History of Algebraic Ideas and Research on Educational Algebra

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0. INTRODUCTION

The history of algebraic ideas is nowadays frequently used on research on educational algebra. As we have stated in Puig and Rojano (2004), there is no need to further discuss the necessity or usefulness of studying the history of mathematics for mathematics education. ICMI established a group more than 20 years ago to study the relationship between the history and pedagogy of mathematics, and during the last few years important publications have appeared that summarise the work done outside and inside this group, especially the ICMI Study *History in Mathematics Education* (Fauvel & van Maanen, 2000), but also Katz (2000), and Jahnke, Knoche, and Otte (1996).

What we are going to present here is a way of using history in educational mathematics pioneered by Eugenio Filloy some 25 years ago[1]. I have been working with him since some 15 years[2], so my work owes him the starting idea and is tangled with his. All good ideas comes from him, the mistakes are my own responsibility.

1. FEATURES OF OUR USE OF HISTORY

Our use of history has two fundamental features. On the one hand, it is concerned with an analysis of algebraic ideas. As a result, there is very little interest for us in, for example, questions of dating or of priority in developing the concepts of algebra. What we are basically interested in is identifying the algebraic ideas that are brought into play in a specific text and the evolution of those ideas, which can be seen by comparing texts; in this context we can consider historical texts as cognitions and analyse them as we analyse the performance of pupils, whose output also constitutes mathematical texts.

The second feature creates a close bond between historical research and research into educational mathematics, which allows us to state that our historical research belongs to research into educational mathematics, and is characterised by a two-way movement between historical texts and school systems (Filloy and Rojano, 1984; Puig, 1994):

1) *From educational algebra to history of algebraic ideas*. The problematics of the teaching and learning of algebra is what determines which texts must be sought out in history and what questions should be addressed to them.
2) *From history of algebraic ideas to educational algebra.* The examination of historical texts leads to (a) considering new items that have to form part of what we call the Competence Model, that is, a description of the competent behaviour considered as the behaviour of the ideal or epistemic subjet, (b) having new ways of understanding the performance of pupils and, therefore, of developing what we called the Model of Cognition, and, lastly, (c) developing Teaching Models. With all these things teaching experiments are organised and the performance of the pupils is analysed.
3) *Back to history*. Attention is redirected to the historical texts in order to question them once again, now using the results obtained with the pupils, that is, the results

derived from the performance of the pupils when all that has been extracted from the analysis of algebraic ideas is incorporated into the teaching model and into the analysis of the teaching and learning processes.

4) *And so on, repeatedly.*

2. SOME PROBLEMS FROM WHICH IT IS WORTH TO GO TO HISTORY

We will start by indicating some of the directions in which we are turning to history as a result of issues present in the current problematics of research in educational algebra.

The use of spreadsheets to teach how to solve problems which pupils were traditionally taught to solve by using some version of the Cartesian method[3], whether with the aim of serving as an intermediary for the teaching of the Cartesian method or with a view to replacing it with a new method, poses the question of the ways of naming unknown quantities.

In fact, in a spreadsheet one can refer to unknown quantities by assigning one or more cells for one or more unknown quantities, like the way in which one assigns one or more letters to one or more unknown quantities in the second step of the Cartesian method, and it is advisable to place the cell beneath a cell in which the name of the quantity is written in the vernacular, either complete or else abbreviated in some way. The relationships between the quantities can then be represented as operations expressed in spreadsheet language. In that language the cells are designated by a matrix code consisting of a letter and a number, e.g. B3, which indicates the column and row of the cell, and the cell name then becomes the name of the corresponding quantity in the spreadsheet language. The cell name can be written explicitly using the keyboard or generated by clicking on it with the mouse.

When the spreadsheet is used to solve word problems its language has considerably more complexity than what we have just described, but this brief outline shows that the way in which the unknown is named in this language is different from that of the language of current school algebra. This has consequences for the use of the spreadsheet in teaching how to solve problems and in the performance of pupils when they are taught in this way. It is therefore worth turning now to historical texts in which the language of current school algebra had not yet been developed, in order to see how unknown quantities are named in them and what effects the way of naming them has on the way of representing the relationships between quantities which are translated from the statements of the problems and the relationships generated in the course of the calculations. I will say something on this issue afterwards.

Therefore, both in the case of the use of the spreadsheet and in that of the teaching of solving problems by using the Cartesian method, at some point it is necessary to teach the use of more than one letter (or cell) to represent unknown quantities, together with the corresponding way of handling expressions in the corresponding languages. We have pointed out in Puig and Rojano (2004) that most of the languages of algebra prior to Vieta were incapable of this, and yet they solved problems which we would now naturally represent by using more than one letter. It would be interesting, therefore, to turn to the historical texts once again in order to examine the ways in which this was done. I will also say something on this issue afterwards.

Finally, the use of graphic calculators, with the possibility of collecting data with CBR [Calculator Based Laboratory] or CBL [Calculator Based Ranger] sensors, to teach the idea of a family of functions as a means for organising phenomena through the modelling of real situations poses the problem of the establishment of canonical forms in which the parameters express properties of each family of functions and, therefore, of the situations that are modelled. Bound up with this is the development of a calculation, that is, a set of algebraic transformations, which makes it possible to reduce the expressions obtained by modelling the real situation to one of the canonical forms. The history of the idea of canonical form (and that of calculation or algebraic transformations which is bound up with it) thus acquires a new perspective which is worth studying, in addition to its relationship with the use of the Cartesian Method that we are going to analyse in what follows.

3. THE CARTESIAN METHOD AS A PARADIGM OF ALGEBRAIC PROBLEM SOLVING

What lies at the heart of algebraic problem solving is the expression of problems in the language of algebra by means of equations. Stacey and MacGregor (2001) have pointed out that a major reason for the difficulty that students have in using algebraic methods for solving problems is not understanding its basic logic—that is, the logic that underlies the Cartesian Method. There is a students' compulsion to calculate, based on their prior experiences with arithmetic problem solving. This tendency to operate backwards rather than forwards (Kieran, 1992) prevents students from finding sense in the actions of analysing the statement of the problem and translating it into equations which express, in algebraic language, the relations among quantities; actions of analysing and translating that are the main features of the Cartesian Method.

Teaching models that take into account these tendencies have been developed, and studied by Filloy, Rojano, and Rubio (2001). They state that in order to give sense to the Cartesian Method users should recognise the algebraic expressions, used in the solution of the problem, as expressions that involve unknowns. Competent use of expressions with unknowns is achieved when it makes sense to perform operations between the unknown and the data of the problem. In steps prior to competent use of the Cartesian Method, the pragmatics of the more concrete sign systems leads to using the letters as variables, passing through a stage in which the letters are only used as names and representations of generalised numbers, and a subsequent stage in which they are only used for representing what is unknown in the problem. These last two stages, both clearly distinct, are predecessors of the use of letters as unknowns and using algebraic expressions as relations between magnitudes, in particular as functional relations.

Furthermore, competent use of the Cartesian Method is linked with the creation of families of problems that are represented in the mathematical sign system (MSS) of algebra as canonical forms. This implies an evolution of the use of symbolisation in which, finally, the competent user can give meaning to a symbolic representation of the problem that arises from the particular concrete examples given in teaching. Students will make sense of the Cartesian Method when they become finally aware that by applying it they can solve families of problems, defined by the same scheme of solution. The integrated conception of the method needs the confidence of the user that the general application of its steps will necessarily lead to the solution of these families of problems.

We are going to present an analysis of algebraic problem solving in history bearing in mind this results from research on its teaching and learning. We will have to examine the characteristics of the Cartesian Method, and the search for canonical forms that represent families of problems and its methods of resolution.

First of all we will examine the formulation of the method as proposed by Descartes. Indeed, the reason for calling this method *Cartesian* is that one part of Descartes' *Regulæ ad directionem ingenii (Rules for the direction of the mind)*[4] can be interpreted as an examination of the nature of the work of translating the verbal statement of an arithmetic-algebraic problem into the mathematical sign system (MSS) of algebra and its solution in that MSS. This is how it was understood by Polya, who, in the chapter "The Cartesian pattern" in his book *Mathematical Discovery*, rewrote the pertinent Cartesian rules in such a way that they could be seen as problem solving principles that use the MSS of algebra. Polya's paraphrase of Descartes's rules is as follows:

(1) First, having well understood the problem, reduce it to the determination of certain unknown quantities (Rules XIII–XVI).[5]
[…]
(2) Survey the problem in the most natural way, taking it as solved and visualizing in suitable order all the relations that must hold between the unknowns and the data according to the condition (Rule XVII).[6]
[…]
(3) Detach a part of the condition according to which you can express the same quantity in two different ways and so obtain an equation between the unknowns. Eventually you should split the condition into as many parts, and so obtain a system of as many equations, as there are unknowns (Rule XIX).[7]
[…]
(4) Reduce the system of equations to one equation (Rule XXI).[8] (Polya, 1966, pp. 27-28)

Broken down into ideal steps, that is, those that would be taken by a competent user (the ideal subject), the first step of the Cartesian Method consists in an analytical reading of the statement of the problem, which reduces it to a list of quantities and relations between quantities.

The second step consists in choosing a quantity that will be represented by a letter (or several quantities that will be represented by different letters), and the third step consists in representing other quantities by means of algebraic expressions that describe the (arithmetic) relation that these quantities have with others that have already been represented by a letter or an algebraic expression. With the MSS of current school algebra this is done by maintaining the representation of each quantity by a different letter and taking care that each letter should represent a different quantity and combining the letters with signs for operations and with delimiters, while also observing certain rules of syntax that express the order in which the operations represented in the expression are performed. Descartes (1701) indicates that one makes an abstraction of the fact that some terms are known and others unknown. Treating known and unknown in the same way is precisely one of the fundamental features of the method's algebraic character, and Descartes himself pointed out that the basic nature of his method consisted in this[9].

The fourth step consists in establishing an equation (or as many equations as the number of different letters that it was decided to introduce in the second step) by equating two of the expressions written in the third step that represent the same quantity. In Descartes' rule nineteenth what gives meaning to the construction of the equation is the expression of a quantity in two different ways[10].

This concludes the part of the method described in the *Regulæ* that corresponds to the translation of the statement of the problem into the MSS of algebra. The continuation of the method, which describes the solving of the equation, must be sought in the *Geometry*[11] that Descartes published as an Appendix to the *Discourse on Method*, which is where he actually develops what he himself calls "his algebra".

In fact, in his *Geometry*, Descartes explains the method in a section called "Comment il faut venir aux Equations qui servent a resoudre les problemes", in which he also emphasises the similar treatment of known and unknown, and the writing of an equation based on the expression of a quantity in two different ways, "until we find it possible to express a single quantity in two ways. This will constitute an equation, since the terms of one of these two expressions are together equal to the terms of the other" (Descartes, 1925, p. 300). However, Descartes goes on from what can be found in the *Regulæ* with the development of the method, explaining that once all the equations have been constructed the equations must be transformed.

Here, Descartes does not expound the rules for the transformation of algebraic expressions. He assumes that they are known, but what he does say is the form that the canonical equation must have, indicating that the transformations must be done in such a way as eventually to obtain an special kind of equation:

so as to obtain a value for each of the unknown lines; and so we must combine them until there remains a single unknown line which is equal to some known line, or whose square, cube, fourth power [lit. square of square], fifth power [sursolide], sixth power [lit. square of cube], etc., is equal to the sum or difference of two or more quantities, one of which is known, while the others consist of mean proportionals between unity and this square, or cube, or fourth power [lit. square of square], etc., multiplied by other known lines. I may express this as follows:

$$\begin{array}{rl} & z = b \\ \text{or} & z^2 = -az + bb, \\ \text{or} & z^3 = +az^2 + bbz - c^3, \\ \text{or} & z^4 = az^3 - c^3z + d^4, \text{ et cetera.} \end{array}$$

That is, z, which I take for the unknown quantity, is equal to b; or, the square of z is equal to the square of b diminished by a multiplied by z … (Descartes, 1925, p. 9; in square brackets we have added the names that Descartes uses for the species and that Smith does not retain in his translation).

Thus the method continues by transforming the written algebraic expressions and the resulting equations in order to reduce them to a canonical form. This implies that it has previously been determined which expressions and which equations will be considered canonical, and that one has a catalogue of all the possible canonical forms and procedures for solving each of them.

We have just shown the ones that Descartes presents specifically, but we could say that they all come down to a single canonical form, which Descartes presents broken down by degrees, for the form is the same in all cases. The breakdown by degrees is justified by the fact that the solving procedure is different for each degree (or else does not exist, depending on the degree). The form of the canonical equation, written in the most general form, is:

$$x^n = a_{n-1}x^{n-1} \pm a_{n-2}x^{n-2} \pm \dots \pm a_2x^2 \pm a_1x \pm a_0$$

Thus Descartes makes the power of the highest degree without a coefficient (so that there is only one unknown and no known quantity on the left side of the equation) equal to the rest of the polynomial. As there are still unknown quantities (the other powers of the unknown) in the rest of the polynomial, he says that this is a known quantity (the monomial of degree zero) and quantities “consisting of certain proportional means between unity and this square or cube, etc.”.

Thus the algebraic expressions that are considered canonical are the polynomials, but not exactly our current polynomials. This is so because the reiteration of the four elementary arithmetic operations, when these operations are performed on unknown quantities, leads to a situation in which all the multiplications (and divisions) produce a quantity *multiplied by itself a certain number of times* and multiplied by a specific number, that is, they produce a monomial, and the reiteration of additions (and subtractions), which can only be performed between monomials of the same degree (and this fact is crucial), produces an addition (and subtraction) of monomials.

Polynomials are, in fact, the conclusion in this history of all the forms that have been considered canonical at one time or another, but first it was necessary that the idea of the search for canonical forms should appear. For this idea to be able to appear it is necessary that problem solving should not be considered with the sole aim of obtaining the result of the specific problem in question, but that the solving process should include, to use Polya's terminology, a fourth "looking back" phase with an epistemic character (Puig, 1996), in which the solving procedure is analysed and problems are generated that can be solved with the same procedure or with variants or generalisations of that solving procedure. But it is also necessary that one should have an MSS in which the analysis of the solution can be made by relinquishing the specific numbers with which the calculations are performed. This requires that in some way one should be able to represent the numbers with which one calculates and the calculations that are performed with them as expressions.

The idea of searching for canonical forms then appears because of the need to reduce the number of expressions (equations) that are produced as a result of the translation of problems into equations that one already knows how to solve.

This idea of reducing to equations that one already knows how to solve leads to two projects as well as the project of identifying what will be called a canonical form: on the one hand, having a catalogue of the equations that one already knows how to solve, and, on the other, developing a calculus with expressions that enables one to transform any equation into one that can be solved.

This project takes a shape that for us is increasingly algebraic when the catalogue of expressions ceases to be constituted by accumulating solved problems, the corresponding expressions, and the techniques and procedures (or algorithms) for solving each of them, and ends up as a catalogue of all possible canonical forms.

Babylonian algebra does not satisfy this criterion, even though 1) there are catalogues of techniques and of problems that they can solve; 2) they use the sumerograms that signify "long" and "wide" to represent quantities that have nothing to do with geometrical figures; 3) the solving procedures are analytic; and 4) configurations are reduced to others that they know how to solve (cf. Høyrup, 2002). But Diophantus' *Arithmetic* also does not satisfy it.

However, the search for all possible canonical forms requires, on the one hand, the availability of an MSS in which expressions are represented precisely enough to make it possible to carry out a search for possibilities. On the other hand, it modifies the project of constructing a catalogue of what one already knows how to solve and converts it into a project of knowing how to solve *all* the canonical forms. This does not mean that the MSS has to be "symbolic" in the sense of the distinction between "rhetorical", "syncopated" and "symbolic" made by Nesselmann (1842). This is testified by the fact that in the *Concise book of the calculation of al-jabr and al-muqâbala*, al-Khwârizmî[12] establishes such a catalogue of canonical forms in an MSS that consists only of natural language, in this case Arabic, and various geometrical figures, which are inserted in the text as representations (*sûra*, "figure", but also "representation" or even "photograph"), always preceded by the words "this is the representation" or "this is the figure".

3. AL-KHWÂRIZMÎ'S ALGEBRA

What al-Khwârizmî did in the *Concise book of the calculation of al-jabr and al-muqâbala*, and what separates him from all the previous works that have been seen as algebra after him, and in this sense al-Khwârizmî's book is the beginning of algebra, was that he began by establishing "all the types or species of numbers that are required for calculations".

The context in which he seems to have examined those species is that of the exchange of money in trading or inheritances, and from it he takes the names that he uses for the species of numbers. The world of commercial problems and inheritances is linear or quadratic: in the course of the calculations there are numbers that are multiplied by themselves, in which case they are "roots" of other numbers, and the numbers that result from multiplying a number by itself are *mâl*, literally "possession" or "treasure"; other numbers are not multiplied by themselves and are not the result of multiplying a number by itself, and therefore they are neither roots nor treasures, they are "simple numbers" or *dirhams* (the monetary unit). Treasures, roots and simple numbers are thus the species of numbers that al-Khwârizmî considers.

In his *Arithmetic*[13], Diophantus had already distinguished different species (*eidei*) of numbers, with a different conceptualisation (ways in which a number may have been given), using the names "monas (units)", "arithmos", "dynamis", "cubos", "dynamodynamis", "dynamocubos", etc., and thus a longer series than al-Khwârizmî's.

Calculating with al-Khwârizmî's or Diophantus's species of numbers follows similar rules: what is obtained is always an expression equivalent to our polynomials or rational expressions, as the numbers of the same species are added together, or are taken that many times, or that many parts are taken, and the result is a certain number of times or a certain number of parts of a number of that species; and if numbers of different species are added, the sum cannot be performed and is simply indicated. Thus,

اربعة اتساع مال وتسعة دراهم الا اربعة اجذار يعدل جذرا

"four ninths of treasure and nine dirhams minus four roots, equal to one root" (Rosen, 1831, p. 41 of the text in Arabic) is an algebraic equation in al-Khwârizmî's book, since al-Khwârizmî's MSS uses vernacular language (Arabic in his case) exclusively; and

$\Delta^{\Upsilon}\overline{\beta}\,\overset{o}{M}\overline{\sigma}\ \iota\sigma\alpha\ \overset{o}{M}\overline{\sigma\eta}$ (Tannery, 1893, vol. I, p. 64, l. 7)

is an equation in Diophantus's MSS, which is read as "dynamis 2 monas (units) 200 equals monas (units) 208", since Diophantus uses abbreviations for the names of the species of numbers, which in this case consist of the first two letters of the word, and the Greek system of numeration uses the letters of the alphabet marked with a horizontal stroke, in a system which is not positional but additive, with codes for the nine units, the nine tens and the nine hundreds. There is almost no conceptual difference between the algebraic expressions and the equations of the two authors, as what is represented in them is the names of the species, the specific numbers that indicate how many of each species there are, the operations between the quantities of each species, and the relationship of equality between quantities.

Al-Khwârizmî's book might thus be seen as more elementary or situated one step behind Diophantus, as the set of species of numbers is smaller and the expression uses only the signs of the vernacular. However, what is new in al-Khwârizmî's book is that it suggests having a complete set of possibilities of combinations of the different kinds of numbers. It is clear that initially the possibilities are infinite, and that therefore it is necessary to reduce them to canonical forms in order to be able to consider the obtaining of a complete set. But al-Khwârizmî's aim then is also to find an algorithmic rule that makes it possible to solve each of the canonical forms, and to establish a set of operations of calculation with the expressions, which makes it possible to reduce any equation consisting of those species of numbers to one of the canonical forms. All the possible equations would then be soluble in his calculation. Moreover, al-Khwârizmî also establishes a method for translating any (quadratic) problem into an equation expressed in terms of those species, so that all quadratic problems would then be soluble in his calculation.

Al-Khwârizmî obtains the set of canonical forms by combining all the possible forms of the three species, taken two at a time and taken three at a time. He thus obtains the three forms which he calls “simple”, making the species equal two at a time:

treasure equal to roots
treasure equal to numbers
roots equal to numbers

and the three forms which he calls “compound”, adding two of them without taking order into account and making them equal to the third:

treasure and roots equal to numbers
treasure and numbers equal to roots
roots and numbers equal to treasure.

As al-Khwârizmî is able to present an algorithm to solve each of these canonical forms simply by collecting and justifying methods which are established and which have been in use since the time of the Babylonians, all that remains is to establish a procedure for translating the statements of the problems into their algebraic expressions and a calculation that makes it possible to transform any equation into one of the canonical forms.

The species of numbers refer to concrete numbers with which calculations are performed, so that in order to be able to translate the statements of the problems into those algebraic expressions it is necessary to be able to refer also to unknown quantities as if they were concrete numbers and calculate with them, that is, it is necessary to name the unknown and treat it like a known number. What al-Khwârizmî does to achieve this is to use the word *shay’*, literally “thing”, to name an unknown quantity. He then uses it to perform the calculations which the analysis of the quantities and relationships present in the problem indicates to him as being necessary, and in the course of the calculations he sees what species of number that thing is: a root if it is multiplied by itself, or a treasure if it is the result of a quantity that has been multiplied by itself, so that he can translate the statement of the problem into two expressions which represent the same quantity and make them equal so as to have an equation. These are in fact the steps of the Cartesian method.

“Thing”, incidentally, is a common name for representing any unknown quantity, not the proper name of a specific unknown quantity, unlike what is established by the Cartesian method; in fact, al-Khwârizmî does not say “the thing” but “thing”, that is, “a thing”, when he refers to the unknown quantity which he calls “thing”. In the course of the construction of the equation which translates the problem, however, “thing” is bound to one of the unknown quantities, functioning as the proper name of that quantity.

The operations in the calculation are algebraic transformations of the equations which seek to obtain one of the canonical forms. However, the canonical forms have three features which characterise them (and which cause the complete set of canonical forms to have 6 items), and the operations are directed at achieving each of those three features.

The first is that there are no negative terms, or, to use al-Khwârizmî’s terminology, there is nothing “that is lacking” on either of the two sides of the equation.

In fact, in al-Khwârizmî’s or Diophantus’s algebraic expressions there are quantities that are being subtracted from other quantities. There are not positive and negative quantities, but quantities that are being added to others (additive quantities) and quantities that are being subtracted from others, and the latter cannot be conceived on their own but only as being subtracted from others. Thus, al-Khwârizmî may even go so far as to speak of “minus thing” when he is explaining the sign rules, but he is always referring to a situation in which that thing is being subtracted from something:

When you say ten minus thing by ten and thing, you say ten by ten, a hundred, and minus thing by ten, ten “subtractive” things, and thing by ten, ten “additive” things, and minus thing by thing, “subtractive” treasure; therefore, the product is a hundred dirhams minus one treasure. (Rosen, 1881, p. 17 of the text in Arabic)

However, as the subtractive quantities are conceived as something that has been subtracted from something, an expression in which there is a subtractive quantity represents a quantity with a defect, a quantity in which something is lacking. Diophantus’s sign system expresses this way of conceiving the subtractive in an especially explicit way, as in his sign system all the additive quantities are written together, juxtaposed in a sequence one after another, and all the subtractive quantities are written afterwards, also juxtaposed, preceded by the word *leipsis* (what is lacking). Thus, the algebraic expression

$$x^3 - 3x^2 + 3x - 1$$

is written as

$$\mathrm{K}^{\Upsilon}\overline{\alpha}\,\varsigma\,\overline{\gamma}\,\Lambda\,\Delta^{\Upsilon}\overline{\gamma}\,\overset{\mathrm{o}}{\mathrm{M}}\,\overline{\alpha}$$ (Tannery, 1893, vol. I, p. 434, l. 10),

an abbreviation of “cubos 1 arithmos 3 what is lacking dynamis 3 monas (units) 1”, in which the expressions corresponding to x^3 and $3x$ are juxtaposed on one side, and x^2 and 1 on the other, separated by the abbreviation for “what is lacking”.

It is precisely this idea that there is something lacking in the quantity that is directly responsible for the form adopted by the operation which eventually gave its name to algebra. In fact, the objective of the operation which al-Khwârizmî calls *al-jabr* is that nothing should be “lacking” on either side of the equation. That is why the operation is called *al-jabr*, literally “restoration”, because it restores what is lacking. In terms of the language of current algebra, *al-jabr* eliminates the negative terms in an equation by adding them to the other side, but *al-jabr* is not equivalent to the transposition of terms because the modern transposition of terms can also transfer a positive term to the other side by making it negative, which goes against the intention of the *al-jabr* operation (but is consistent with the fact that the canonical form that one now seeks to attain with algebraic transformations is $ax^2 + bx + c = 0$, with a, b and c being real numbers, and not al-Khwârizmî’s canonical forms).

The second characteristic feature of al-Khwârizmî’s canonical forms is that each species of number appears only once. The algebraic transformation that this pursues is *al-muqâbala*, literally “opposition”. As al-Khwârizmî always performs this operation after *al-jabr*, at this point there is nothing lacking, there are no negative terms in the equation. The operation consists in compensating for the number of times that a given species of number appears on each side of the equation, leaving the difference on the appropriate side.

Lastly, the third characteristic is that there is only one treasure, or, in modern terms, that the coefficient of the treasure is 1. This is achieved by means of two operations which al-Khwârizmî calls “reduction” (*radd*) and “completion” (*ikmâl* or *takmîl*). “Reduction” is used when the coefficient of the treasure is greater than one, and it consists in dividing the complete equation by the coefficient; and “completion” is used when the coefficient of the treasure is less than one (it is “part of a treasure”, in al-Khwârizmî’s words), and it consists in multiplying the complete equation by the inverse of the coefficient.

The first two operations, *al-jabr* and *al-muqâbala*, appear in the title of al-Khwârizmî’s book as the characteristic operations of the calculation, and they are also mentioned, although not by name, in the introduction to Diophantus's *Arithmetic* (Tannery, 1893, vol. I, p. 14, ll. 16-20).

What makes all these calculations meaningful, therefore, is the idea of the establishment of a complete set of canonical forms, which then organises algebraic expressions through transformations, and it organises problems into families of problems that are solved in the same way.

Al-Khwârizmî's complete set of canonical forms was complete only with the condition of restricting the species of numbers to the three that he considered. The continuation, including the cube as the fourth species, was developed by ᶜUmar al-Khayyâm[14], who established that the complete set of canonical forms had 25 items, but that he could not find an algorithm for solving the 25. What al-Khayyâm did as a result of his inability to give a strictly algebraic solution for the matter was to show how the solution of the canonical forms could be constructed in the cases that resisted him by means of intersecting conical sections. As a response to the same inability, Sharaf al-Dîn al-Tûsî added to this the establishment of procedures for the approximate calculation of roots[15]. For our present purpose, these non-algebraic responses to the lack of ability to find algorithms for all the canonical forms do not interest us. Nor are we interested in the fact that eventually algorithms were found not only for al-Khayyâm's 25 canonical forms but also for fourth-degree equations; and that it was proved by Lagrange, Abel and Galois the impossibility of finding such an algorithm for higher-degree equations.

4. A COMPONENT OF THE HISTORY OF SYMBOLISATION: THE REPRESENTATION OF UNKNOWN QUANTITIES AND OF SPECIES OF NUMBERS

In Puig and Rojano (2004), there is an analysis of how the central core of the evolution of the language of algebra has to do with the way in which unknown quantities, on the one hand, and species of numbers, on the other, are represented in algebraic expressions and therefore in equations.

As on al-Khwârizmî's sign system, in most of the sign systems of mediaeval algebra there is only one name to represent the unknown, "thing", which is in fact a common name although used as a proper name. Consequently, those MSSs cannot represent different unknown quantities with different proper names. Instead, once an unknown quantity has been named as "a thing", the others have to be named with compound names constructed more or less algorithmically from the relationships between it and each new unknown quantity (for example, "ten minus thing" is the name that one could give to an unknown quantity of which it is known that when it is added to "thing" the result is ten). However, the network of relationships between the quantities in the problem might be so complex that it is extremely intricate, or even impossible, to name all the quantities with compound names: for these problems, the fact that only the term "thing" is available makes the sign system not very efficient.

Mediaeval algebraists resorted to various devices to get round this. Sometimes they used the term "thing" again, but with a qualifier. This is the case with Abû Kâmil, who in one problem in his book of algebra (cf. Levey, ed. 1966, pp. 142-144) uses the names "large thing" and "small thing" ("res magna" and "res parva" in the Latin version edited by Sesiano, 1993, p. 388). Sometimes they used names of coins for the other unknown quantities. This is also the case with Abû Kâmil, who uses *dînâr* and *obolos* (cf. Levey, ed. 1966, p. 133, n. 140, although on this occasion Abû Kâmil is expounding a different solution for a problem that has already been solved using "thing" on its own), or with Leonardo de Pisa, who uses *denaro*, as well as *res* (cf. Boncompagni, 1847, pp. 435-436 and p. 455). In the part devoted to inheritances in al-Khwârizmî's book, at one point he does not even use the term “thing” but calls the inheritance *mâl*, treasure, using it in its vernacular sense, and he calls what corresponds to each of the heirs “share” or “part share”, and he constructs the indeterminate linear equation “five shares and two parts of eleven of share equal to the treasure”. According to Anbouba (1978), in the same part of al-Khwârizmî’s book there is also a problem in which he constructs a linear system of two equations using “thing” and “part of thing” to name two different unknown quantities.[16]

Moreover, what appears in the algebraic expressions is the names of the species of numbers (simple number or *dirham*, root, treasure, cube, etc.; or, in the translation into Latin, *numerus*, *radix*, *census*[17], *cubus*, etc.), but the quantity which is qualified with this species is not named. From the identification of “thing” with “root” it is assumed that the treasure is the thing multiplied by itself, but there is no way of expressing another quantity represented with another proper name which has been multiplied by itself. The algebraic expressions of these sign systems do not say “five treasures of thing” but just “five treasures”, unlike the sign system of current algebra, which uses $5x^2$ to say “five times the square of x”, and, therefore, is structurally prepared for designating another unknown quantity with another proper name, y, and saying “five times the square of y”, $5y^2$.

The sign system of Indian algebra does have proper names for different unknown quantities (it uses names of colours for this purpose), and it forms algebraic expressions by juxtaposing the name of the unknown quantity and the name of the species (cf. Colebrooke, ed., 1817), but this system did not have any impact on mediaeval Arabic algebra, nor, therefore, on algebra in the Christian West. It was not until Vieta that a sign system was developed in which there were proper names for different quantities, together with the names of the species. But Vieta’s sign system also used letters as proper names, and not just for unknown quantities but also for known quantities. This freed the algebraic expressions from ambiguities and made them capable of providing a direct representation of the quantities analysed in the statements of the problems.

However, we have shown (Puig and Rojano, 2004) that Vieta’s sign system lacks full operational capacity on the syntactic level because the species of numbers are represented by words or abbreviations of them, although these words are constructed algorithmically from certain basic words. We have also shown that this syntactic operativity is attained when one combines the representation of quantities by letters, introduced by Vieta, with the representation of species by means of numbers which indicate the position of the species in the series of species, introduced by Chuquet and Bombelli[18]. The algorithmic rules for the construction of the names of the species can then be replaced by those numbers and converted into part of the calculation.

5. BY WAY OF CONCLUSION

In the “Variables in the Vernacular” section of his phenomenological analysis of the language of algebra, Hans Freudenthal recounts that

> When my daughter was at the age when children play the game “what does this mean?” and I asked her what is “thing” she answered: Thing is if you mean something and you do not know what is its name. (Freudenthal, 1983, p. 474)

Students taught to name the unknown of a word problem with an x, frequently see the x as a common name meaning “unknown” and not a proper name referring to a specific unknown quantity, labelling then any unknown quantity with an x. The meaning they give to x does not correspond to its meaning in the current sign system of algebra, and is not efficient enough to be of use in the Cartesian Method. The x is seen by pupils as “thing” in al-Khwârizmî's sign system, but without taking care to bind it during (a part of) the calculations to a specific unknown quantity like al-Khwârizmî does. However, the only way to use a variable as a proper name is to bind it by using some device, logical or context dependent, like a demonstrative "this thing [means that quantity]", "this *x* [means that quantity]". But this demonstrative binding is not efficient when one has to build algebraic expressions in which different "things" are of different species, and these algebraic expressions have to be manipulated syntactically. Furthermore, syntactic manipulation of algebraic expressions is an automatic, meaningless task for the majority of pupils.

We have sketched that history tells us how both syntactic manipulation of algebraic expressions and the use of the Cartesian Method make sense thanks to the idea of canonical forms and to the development of a calculus that warrants us to solve families of problems or, as stated by Vieta as a conclusion of his *Introduction to the Analytical Art*, "nullum non problema solvere", "to solve every problem" (Van Schooten, ed., p. 12; Witmer, ed., p. 32).

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NOTES

[1] See Filloy (1980) as the earlier presentation of his way of using history in research on mathematics education.

[2] See Filloy, Puig and Rojano (1993) as an early account in Spanish of our joint venture.

[3] Research on the use of spreadsheets in algebra teaching and learning has been recently summarized in Kieran and Yerushalmy (2004).

[4] The canonical edition of Descartes' works is the one by Charles Adam and Paul Tannery, *Œuvres de Descartes*, volume X of which contains the original Latin of the rules. These *Regulæ ad*

directionem ingenii were not published in Descartes' lifetime and appeared in print for the first time in a collection of previously unpublished texts in Holland in 1701, with the title *Opuscula posthuma physica et mathematica*. The first French translation is contained in volume eleven of the 1826 edition by Victor Cousin, *Œuvres de Descartes*.

[5] Although Polya says that this sentence paraphrases four of Descartes's rules, rule XIII really contains all that it paraphrases: "Quand nous comprenons parfaitement une question, il faut la dégager de toute conception superflue, la réduire au plus simple, la subdiviser le plus possible au moyen de l'énumération." (Descartes, 1826, p. 284). Previously (rule VII) Descartes has already stated the importance of "énumération", which he defines as "la recherche attentive et exacte de tout ce qui a rapport à la question proposée. […] cette recherche doit être telle que nous puissions conclure avec certitude que nous n'avons rien mis à tort" (Descartes, 1826, p. 235). Rule XIV speaks of the understanding of "l'étendue réelle des corps" and says that the preceding rule also applies to it. Rules XV and XVI are advice for the mind to pay attention to the essential and for the memory not to weary itself with what may be necessary but does not require the attention of the mind. Rule XV recommends drawing figures to keep the mind attentive: "Souvent il est bon de tracer ces figures, et de les montrer aux sens externes, pour tenir plus facilement notre esprit attentif." (Descartes, 1826, p. 313). Rule XVI recommends not using complete figures, but mere jottings to unburden the memory, when the attention of the mind is not needed: "Quant à ce qui n'exige pas l'attention de l'esprit, quoique nécessaire pour la conclusion, il vaut mieux le designer par de courtes notes que par des figures entières. Par ce moyen la mémoire ne pourra nous faire défaut, et cependant la pensée ne sera pas distraite, pour le retenir, des autres opérations auxquelles elle est occupée." (Descartes, 1826, p. 313)

[6] "Il faut parcourir directement la difficulté proposée, en faisant abstraction de ce que quelques uns de ses termes sont connus et les autres inconnus, et en suivant, par la marche véritable, la mutuelle dépendance des unes et des autres." (Descartes, 1826, p. 319)

[7] "C'est par cette méthode qu'il faut chercher autant de grandeurs exprimées de deux manières différentes que nous supposons connus de termes inconnus, pour parcourir directement la difficulté; car, par ce moyen, nous aurons autant de comparaisons entre deux choses égales." (Descartes, 1826, p. 328)

[8] "S'il y a plusieurs équations de cette espèce, il faudra les réduire toutes à une seule, savoir à celle dont les termes occuperont le plus petit nombre de degrés, dans la série des grandeurs en proportion continue, selon laquelle ces termes eux-mêmes doivent être disposés." (Descartes, 1826, p. 329)

[9] "[…] totum huius loci artificium consistet in eo, quod ignota pro cognitis supponendo possimus facilem & directam quærendi viam nobis proponere, etiam in difficultatibus quantumcumque intricatis." (Descartes, 1701, pp. 61-62)

[10] "Per hanc ratiocinandi methodum quarenda sunt tot magnitudines duobus modis differentibus expressa." (Descartes, 1701, p. 66)

[11] We quote from the facsimile of the first edition, edited and translated by D. E. Smith and M. L. Latham (Descartes, 1925).

[12] Al-Khwârizmî wrote this book in the 9th century. The only extant copy of it was edited and translated into English by Rosen (1831). According to Høyrup (1991), Gerardo de Cremona Latin mediaeval translation, edited by Hughes (1986), was made with a better Arab manuscript, and is therefore more reliable. We have used both Rosen's edition of the Arab text and Hughes's edition of the Latin translation on our English version of the quotes from al-Khwârizmî's text.

[13] The canonical edition of the six extant Greek books of Diophantus' *Arithmetic* is Tannery (1893). There are two editions of the four books only extant in the Arabic translation of Qustâ ibn Lûqâ: Sesiano (1982) and Rashed (1984).

[14] There is a recent edition of the Arabic text of al-Khayyâm's *Treatise on Algebra*, accompanied by a translation into French, in Rashed and Vahebzadeh (1999). One can also consult the English translation by Kasir (1931).

[15] There is an edition by Roshdi Rashed of the Arabic text of Sharaf al-Dîn al-Tûsî's *Treatise on Equations*, accompanied by a translation into French, in al-Tûsî (1986).

[16] Diophantus also has a single name for unknown quantities (*arithmos*). In problem 28 in Book II of his *Arithmetic* (Tannery, 1893, vol. I, pp. 124-127), he resorts to the device of saying that a second

unknown quantity is one unit (monas 1), performing the calculations using this supposition, and then in the result changing the units to *arithmos* and calculating again.

[17] "Census" was the term chosen by Gerardo de Cremona for *mâl*, treasure, in his translation of al-Khwârizmî's book of algebra, and it was the one that caught on in the Christian Mediaeval West (cf. the edition by Hughes, 1986).

[18] This is already present in Chuquet's *Triparty*, written in French in 1484. However, this book by Chuquet remained unpublished and was therefore scarcely known until the end of the nineteenth century, when Aristide Marre published it (Marre, 1880). Bombelli used the same kind of representation in his *Algebra*, from which it became more widely known among algebraists (see Bortolotti's 1966 edition of Bombelli's *Algebra*).