# PUPILS' PROMPTED PRODUCTION OF A MEDIEVAL MATHEMATICAL SIGN SYSTEM

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## Introduction.

I have faced the task of making sense of the protocol that Terezinha provided us in a way I daresay that makes use of Foucault's way of dealing with historical documents. Having meager information on the aims and context of the research where the protocol was produced, I have stuck to the text I had, trying to turn this document into a monument. That means, in Foucault's words, "not the interpretation of the document, nor the attempt to decide whether it is telling the truth or what is its expressive value, but to work on it from within and to develop it" (Foucault, 1969, p. 14). Moreover, I have confronted this text in a dialogue with a text from the 13th century —*De Numeris Datis*, by Jordanus de Nemore—, using the history of mathematical ideas like Filloy does (see, for instance, Filloy, 1990), i. e., like a way to shed light on pupils' productions. I don't claim then to have found the meaning of the protocol, instead I'd rather say that my work on this text has produced new senses that I would like to be fortunate.

By making this statement I'm also merely introducing the semiotic idiom in which I feel more at ease. In it, actually, each new reading of a text takes it as a textual space whose transformation through the act of reading produces a new text together with new senses. A new sense for a sign or a text becomes a new meaning if this sense is fortunate, that is, introduces a new use for the sign or text that get to be shared by a community, entering therefore the encyclopedia[1].

## A problem and several tasks.

Two pairs of pupils were given a problem to solve and several tasks to perform. The first pair had to solve the problem and to give the solution to the interviewer. Once they gave the solution to her, they were asked to write a message explaining their solution to some friends that had to solve a similar problem. They were requested not to use numbers in this written description. Once they produced such a description, as it happened to have been written in plain English, the interviewer gave them another task: to write it anew "in

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[1] See Talens & Company (1984) for a discussion of the terms 'text' and 'textual space', 'meaning' and 'sense' that is in the origin of the use of them I'm making here. See Voloshinov (1992) for the stress on the social character of the process of signification – the original Russian edition dates from 1929 and it seems to have been written at least partially by M. Bakhtin.

mathematics. Using signs like in mathematics". She prompted them also by saying "you can call the total number of books *A*".

The second pair of pupils were given the task to solve the problem using the message sent by the first one. I am not going to analyze this part of the protocol, but only the tasks accomplished by the first pair.

A problem or a task, considered as a text / textual space, conveys meanings as a result of all uses of this problem or task in the history of a community. The movement between meaning and sense that I have sketched accounts for the process of semiosis in general, and the way in which semiotic practices produce an *episteme*. However, when a problem or a task is given to some pupils by a teacher, we have to deal with this process in a teaching-learning situation within a school system. As a consequence it becomes necessary to consider the meanings the problem or task conveys to these pupils as a result of their personal history, that is, the uses they have done of the set of notions involved and the social practices, both discursive and non-discursive, in which they have encountered this kind of problem or task. Moreover, it becomes necessary to consider also the competence model (Filloy, 1990) as the good fortune of the senses produced by pupils heavily hangs on their adequacy to the competence model. In the case of the situation described in the protocol, we can assume that pupils are supposed to become competent in the use of the mathematical sign system of school algebra, specially in its use when solving arithmetic problems of several combined operations.

## Mathematical sign systems in the school system.

It's not unusual that a description of the language in which mathematical texts are written distinguishes between two subsets of signs. One subset is seen as containing the signs that are considered really mathematical —often qualified as "artificial"—, and to the other belong the "natural" signs from some vernacular tongue. Even if this distinction can be done, it leads usually to focus the study of mathematical sing systems on the study of mathematical signs, and their teaching on the teaching of the use of these signs. It should not be then a surprise if pupils end up identifying mathematics with the use of letters.

I'd rather like to consider instead that what is qualified as mathematical in the expression 'mathematical sign systems' are the systems and not the signs, i. e., that mathematical sign systems means mathematical systems of signs and not systems of mathematical signs. Thus, for instance, Freudenthal shown in the chapter *The Algebraic Language* of his *Didactical Phenomenology* how the usual expression in German arithmetic classrooms 'sieben minus vier' is not "normal German" and that there are similar expressions in Dutch that are not "normal Dutch" either (Freudenthal, 1983, pp. 486-487). This is the case also in Spanish, and is more conspicuous in not Indo-European languages, of which we have one in Spain - the Basque language. A mathematical sign system for arithmetic can have segments with the same matter of expression that a vernacular tongue, but even when the very words of this tongue are used, grammar, semantics and pragmatics are not exactly the same as those of it.

Mathematical sign systems do have of course segments whose signs don't belong to any vernacular tongue, but mainly or only to mathematical sign systems; what I want to stress is that we should not identify mathematical language with these signs, since we are not interested in analyzing mathematical texts that only exist in Bourbaki's heaven —they not even exist in Bourbaki's books—, but actual mathematical texts. Furthermore, we are specially interested in analyzing mathematical texts that are produced by pupils in the school system while they are taught mathematics, and, as Filloy pointed out, we need a notion of mathematical sign system wider enough to account for this kind of texts.

From this point of view, I have discuss in Puig (1994a) some characteristics of mathematical sign systems that are worth considering here, briefly stated:

1) Mathematical texts are produced by means of stratified mathematical sign systems whose matter of the expression is heterogeneous.

2) The heterogeneity of the matter of the expression is shown by the existence in mathematical texts of segments that, when seen isolated, seem to have been produced by different languages. Nevertheless, these segments are not ruled separately by the rules of those languages, but by new rules that result from a specific combination of them.

3) There are pointers that refer mutually signs from segments of different matter of expression.

4) Both in the history of mathematics and in the history of individuals, mathematical sign systems are the product of a process of progressive abstraction. As a consequence, those that are actually used are stratified. The strata come from different moments of the process of abstraction, and are related among them by the correspondences established by this process.

5) The autonomy of the transformations of the expression from the content plays an important role in the process of abstraction that leads to the production of a new mathematical sign system.

## A medieval mathematical sign system.

The earlier known mathematical texts that use letters to stand for quantities are those written by Jordanus de Nemore in the 13th century, namely *De Numeris Datis* and *De Elementis Arithmetice Artis*. The critical edition of *De Numeris Datis* has been published by Hughes (1981), who gives 1225 as the more likely date of publication.

*De Numeris Datis* is written in Latin and is straightforwardly organized: three definitions at the beginning and 115 propositions distributed in four books, without any explanation of the aims of the book, neither an introduction nor transitions between books.

The propositions are presented always in the same three parts:

1) A statement asserting that if some numbers (or ratios) have been given, along with some relations between them, then some other numbers (or ratios) have also been given.

2) A series of transformations of the numbers (or ratios) and the relations that either show that the numbers have indeed been given or convert them into the numbers and relations of the hypothesis of some previous proposition.

3) The calculation of an example with concrete numbers.

Like in Diofanto's *Arithmetic*, the statement does not involve concrete numbers. Unlike in Diofanto's *Arithmetic*, the argument does not involve either concrete numbers, moreover the quantities mentioned are often represented with letters. This use of letters in the arguments of the propositions is the main reason why *De Numeris Datis* has been considered as the first medieval advanced algebra. However, neither Jordanus de Nemore uses letters always in *De Numeris Datis* —but only in two thirds of the arguments—, nor he uses them only in his so called advanced algebra treatise —but also in his elementary arithmetic, *De Elementis Arismetice Artis*, whose Latin text has been recently edited by Busard, with a paraphrase in English (Busard, 1991).

I have reported an analysis of the mathematical sign system of *De Numeris Datis* in Puig (1994b). I will present here two propositions from the beginning of the book and a summary of the part of my account of the characteristics of its mathematical sign system that is relevant to the discussion of the protocol we are analyzing here.

The statement of I-1 is "Si numerus datus in duo dividatur quorum differentia data, erit utrumque eorum datum", that can be translated[2] as follows: "If a number that has been given is divided in two parts whose difference has been given, then each of the parts has been given".

The argument is: "Since the lesser part and the difference equal the larger, the lesser with another equal to itself together with the difference make the given number. Subtracting therefore the difference from the total, what remains is twice the lesser. Halving this yields the smaller and, consequently, the greater part".

The statement of I-3 is "If a number that has been given is divided in two parts whose product has been given, then each of the parts has been given".

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[2] The translation of Hughes is more liberal than mine, but even if the literality gives a coarse English, his liberality makes impossible to see some of the characteristic of the original text that are essential to its description.

The argument to prove it can be schematized, preserving Jordanus de Nemore use of letters, as follows:

"Let *abc* be the number that has been given, divided in *ab* and *c*[3].

*ab* by *c* makes *d*, given.

*abc* by himself makes *e*.

Let the quadruple of *d* be *f*.

Taking *f* from *e* remains *g*.

*g* is the square of the difference between *ab* and *c*.

The square root of *g* is *b*.

And *b* is the difference between *ab* and *c*.

Since *b* has been given, *c* and *ab* have been given".

What follows is the relevant part of my account, that is grounded on the analysis reported in Puig (1994b).

The quantities that appear in the arguments are named sometimes, like in proposition I.1, using its meaning by reference to an initial number divided in two parts (the lesser part, the larger part, etc.) and sometimes, like in proposition I.3, with a letter (or some letters put together).

Whenever letters are used, all quantities, both known and unknown, are represented by letters; the letters are marks to denote the quantities that are built in the course of the argument and appear in alphabetical order, without any distinction between known and unknown quantities.

Besides, a quantity can be denoted by more than one letter. Each letter does not represent then a number, but the instance of appearance of a number in the course of the argument.

There is a lack of syntactic operativity, excepting for juxtaposition to mean addition; thus, when a new quantity is built using quantities already denoted by letters with an operation different from addition, the only way to denote the new quantity is to introduce a new letter to do it, there is no way of using the letters denoting the quantities involved (i. e., to denote the product of *a* by itself the *a* is useless: it is necessary to introduce a *b*).

Moreover, the relations between quantities can not be produced on the expression level of the sign system, they have to be produced on the content level instead.

[3] Hughes translates "Let the given number *a* be separated into *x* and *y*", making impossible to see in its translation that Jordanus de Nemore does not distinguish between known and unknown quantities.

## The problem and pupils' solution.

The problem posed to the pupils corresponds to the first problem in Diofanto's *Arithmetic* and to proposition I.1 of *De Numeris Datis*. They are not however the same. The problem posed to the pupils tells a story about classes and books to share and the quantities involved are concrete quantities, namely "numbers of books". Diofanto's first problem tells a story about pure numbers and arithmetic operations on them. Proposition I.1 of *De Numeris Datis* is actually a theorem – but a theorem on the possibility of solution of a problem (or a class of problems) and contains then its solution. Moreover, one of the quantities involved is expressed in Diofanto and Jordanus de Nemore as a "difference", while in the problem posed to the pupils is expressed by means of the comparative "more than".

Nevertheless, the page with the computations shows that the pupils gave the same solution that Jordanus de Nemore includes in the argument "Subtracting therefore the difference from the total, what remains is twice the lesser. Halving this yields the smaller and, consequently, the greater part".

## The task of writing a message without numbers.

To write the solution without numbers they have to name the quantities involved in the operations that lead from the data to the unknowns, to express the operations and to construct sentences that link the appropriate quantities and operations in due order. In a word problem of several combined operations, the data and the unknowns have names in the statement of the problem, but the auxiliary quantities can be mentioned in the statement or not.

Jane begins by writing the name of the quantity that correspond to the first number they have used in the solution ("Get the number of books and…") and expresses the subtraction by writing "take away". Then Ann and Jane look for an expression for the quantity taked away, ("Take away what?"). They try "the number that one class had more" (Ann) and "the amount that is bigger" (Jane) till Ann writes the complex expression "the amount of how many more books one class had". Both quantities are data named in the statement of the problem, and the names that they have given them are only slight modifications of those that appear in the statement.

The next quantity they have to name is an auxiliary one that is not mentioned in the statement. Jane simply calls it "the amount" writing "split the amount into half and add the number of how many more", completing the sentence after an interchange with Ann: "books one class had".

They write also a final sentence that corresponds to the checking of the correctness of the answers.

Jane and Ann do not seem to have got into much trouble in writing their solution without numbers – nor letters. Jordanus de Nemore did not use letters either, although he did know to use them.

**The task of writing a message with letters.**

The nature of this task for the pupils is very different from the previous one. Pupils are requested to write a text with a mathematical sign system that is new for them. They have to produce a new mathematical sign system at the same time that they are translating the text they have written to the mathematical sign system they are producing.

The interviewer introduces the substitution of the names of the quantities (that already stand for the numbers in the first text of the solution written by the pupils) by letters, saying that it is “like a code” (i. e., each different number or quantity, a different letter).

After some questioning, Jane seems to understand (“Oh, I get it”) and writes the exact words said by the interviewer (“Call the number of books they have *A*”), but she does not know what to do next.

The interviewer goes on introducing the next quantity, but she changes its name (“the difference”, a pure arithmetic name, instead of “the amount of how many more books one class had”). Jane complete the name of the quantity (“the difference between the classes” referring to its meaning in the context of the story) and ask if she has to call it *B* (“What? *B*?”). The interviewer agrees opening up the way to other letters in alphabetical order. Jane writes “call the difference between the two classes *B*”, and Ann writes the first sentence to express an operation (“Get *A* and take away *B*”).

The crucial step is given by Jane. She begins saying “Split *B*”, making the next operation to the last mentioned letter in a kind of sequence. But as Ann wants to split *A* instead, and explain to her questioning “*A*? Take away *B and* split” (my stress), she interprets ‘and’ as ‘and then’, and produces an unexpected mathematical sign system: “call *A* take away $B = C$”.

Ann has not yet understood the rules of the new game and says “Split in half”. Jane corrects her and states clearly “Split *C* in half”.

The new game is definitively established when some items later Jane writes “and add *B*”, Ann asks “To what”, and as Ann does not agree with the answer of Jane (“To *C*”), Jane solves the problem by saying the next letter (“To *D*”) and looking afterwards for its meaning (“Get the second answer. Call the second answer. Half of *C*.” “$C$:2=$D$”).

A medieval mathematical sign system has been produced by Jane. We know from the analysis of *De Numeris Datis* its differences with the mathematical sign system of school algebra. I wonder whether these differences are only a step to it or whether its lack of operativity can be a serious obstacle.

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