, SPAIN.
Email address: `josabina@ull.edu.es`

Sergio Segura de León
Departament d'Anàlisi Matemàtica, Universitat de València,
Dr. Moliner 50, 46100 Burjassot, València, SPAIN.
Email address: `sergio.segura@uv.es`